ขอบเขตแบบไม่สม่ำเสมอในการประมาณด้วยการแจกแจงปกติสำหรับค่าสถิติสหสัมพันธ์เมทริกซ์
และตัวแปรสุ่มที่มีขอบเขตและเป็นอิสระต่อกัน

นางสาวณัฐกาญจน์ ใจดี

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NON-UNIFORM BOUNDS IN NORMAL APPROXIMATION FOR MATRIX CORRELATION STATISTICS AND INDEPENDENT BOUNDED RANDOM VARIABLES

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วิทยานิพนธ์ฉบับนี้ประกอบด้วยสองส่วนหลัก

ในส่วนแรก ให้ (a_{ij}) และ (b_{ij}) เป็นเมทริกซ์ขนาด $n \times n$ ของจำนวนจริง และ π เป็นการเรียงสับเปลี่ยนเชิงสุ่มบน $\{1, 2, ..., n\}$

ให้

$$V = \frac{\sum_i a_{i\pi(i)} - E(\sum_i a_{i\pi(i)})}{\sqrt{Var(\sum_i a_{i\pi(i)})}} \quad \text{และ} \quad W = \sum_{\substack{i,j \\ i \neq j}} a_{ij} b_{\pi(i)\pi(j)}$$

ภายใต้เงื่อนไขการมีขอบเขต เราให้ขอบเขตแบบไม่สม่ำเสมอในการประมาณด้วยการแจกแจงปกติสำหรับ V และ W โดยใช้วิธีอสมการความเข้มข้นของวิธีการของสไตน์

ในส่วนที่สอง ให้ $X_1, X_2, ..., X_n$ เป็นตัวแปรสุ่มที่มีขอบเขตและเป็นอิสระต่อกัน เราให้ขอบเขตแบบไม่สม่ำเสมอในการประมาณด้วยการแจกแจงปกติสำหรับ $X_1 + X_2 + \cdots + X_n$ โดยใช้วิธีการของสไตน์โดยไม่ใช้อสมการความเข้มข้น

ภาควิชา ...**คณิตศาสตร์**...	ลายมือชื่อนิสิต..
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This thesis contains two main parts.

In the first part, let (a_{ij}) and (b_{ij}) be $n \times n$ matrices of real numbers and π a random permutation of $\{1, 2, \ldots, n\}$. Let

$$V = \frac{\sum\limits_i a_{i\pi(i)} - E(\sum\limits_i a_{i\pi(i)})}{\sqrt{\mathrm{Var}(\sum\limits_i a_{i\pi(i)})}} \quad \text{and} \quad W = \sum_{\substack{i,j \\ i \neq j}} a_{ij} b_{\pi(i)\pi(j)}.$$

Under a boundedness condition, we establish non-uniform bounds in normal approximation for V and W by using a concentration inequality approach of Stein's method.

In the second part, let $X_1, X_2, \ldots, X_n$ be independent bounded random variables. We give a non-uniform bound in normal approximation for $X_1 + X_2 + \cdots + X_n$ by using Stein's method without using the concentration inequality approach.

Department	...**Mathematics**...	Student's signature............................	
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CONTENTS

CHAPTER I
INTRODUCTION

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be pairs of n sample values and assign to each pair (x_i, x_j) a score a_{ij} and to each (y_i, y_j) a score b_{ij}. In the first part of this work, we investigate the statistics

$$W := \sum_{\substack{i,j \\ i \neq j}} a_{ij} b_{\pi(i)\pi(j)}$$

where π is a random permutation of $\{1, 2, \ldots, n\}$. This W is always called "matrix correlation statistics"and first introduced by Daniels([1]) which has many applications in geography and epidemiology.

In 2005, Barbour and Chen ([2]) established a uniform bound in normal approximation for W when $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ symmetric matrices. Their argument is based on a concentration inequality approach of Stein's method and is in the spirit of Chen and Shao([3]).

Let

$$A_0 := \frac{1}{n(n-1)} \sum_{\substack{i,j \\ i \neq j}} a_{ij},$$

$$A_{12} := \frac{1}{n} \sum_i \{a_i^*\}^2,$$

$$A_{22} := \frac{1}{n(n-1)} \sum_{\substack{i,j \\ i \neq j}} \widetilde{a}_{ij}^2,$$

$$A_{13} := \frac{1}{n} \sum_i |a_i^*|^3,$$

where

$$a_i^* := \frac{1}{n-2} \sum_{\substack{j \\ j \neq i}} (a_{ij} - A_0)$$

and

$$\widetilde{a}_{ij} := a_{ij} - a_i^* - a_j^* - A_0,$$

and make similar definitions for B.

Let Φ be the standard normal distribution function. The following is their result.

Theorem 1.1. *For* $\mu = \frac{1}{n(n-1)} \sum_{\substack{i,j \\ i \neq j}} \sum_{\substack{l,m \\ l \neq m}} a_{ij} b_{lm}$ *and* $\sigma^2 = \frac{4n^2(n-2)^2}{n-1} A_{12}B_{12}$*, there exists a constant* C *such that*

$$\sup_{z \in \mathbb{R}} \Big| P(\frac{W-\mu}{\sigma} \leq z) - \Phi(z) \Big| \leq C(\delta + \delta^2 + \delta_2)$$

where $\delta = 128\delta_1, \quad \delta_1 = n^4 \sigma^{-3} A_{13}B_{13}$ *and* $\delta_2^2 = \frac{(n-1)^3}{2n(n-2)^2(n-3)} \cdot \frac{A_{22}B_{22}}{A_{12}B_{12}}$.

In the first part, we give a non-uniform bound in normal approximation for W in case A and B are symmetric. To do this, we need a non-uniform bound on combinatorial central limit theorem which is stated in Theorem 1.2.

Theorem 1.2. *If* $\frac{1}{\sqrt{\mathrm{Var}(\sum_i a_{i\pi(i)})}} \sup_{i,j} \Big| a_{ij} - \frac{E(\sum_i a_{i\pi(i)})}{n} \Big| \leq \frac{K}{\sqrt{n}}$ *for some positive real number* K*, then there exists a constant* C *such that for every real number* z*,*

$$\Big| P\Big(\frac{\sum_i a_{i\pi(i)} - E(\sum_i a_{i\pi(i)})}{\sqrt{\mathrm{Var}(\sum_i a_{i\pi(i)})}} \leq z \Big) - \Phi(z) \Big| \leq \frac{CK}{(1+|z|)^3\sqrt{n}}.$$

The following are our main results of a non-uniform bound for matrix correlation statistics.

Theorem 1.3. *There exists a constant* C *such that for every real number* z*,*

$$\begin{aligned} &\Big| P\Big(\frac{W-\mu}{\sigma} \leq z\Big) - P(V(\pi) \leq z) \Big| \\ &\leq \frac{C}{1+|z|^3} \Big\{ \delta + \delta^2 + \delta_2 + \delta\delta_2 + \delta(n^{\frac{1}{2}}\delta)^2 + \delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}} + \delta_2(n^{\frac{1}{2}}\delta) + \delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}} + \delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}} \Big\} \\ &\quad + \frac{C\delta_2^2}{1+z^2} \end{aligned}$$

where $V(\pi) := \frac{2(n-2)}{\sigma} \sum_i a_i^* b_{\pi(i)}^*$.

In particular, if $\frac{2(n-2)}{\sigma} \sup_{i,j} |a_i^* b_j^*| \leq \frac{K}{\sqrt{n}}$ *for some positive real number* K*, then*

$$\begin{aligned} &\Big| P\Big(\frac{W-\mu}{\sigma} \leq z\Big) - \Phi(z) \Big| \\ &\leq \frac{CK}{1+|z|^3} \Big\{ \delta + \delta^2 + \delta_2 + \delta\delta_2 + \delta(n^{\frac{1}{2}}\delta)^2 + \delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}} + \delta_2(n^{\frac{1}{2}}\delta) + \delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}} \\ &\quad + \delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}} + \frac{1}{\sqrt{n}} \Big\} + \frac{C\delta_2^2}{1+z^2}. \end{aligned}$$

Corollary 1.4. *If $\delta_1 \sim n^{-1/2}$ and $\delta_2 \sim n^{-1/2}$, then there exists a constant C such that for every real number z,*

$$\Big|P\Big(\frac{W-\mu}{\sigma} \le z\Big) - P(V(\pi) \le z)\Big| \le \frac{C}{(1+z^2)\sqrt{n}}.$$

Furthermore, if $\frac{2(n-2)}{\sigma}\sup_{i,j}|a_i^* b_j^*| \le \frac{K}{\sqrt{n}}$ for some positive real number K, then

$$\Big|P\Big(\frac{W-\mu}{\sigma} \le z\Big) - \Phi(z)\Big| \le \frac{CK}{(1+z^2)\sqrt{n}}.$$

To simplify the theorem, we let $a_{ij} \sim \frac{1}{n^p}$ and $b_{ij} \sim \frac{1}{n^q}$ for every $i, j \in \{1, 2, ..., n\}$. Then we have

$$\begin{aligned}
&a_i^* \sim \frac{1}{n^p}, \quad A_{12} \sim \frac{1}{n^{2p}}, \quad A_{13} \sim \frac{1}{n^{3p}}, \quad A_{22} \sim \frac{1}{n^{2p}}, \\
&b_j^* \sim \frac{1}{n^p}, \quad B_{12} \sim \frac{1}{n^{2q}}, \quad B_{13} \sim \frac{1}{n^{3q}}, \quad B_{22} \sim \frac{1}{n^{2q}} \quad \text{and} \quad \sigma^2 \sim \frac{n^3}{n^{2(p+q)}}
\end{aligned}$$

which implies

$$\frac{2(n-2)}{\sigma}\sup_{i,j}|a_i^* b_j^*| \le \frac{K}{\sqrt{n}}, \quad \delta_1 \sim n^{-1/2} \quad \text{and} \quad \delta_2 \sim n^{-1/2}$$

so by Corollary 1.4, we have

$$\Big|P\Big(\frac{W-\mu}{\sigma} \le z\Big) - \Phi(z)\Big| \le \frac{CK}{(1+z^2)\sqrt{n}}.$$

For the second part, we give a non-uniform bound for sum of independent bounded random variables by Stein's method without using the concentration inequality approach.

Let $X_1, X_2, \ldots, X_n$ be independent and not necessarily identically distributed random variables with zero means and finite variances.

Define

$$W = \sum_{i=1}^{n} X_i$$

and assume that $\mathrm{Var}(W) = 1$.

If $E|X_i|^3 < \infty$, then we have the uniform Berry-Esseen theorem

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le C_0 \sum_{i=1}^{n} E|X_i|^3, \tag{1.1}$$

and the non-uniform Berry-Esseen theorem,

$$|P(W \le z) - \Phi(z)| \le \frac{C_1}{1+|z|^3}\sum_{i=1}^{n} E|X_i|^3, \tag{1.2}$$

where both C_0 and C_1 are absolute constants.

Note that if X_i's are identically distributed, (1.1) and (1.2) were first obtained by Esseen([4]) and Nagaev([5]) respectively. Bikelis([6]) generalized Nagaev's result to the case that X_i's are not necessarily identically distributed random variables. The best constant $C_0 = 0.7975$ is done by Van Beeck([7]). In case of non-uniform bound, Paditz([8], [9]) calculated C_1 to be 114.7 and 32 in 1977 and 1989, respectively.

The standard tool used in Esseen([4]), Nagaev([5]), Bikelis([6]) and Paditz([8], [9]) is the Fourier analytic method. In 1972, Stein introduced a powerful and general method which is free from Fourier method and relied instead on the elementary differential equation. The technique used was novel. Stein's ideas have been applied with much success in the area of normal approximation (See, for examples, Bolthausen([10]) and Chen and Shao([11])).

Chen and Shao([11]) gave a non-uniform Berry-Esseen bound without assuming the existence of third moments. Their argument is based on a concentration inequality approach of Stein's method. In particular, if the random variables are bounded, they simplify the proof of uniform bound without using the concentration inequality approach(see Chen and Shao([12])). Their theorem is as follows.

Theorem 1.5. *Assume that* $|X_i| \le \delta$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le 3.3\delta.$$

In this work, we use the idea of Chen and Shao([12]) to prove a non-uniform Berry-Esseen bound for independent bounded random variables without using the concentration inequality approach. The followings are our main results.

Theorem 1.6. *Assume that* $|X_i| \le \delta$ *for* $i = 1, 2, \ldots, n$. *Then there exists a constant* C_δ *which depends on* δ *such that for every real number* z,

$$|P(W \le z) - \Phi(z)| \le C_\delta e^{-\frac{|z|}{2}}\delta$$

where $C_\delta = 4.45 + 2.21e^{2\delta + (\delta^{-2}(e^{2\delta}-1-2\delta))}$.

Theorem 1.7. *Assume that* $|X_i| \leq \delta$ *for* $i = 1, 2, \ldots, n$. *Then there exists a constant* C *which does not depend on* δ *such that for every real number* z,

$$|P(W \leq z) - \Phi(z)| \leq \frac{C\delta}{1 + |z|^3}.$$

Observe from Theorem 1.6 that if $\delta \to 0$ we have $C_\delta \to 20.78$. This result and the fact that $\lim\limits_{z\to\infty} \frac{1 + z^3}{e^{\frac{z}{2}}} = 0$ lead us to conclude that the non-uniform bound in Theorem 1.6 is better than the result of Paditz([9]).

To illustrate the results, we give an example of non-uniform bound in normal approximation of the binary expansion of a random integer.

Let $n \geq 2$ and X be a random variable uniformly distributed over $\{0, 1, \ldots, n-1\}$.

Let k be such that $2^{k-1} < n \leq 2^k$. Write the binary expansion of X

$$X = \sum_{i=1}^{k} X_i 2^{k-i}$$

and let $S = X_1 + X_2 + \cdots + X_k$ be the number of ones in the binary expansion of X.

If $n = 2^k$, by Theorem 1.6 and Theorem 1.7, then for every real number z,

$$\Big| P\Big(\frac{S - (k/2)}{\sqrt{k/4}} \leq z \Big) - \Phi(z) \Big| \leq \frac{C_k e^{-\frac{|z|}{2}}}{k^{1/2}}$$

and

$$\Big| P\Big(\frac{S - (k/2)}{\sqrt{k/4}} \leq z \Big) - \Phi(z) \Big| \leq \frac{C}{(1 + |z|^3) k^{1/2}}$$

where $C_k = 4.45 + 2.21 e^{\frac{2}{\sqrt{k}} + (k(e^{\frac{2}{\sqrt{k}}} - 1 - \frac{2}{\sqrt{k}}))}$.

This thesis is organized as follows. Preliminaries are in Chapter 2. A non-uniform bound in a combinatorial central limit theorem is in Chapter 3 while non-uniform bounds in normal approximation for matrix correlation statistics and sum of independent bounded random variables are considered in Chapter 4 and Chapter 5, respectively.

CHAPTER II
PRELIMINARIES

In this chapter, we review some basic knowledges in probability and the idea of Stein's method and its approximation.

2.1 Basic Knowledge in Probability

In this section, we give some basic knowledges in probability which will be used in our work. The proof is omited but can be found in [13], [14] and [15].

A **probability space** is a measure space $(\Omega, \mathcal{F}, P)$ for which $P(\Omega) = 1$. The measure P is called a **probability measure**. The set Ω will be referred to as a **sample space** and its elements are called **points** or **elementary events**. The elements of $\mathcal{F}$ are called **events**. For any event A, the value $P(A)$ is called the **probability of** A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if for every Borel set B in $\mathbb{R}$, $X^{-1}(B)$ belongs to $\mathcal{F}$. We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

A random variable X with the distribution function F is said to be a **discrete random variable** if the image of X is countable and it is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability function** of X.

Now we will give some examples of random variables.

We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2 \Big).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

We say that X is a **uniform** random variable with parameter n if there exist $x_1, x_2, \dots, x_n$ such that $P(X = x_i) = \dfrac{1}{n}$ for any $i = 1, 2, \dots, n$ and denoted by $X \sim U(n)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_\alpha$ is a sub σ-algebra of $\mathcal{F}$ for each $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J = \{1, 2, \dots, k\}$ of Λ,

$$P(\bigcap_{m=1}^{k} A_m) = \prod_{m=1}^{k} P(A_m)$$

where $A_m \in \mathcal{F}_m$ for $m = 1, 2, \dots, k$.

Let $\mathcal{E}_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{E}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\mathcal{E}_\alpha) | \alpha \in \Lambda\}$ is independent where $\sigma(\mathcal{E}_\alpha)$ is the smallest σ-algebra with $\mathcal{E}_\alpha \subseteq \sigma(\mathcal{E}_\alpha)$.

We say that the set of random variables $\{X_\alpha |\ \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha)|\ \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B \text{ is a Borel subset of } \mathbb{R}\}$.

Theorem 2.1. *Random variables $X_1, X_2, \dots, X_n$ are* **independent** *if for any Borel sets $B_1, B_2, \dots, B_n$, we have*

$$P(\bigcap_{i=1}^{n} \{X_i \in B_i\}) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2. *If X_{ij} ; $i = 1, 2, \dots, n,\ j = 1, 2, \dots, m_i$ are independent and $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ are measurable, then $\{f_i(X_{i1}, X_{i2}, \dots, X_{im_i}),\ i = 1, 2, \dots, n\}$ is independent.*

Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$. If $\displaystyle\int_\Omega |X| dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.3.

1. *If X is a discrete random variable, then $E(X) = \displaystyle\sum_{x \in \operatorname{Im} X} xP(X = x)$.*

2. If X is a continuous random variable with probability function f, then

$$E(X) = \int_{\mathbb{R}} xf(x)dx.$$

Proposition 2.4. *Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$ and $a, b \in \mathbb{R}$. Then we have the followings:*

1. *$E(aX + bY) = aE(X) + bE(Y)$.*
2. *If $X \leq Y$, then $E(X) \leq E(Y)$.*
3. *$|E(X)| \leq E(|X|)$.*

Let X be a random variable which $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the ***k*-th moment** of X about the origin and call $E[(X - E(X))^k]$ the ***k*-th moment** of X about the mean.

We call the second moment of X about the mean, the **variance** of X, denoted by $\mathrm{Var}(X)$. Then

$$\mathrm{Var}(X) = E[X - E(X)]^2.$$

We note that

1. $\mathrm{Var}(X) = E(X^2) - E^2(X)$.
2. If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

Proposition 2.5. *If $X_1, X_2, \ldots, X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, \ldots, n$, then*

1. *$E(X_1X_2 \cdots X_n) = E(X_1)E(X_2) \cdots E(X_n)$,*
2. $\mathrm{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2 \mathrm{Var}(X_1) + a_2^2 \mathrm{Var}(X_2) + \cdots + a_n^2 \mathrm{Var}(X_n)$ *for any real number $a_1, a_2, \ldots, a_n$.*

The following inequalities are useful in our work.

1. **Hölder's inequality** :

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p)E^{\frac{1}{q}}(|Y|^q)$$

where $1 \leq p, q \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $E(|X|^p) < \infty$, $E(|Y|^q) < \infty$.

2. **Chebyshev's inequality** :

$$P(\{|X - E(X)| \geq \varepsilon\}) \leq \frac{\text{Var}(X)}{\varepsilon^2} \quad \text{for all} \quad \varepsilon > 0$$

where $E(X^2) < \infty$.

3. **Rosenthal's inequality** : If $X_1, X_2, \ldots, X_n$ are independent random variables such that $EX_i = 0$, then for $p \geq 2$,

$$E|\sum_{i=1}^{n} X_i|^p \leq C(p)\big(\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}\big)$$

where $C(p)$ is a positive constant depending only on p.

Let X be a finite expected value random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0,1]$ by

$$P_{\mathcal{D}}(E) = P(E)$$

and a sign-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP.$$

Then, by Radon-Nikodym theorem we have $\mathcal{Q}_X \ll P_{\mathcal{D}}$ and there exists a unique measurable function $E^{\mathcal{D}}(X)$ on $(\Omega, \mathcal{F}, P)$ such that

$$\int_E E^{\mathcal{D}}(X) dP_{\mathcal{D}} = \mathcal{Q}_X(E) = \int_E X dP \quad \text{for any } E \in \mathcal{D}.$$

We call $E^{\mathcal{D}}(X)$ the **conditional expectation** of X with respect to $\mathcal{D}$.

Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$ by $E^Y(X)$.

Theorem 2.6. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.*

1. *If X is random variable on $(\Omega, \mathcal{D}, P_{\mathcal{D}})$, then $E^{\mathcal{D}}(X) = X$ a.s.$[P_{\mathcal{D}}]$.*

2. *$E^{\mathcal{F}}(X) = X$ a.s.$[P]$.*

3. *If $\sigma(X)$ and $\mathcal{D}$ are independent, then $E^{\mathcal{D}}(X) = E(X)$ a.s.$[P_{\mathcal{D}}]$.*

Theorem 2.7. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|)$ and $E(|Y|)$ are finite. Then for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$ the followings hold.*

1. *If $X \leq Y$, then $E^{\mathcal{D}}(X) \leq E^{\mathcal{D}}(Y)$ a.s. $[P_{\mathcal{D}}]$.*

2. *$E^{\mathcal{D}}(aX + bY) = aE^{\mathcal{D}}(X) + bE^{\mathcal{D}}(X)$ a.s. $[P_{\mathcal{D}}]$ for any $a, b \in \mathbb{R}$.*

Theorem 2.8. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|XY|)$ and $E(|Y|)$ are finite and $\mathcal{D}_1, \mathcal{D}_2$ be sub σ-algebras of $\mathcal{F}$. If X is a random variable with respect to $\mathcal{D}_1$, then*

1. *$E^{\mathcal{D}_1}(XY) = XE^{\mathcal{D}_1}(Y)$ a.s. $[P_{\mathcal{D}_1}]$.*

2. *$E^{\mathcal{D}_2}(XY) = E^{\mathcal{D}_2}(XE^{\mathcal{D}_1}(Y))$ a.s. $[P_{\mathcal{D}_2}]$.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. For any event A on $\mathcal{F}$, we define the **conditional probability of A given $\mathcal{D}$** by

$$P(A|\mathcal{D}) = E^{\mathcal{D}}(1_A)$$

where 1_A is defined by

$$1_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A. \end{cases}$$

2.2 Stein's Method and Distributional Approximations

A common theme in probability theory is the approximation of complicated probability distributions by simpler ones. Stein([16]) introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Stein's technique is free of Fourier methods and relied instead on the elementary differential equation. Stein's method has been applied with much success in the area of normal approximation(See, for examples, Erickson([17]), Bolthausen([10]), Baldi, Rinott, and Stein([18]), and Barbour([19])). This method was extended from the normal distribution to the Poisson distribution by Chen([20]). Chen's work has resulted in advances in the theory of Poisson approximation and has helped to develop and improve upon a body of interesting applications and examples for theoretical developments, see Barbour and Eagleson([21]), Holst and Janson([22]).

In this section, we give two examples which use Stein's method for distributional appproximation.

2.2.1 Stein's Method for Normal Approximation

Let Z be a standard normally distributed random variable, and let $\mathcal{C}_{bd}$ be the set of continuous and piecewise continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$. Stein's method rests on the following characterization.

Lemma 2.9. *Let W be a real valued random variables. Then W has a standard normal distribution if and only if*

$$Ef'(W) = EWf(W)$$

for all $f \in \mathcal{C}_{bd}$.

For a real valued measurable function h with $E|N(h)| < \infty$,

$$f'(w) - wf(w) = h(w) - N(h), \quad w \in \mathbb{R} \tag{2.1}$$

is Stein equation for normal distribution where

$$N(h) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(x) e^{-x^2/2} dx.$$

If $h_{w_0} = 1_{(-\infty, w_0]}$, then the solution $f = f_{w_0}$ is given by

$$f_{w_0}(w) = \begin{cases} \sqrt{2\pi} e^{\frac{w^2}{2}} \Phi(w)[1 - \Phi(w_0)] & \text{if } \ w \leq w_0 \\ \sqrt{2\pi} e^{\frac{w^2}{2}} \Phi(w_0)[1 - \Phi(w)] & \text{if } \ w \geq w_0. \end{cases} \tag{2.2}$$

In [23], Stein used the exchangeable pairs and antisymmetric function to find a bound for normal approximation.

Let W be a random variable that is not necessarily the partial sum of independent random variables. Suppose that W is approximately normal, and that we want to find how accurate the approximation is. Another basic approach to Stein's method is to introduce a second random variable W' on the same probability space, in such a way that (W, W') is **an exchangeable pair**; that is

$$(W, W') \quad \text{and} \quad (W', W)$$

have the same distribution. The approach makes essential use of the following proposition.

Proposition 2.10. ([23], p.10) *Let (X,Y) be an exchangeable pair. Then, for all measurable functions $F:\mathbb{R}^2\to\mathbb{R}$ which is antisymmetric in the sense that, for all $x,x'\in\mathbb{R}$*

$$F(x,x')=-F(x',x),$$

we have

$$EE^X F(X,Y)=0,$$

provided that

$$E|F(X,Y)|<\infty.$$

Stein's result is as follows.

Theorem 2.11. *Let X_i be independent random variables with zero means and $\sum_{i=1}^{n} EX_i^2 = 1$, and put $W=\sum_{i=1}^{n} X_i$. Let $\{\widetilde{X}_i\,|\,1\le i\le n\}$ be an independent copy of $\{X_i\,|\,1\le 1\le n\}$, and let I have uniform distribution on $\{1,2,\ldots,n\}$, independent of $\{X_i\}$ and $\{\widetilde{X}_i\}$. Define $W'=W-X_I+\widetilde{X}_I$. Then*

$$|Eh(W)-EN(h)|\le |Ef_h'(W)(1-\frac{n}{2}(W-W')^2)|+\frac{n}{2}\|h'\|E|W-W'|^3$$

where h is a real valued measurable function with $E|N(h)|<\infty$ and f_h is the stein solution of (2.1).

2.2.2 Stein's Method for Cauchy Approximation

In this section, we give the second example that use Stein's method to give bounds. In this example, the limit distribution function is Cauchy and the random variables X_i's are need not independent.

At the heart of Stein's method lies a Stein equation. For a real valued measurable function h with $E|Cau(h)|<\infty$,

$$f'(w)-\frac{2wf(w)}{1+w^2}=h(w)-Cau(h),\quad w\in\mathbb{R} \tag{2.3}$$

is Stein equation for Cauchy distribution where

$$Cau(h)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{h(x)}{1+x^2}dx.$$

If $h_{w_0} = 1_{(-\infty,w_0]}$, then the solution of Stein equation for Cauchy distribution is

$$f_{w_0}(w) = \begin{cases} \pi(1+w^2)F(w)(1-F(w_0)) & \text{if } w \le w_0 \\ \pi(1+w^2)F(w_0)(1-F(w)) & \text{if } w \ge w_0 \end{cases} \tag{2.4}$$

where

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+t^2}dt. \tag{2.5}$$

We use Stein equation to find necessary and sufficient conditions for a random variable W to be Cauchy.

Theorem 2.12. *Let W be a random variable. Then W has a Cauchy distribution if and only if for all functions $f:\mathbb{R}\to\mathbb{R}$ such that f' exists a.e., continuous a.e., and $E|f'(W)|<\infty$, we have*

$$Ef'(W) = 2E\frac{Wf(W)}{1+W^2}.$$

Proof. To prove the necessity, we assume that W has a Cauchy distribution and let $f:\mathbb{R}\to\mathbb{R}$ be such that f' exists a.e., continuous a.e. and $E|f'(W)|<\infty$. Then

$$\begin{aligned}
&2E\frac{Wf(W)}{1+W^2}\\
&= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{wf(w)}{(1+w^2)^2}dw\\
&= \frac{2}{\pi}\Big\{\int_{-\infty}^{0}\frac{wf(w)}{(1+w^2)^2}dw + \int_{0}^{\infty}\frac{wf(w)}{(1+w^2)^2}dw\Big\}\\
&= \frac{2}{\pi}\Big\{\int_{-\infty}^{0}\frac{w}{(1+w^2)^2}(f(0)-\int_{w}^{0}f'(t)dt)dw + \int_{0}^{\infty}\frac{w}{(1+w^2)^2}(f(0)+\int_{0}^{w}f'(t)dt)dw\Big\}\\
&= \frac{2}{\pi}\Big\{\int_{-\infty}^{0}\int_{w}^{0}\frac{(-w)f'(t)}{(1+w^2)^2}dtdw + \int_{0}^{\infty}\int_{0}^{w}\frac{wf'(t)}{(1+w^2)^2}dtdw + f(0)\int_{-\infty}^{\infty}\frac{w}{(1+w^2)^2}dw\Big\}\\
&= \frac{2}{\pi}\Big\{\int_{-\infty}^{0}f'(t)\int_{\infty}^{t}\frac{(-w)}{(1+w^2)^2}dwdt + \int_{0}^{\infty}f'(t)\int_{t}^{\infty}\frac{w}{(1+w^2)^2}dwdt\Big\}\\
&= \frac{1}{\pi}\Big\{\int_{-\infty}^{0}\frac{f'(t)}{1+t^2}dt + \int_{0}^{\infty}\frac{f'(t)}{1+t^2}dt\Big\}\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f'(t)}{1+t^2}dt\\
&= Ef'(W).
\end{aligned}$$

Conversely, let $w_0 \in \mathbb{R}$. From Neammanee([24]), we have $E|f'_{w_0}(W)| < \infty$. Then

$$\begin{aligned} 0 &= E\Big[f'_{w_0}(W) - 2\frac{Wf_{w_0}(W)}{1+W^2}\Big] \\ &= E[\mathit{1}_{(-\infty,w_0]}(W) - Cau(\mathit{1}_{(-\infty,w_0]})] \\ &= P(W \le w_0) - F(w_0) \end{aligned}$$

where we have used Stein equation (2.3) in the first equality.
Hence W has a Cauchy distribution. □

To illustrate the technique, we give examples which used Stein method to find a bound in the approximation by Cauchy distribution.

Theorem 2.13. *Let $X_1, X_2, \ldots, X_n$ be random variables and S_i a subset of $\{1,2,\ldots,n\}$ such that $EX_i = 0$ and $EX_i^4 < \infty$ for each $i = 1,2,\ldots,n$. Let $W = X_1 + X_2 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,*

$$\begin{aligned} &\Big|P(W \le w_0) - F(w_0)\Big| \\ &\le 3\sqrt{E[1 - \frac{2}{1+W^2}\sum_{i=1}^{n}\sum_{j\in S_i} X_iX_j]^2} \\ &\quad + 4\pi \min\Big\{E\sum_{i=1}^{n}|X_i|\Big|\sum_{j\in S_i} X_i\Big|, \sqrt{E\sum_{i=1}^{n}X_i^2 E\Big|\sum_{i=1}^{n}X_i\sum_{j\in S_i}X_j\Big|^2}\Big\}F(w_0)(1-F(w_0)) \\ &\quad + 2\pi E\sum_{i=1}^{n}\Big|E^{\{X_j|j\notin S_i\}}X_i\Big| + 12(\pi+1)E\sum_{i=1}^{n}|X_i|\Big(\sum_{j\in S_i}X_j\Big)^2. \end{aligned}$$

where F is defined as in (2.5).

Proof. Let I be a random variable that is uniformly distributed over $\{1,2,\ldots,n\}$ and independent of $\{X_1, X_2, \ldots, X_n\}$. Let $\widetilde{\mathcal{B}}$ be a σ-algebra in which the random variables I and $\{X_1, X_2, \ldots, X_n\}$ are measurable, $\mathcal{B}$ the sub-σ-algebra of $\widetilde{\mathcal{B}}$ generated by $\{X_1, X_2, \ldots, X_n\}$ and $\mathcal{C}$ the σ-algebra generated by I and $\{X_j \mid j \notin S_I\}$.

Let

$$G = nX_I \quad\text{and}\quad \widetilde{W} = W - \sum_{j\in S_I} X_j.$$

By the fact that

$$E^{\mathcal{B}}G = E^{\mathcal{B}}nX_I = \sum_{i=1}^{n} X_i = W \quad\text{and}$$

$E\dfrac{Gf(\widetilde{W})}{1+\widetilde{W}^2} = EE^{\mathcal{C}}\dfrac{Gf(\widetilde{W})}{1+\widetilde{W}^2} = E(E^{\mathcal{C}}G)\dfrac{f(\widetilde{W})}{1+\widetilde{W}^2}$ for any $f:\mathbb{R}\to\mathbb{R}$, we have

$$\begin{aligned} E\frac{Wf(W)}{1+W^2} &= E(E^{\mathcal{B}}G)\frac{f(W)}{1+W^2} \\ &= EE^{\mathcal{B}}\Big(\frac{Gf(W)}{1+W^2}\Big) \\ &= E\frac{Gf(W)}{1+W^2} \\ &= EG\Big[\frac{f(W)}{1+W^2} - \frac{f(\widetilde{W})}{1+\widetilde{W}^2}\Big] + E(E^{\mathcal{C}}G)\frac{f(\widetilde{W})}{1+\widetilde{W}^2}. \end{aligned}$$

Applying (2.3) with $h = 1_{(-\infty,w_0]}$, we have

$$\begin{aligned} &|P(W\le w_0) - F(w_0)| \\ &= \Big|E\Big[f'_{w_0}(W) - 2G\Big(\frac{f_{w_0}(W)}{1+W^2} - \frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big)\Big] - 2E\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big| \\ &= \Big|E\Big[f'_{w_0}(W) - 2\frac{G(W-\widetilde{W})f'_{w_0}(W)}{1+W^2} + 2\frac{G(W-\widetilde{W})f'_{w_0}(W)}{1+W^2} \\ &\quad + 4\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2} - 4\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2} \\ &\quad - 2G\Big(\frac{f_{w_0}(W)}{1+W^2} - \frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big)\Big] - 2E\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big| \\ &\le E|f'_{w_0}(W)|\Big|1 - \frac{2G(W-\widetilde{W})}{1+W^2}\Big| \\ &\quad + 2E|G|\Big|\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2} - \frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2} + \frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| \\ &\quad + 4E\Big|\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}\Big| + 2E\Big|\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big| \end{aligned} \tag{2.6}$$

where f_{w_0} is defined as in (2.4). We note that for $W < \widetilde{W}$,

$$\begin{aligned} &\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2} - \frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2} + \frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2} \\ &= \int_W^{\widetilde{W}} \Big[\Big(\frac{f_{w_0}(w)}{1+w^2}\Big)' - \frac{f'_{w_0}(W)}{1+W^2} + \frac{2Wf_{w_0}(W)}{(1+W^2)^2}\Big]dw \\ &= \int_W^{\widetilde{W}} \Big[\frac{f'_{w_0}(w)}{1+w^2} - \frac{2wf_{w_0}(w)}{(1+w^2)^2} - \frac{f'_{w_0}(W)}{1+W^2} + \frac{2Wf_{w_0}(W)}{(1+W^2)^2}\Big]dw \\ &= \int_W^{\widetilde{W}}\int_W^w \Big(\frac{f'_{w_0}(y)}{1+y^2}\Big)'dydw - 2\int_W^{\widetilde{W}}\int_W^w \Big(\frac{yf_{w_0}(y)}{(1+y^2)^2}\Big)'dydw \end{aligned}$$

$$= \int_W^{\widetilde{W}} \int_y^{\widetilde{W}} \Big(\frac{f'_{w_0}(y)}{1+y^2}\Big)' dwdy - 2\int_W^{\widetilde{W}} \int_y^{\widetilde{W}} \Big(\frac{yf_{w_0}(y)}{(1+y^2)^2}\Big)' dwdy$$
$$= \int_W^{\widetilde{W}} (\widetilde{W}-y)\Big(\frac{f'_{w_0}(y)}{1+y^2}\Big)' dy - 2\int_W^{\widetilde{W}} (\widetilde{W}-y)\Big(\frac{yf_{w_0}(y)}{(1+y^2)^2}\Big)' dy.$$

By the same argument, we can show that

$$\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2} - \frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2} + \frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}$$
$$= \int_{\widetilde{W}}^{W} (w-\widetilde{W})\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)' dw - 2\int_{\widetilde{W}}^{W} (w-\widetilde{W})\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)' dw$$

for $\widetilde{W} < W$.

Hence

$$\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2} - \frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2} + \frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}$$
$$= \int_{-\infty}^{\infty} (\widetilde{W}-w)[I(w\le\widetilde{W}) - I(w\le W)]\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)' dw$$
$$- 2\int_{-\infty}^{\infty} (\widetilde{W}-w)[I(w\le\widetilde{W}) - I(w\le W)]\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)' dw. \qquad (2.7)$$

Therefore, by (2.6) and (2.7),

$$|P(W\le w_0) - F(w_0)|$$
$$\le E|f'_{w_0}(W)|\Big|1 - \frac{2G(W-\widetilde{W})}{1+W^2}\Big|$$
$$+ 2E|G|\int_{-\infty}^{\infty} |(\widetilde{W}-w)||[I(w\le\widetilde{W}) - I(w\le W)]|\Big|\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)'\Big| dw$$
$$+ 4E|G|\int_{-\infty}^{\infty} |(\widetilde{W}-w)||[I(w\le\widetilde{W}) - I(w\le W)]|\Big|\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)'\Big| dw$$
$$+ 4E\Big|\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}\Big| + 2E\Big|\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big|$$
$$\le \sup_{w\in\mathbb{R}}|f'_{w_0}(w)|E\Big|1 - 2\frac{G(W-\widetilde{W})}{1+W^2}\Big|$$
$$+ 2\sup_{w\in\mathbb{R}}\Big|\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)'\Big|E|G|\int_{-\infty}^{\infty}|\widetilde{W}-w||[I(w\le\widetilde{W}) - I(w\le W)]|dw$$
$$+ 4\sup_{w\in\mathbb{R}}\Big|\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)'\Big|E|G|\int_{-\infty}^{\infty}|\widetilde{W}-w||[I(w\le\widetilde{W}) - I(w\le W)]|dw$$
$$+ 4E\Big|\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}\Big| + 2\sup_{w\in\mathbb{R}}\Big|\frac{f_{w_0}(w)}{1+w^2}\Big|E|E^{\mathcal{C}}G|$$

$$
\begin{aligned}
&\leq \sup_{w\in\mathbb{R}} |f'_{w_0}(w)| \sqrt{E\Big[1-\frac{2}{1+W^2}E^{\mathcal{B}}G(W-\widetilde{W})\Big]^2} \\
&\quad + \Big(\sup_{w\in\mathbb{R}}\Big|\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)'\Big| + 2\sup_{w\in\mathbb{R}}\Big|\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)'\Big|\Big)E|G|(\widetilde{W}-W)^2 \\
&\quad + 4E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| + 2\sup_{w\in\mathbb{R}}\Big|\frac{f_{w_0}(w)}{1+w^2}\Big|E|E^{\mathcal{C}}G| \\
&\leq 3\sqrt{E\Big[1-\frac{2}{1+W^2}E^{\mathcal{B}}G(W-\widetilde{W})\Big]^2} + 4E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| \\
&\quad + 2\pi E|E^{\mathcal{C}}G| + 12(\pi+1)E|G|(\widetilde{W}-W)^2 \qquad (2.8)
\end{aligned}
$$

where we have used the facts that

$$
\begin{aligned}
&|f_{w_0}(w)/(1+w^2)| \leq \pi F(w_0)(1-F(w_0)), \qquad (2.9)\\
&|f'_{w_0}(w)| \leq 3,\\
&|(f'_{w_0}(w)/(1+w^2))'| \leq 6+2\pi \quad \text{and}\\
&|(wf_{w_0}(w)/(1+w^2)^2)'| \leq 3+5\pi
\end{aligned}
$$

for all $w_0, w \in \mathbb{R}$ (see [24]).
Note that

$$
E|E^{\mathcal{C}}G| = E\sum_{i=1}^{n}\Big|E^{\{X_j \mid j\notin S_i\}}X_i\Big|, \qquad (2.10)
$$

(see [23], eq. 21, p. 109)

$$
E|G|(\widetilde{W}-W)^2 = E\Big|\sum_{i=1}^{n}X_i\Big(\sum_{j\in S_i}X_j\Big)^2\Big| \leq E\sum_{i=1}^{n}|X_i|\Big(\sum_{j\in S_i}X_j\Big)^2, \qquad (2.11)
$$

and

$$
\begin{aligned}
\sqrt{E\Big[1-\frac{2}{1+W^2}E^{\mathcal{B}}G(W-\widetilde{W})\Big]^2} &= \sqrt{E\Big[1-\frac{2}{1+W^2}E^{\mathcal{B}}nX_I\sum_{j\in S_I}X_j\Big]^2} \\
&= \sqrt{E\Big[1-\frac{2}{1+W^2}\sum_{i=1}^{n}E^{\mathcal{B}}X_i\sum_{j\in S_i}X_j\Big]^2} \\
&= \sqrt{E\Big[1-\frac{2}{1+W^2}\sum_{i=1}^{n}\sum_{j\in S_i}X_iX_j\Big]^2}. \qquad (2.12)
\end{aligned}
$$

By (2.9), we have

$$
\begin{aligned}
E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| &\le E\Big|\frac{G(\widetilde{W}-W)f_{w_0}(W)}{(1+W^2)}\Big| \\
&\le \pi F(w_0)(1-F(w_0))E|G||W-\widetilde{W}| \\
&= \pi F(w_0)(1-F(w_0))E\Big|\sum_{i=1}^{n} X_i\Big(\sum_{j\in S_i} X_j\Big)\Big| \\
&\le \pi F(w_0)(1-F(w_0))E\sum_{i=1}^{n} |X_i|\Big|\Big(\sum_{j\in S_i} X_j\Big)\Big| \qquad (2.13)
\end{aligned}
$$

and

$$
\begin{aligned}
E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| &\le \pi F(w_0)(1-F(w_0))E|G||W-\widetilde{W}||W| \\
&\le \pi F(w_0)(1-F(w_0))\sqrt{E|G|^2|W-\widetilde{W}|^2}\sqrt{EW^2} \\
&= \pi F(w_0)(1-F(w_0))\sqrt{E\Big|\sum_{i=1}^{n} X_i\sum_{j\in S_i} X_j\Big|^2 E\sum_{i=1}^{n} X_i^2}. \qquad (2.14)
\end{aligned}
$$

Therefore, by (2.8) – (2.14) the theorem is proved. □

In the case that X_n's are independent we have the following corollary and example.

Corollary 2.14. *Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $EX_i = 0$ and $EX_i^4 < \infty$ for all $i = 1, 2, \ldots, n$. Let $W = X_1 + X_2 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,*

$$
\begin{aligned}
&\Big|P(W \le w_0) - F(w_0)\Big| \\
&\le 3\sqrt{E\Big[1-\frac{2}{1+W^2}\sum_{i=1}^{n} X_i^2\Big]^2} \\
&\quad + 4\pi \min\Big\{\sum_{i=1}^{n} EX_i^2, \sqrt{n\Big(\sum_{i=1}^{n} EX_i^2\Big)\Big(\sum_{i=1}^{n} EX_i^4\Big)}\Big\}F(w_0)(1-F(w_0)) \\
&\quad + 12(\pi+1)\sum_{i=1}^{n} E|X_i|^3.
\end{aligned}
$$

Proof. It follows from Theorem 2.13 by choosing $S_i = \{i\}$ and the fact that

$$|E^{\{X_j|j\notin S_i\}}X_i| = |E^{\{X_j|j\neq i\}}X_i| = |EX_i| = 0.$$

□

Example 2.15. *Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables with zero means, $EY_i^2 = \frac{1}{2}$, and $E|Y_i|^5 < \infty$. Let $X_i = \frac{Y_i}{\sqrt{n}}$ and $W = X_1 + X_2 + \cdots + X_n$. Then, for all $w_0 \in \mathbb{R}$,*

$$|P(W \leq w_0) - F(w_0)| \leq \frac{C}{\sqrt[4]{n}} + C \min\{\frac{1}{2}, \frac{1}{\sqrt{2}}\sqrt{E|Y_i|^4}\} F(w_0)(1 - F(w_0)).$$

CHAPTER III
A NON-UNIFORM BOUND IN A COMBINATORIAL CENTRAL LIMIT THEOREM

Let (a_{ij}) be an $n \times n$ matrix of real numbers and π a random permutation of $\{1, 2, \ldots, n\}$.

Let

$$\mu := E(\sum_i a_{i\pi(i)}) \;\text{ and }\; \sigma^2 := \mathrm{Var}(\sum_i a_{i\pi(i)}).$$

A theorem which has been proved under various conditions by Hoeffding([25]), Motoo ([26]) and others states that

$$V(\pi) := \frac{1}{\sigma}(\sum_i a_{i\pi(i)} - \mu)$$

is approximately standard normally distributed. This theorem is always called a combinatorial central limit theorem. In this chapter, we shall investigate the rate of convergence.

With these definitions, it was shown in Bolthasen([10]) that

$$V(\pi) = \sum_i x_{i\pi(i)}$$

for some x_{ij} such that

$$\sum_i x_{ij} = 0 \quad \text{for each } \; j = 1, 2, \ldots, n \; \text{ and}$$
$$\sum_j x_{ij} = 0 \quad \text{for each } \; i = 1, 2, \ldots, n.$$

For example, we can choose

$$x_{ij} = \frac{a_{ij} - a_{i.} - a_{.j} + a_{..}}{\sigma}$$

where $a_{i.} = \sum_j \frac{a_{ij}}{n}$, $a_{.j} = \sum_i \frac{a_{ij}}{n}$ and $a_{..} = \sum_{i,j} \frac{a_{ij}}{n^2}$.

Uniform estimates have been obtained by Von Bahr([27]) and Ho and Chen([28]), but they yield the rate $O\Big(\frac{1}{\sqrt{n}}\Big)$ only under some boundedness conditions like

$$\sup_{i,j} |x_{ij}| \leq \frac{K}{\sqrt{n}}. \tag{3.1}$$

The best uniform bound is obtained by Bolthausen([10]) in 1984. His argument is based on an inductive approach of Stein's method. The following is his result.

Theorem 3.1. *There exists a constant C such that*

$$\sup_{z\in\mathbb{R}} |P(V(\pi) \leq z) - \Phi(z)| \leq C\beta$$

where

$$\beta := \frac{1}{n} \sum_{i,j} |x_{ij}|^3.$$

We note that under the condition (3.1),

$$\beta \leq \frac{K}{\sqrt{n}}.$$

In this chapter, we give a non-uniform bound in a combinatorial central limit theorem of $V(\pi)$ under the boundedness condition (3.1). The argument is based on a concentration inequality approach. Our result is the following.

Theorem 3.2. *Assume that* (3.1) *hold. Then there exists a positive constant C such that for every real number z,*

$$|P(V(\pi) \leq z) - \Phi(z)| \leq \frac{CK}{(1 + |z|)^3\sqrt{n}}.$$

This chapter is organized as follows. Auxiliary results are in section 3.1. The concentration inequality are proved in section 3.2 while the proof of main result is given in section 3.3.

From now on, C stands for a positive constant with possibly different values in different places and we assume (3.1) holds.

3.1 Auxiliary Results

In this section, we give auxiliary results which used in section 3.2 and section 3.3. To do these, we need the following construction from Ho and Chen([28]).

Let I, J, L, M be uniformly distributed random variables on $\{1, 2, \ldots, n\}$ and ρ, τ random permutations of $\{1, 2, \ldots, n\}$ such that

(I, J) and (L, M) are uniformly distributed on $\{(i, j)|i, j = 1, 2, \ldots, n$ and $i \neq j\}$

(I,J), (L,M) and τ are independent,

(I,J) and ρ are independent and

$$\rho(\alpha) = \begin{cases} \tau(\alpha) & \text{if} \quad \alpha \neq I, J, \tau^{-1}(L), \tau^{-1}(M), \\ L & \text{if} \quad \alpha = I, \\ M & \text{if} \quad \alpha = J, \\ \tau(I) & \text{if} \quad \alpha = \tau^{-1}(L), \\ \tau(J) & \text{if} \quad \alpha = \tau^{-1}(M) \end{cases}$$

where $\rho(\rho^{-1}(\alpha)) = \rho^{-1}(\rho(\alpha)) = \alpha$.

Next, we let

$$V(\rho) := \sum_i x_{i\rho(i)}, \qquad V(\tau) := \sum_i x_{i\tau(i)},$$
$$\widehat{V}(\rho) := V(\rho) - x_{I\rho(I)} - x_{J\rho(J)} + x_{I\rho(J)} + x_{J\rho(I)},$$

it follows that $(\widehat{V}(\rho), V(\rho))$ is an exchangeable pair and

$$\widehat{V}(\rho) - V(\rho) = x_{IM} + x_{JL} - x_{IL} - x_{JM} \text{ are independent with } V(\tau). \tag{3.2}$$

By the same argument of Lemma 2.1 in Barbour and Chen([2]), we have

$$E^{\rho}(\widehat{V}(\rho)) = (1 - \frac{2}{n-1})V(\rho) \tag{3.3}$$

and

$$E(\widehat{V}(\rho) - V(\rho))^2 = \frac{4}{n-1}. \tag{3.4}$$

Lemma 3.3. *There exists a constant C such that*

(i) $|EV^3(\rho)| \leq \dfrac{CK}{\sqrt{n}}$,

(ii) $|EV^4(\rho)| \leq CK$.

Proof. For $m \in \mathbb{N}$, let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h(\widehat{v}, v) = (\widehat{v} - v)(\widehat{v}^{m-1} + v^{m-1}).$$

Then h is antisymmetric. Since $(\widehat{V}(\rho), V(\rho))$ is an exchangeable pair, by Proposition 2.10, we have

$$
\begin{aligned}
0 &= E(\widehat{V}(\rho) - V(\rho))(\widehat{V}^{m-1}(\rho) + V^{m-1}(\rho)) \\
&= E(\widehat{V}(\rho) - V(\rho))\Big\{2V^{m-1}(\rho) + (\widehat{V}^{m-1}(\rho) - V^{m-1}(\rho))\Big\} \\
&= 2E(\widehat{V}(\rho) - V(\rho))V^{m-1}(\rho) + E(\widehat{V}(\rho) - V(\rho))(\widehat{V}^{m-1}(\rho) - V^{m-1}(\rho)) \\
&= 2E(E^{\rho}(\widehat{V}(\rho)) - V(\rho))V^{m-1}(\rho) + E(\widehat{V}(\rho) - V(\rho))(\widehat{V}^{m-1}(\rho) - V^{m-1}(\rho)) \\
&= 2E\Big\{(-\frac{2}{n-1}V(\rho))V^{m-1}(\rho)\Big\} + E(\widehat{V}(\rho) - V(\rho))(\widehat{V}^{m-1}(\rho) - V^{m-1}(\rho)) \qquad \text{(by (3.3))} \\
&= -\frac{4}{n-1}EV^{m}(\rho) + E(\widehat{V}(\rho) - V(\rho))(\widehat{V}^{m-1}(\rho) - V^{m-1}(\rho)).
\end{aligned}
$$

Hence

$$
EV^{m}(\rho) = \frac{n-1}{4}E(\widehat{V}(\rho) - V(\rho))(\widehat{V}^{m-1}(\rho) - V^{m-1}(\rho)). \tag{3.5}
$$

Let

$$
\Delta V := V(\rho) - V(\tau) \quad \text{and} \quad \Delta\widehat{V} := \widehat{V}(\rho) - V(\tau).
$$

Then

$$
\begin{aligned}
|\Delta V| &= |x_{\tau^{-1}(L)\tau(I)} + x_{\tau^{-1}(M)\tau(J)} + x_{IL} + x_{JM} \\
&\qquad - x_{I\tau(I)} - x_{J\tau(J)} - x_{\tau^{-1}(L)L} - x_{\tau^{-1}(M)M}| \\
&\leq \frac{CK}{\sqrt{n}}
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
|\Delta\widehat{V}| &= |x_{\tau^{-1}(L)\tau(I)} + x_{\tau^{-1}(M)\tau(J)} + x_{IM} + x_{JL} \\
&\qquad - x_{I\tau(I)} - x_{J\tau(J)} - x_{\tau^{-1}(L)L} - x_{\tau^{-1}(M)M}| \\
&\leq \frac{CK}{\sqrt{n}}.
\end{aligned} \tag{3.7}
$$

(i) By (3.2), (3.4) – (3.7), we have

$$
\begin{aligned}
|EV^{3}(\rho)| &= \frac{n-1}{4}|E(\widehat{V}(\rho) - V(\rho))(\widehat{V}^{2}(\rho) - V^{2}(\rho))| \\
&= \frac{n-1}{4}|E(\widehat{V}(\rho) - V(\rho))^{2}(\widehat{V}(\rho) + V(\rho))| \\
&= \frac{n-1}{4}|E(\widehat{V}(\rho) - V(\rho))^{2}(2V(\tau) + \Delta\widehat{V} + \Delta V)|
\end{aligned}
$$

$$\leq \frac{n-1}{2}|E(\widehat{V}(\rho)-V(\rho))^2V(\tau)| + \frac{n-1}{4}|E(\widehat{V}(\rho)-V(\rho))^2(\Delta\widehat{V}+\Delta V)|$$
$$\leq \frac{n-1}{2}E(\widehat{V}(\rho)-V(\rho))^2|EV(\tau)|$$
$$+ CK\sqrt{n}E|\widehat{V}(\rho)-V(\rho)|^2$$
$$= \frac{CK}{\sqrt{n}}.$$

(ii) By (3.4) – (3.7), we have

$$\begin{aligned}
|EV^4(\rho)| &= \frac{n-1}{4}|E(\widehat{V}(\rho)-V(\rho))(\widehat{V}^3(\rho)-V^3(\rho))| \\
&= \frac{n-1}{4}|E(\widehat{V}(\rho)-V(\rho))^2(\widehat{V}^2(\rho)+\widehat{V}(\rho)V(\rho)+V^2(\rho))| \\
&= \frac{n-1}{4}|E(\widehat{V}(\rho)-V(\rho))^2(3V^2(\tau)+3V(\tau)\Delta\widehat{V}+3V(\tau)\Delta V \\
&\qquad + \Delta\widehat{V}\Delta V + \Delta\widehat{V}^2 + \Delta V^2)| \\
&\leq CKn|E(\widehat{V}(\rho)-V(\rho))^2(EV^2(\tau)+E|V(\tau)|+1)| \\
&\leq CKnE(\widehat{V}(\rho)-V(\rho))^2 \\
&= CK.
\end{aligned}$$

□

To prove the concentration inequality(Lemma 3.6), we construct the following system.

Let $\overline{I}, \overline{J}, \overline{L}, \overline{M}$ be uniformly distributed random variables on $\{1, 2, \ldots, n\}$ which satisfy the following.

$(\overline{I}, \overline{J})$ and $(\overline{L}, \overline{M})$ are uniformly distributed random variables on

$$\{(i,j)|\ i,j = 1,2,\ldots,n \text{ and } i \neq j\},$$

$[(\overline{I}, \overline{J}), (\overline{L}, \overline{M})]$ is uniformly distributed on
$\{[(i,j),(l,m)]|\ i,j,l,m = 1,2,\ldots,n$ and $i \neq j, l \neq m$ and $(i,j) \neq (l,m)\}$ and

$[(\overline{I}, \overline{J}), (\overline{L}, \overline{M})]$ and ρ are independent.

Hence

$$P\big([(\overline{I}, \overline{J}), (\overline{L}, \overline{M})] = [(i,j),(l,m)]\big) = \frac{1}{n(n-1)(n(n-1)-1)}$$

for $i,j,l,m = 1,2,\ldots,n, i \neq j, l \neq m$ and $(i,j) \neq (l,m)$.

Let $\gamma = 16\beta$ and

$$
\begin{aligned}
c[(i,j),(l,m)] &:= |x_{il}+x_{jm}-x_{im}-x_{jl}|\min(|x_{il}+x_{jm}-x_{im}-x_{jl}|,\gamma),\\
c'[(i,j),(l,m)] &:= c[(i,j),(l,m)] - Ec[(i,j),(\rho(i),\rho(j))],\\
C(\rho) &:= \sum_{\substack{i,j\\ i\neq j}} c'[(i,j),(\rho(i),\rho(j))],\\
C(\tau) &:= \sum_{\substack{i,j\\ i\neq j}} c'[(i,j),(\tau(i),\tau(j))]
\end{aligned}
$$

and
$$
\begin{aligned}
\widehat{C}(\rho) &:= C(\rho) - c'[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] - c'[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]\\
&\quad + c'[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + c'[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))].
\end{aligned}
$$

It follows that $(\widehat{C}(\rho), C(\rho))$ is an exchangeable pair.

Lemma 3.4. *There exists a constant C such that*

(i) $E(\widehat{C}(\rho)-C(\rho))^2 \le \dfrac{CK\gamma^2}{n}$,

(ii) $E|\widehat{C}(\rho)-C(\rho)|^3 \le \dfrac{CK\gamma^3}{n^{3/2}}$,

(iii) $EC^2(\rho) \le CKn\gamma^2$.

Proof. (i) Follows from the fact that

$$
\begin{aligned}
&E(\widehat{C}(\rho)-C(\rho))^2\\
&= E\Big\{c'[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + c'[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))]\\
&\qquad - c'[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] - c'[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]\Big\}^2\\
&= E\Big\{c[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] - Ec[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))]\\
&\qquad + c[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))] - Ec[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))]\\
&\qquad - c[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + Ec[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))]\\
&\qquad - c[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))] + Ec[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]\Big\}^2\\
&= E\Big\{c[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + c[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))]\\
&\qquad - c[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] - c[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]\Big\}^2\\
&\le 4\Big\{Ec^2[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + Ec^2[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))]\\
&\qquad + Ec^2[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] + Ec^2[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]\Big\}\\
&= 8Ec^2[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + 8Ec^2[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] \qquad (3.8)
\end{aligned}
$$

and $\quad c[(i,j),(l,m)] \le \dfrac{CK\gamma}{\sqrt{n}}$.

(ii) Using the same argument of (i).

(iii) By the fact that

$$\begin{aligned}
&\sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{l,m\\ l\neq m}} c'[(i,j),(l,m)]\\
&= \sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{l,m\\ l\neq m}} \Big(c[(i,j),(l,m)] - Ec[(i,j),(\rho(i),\rho(j))]\Big)\\
&= \sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{l,m\\ l\neq m}} c[(i,j),(l,m)] - n(n-1)\sum_{\substack{i,j\\ i\neq j}} Ec[(i,j),(\rho(i),\rho(j))]\\
&= \sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{l,m\\ l\neq m}} c[(i,j),(l,m)] - \sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{s,t\\ s\neq t}} c[(i,j),(s,t)]\\
&= 0,
\end{aligned}$$

we have

$$\begin{aligned}
E^{\rho}\widehat{C}(\rho) &= E^{\rho}\Big\{C(\rho) - c'[(\bar{I},\bar{J}),(\rho(\bar{I}),\rho(\bar{J}))] - c'[(\bar{L},\bar{M}),(\rho(\bar{L}),\rho(\bar{M}))]\\
&\qquad + c'[(\bar{I},\bar{J}),(\rho(\bar{L}),\rho(\bar{M}))] + c'[(\bar{L},\bar{M}),(\rho(\bar{I}),\rho(\bar{J}))]\Big\}\\
&= C(\rho) - E^{\rho}\Big\{c'[(\bar{I},\bar{J}),(\rho(\bar{I}),\rho(\bar{J}))] + c'[(\bar{L},\bar{M}),(\rho(\bar{L}),\rho(\bar{M}))]\\
&\qquad - c'[(\bar{I},\bar{J}),(\rho(\bar{L}),\rho(\bar{M}))] - c'[(\bar{L},\bar{M}),(\rho(\bar{I}),\rho(\bar{J}))]\Big\}\\
&= C(\rho) - \frac{1}{n(n-1)}\sum_{\substack{i,j\\ i\neq j}} c'[(i,j),(\rho(i),\rho(j))] - \frac{1}{n(n-1)}\sum_{\substack{l,m\\ l\neq m}} c'[(l,m),(\rho(l),\rho(m))]\\
&\quad + \frac{1}{n(n-1)(n(n-1)-1)}\sum_{\substack{i\neq j, l\neq m\\ (i,j)\neq(l,m)}} c'[(i,j),(\rho(l),\rho(m))]\\
&\quad + \frac{1}{n(n-1)(n(n-1)-1)}\sum_{\substack{i\neq j, l\neq m\\ (i,j)\neq(l,m)}} c'[(l,m),(\rho(i),\rho(j))]\\
&= C(\rho) - \frac{2}{n(n-1)}\sum_{\substack{i,j\\ i\neq j}} c'[(i,j),(\rho(i),\rho(j))]\\
&\quad + \frac{2}{n(n-1)(n(n-1)-1)}\sum_{\substack{i\neq j, l\neq m\\ (i,j)\neq(l,m)}} c'[(i,j),(\rho(l),\rho(m))]
\end{aligned}$$

$$= C(\rho) - \frac{2}{n(n-1)}C(\rho) + \frac{2}{n(n-1)(n(n-1)-1)} \sum_{\substack{i,j \\ i\neq j}} \sum_{\substack{l,m \\ l\neq m}} c'[(i,j),(\rho(l),\rho(m))]$$
$$- \frac{2}{n(n-1)(n(n-1)-1)} \sum_{\substack{i,j \\ i\neq j}} c'[(i,j),(\rho(i),\rho(j))]$$
$$= C(\rho) - \frac{2}{n(n-1)}C(\rho) + \frac{2}{n(n-1)(n(n-1)-1)} \sum_{\substack{i,j \\ i\neq j}} \sum_{\substack{l,m \\ l\neq m}} c'[(i,j),(l,m)]$$
$$- \frac{2}{n(n-1)(n(n-1)-1)} C(\rho)$$
$$= \Big(1 - \frac{2}{n(n-1)-1}\Big) C(\rho). \tag{3.9}$$

By (3.9), we can apply the argument of (3.5), by using (3.9) and $\widehat{V}(\rho) = \widehat{C}(\rho)$, $V(\rho) = C(\rho)$ and $m = 2$, we have

$$EC^2(\rho) = \frac{n(n-1)-1}{4} E(\widehat{C}(\rho) - C(\rho))^2. \tag{3.10}$$

Therefore, (iii) follows from (i). □

Lemma 3.5. *Let $\gamma = 16\beta$. Then there exists a constant C such that*

$$E(\eta(\gamma) - E\eta(\gamma))^4 \leq \frac{CK\gamma^4}{n^2}$$

where $\eta(\gamma) := \frac{n-1}{4} E^{\rho}\big\{|\widehat{V}(\rho) - V(\rho)| \min(|\widehat{V}(\rho) - V(\rho)|, \gamma)\big\}$.

Proof. From the fact that

$$\eta(\gamma) = \frac{n-1}{4} E^{\rho}\{|x_{I\rho(I)} + x_{J\rho(J)} - x_{I\rho(J)} - x_{J\rho(I)}|$$
$$\times \min(|x_{I\rho(I)} + x_{J\rho(J)} - x_{I\rho(J)} - x_{J\rho(I)}|, \gamma)\}$$
$$= \frac{1}{4n} \sum_{\substack{i,j \\ i\neq j}} \{|x_{i\rho(i)} + x_{j\rho(j)} - x_{i\rho(j)} - x_{j\rho(i)}|$$
$$\times \min(|x_{i\rho(i)} + x_{j\rho(j)} - x_{i\rho(j)} - x_{j\rho(i)}|, \gamma)\}$$
$$= \frac{1}{4n} \sum_{\substack{i,j \\ i\neq j}} c[(i,j),(\rho(i),\rho(j))],$$

we have

$$E(\eta(\gamma) - E\eta(\gamma))^4 = E\Big\{\frac{1}{4n} \sum_{\substack{i,j \\ i\neq j}} c'[(i,j),(\rho(i),\rho(j))]\Big\}^4$$
$$= \frac{1}{256n^4} EC^4(\rho). \tag{3.11}$$

Hence, to prove the lemma, it suffices to show that

$$EC^4(\rho) \le CKn^2\gamma^4.$$

$$\begin{aligned} \text{Let} \quad \Delta\widehat{C} &= \widehat{C}(\rho) - C(\tau), \\ \Delta C &= C(\rho) - C(\tau), \\ \kappa &= \{(i,j)|\ i,j = 1,2,\ldots,n,\ i \ne j\} \quad \text{and} \\ \Gamma &= \{(i,j) \in \kappa|\ i = I, J, \tau^{-1}(L), \tau^{-1}(M) \text{ or } j = I, J, \tau^{-1}(L), \tau^{-1}(M)\}. \end{aligned}$$

Then

$$|\Gamma| \le Cn$$

and $c'[(i,j),(\rho(i),\rho(j))] = c'[(i,j),(\tau(i),\tau(j))]$ on $\kappa - \Gamma$. Hence

$$\Delta C = \sum_{(i,j)\in\Gamma} c'[(i,j),(\rho(i),\rho(j))] - \sum_{(i,j)\in\Gamma} c'[(i,j),(\tau(i),\tau(j))],$$

which implies that

$$\begin{aligned} |\Delta C| &= \Big| \sum_{(i,j)\in\Gamma} c'[(i,j),(\rho(i),\rho(j))] - \sum_{(i,j)\in\Gamma} c'[(i,j),(\tau(i),\tau(j))] \Big| \\ &\le \sum_{(i,j)\in\Gamma} c[(i,j),(\rho(i),\rho(j))] + \sum_{(i,j)\in\Gamma} c[(i,j),(\tau(i),\tau(j))] \\ &\le \frac{CK\gamma}{\sqrt{n}}|\Gamma| \\ &\le CKn^{\frac{1}{2}}\gamma \end{aligned} \tag{3.12}$$

where we have used the fact that $c[(i,j),(l,m)] \le \dfrac{CK\gamma}{\sqrt{n}}$ in the second inequality. We observe that

$$|\Delta\widehat{C}| = |\Delta C + \widehat{C}(\rho) - C(\rho)| \le |\Delta C| + |\widehat{C}(\rho) - C(\rho)| \le CKn^{\frac{1}{2}}\gamma. \tag{3.13}$$

Using the same technique of (3.10), we have

$$EC^4(\rho) = \frac{n(n-1)-1}{4} E(\widehat{C}(\rho) - C(\rho))(\widehat{C}^3(\rho) - C^3(\rho)).$$

By Lemma 3.4, (3.12) and (3.13), we have

$$\begin{aligned}
&E(\widehat{C}(\rho)-C(\rho))(\widehat{C}^3(\rho)-C^3(\rho))\\
&= E[(\widehat{C}(\rho)-C(\rho))^2\{(\widehat{C}(\rho)-C(\rho))^2+3\widehat{C}(\rho)C(\rho)\}]\\
&= E(\widehat{C}(\rho)-C(\rho))^4+3E(\widehat{C}(\rho)-C(\rho))^2C^2(\tau)\\
&\quad +3E(\widehat{C}(\rho)-C(\rho))^2C(\tau)(\Delta C+\Delta\widehat{C})+3E(\widehat{C}(\rho)-C(\rho))^2\Delta C\Delta\widehat{C}\\
&\le \frac{CK\gamma}{\sqrt{n}}E|\widehat{C}(\rho)-C(\rho)|^3+3E(\widehat{C}(\rho)-C(\rho))^2EC^2(\tau)\\
&\quad +CKn^{\frac{1}{2}}\gamma E(\widehat{C}(\rho)-C(\rho))^2\{E|C(\tau)|^2\}^{\frac{1}{2}}+CKn\gamma^2E(\widehat{C}(\rho)-C(\rho))^2\\
&\le CK\gamma^4.
\end{aligned}$$

Therefore

$$EC^4(\rho)\le CKn^2\gamma^4.$$

This completes the proof. □

3.2 Concentration Inequality

In this section, we give a concentration inequality lemma which will be used in the next section.

Lemma 3.6. (Concentration inequality) *Let $0<a\le c\le d$ and $(1+a)\beta\le 1$. Then*

$$P(c\le V(\rho)\le d)\le\frac{CK}{(1+a)^3}(d-c+\beta)$$

for some constant C.

Proof. Let $\gamma=16\beta$ and define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(t)=\begin{cases}0 & \text{for } t<c-\gamma\\ (1+t+\gamma)^3(t-c+\gamma) & \text{for } c-\gamma\le t\le d+\gamma\\ (1+t+\gamma)^3(d-c+2\gamma) & \text{for } t>d+\gamma.\end{cases}$$

Therefore g is non-decreasing and

$$g'(t) = \begin{cases} 0 & \text{for } t < c-\gamma \\ (1+t+\gamma)^3 + 3(1+t+\gamma)^2(t-c+\gamma) & \text{for } c-\gamma < t < d+\gamma \\ 3(1+t+\gamma)^2(d-c+2\gamma) & \text{for } t > d+\gamma \end{cases}$$

$$\geq \begin{cases} 0 & \text{for } t < c-\gamma \text{ or } t > d+\gamma \\ (1+t+\gamma)^2(1+4t+4\gamma-3c) & \text{for } c-\gamma < t < d+\gamma \end{cases}$$

$$\geq \begin{cases} 0 & \text{for } t < c-\gamma \text{ or } t > d+\gamma \\ (1+c)^3(1+4c-3c) & \text{for } c-\gamma < t < d+\gamma \end{cases}$$

$$= \begin{cases} 0 & \text{for } t < c-\gamma \text{ or } t > d+\gamma \\ (1+c)^3 & \text{for } c-\gamma < t < d+\gamma. \end{cases} \tag{3.14}$$

Let

$$K(t) := \frac{n-1}{4}(\widehat{V}(\rho) - V(\rho))[\mathit{1}(0 \leq t \leq \widehat{V}(\rho) - V(\rho)) - \mathit{1}(\widehat{V}(\rho) - V(\rho) \leq t < 0)]$$

and $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h(\widehat{v}, v) = (\widehat{v} - v)(g(\widehat{v}) + g(v)).$$

By the same argument of (3.5) and (3.14), we have

$$\begin{aligned} & EV(\rho)g(V(\rho)) \\ &= \frac{n-1}{4}E(\widehat{V}(\rho) - V(\rho))(g(\widehat{V}(\rho)) - g(V(\rho))) \\ &= \frac{n-1}{4}E\big[(\widehat{V}(\rho) - V(\rho))\int_0^{\widehat{V}(\rho)-V(\rho)} g'(V(\rho)+t)dt\big] \\ &= \frac{n-1}{4}E\big[(\widehat{V}(\rho) - V(\rho)) \\ &\quad \times \int_{\mathbb{R}} g'(V(\rho)+t)[\mathit{1}(0 \leq t \leq \widehat{V}(\rho) - V(\rho)) - \mathit{1}(\widehat{V}(\rho) - V(\rho) \leq t < 0)]dt\big] \\ &= E[\int_{\mathbb{R}} g'(V(\rho)+t)K(t)dt] \qquad (3.15) \\ &\geq (1+c)^3 E\big[\mathit{1}(c \leq V(\rho) \leq d)\int_{|t|<\gamma} K(t)dt\big] \end{aligned}$$

$$
\begin{aligned}
&= (1+c)^3 \frac{n-1}{4} E[\mathit{1}(c \le V(\rho) \le d)|\widehat{V}(\rho) - V(\rho)| \min(\gamma, |\widehat{V}(\rho) - V(\rho)|)] \\
&= (1+c)^3 \frac{n-1}{4} E\big\{E^{\rho}[\mathit{1}(c \le V(\rho) \le d)|\widehat{V}(\rho) - V(\rho)| \min(\gamma, |\widehat{V}(\rho) - V(\rho)|)]\big\} \\
&= (1+c)^3 E[\mathit{1}(c \le V(\rho) \le d)\eta(\gamma)] \\
&\ge (1+c)^3 E[\mathit{1}(c \le V(\rho) \le d)\eta(\gamma)\mathit{1}(\eta(\gamma) > \frac{1}{4})] \\
&\ge \frac{(1+c)^3}{4} E[\mathit{1}(c \le V(\rho) \le d)\mathit{1}(\eta(\gamma) > \frac{1}{4})] \\
&= \frac{(1+c)^3}{4} E[\mathit{1}(c \le V(\rho) \le d) - \mathit{1}(\mathit{1}(c \le V(\rho) \le d), \eta(\gamma) \le \frac{1}{4})] \\
&\ge \frac{(1+c)^3}{4} E[\mathit{1}(c \le V(\rho) \le d) - \mathit{1}(\eta(\gamma) \le \frac{1}{4})] \\
&\ge \frac{(1+c)^3}{4} [P(c \le V(\rho) \le d) - P(\eta(\gamma) \le \frac{1}{4})]
\end{aligned}
$$

which implies that

$$
\begin{aligned}
P(c \le V(\rho) \le d) &\le \frac{4}{(1+c)^3} EV(\rho)g(V(\rho)) + P(\eta(\gamma) \le \frac{1}{4}) \\
&\le \frac{4}{(1+a)^3} EV(\rho)g(V(\rho)) + P(\eta(\gamma) \le \frac{1}{4}). \qquad (3.16)
\end{aligned}
$$

By definition of g and Lemma 3.3(ii), we have

$$
\begin{aligned}
EV(\rho)g(V(\rho)) &\le E|V(\rho)||g(V(\rho))| \\
&\le (d-c+2\gamma)E|V(\rho)||1+V(\rho)+\gamma|^3 \\
&\le C(d-c+2\gamma)E|V(\rho)|(|V(\rho)|^3+1) \\
&= C(d-c+2\gamma)(E|V(\rho)|^4 + E|V(\rho)|) \\
&\le C(d-c+2\gamma)(E|V(\rho)|^4 + \{E|V(\rho)|^2\}^{\frac{1}{2}}) \\
&\le CK(d-c+2\gamma) \\
&\le CK(d-c+\beta) \qquad (3.17)
\end{aligned}
$$

where we have used the fact that $\gamma \le 16$ in the third inequality. Observe that

$$
\begin{aligned}
E|\widehat{V}(\rho) - V(\rho)|^3 &= E|x_{IL} + x_{JM} - x_{IM} - x_{JL}|^3 \\
&\le 16E\big(|x_{IL}|^3 + |x_{JM}|^3 + |x_{IM}|^3 + |x_{JL}|^3\big) \\
&= 64E|x_{IL}|^3 \\
&= \frac{64}{n^2}\sum_{i,j}|x_{ij}|^3 \\
&= \frac{64\beta}{n}. \qquad (3.18)
\end{aligned}
$$

To bound $P(\eta(\gamma) \leq \frac{1}{4})$, we note that

$$\min(x, y) \geq x - \frac{x^2}{4y} \qquad \text{for } \; x \geq 0, y > 0.$$

From this fact and by (3.4) and (3.18), we have

$$\begin{aligned}
E(\eta(\gamma)) &= \frac{n-1}{4} E\{|\widehat{V}(\rho) - V(\rho)| \min(|\widehat{V}(\rho) - V(\rho)|, 16\beta)\} \\
&\geq \frac{n-1}{4} E\Big\{|\widehat{V}(\rho) - V(\rho)|\Big(|\widehat{V}(\rho) - V(\rho)| - \frac{|\widehat{V}(\rho) - V(\rho)|^2}{64\beta}\Big)\Big\} \\
&= \frac{n-1}{4}\{E|\widehat{V}(\rho) - V(\rho)|^2 - \frac{1}{64\beta} E|\widehat{V}(\rho) - V(\rho)|^3\} \\
&\geq \frac{n-1}{4}\{\frac{4}{n-1} - \frac{1}{64\beta}(\frac{64\beta}{n})\} \\
&= 1 - \frac{n-1}{4n} \\
&\geq 1 - \frac{1}{4} \\
&= \frac{3}{4}.
\end{aligned}$$

Hence

$$\begin{aligned}
P(\eta(\gamma) \leq \frac{1}{4}) &\leq P(E\eta(\gamma) - \eta(\gamma) \geq \frac{1}{2}) \\
&\leq CE(\eta(\gamma) - E\eta(\gamma))^4 \\
&\leq \frac{CK\gamma^4}{n^2} \\
&\leq CK\beta^4 \\
&\leq \frac{CK\beta}{(1+a)^3} \qquad (3.19)
\end{aligned}$$

where we have used Lemma 3.5 in the third inequality and the fact that $(1+a)\beta \leq 1$ in the last inequality. From (3.16), (3.17) and (3.19), we have

$$P(c \leq V(\rho) \leq d) \leq \frac{CK(d - c + \beta)}{(1+a)^3}.$$

□

Lemma 3.7. *Let $z \in \mathbb{R}$ and f_z be the solution of Stein's equation. For $z \geq 12$, $|u| < 2(1 + \frac{z}{4})$ and $(1+z)\beta < 1$, we have*

$$Eg(V(\rho) + u) \leq \frac{C}{(1+\frac{z}{4})^3}(1 + \frac{z}{4}\beta).$$

where $g : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$g(w) = (wf_z(w))'.$$

Proof. Use the same argument as in Lemma 5.1 and Lemma 5.2 in Chen and Shao([11]). □

3.3 Proof of the Main Results

It suffice to consider $z \geq 0$ as we can apply the result to $-W$ when $z < 0$. Let $z \geq 0$.

If $0 \leq z < 12$, by Theorem 3.1, we have

$$|P(V(\pi) \leq z) - \Phi(z)| \leq C\beta \leq \frac{CK}{\sqrt{n}} \leq \frac{CK}{(1+z)^3\sqrt{n}}.$$

Assume that $z \geq 12$.

Case 1. $(1+z)\beta > 1$.

By Lemma 3.3(2),

$$\begin{aligned} P(V(\pi) \geq z) &= P(1 + V(\pi) \geq 1 + z) \\ &\leq \frac{E(1+V(\pi))^4}{(1+z)^4} \\ &\leq \frac{CE(1+V^4(\pi))}{(1+z)^4} \\ &\leq \frac{CK}{(1+z)^4}. \end{aligned}$$

Hence

$$\begin{aligned} |P(V(\pi) \leq z) - \Phi(z)| &= |(1 - P(V(\pi) \leq z)) - (1 - \Phi(z))| \\ &\leq P(V(\pi) > z) - (1 - \Phi(z)) \\ &\leq \frac{CK}{(1+z)^4} \\ &\leq \frac{CK\beta}{(1+z)^3} \\ &\leq \frac{CK}{(1+z)^3\sqrt{n}} \end{aligned}$$

where we have used the fact $1 - \Phi(z) \leq \frac{e^{-\frac{1}{2}z^2}}{z\sqrt{2\pi}} < \frac{C}{(1+z)^4}$ (see [23], eq. 25, p. 23) in the second inequality.

Case 2. $(1+z)\beta \leq 1$.

Let f be a real valued function. By the same argument of (3.15), we have

$$E[V(\rho)f(V(\rho))] = E\int_{\mathbb{R}} f'(V(\rho)+t)K(t)dt \tag{3.20}$$

where $K(t)$ is defined as in Lemma 3.6. If $f(t) = t$, by (3.20), we have

$$E\int_{\mathbb{R}} K(t)dt = EV^2(\rho) = 1. \tag{3.21}$$

Let f_z be the solution of Stein's equation defined by (2.1). Hence, by Stein's equation (2.1), (3.20) and (3.21)

$$\begin{aligned} P(V(\pi) \leq z) - \Phi(z) &= P(V(\rho) \leq z) - \Phi(z) \\ &= Ef_z'(V(\rho)) - EV(\rho)f_z(V(\rho)) \\ &= Ef_z'(V(\rho)) - E\int_{\mathbb{R}} f_z'(V(\rho)+t)K(t)dt \\ &= Ef_z'(V(\tau)) - E\int_{\mathbb{R}} f_z'(V(\rho)+t)K(t)dt \\ &= E\int_{\mathbb{R}} [f_z'(V_a(\tau)) - f_z'(V_a(\rho)+t)]K(t)dt \\ &= R \end{aligned} \tag{3.22}$$

where

$$R = E\int_{\mathbb{R}} [f_z'(V(\tau)) - f_z'(V(\tau)+\Delta V + t)]K(t)dt.$$

Since $|\widehat{V}(\rho) - V(\rho)| = |x_{IM} + x_{JL} - x_{IL} - x_{JM}| \leq \frac{CK}{\sqrt{n}}$, $K(t) = 0$ for $|t| > \frac{CK}{\sqrt{n}}$ which implies that

$$R = E\int_{|t|\leq \frac{CK}{\sqrt{n}}} [f_z'(V(\tau)) - f_z'(V(\tau)+\Delta V + t)]K(t)dt.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(w) = (wf_z(w))'$. By the fact that

$$|f_z'(w+s) - f_z'(w+t) - \int_t^s g(w+u)du| \leq \mathit{1}(z - \max(s,t) < w \leq z - \min(s,t))$$

(see [11], p. 250),

we have

$$|R| \le R_1 + R_2$$

where

$R_1 = E\int_{|t|\le\frac{CK}{\sqrt{n}}} 1(z - \max(0, \Delta V + t) < V(\tau) \le z - \min(0, \Delta V + t))K(t)dt$ and

$R_2 = \Big|E\int_{|t|\le\frac{CK}{\sqrt{n}}} K(t)\int_{\Delta V+t}^{0} g(V(\tau)+u)du\,dt\Big|.$

For $|t| \le \dfrac{CK}{\sqrt{n}}$, we see that for sufficiently large n,

$$z - \max(0, \frac{CK}{\sqrt{n}} + t) \ge z - \frac{CK}{\sqrt{n}} - |t| \ge z - \frac{CK}{\sqrt{n}} \ge z - 1 > 0 \quad \text{for} \quad z \ge 12.$$

Hence, by Lemma 3.6,

$$\begin{aligned}
R_1 &\le E\int_{|t|\le\frac{CK}{\sqrt{n}}} 1(z - \max(0, \frac{CK}{\sqrt{n}} + t) < V(\tau) \le z - \min(0, \frac{CK}{\sqrt{n}} + t))K(t)dt \\
&= E\int_{|t|\le\frac{CK}{\sqrt{n}}} E^{\{I,J,M,L\}} 1(z - \max(0, \frac{CK}{\sqrt{n}} + t) < V(\tau) \le z - \min(0, \frac{CK}{\sqrt{n}} + t))K(t)dt \\
&= E\int_{|t|\le\frac{CK}{\sqrt{n}}} P(z - \max(0, \frac{CK}{\sqrt{n}} + t) < V(\tau) \le z - \min(0, \frac{CK}{\sqrt{n}} + t))K(t)dt \\
&\le \frac{C}{z^3}E\int_{\mathbb{R}}(\frac{CK}{\sqrt{n}} + |t| + \beta)K(t)dt \qquad \text{(by Lemma 3.6)} \\
&\le \frac{CK}{z^3\sqrt{n}}E\int_{\mathbb{R}}K(t)dt + \frac{CK}{z^3}E\int_{\mathbb{R}}|t|K(t)dt \\
&\le \frac{CK}{(1+z)^3\sqrt{n}} + \frac{CK}{(1+z)^3}E\int_{\mathbb{R}}|t|K(t)dt \qquad \text{(by (3.21))} \\
&\le \frac{CK}{(1+z)^3\sqrt{n}} \qquad (3.23)
\end{aligned}$$

where we have used the fact that

$$E\int_{\mathbb{R}}|t|K(t)dt = \frac{n-1}{8}E|\widehat{V}(\rho) - V(\rho)|^3 \le \frac{CK}{\sqrt{n}} \qquad (3.24)$$

in the last inequality. By Lemma 3.7, we have

$$\begin{aligned}
R_2 &\le \Big|E\int_{|t|\le\frac{CK}{\sqrt{n}}} K(t)\int_{\frac{CK}{\sqrt{n}}+t}^{0} g(V(\tau)+u)du\,dt\Big| \\
&= \Big|EE^{\{I,J,M,L\}}\int_{|t|\le\frac{CK}{\sqrt{n}}} K(t)\int_{\frac{CK}{\sqrt{n}}+t}^{0} g(V(\tau)+u)du\,dt\Big|
\end{aligned}$$

$$
\begin{aligned}
&= \left| E \int_{|t| \le \frac{CK}{\sqrt{n}}} K(t) \int_{\frac{CK}{\sqrt{n}}+t}^{0} Eg(V(\tau)+u)du\, dt \right| \\
&\le \frac{CK}{(1+\frac{z}{4})^3}(1+\frac{z}{4}\beta)E\int_{\mathbb{R}} (\frac{CK}{\sqrt{n}} + |t|)K(t)dt \\
&\le \frac{CK}{(1+z)^3\sqrt{n}} \qquad (3.25)
\end{aligned}
$$

where we have used the fact that $(1+\frac{z}{4}\beta) \le 1+(1+z)\beta \le CK$, (3.21) and (3.24) in the last inequality. Hence, by (3.22), (3.23) and (3.25), we have

$$
|P(V(\rho) \le z) - \Phi(z)| \le \frac{CK}{(1+z)^3\sqrt{n}}.
$$

□

CHAPTER IV
A NON-UNIFORM BOUND IN NORMAL APPROXIMATION FOR MATRIX CORRELATION STATISTICS

Let $A = (a_{ij})$ and $B = (b_{ij})$ be symmetric $n \times n$ matrices, and define

$$W := W(\pi) := \sum_{\substack{i,j \\ i \neq j}} a_{ij} b_{\pi(i)\pi(j)},$$

where π is a random permutation of $\{1, 2, \ldots, n\}$. This W is always called "matrix correlation statistics" and first introduced by Daniels([1]). In the definition of W, the diagonal elements play no part, so that we may assume that $a_{ii} = b_{ii} = 0$ for all $i \in \{1, 2, \ldots, n\}$.

Let

$$\mu := E(W) = \frac{1}{n(n-1)} \sum_{\substack{i,j \\ i \neq j}} \sum_{\substack{l,m \\ l \neq m}} a_{ij} b_{lm},$$

$$A_0 := \frac{1}{n(n-1)} \sum_{\substack{i,j \\ i \neq j}} a_{ij},$$

$$A_{12} := \frac{1}{n} \sum_i \{a_i^*\}^2,$$

$$A_{22} := \frac{1}{n(n-1)} \sum_{\substack{i,j \\ i \neq j}} \widetilde{a}_{ij}^2,$$

$$A_{13} := \frac{1}{n} \sum_i |a_i^*|^3,$$

where
$$a_i^* := \frac{1}{n-2} \sum_{\substack{j \\ j \neq i}} (a_{ij} - A_0)$$

and
$$\widetilde{a}_{ij} := a_{ij} - a_i^* - a_j^* - A_0,$$

and make similar definitions for B. With these definitions and their analogues for the matrix B, it was shown in Barbour and Eagleson([29]) that

$$\begin{aligned} W &= \mu + \widetilde{V}(\pi) + \widetilde{\Delta}(\pi), \qquad (4.1)\\ \sigma^2 &:= \operatorname{Var} \widetilde{V}(\pi) = \frac{4n^2(n-2)^2}{n-1} A_{12}B_{12},\\ \operatorname{Var} \widetilde{\Delta}(\pi) &= \frac{2n(n-1)^2}{n-3} A_{22}B_{22},\\ Cov(\widetilde{V}(\pi), \widetilde{\Delta}(\pi)) &= 0, \end{aligned}$$

where

$$\widetilde{V}(\pi) := 2(n-2)\sum_i a_i^* b_{\pi(i)}^*$$

and
$$\widetilde{\Delta}(\pi) := \sum_{\substack{i,j\\ i\neq j}} \widetilde{a}_{ij}\widetilde{b}_{\pi(i)\pi(j)}.$$

To avoid trivial exceptions, we assume that $A_{12}B_{12} > 0$, otherwise, $\widetilde{V}(\pi) = 0$ a.s.

In 2005, Barbour and Chen([2]) gave a uniform Berry-Esseen bound for the normal approximation to W. The following is their result.

Theorem 4.1. *There exists a constant C such that*

$$\sup_{z\in\mathbb{R}} \Big| P\Big(\frac{W-\mu}{\sigma} \le z\Big) - P(V(\pi) \le z)\Big| \le C(\delta + \delta^2 + \delta_2)$$

and

$$\sup_{z\in\mathbb{R}} \Big| P\Big(\frac{W-\mu}{\sigma} \le z\Big) - \Phi(z)\Big| \le C(\delta + \delta^2 + \delta_2)$$

where $V(\pi) := \dfrac{\widetilde{V}(\pi)}{\sigma}$, $\Delta(\pi) := \dfrac{\widetilde{\Delta}}{\sigma}$, $\delta = 128\delta_1$, $\delta_1 = n^4\sigma^{-3}A_{13}B_{13}$ *and*

$$\delta_2^2 := E\Delta^2(\pi) = \frac{\operatorname{Var}\widetilde{\Delta}(\pi)}{\operatorname{Var}\widetilde{V}(\pi)} = \frac{(n-1)^3}{2n(n-2)^2(n-3)} \cdot \frac{A_{22}B_{22}}{A_{12}B_{12}}.$$

In this chapter, we use the idea of Barbour and Chen([2]) and Chen and Shao([3]) to prove a non-uniform Berry-Esseen bound for W. The following are our main results.

Theorem 4.2. *There exists a constant C such that for every real number z,*

$$\begin{aligned} &\Big| P\Big(\frac{W-\mu}{\sigma} \le z\Big) - P(V(\pi) \le z)\Big|\\ &\le \frac{C}{1+|z|^3}\Big\{\delta + \delta^2 + \delta_2 + \delta\delta_2 + \delta(n^{\frac{1}{2}}\delta)^2 + \delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}} + \delta_2(n^{\frac{1}{2}}\delta) + \delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}} + \delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}\\ &\quad + \frac{C\delta_2^2}{1+z^2}. \qquad (4.2) \end{aligned}$$

In particular, if $\frac{2(n-2)}{\sigma}\sup_{i,j}|a_i^*b_j^*| \le \frac{K}{\sqrt{n}}$ *for some positive real number* K*, then*

$$\begin{aligned}&\Big|P\Big(\frac{W-\mu}{\sigma}\le z\Big)-\Phi(z)\Big|\\ &\le \frac{CK}{1+|z|^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\\ &\quad+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\frac{1}{\sqrt{n}}\Big\}+\frac{C\delta_2^2}{1+z^2}. \qquad (4.3)\end{aligned}$$

Corollary 4.3. *If* $\delta_1 \sim n^{-1/2}$ *and* $\delta_2 \sim n^{-1/2}$*, then for any real number* z

$$\Big|P\Big(\frac{W-\mu}{\sigma}\le z\Big)-P(V(\pi)\le z)\Big| \le \frac{C}{(1+z^2)\sqrt{n}}.$$

Furthermore, if $\frac{2(n-2)}{\sigma}\sup_{i,j}|a_i^*b_j^*| \le \frac{K}{\sqrt{n}}$ *for some positive real number* K*, then*

$$\Big|P\Big(\frac{W-\mu}{\sigma}\le z\Big)-\Phi(z)\Big| \le \frac{CK}{(1+z^2)\sqrt{n}}.$$

To simplify the theorem, we let $a_{ij}\sim\frac{1}{n^p}$ and $b_{ij}\sim\frac{1}{n^q}$ for every $i,j\in\{1,2,\dots,n\}$. Then we have

$$\begin{aligned}&a_i^*\sim\frac{1}{n^p}, A_{12}\sim\frac{1}{n^{2p}},\quad A_{13}\sim\frac{1}{n^{3p}},\quad A_{22}\sim\frac{1}{n^{2p}},\\ &b_j^*\sim\frac{1}{n^p}, B_{12}\sim\frac{1}{n^{2q}},\quad B_{13}\sim\frac{1}{n^{3q}},\quad B_{22}\sim\frac{1}{n^{2q}}\quad\text{and}\quad \sigma^2\sim\frac{n^3}{n^{2(p+q)}}\end{aligned}$$

which implies

$$\frac{2(n-2)}{\sigma}\sup_{i,j}|a_i^*b_j^*|\le\frac{K}{\sqrt{n}},\quad \delta_1\sim n^{-1/2}\quad\text{and}\quad\delta_2\sim n^{-1/2}$$

so by Corollary 4.3

$$\Big|P\Big(\frac{W-\mu}{\sigma}\le z\Big)-\Phi(z)\Big| \le \frac{CK}{(1+z^2)\sqrt{n}}.$$

This chapter is organized as follows. Auxiliary results are in section 4.1. The concentration inequality for non-uniform is proved in section 4.2, the proof of main result is given in section 4.3 while the more general array is considered in section 4.4.

4.1 Auxiliary Results

In this section, we give auxiliary results which used in section 4.2 and section 4.3.

For each $i, j = 1, 2, \ldots, n$ and $a \geq 0$, let

$$x_{ij} := \frac{2(n-2)}{\sigma} a_i^* b_j^* \quad \text{and} \quad x_{a,ij} := x_{ij} \mathit{1}(|x_{ij}| \leq 1 + a).$$

We note also that, with a_i^* and b_i^* defined as before,

$$\begin{aligned}
\sum_i a_i^* &= \frac{1}{n-2} \sum_{\substack{i,j \\ i \neq j}} (a_{ij} - A_0) \\
&= \frac{1}{n-2} \{ \sum_{\substack{i,j \\ i \neq j}} a_{ij} - \sum_{\substack{i,j \\ i \neq j}} A_0 \} \\
&= \frac{1}{n-2} \{ n(n-1)A_0 - n(n-1)A_0 \} \\
&= 0
\end{aligned}$$

and

$$\sum_j b_j^* = 0,$$

which implies

$$\begin{aligned}
&\sum_i x_{ij} = 0 \quad \text{for each } j = 1, 2, \ldots, n \quad \text{and} \\
&\sum_j x_{ij} = 0 \quad \text{for each } i = 1, 2, \ldots, n. \qquad (4.4)
\end{aligned}$$

Let

$$\beta := \frac{1}{n} \sum_{i,j} |x_{ij}|^3.$$

We note that

$$\begin{aligned}
\beta &= \frac{8(n-2)^3}{n\sigma^3} \sum_{i,j} |a_i^* b_j^*|^3 \\
&\leq 8\sigma^{-3} n^2 (\sum_i |a_i^*|^3)(\sum_j |b_j^*|^3) \\
&= 8\sigma^{-3} n^2 (nA_{13})(nB_{13}) \\
&= 8\delta_1 \\
&= \frac{\delta}{16}. \qquad (4.5)
\end{aligned}$$

Let I, J, L, M, ρ and τ be defined as in Chapter 3.

The first step is to construct an exchangeable pair. We then set

$$
\begin{aligned}
V(\pi) &:= \frac{1}{\sigma}\widetilde{V}(\pi) = \sum_i x_{i\pi(i)}, \quad V(\rho) := \sum_i x_{i\rho(i)}, \quad V(\tau) := \sum_i x_{i\tau(i)},\\
V_a(\pi) &:= \sum_i x_{a,i\pi(i)}, \quad V_a(\rho) := \sum_i x_{a,i\rho(i)}, \quad V_a(\tau) := \sum_i x_{a,i\tau(i)},\\
\widehat{V}(\rho) &:= V(\rho) - x_{I\rho(I)} - x_{J\rho(J)} + x_{I\rho(J)} + x_{J\rho(I)},\\
\widehat{V}_a(\rho) &:= V_a(\rho) - x_{a,I\rho(I)} - x_{a,J\rho(J)} + x_{a,I\rho(J)} + x_{a,J\rho(I)},\\
\Delta(\pi) &:= \frac{1}{\sigma}\widetilde{\Delta}(\pi) = \frac{1}{\sigma}\sum_{\substack{i,j\\ i\neq j}} \widetilde{a}_{ij}\widetilde{b}_{\pi(i)\pi(j)},\\
\Delta(\rho) &:= \frac{1}{\sigma}\sum_{\substack{i,j\\ i\neq j}} \widetilde{a}_{ij}\widetilde{b}_{\rho(i)\rho(j)}
\end{aligned}
$$

and
$$
\begin{aligned}
\widehat{\Delta}(\rho) := \frac{1}{\sigma}\Big\{\Delta(\rho) &+ \sum_{\substack{j\\ j\neq I,J}} \{\widetilde{a}_{Ij}(\widetilde{b}_{\rho(I)\rho(j)} - \widetilde{b}_{\rho(J)\rho(j)}) + \widetilde{a}_{Jj}(\widetilde{b}_{\rho(J)\rho(j)} - \widetilde{b}_{\rho(I)\rho(j)})\}\\
&+ \sum_{\substack{i\\ i\neq I,J}} \{\widetilde{a}_{iI}(\widetilde{b}_{\rho(i)\rho(I)} - \widetilde{b}_{\rho(i)\rho(J)}) + \widetilde{a}_{iJ}(\widetilde{b}_{\rho(i)\rho(J)} - \widetilde{b}_{\rho(i)\rho(I)})\}\\
&+ \widetilde{a}_{JI}(\widetilde{b}_{\rho(J)\rho(I)} - \widetilde{b}_{\rho(I)\rho(J)}) + \widetilde{a}_{IJ}(\widetilde{b}_{\rho(I)\rho(J)} - \widetilde{b}_{\rho(J)\rho(I)})\Big\}
\end{aligned}
$$

it follows that $(\widehat{V}(\rho), V(\rho)), (\widehat{V}_a(\rho), V_a(\rho))$ and $(\widehat{\Delta}(\rho), \Delta(\rho))$ are exchangeable pairs (see [2]),

$$\widehat{V}(\rho) - V(\rho) = x_{IM} + x_{JL} - x_{IL} - x_{JM} \quad \text{and independent with } V(\tau)$$

and

$$\widehat{V}_a(\rho) - V_a(\rho) = x_{a,IM} + x_{a,JL} - x_{a,IL} - x_{a,JM} \quad \text{and independent with } V_a(\tau). \tag{4.6}$$

Lemma 4.4. *For any $a \geq 0$, we have*

$$E^{\rho}(\widehat{V}_a(\rho)) = (1 - \frac{2}{n-1})V_a(\rho) + \frac{2}{n(n-1)}\sum_{i,j} x_{a,ij}.$$

Proof.

$$
\begin{aligned}
E^{\rho}(\widehat{V}_a(\rho)) &= E^{\rho}(V_a(\rho) - x_{a,I\rho(I)} - x_{a,J\rho(J)} + x_{a,I\rho(J)} + x_{a,J\rho(I)})\\
&= V_a(\rho) - E^{\rho}(x_{a,I\rho(I)} + x_{a,J\rho(J)} - x_{a,I\rho(J)} - x_{a,J\rho(I)})
\end{aligned}
$$

$$
\begin{aligned}
&= V_a(\rho) - \frac{1}{n}\sum_i x_{a,i\rho(i)} - \frac{1}{n}\sum_j x_{a,j\rho(j)} \\
&\qquad + \frac{1}{n(n-1)}\sum_{\substack{i,j\\ i\neq j}} x_{a,i\rho(j)} + \frac{1}{n(n-1)}\sum_{\substack{i,j\\ i\neq j}} x_{a,j\rho(i)} \\
&= V_a(\rho) - \frac{2}{n}\sum_i x_{a,i\rho(i)} + \frac{2}{n(n-1)}\sum_{\substack{i,j\\ i\neq j}} x_{a,i\rho(j)} \\
&= V_a(\rho) - \frac{2}{n}V_a(\rho) + \frac{2}{n(n-1)}\sum_{i,j} x_{a,i\rho(j)} - \frac{2}{n(n-1)}\sum_i x_{a,i\rho(i)} \\
&= V_a(\rho) - \frac{2}{n}V_a(\rho) + \frac{2}{n(n-1)}\sum_{i,j} x_{a,ij} - \frac{2}{n(n-1)}\sum_i x_{a,i\rho(i)} \\
&= V_a(\rho) - \frac{2}{n}V_a(\rho) + \frac{2}{n(n-1)}\sum_{i,j} x_{a,ij} - \frac{2}{n(n-1)}V_a(\rho) \\
&= (1 - \frac{2}{n-1})V_a(\rho) + \frac{2}{n(n-1)}\sum_{i,j} x_{a,ij}.
\end{aligned}
$$

□

Lemma 4.5.

(i) $E(\widehat{V}(\rho) - V(\rho))^2 = \frac{4}{n-1}$,

(ii) $E(\widehat{\Delta}(\rho) - \Delta(\rho))^2 = 8n^{-1}E\Delta^2(\rho) = 8n^{-1}\delta_2^2$.

Proof. Follows the argument of Lemma 2.1 in Chen and Barbour([2]). □

Lemma 4.6. *Let $a \geq 0$.*

(i) *If* $\frac{\delta}{(1+a)^3} < 1$, *then*
$\frac{4}{n-1} - \frac{17\delta}{16(n-1)(1+a)} \leq E(\widehat{V}_a(\rho) - V_a(\rho))^2 \leq \frac{4}{n-1}$.

(ii) $E|\widehat{V}_a(\rho) - V_a(\rho)|^3 \leq \frac{4\delta}{n}$.

(iii) $E|\Delta V_a|^3 \leq \frac{32\delta}{n}$ *where* $\Delta V_a := V_a(\rho) - V_a(\tau)$.

(iv) $E|\Delta\widehat{V}_a|^3 \leq \frac{32\delta}{n}$ *where* $\Delta\widehat{V}_a := \widehat{V}_a(\rho) - V_a(\tau)$.

Proof. Let $a \geq 0$ and let

$$\hat{y}_{a,ij} := x_{ij}1(|x_{ij}| > 1+a), \quad \alpha_a := \sum_{i,j} x_{ij}^2 1(|x_{ij}| > 1+a),$$

and $$\Lambda_a := \{(i,j) \mid |x_{ij}| > 1+a\}.$$

Then, by (4.5),

$$|\Lambda_a| \le \frac{1}{(1+a)^3} \sum_{(i,j)\in\Lambda_a} |x_{ij}|^3 \le \frac{1}{(1+a)^3} \sum_{i,j} |x_{ij}|^3 = \frac{n\beta}{(1+a)^3} \le \frac{n\delta}{16(1+a)^3}, \tag{4.7}$$

and

$$\alpha_a \le \frac{1}{1+a} \sum_{i,j} |x_{ij}|^3 \mathit{1}(|x_{ij}| > 1+a) \le \frac{n\beta}{1+a} \le \frac{n\delta}{16(1+a)}. \tag{4.8}$$

By the fact that

$$\begin{aligned}
&E(x_{IM} + x_{JL} - x_{IL} - x_{JM})(\hat{y}_{a,IM} + \hat{y}_{a,JL} - \hat{y}_{a,IL} - \hat{y}_{a,JM}) \\
&= E\big[x_{IM}\hat{y}_{a,IM} + x_{IM}\hat{y}_{a,JL} - x_{IM}\hat{y}_{a,IL} - x_{IM}\hat{y}_{a,JM} \\
&\qquad + x_{JL}\hat{y}_{a,IM} + x_{JL}\hat{y}_{a,JL} - x_{JL}\hat{y}_{a,IL} - x_{JL}\hat{y}_{a,JM} \\
&\qquad - x_{IL}\hat{y}_{a,IM} - x_{IL}\hat{y}_{a,JL} + x_{IL}\hat{y}_{a,IL} + x_{IL}\hat{y}_{a,JM} \\
&\qquad - x_{JM}\hat{y}_{a,IM} - x_{JM}\hat{y}_{a,JL} + x_{JM}\hat{y}_{a,IL} + x_{JM}\hat{y}_{a,JM}\big] \\
&= E\big[4x_{IM}\hat{y}_{a,IM} + 4x_{IM}\hat{y}_{a,JL} - 4x_{IM}\hat{y}_{a,IL} - 4x_{IM}\hat{y}_{a,JM}\big]
\end{aligned}$$

and

$$\begin{aligned}
&E(\hat{y}_{a,IM} + \hat{y}_{a,JL} - \hat{y}_{a,IL} - \hat{y}_{a,JM})^2 \\
&= E\big[4\hat{y}^2_{a,IM} + 4\hat{y}_{IM}\hat{y}_{a,JL} - 4\hat{y}_{IM}\hat{y}_{a,IL} - 4\hat{y}_{IM}\hat{y}_{a,JM}\big] \\
&= E\big[4x_{IM}\hat{y}_{a,IM} + 4\hat{y}_{IM}\hat{y}_{a,JL} - 4\hat{y}_{IM}\hat{y}_{a,IL} - 4\hat{y}_{IM}\hat{y}_{a,JM}\big],
\end{aligned}$$

we have

$$\begin{aligned}
&E(\widehat{V}_a(\rho) - V_a(\rho))^2 \\
&= E\big\{(x_{IM} + x_{JL} - x_{IL} - x_{JM}) - (\hat{y}_{a,IM} + \hat{y}_{a,JL} - \hat{y}_{a,IL} - \hat{y}_{a,JM})\big\}^2 \\
&= E(\widehat{V}(\rho) - V(\rho))^2 \\
&\qquad - 2E(x_{IM} + x_{JL} - x_{IL} - x_{JM})(\hat{y}_{a,IM} + \hat{y}_{a,JL} - \hat{y}_{a,IL} - \hat{y}_{a,JM}) \\
&\qquad + E(\hat{y}_{a,IM} + \hat{y}_{a,JL} - \hat{y}_{a,IL} - \hat{y}_{a,JM})^2 \\
&= E(\widehat{V}(\rho) - V(\rho))^2 - 4E(x_{IM}\hat{y}_{a,IM}) - 8E(x_{IM}\hat{y}_{a,JL}) + 8E(x_{IM}\hat{y}_{a,IL}) \\
&\qquad + 8E(x_{IM}\hat{y}_{a,JM}) + 4E(\hat{y}_{a,IM}\hat{y}_{a,JL}) - 4E(\hat{y}_{a,IM}\hat{y}_{a,IL}) - 4E(\hat{y}_{a,IM}\hat{y}_{a,JM}).
\end{aligned} \tag{4.9}$$

To give a bound of $E(\widehat{V}_a(\rho) - V_a(\rho))^2$, we have to bound every terms on the right hand side of (4.9). Note that

$$
\begin{aligned}
E(x_{IM}\hat{y}_{a,IM}) &= \frac{1}{n^2}\sum_{i,m} x_{im}\hat{y}_{a,im} \\
&= \frac{1}{n^2}\sum_{i,m} x_{im}^2 \mathit{1}(|x_{im}| > 1+a) \\
&= \frac{\alpha_a}{n^2}, \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
E(x_{IM}\hat{y}_{a,JL}) &= Ex_{IM}x_{JL}\mathit{1}(|x_{JL}| > 1+a) \\
&= \frac{1}{n^2(n-1)^2}\sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{l,m\\ l\neq m}} x_{im}x_{jl}\mathit{1}(|x_{jl}| > 1+a) \\
&= \frac{1}{n^2(n-1)^2}\Big[\sum_{i,j}\sum_{\substack{l,m\\ l\neq m}} x_{im}x_{jl}\mathit{1}(|x_{jl}| > 1+a) \\
&\qquad - \sum_{i}\sum_{\substack{l,m\\ l\neq m}} x_{im}x_{il}\mathit{1}(|x_{il}| > 1+a)\Big] \\
&= -\frac{1}{n^2(n-1)^2}\sum_{i}\sum_{\substack{l,m\\ l\neq m}} x_{im}x_{il}\mathit{1}(|x_{il}| > 1+a) \qquad \text{(by (4.4))} \\
&= -\frac{1}{n^2(n-1)^2}\Big[\sum_{i}\sum_{l,m} x_{im}x_{il}\mathit{1}(|x_{il}| > 1+a) \\
&\qquad - \sum_{i}\sum_{m} x_{im}^2\mathit{1}(|x_{im}| > 1+a)\Big] \\
&= \frac{1}{n^2(n-1)^2}\sum_{i,m} x_{im}^2\mathit{1}(|x_{im}| > 1+a) \qquad \text{(by (4.4))} \\
&= \frac{\alpha_a}{n^2(n-1)^2}, \quad \text{and} \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
E(x_{IM}\hat{y}_{a,JM}) &= \frac{1}{n^2(n-1)}\sum_{m}\sum_{\substack{i,j\\ i\neq j}} x_{im}\hat{y}_{a,jm} \\
&= \frac{1}{n^2(n-1)}\Big[\sum_{m}\sum_{i,j} x_{im}\hat{y}_{a,jm} - \sum_{m}\sum_{i} x_{im}\hat{y}_{a,im}\Big] \\
&= -\frac{1}{n^2(n-1)}\sum_{i,m} x_{im}^2\mathit{1}(|x_{im}| > 1+a) \qquad \text{(by (4.4))} \\
&= -\frac{\alpha_a}{n^2(n-1)}. \qquad (4.12)
\end{aligned}
$$

Similarly, we can show that $$E(x_{IM}\hat{y}_{a,IL}) = -\frac{\alpha_a}{n^2(n-1)}. \tag{4.13}$$

By definition of Λ_a, we have

$$\begin{aligned}
|E(\hat{y}_{a,IM}\hat{y}_{a,JL})| &\leq \frac{1}{n^2(n-1)^2}\sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{l,m\\ l\neq m}}|\hat{y}_{a,im}\hat{y}_{a,jl}| \\
&\leq \frac{1}{n^2(n-1)^2}\Big(\sum_{i,j}|\hat{y}_{ij}|\Big)^2 \\
&= \frac{1}{n^2(n-1)^2}\Big[\sum_{i,j}|x_{ij}|\mathit{1}(|x_{ij}|>1+a)\Big]^2 \\
&\leq \frac{|\Lambda_a|}{n^2(n-1)^2}\sum_{i,j}|x_{ij}|^2\mathit{1}(|x_{ij}|>1+a) \\
&= \frac{|\Lambda_a|\alpha_a}{n^2(n-1)^2}, \qquad (4.14)
\end{aligned}$$

and $|E(\hat{y}_{a,IM}\hat{y}_{a,IL})|$ and $|E(\hat{y}_{a,IM}\hat{y}_{a,JM})|$ are also less than or equal $\dfrac{|\Lambda_a|\alpha_a}{n^2(n-1)}$. (4.15)

By (4.7) – (4.15) and Lemma 4.5, we have

$$\begin{aligned}
&E(\widehat{V}_a(\rho)-V_a(\rho))^2 \\
&\leq \frac{4}{n-1}-\frac{4\alpha_a}{n^2}-\frac{8\alpha_a}{n^2(n-1)^2}-\frac{16\alpha_a}{n^2(n-1)}+\frac{4|\Lambda_a|\alpha_a}{n^2(n-1)^2}+\frac{8|\Lambda_a|\alpha_a}{n^2(n-1)} \\
&= \frac{4}{n-1}+\frac{4\alpha_a}{n^2}\Big(-1-\frac{2}{(n-1)^2}-\frac{4}{n-1}+\frac{|\Lambda_a|}{(n-1)^2}+\frac{2|\Lambda_a|}{n-1}\Big) \\
&\leq \frac{4}{n-1}+\frac{4\alpha_a}{n^2}\Big(-1-\frac{2}{(n-1)^2}-\frac{4}{n-1}+\frac{n\delta}{16(1+a)^3(n-1)^2}+\frac{n\delta}{8(1+a)^3(n-1)}\Big).
\end{aligned} \tag{4.16}$$

(i) Since $\dfrac{\delta}{(1+a)^3}<1$ and (4.16), we have

$$\begin{aligned}
&E(\widehat{V}_a(\rho)-V_a(\rho))^2 \\
&\leq \frac{4}{n-1}+\frac{4\alpha_a}{n^2}\Big(-1-\frac{2}{(n-1)^2}-\frac{4}{n-1}+\frac{n}{16(n-1)^2}+\frac{n}{8(n-1)}\Big) \\
&= \frac{4}{n-1}+\frac{4\alpha_a}{n^2}\Big(\frac{-16n^2-29n+14}{16(n-1)^2}\Big) \qquad (4.17)\\
&\leq \frac{4}{n-1}.
\end{aligned}$$

Similarly, by (4.7)-(4.15) and Lemma 4.5, we have

$$
\begin{aligned}
&E(\widehat{V}_a(\rho) - V_a(\rho))^2 \\
&\geq \frac{4}{n-1} - \frac{4\alpha_a}{n^2} - \frac{8\alpha_a}{n^2(n-1)^2} - \frac{16\alpha_a}{n^2(n-1)} - \frac{4|\Lambda_a|\alpha_a}{n^2(n-1)^2} - \frac{8|\Lambda_a|\alpha_a}{n^2(n-1)} \\
&= \frac{4}{n-1} - \frac{4\alpha_a}{n^2(n-1)^2}\Big((n-1)^2 + 2 + 4(n-1) + |\Lambda_a| + 2|\Lambda_a|(n-1)\Big) \\
&\geq \frac{4}{n-1} - \frac{4\alpha_a}{n^2(n-1)^2}\Big((n-1)^2 + 2 + 4(n-1) + \frac{n\delta}{16(1+a)^3} + \frac{n(n-1)\delta}{8(1+a)^3}\Big) \\
&\geq \frac{4}{n-1} - \frac{4\alpha_a}{n^2(n-1)^2}\Big((n-1)^2 + 2 + 4(n-1) + \frac{n}{16} + \frac{n(n-1)}{8}\Big) \\
&= \frac{4}{n-1} - \frac{4\alpha_a}{n^2(n-1)^2}\Big(\frac{9n^2}{8} + \frac{31n}{16} - 1\Big) \\
&\geq \frac{4}{n-1} - \frac{17n^2\alpha_a}{2n^2(n-1)^2} \\
&= \frac{4}{n-1} - \frac{17\alpha_a}{2(n-1)^2} \\
&\geq \frac{4}{n-1} - \frac{17n\delta}{32(n-1)^2(1+a)} \\
&\geq \frac{4}{n-1} - \frac{17\delta}{16(n-1)(1+a)},
\end{aligned}
$$

which proves (i).

$$
\begin{aligned}
(ii)\quad E|\widehat{V}_a(\rho) - V_a(\rho)|^3 &= E|x_{a,IL} + x_{a,JM} - x_{a,IM} - x_{a,JL}|^3 \\
&\leq 16E\big(|x_{a,IL}|^3 + |x_{a,JM}|^3 + |x_{a,IM}|^3 + |x_{a,JL}|^3\big) \\
&= 64E|x_{a,IL}|^3 \\
&= \frac{64}{n^2}\sum_{i,j}|x_{a,ij}|^3 \\
&\leq \frac{64}{n^2}\sum_{i,j}|x_{ij}|^3 \\
&= \frac{64\beta}{n} \\
&\leq \frac{4\delta}{n}.
\end{aligned}
$$

$$
\begin{aligned}
(iii)\quad E|\Delta V_a|^3 = E|&x_{a,\tau^{-1}(L)\tau(I)} + x_{a,\tau^{-1}(M)\tau(J)} + x_{a,IL} + x_{a,JM} \\
&- x_{a,I\tau(I)} - x_{a,J\tau(J)} - x_{a,\tau^{-1}(L)L} - x_{a,\tau^{-1}(M)M}|^3
\end{aligned}
$$

$$\leq 64E\big(|x_{a,\tau^{-1}(L)\tau(I)}|^3 + |x_{a,\tau^{-1}(M)\tau(J)}|^3 + |x_{a,IL}|^3 + |x_{a,JM}|^3 + |x_{a,I\tau(I)}|^3 + |x_{a,J\tau(J)}|^3 + |x_{a,\tau^{-1}(L)L}|^3 + |x_{a,\tau^{-1}(M)M}|^3\big). \tag{4.18}$$

To bound (4.18), we note that

$$\begin{aligned} E|x_{a,IL}|^3 &= \frac{1}{n^2}\sum_{i,l}|x_{a,il}|^3 \\ &\leq \frac{1}{n^2}\sum_{i,l}|x_{il}|^3 \\ &= \frac{\beta}{n}. \end{aligned} \tag{4.19}$$

Similarly, we have

$$E|x_{a,JM}|^3 \leq \frac{\beta}{n}. \tag{4.20}$$

Since I and τ are independent, so

$$\begin{aligned} E|x_{a,I\tau(I)}|^3 &= \frac{1}{n}\sum_{i}E|x_{a,i\tau(i)}|^3 \\ &= \frac{1}{n^2}\sum_{i,j}|x_{a,ij}|^3 \\ &\leq \frac{\beta}{n}. \end{aligned} \tag{4.21}$$

By the same argument and using the fact that I, L and τ are independent, we have

$$E|X_{a,\tau^{-1}(L)L}|^3 \text{ and } E|X_{a,\tau^{-1}(L)\tau(J)}|^3 \text{ are less than } \frac{\beta}{n}. \tag{4.22}$$

By the same argument and using the fact that J, M and τ are independent, we have

$$E|X_{a,J\tau(J)}|^3, E|X_{a,\tau^{-1}(M)M}|^3 \text{ and } E|X_{a,\tau^{-1}(M)\tau(J)}|^3 \text{ are less than } \frac{\beta}{n}. \tag{4.23}$$

Therefore, by (4.18) – (4.23),

$$E|\Delta V_a|^3 \leq \frac{512\beta}{n} \leq \frac{32\delta}{n}.$$

(iv) We can prove by the same argument as (iii). □

Lemma 4.7. *Let $a \geq 0$ be such that $\frac{\delta}{(1+a)^3} < 1$. Then*

(i) $|EV_a(\rho)| \leq \frac{\delta}{16(1+a)^2}$,

(ii) $EV_a^2(\rho) \leq 1$,

and there exists a constant C such that

(iii) $|EV_a^3(\rho)| \leq C\delta$,

(iv) $|EV_a^4(\rho)| \leq 3 + C(1+a)\delta$.

Proof. For $m \in \mathbb{N}$, let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h(\widehat{v}, v) = (\widehat{v} - v)(\widehat{v}^{m-1} + v^{m-1}).$$

Then h is antisymmetric. Since $(\widehat{V}_a(\rho), V_a(\rho))$ is an exchangeable pair, by Proposition 2.10, we have

$$\begin{aligned}
0 &= E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{m-1}(\rho) + V_a^{m-1}(\rho)) \\
&= E(\widehat{V}_a(\rho) - V_a(\rho))\Big\{2V_a^{m-1}(\rho) + (\widehat{V}_a^{m-1}(\rho) - V_a^{m-1}(\rho))\Big\} \\
&= 2E(\widehat{V}_a(\rho) - V_a(\rho))V_a^{m-1}(\rho) + E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{m-1}(\rho) - V_a^{m-1}(\rho)) \\
&= 2E(E^{\rho}(\widehat{V}_a(\rho)) - V_a(\rho))V_a^{m-1}(\rho) + E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{m-1}(\rho) - V_a^{m-1}(\rho)) \quad (4.24) \\
&= 2E\Big\{(-\frac{2}{n-1}V_a(\rho) + \frac{2}{n(n-1)}\sum_{i,j} x_{a,ij})V_a^{m-1}(\rho)\Big\} \\
&\quad + E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{m-1}(\rho) - V_a^{m-1}(\rho)) \qquad \text{(by Lemma 4.4)} \\
&= -\frac{4}{n-1}EV_a^{m}(\rho) + \frac{4}{n(n-1)}\sum_{i,j} x_{a,ij}EV_a^{m-1}(\rho) \\
&\quad + E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{m-1}(\rho) - V_a^{m-1}(\rho)).
\end{aligned}$$

Hence

$$EV_a^m(\rho) = \frac{n-1}{4}E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{m-1}(\rho) - V_a^{m-1}(\rho)) + \frac{1}{n}\sum_{i,j} x_{a,ij}EV_a^{m-1}(\rho). \quad (4.25)$$

(i) Since

$$\begin{aligned}
|\frac{1}{n}\sum_{i,j} x_{a,ij}| &= \frac{1}{n}|\sum_{i,j} x_{ij}\mathit{1}(|x_{ij}| \le 1 + a)| \\
&= \frac{1}{n}|\sum_{i,j} x_{ij}\mathit{1}(|x_{ij}| > 1 + a)| \qquad \text{(by (4.4))} \\
&\le \frac{1}{n}\sum_{i,j} |x_{ij}|\mathit{1}(|x_{ij}| > 1 + a) \\
&\le \frac{1}{n(1+a)^2}\sum_{i,j} |x_{ij}|^3 \\
&= \frac{\beta}{(1+a)^2} \\
&\le \frac{\delta}{16(1+a)^2} \qquad (4.26)
\end{aligned}$$

and (4.25), we have $|EV_a(\rho)| \le \dfrac{\delta}{16(1+a)^2}$.

(ii) We note that

$$
\begin{aligned}
|\frac{1}{n}\sum_{i,j} x_{a,ij} EV_a(\rho)| &= (\frac{1}{n}\sum_{i,j} x_{a,ij})^2 \\
&= \frac{1}{n^2}(\sum_{(i,j)\in\Lambda_a} x_{ij}\mathit{1}(|x_{ij}|>1+a))^2 \\
&\le \frac{|\Lambda_a|}{n^2}\sum_{(i,j)\in\Lambda_a} |x_{ij}|^2\mathit{1}(|x_{ij}|>1+a) \\
&\le \frac{|\Lambda_a|\alpha_a}{n^2}. \qquad (4.27)
\end{aligned}
$$

By (4.8), (4.17), (4.25) and (4.27), we have

$$
\begin{aligned}
EV_a^2(\rho) &= \frac{n-1}{4}E(\widehat{V}_a(\rho)-V_a(\rho))^2 + \frac{1}{n}\sum_{i,j} x_{a,ij}EV_a(\rho) \\
&\le 1 + \frac{\alpha_a}{n}\Big(\frac{-16n^2-29n+14}{16(n-1)^2}\Big) + \frac{\alpha_a}{16n} \\
&\le 1.
\end{aligned}
$$

(iii) By (4.6), (4.25), (4.26), Lemma 4.6, (i) and (ii), we have

$$
\begin{aligned}
|EV_a^3(\rho)| &\le \frac{n-1}{4}|E(\widehat{V}_a(\rho)-V_a(\rho))(\widehat{V}_a^2(\rho)-V_a^2(\rho))| + |\frac{1}{n}\sum_{i,j} x_{a,ij}EV_a^2(\rho)| \\
&\le \frac{n-1}{4}|E(\widehat{V}_a(\rho)-V_a(\rho))^2(\widehat{V}_a(\rho)+V_a(\rho))| + |\frac{1}{n}\sum_{i,j} x_{a,ij}| \\
&\le \frac{n-1}{4}|E(\widehat{V}_a(\rho)-V_a(\rho))^2(2V_a(\tau)+\Delta\widehat{V}_a+\Delta V_a)| + \frac{\delta}{16(1+a)^2} \\
&\le \frac{n-1}{2}|E(\widehat{V}_a(\rho)-V_a(\rho))^2V_a(\tau)| + \frac{n-1}{4}E(\widehat{V}_a(\rho)-V_a(\rho))^2|\Delta\widehat{V}_a| \\
&\quad + \frac{n-1}{4}E(\widehat{V}_a(\rho)-V_a(\rho))^2|\Delta V_a| + \frac{\delta}{16(1+a)^2} \\
&\le \frac{n-1}{2}E(\widehat{V}_a(\rho)-V_a(\rho))^2|EV_a(\tau)| \\
&\quad + \frac{n-1}{4}\{E|\widehat{V}_a(\rho)-V_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta\widehat{V}_a|^3\}^{\frac{1}{3}} \\
&\quad + \frac{n-1}{4}\{E|\widehat{V}_a(\rho)-V_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta V_a|^3\}^{\frac{1}{3}} + \delta \\
&\le C\delta.
\end{aligned}
$$

(*iv*) By (4.25), (4.26) and (*iii*), we have

$$\begin{aligned} EV_a^4(\rho) &\le \frac{n-1}{4}|E(\widehat{V}_a(\rho)-V_a(\rho))(\widehat{V}_a^3(\rho)-V_a^3(\rho))| + |\frac{1}{n}\sum_{i,j} x_{a,ij}EV_a^3(\rho)| \\ &\le \frac{n-1}{4}|E(\widehat{V}_a(\rho)-V_a(\rho))(\widehat{V}_a^3(\rho)-V_a^3(\rho))| + \frac{C\delta^2}{(1+a)^2} \\ &\le \frac{n-1}{4}|E(\widehat{V}_a(\rho)-V_a(\rho))(\widehat{V}_a^3(\rho)-V_a^3(\rho))| + C(1+a)\delta. \end{aligned} \tag{4.28}$$

Then (*iv*) follows from Lemma 4.6, (4.28) and the following.

$$\begin{aligned} &|E(\widehat{V}_a(\rho)-V_a(\rho))(\widehat{V}_a^3(\rho)-V_a^3(\rho))| \\ &= |E(\widehat{V}_a(\rho)-V_a(\rho))^2\{(\widehat{V}_a(\rho)-V_a(\rho))^2 + 3\widehat{V}_a(\rho)V_a(\rho)\}| \\ &\le E(\widehat{V}_a(\rho)-V_a(\rho))^4 + 3E(\widehat{V}_a(\rho)-V_a(\rho))^2V_a^2(\tau) \\ &\quad + 3E(\widehat{V}_a(\rho)-V_a(\rho))^2|V_a(\tau)(\Delta V_a + \Delta\widehat{V}_a)| + 3E(\widehat{V}_a(\rho)-V_a(\rho))^2|\Delta V_a\Delta\widehat{V}_a| \\ &\le 4(1+a)E|\widehat{V}_a(\rho)-V_a(\rho)|^3 + 3E(\widehat{V}_a(\rho)-V_a(\rho))^2EV_a^2(\tau) \\ &\quad + 3E(\widehat{V}_a(\rho)-V_a(\rho))^2|V_a(\tau)||(\Delta V_a + \Delta\widehat{V}_a)| \\ &\quad + 24(1+a)E(\widehat{V}_a(\rho)-V_a(\rho))^2|\Delta V_a| \\ &\le 4(1+a)E|\widehat{V}_a(\rho)-V_a(\rho)|^3 + \frac{12}{n-1} \\ &\quad + 3\{E|\widehat{V}_a(\rho)-V_a(\rho)|^3|V_a(\tau)|^{\frac{3}{2}}\}^{\frac{2}{3}}\{E|\Delta V_a + \Delta\widehat{V}_a|^3\}^{\frac{1}{3}} \\ &\quad + 24(1+a)\{E|\widehat{V}_a(\rho)-V_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta V_a|^3\}^{\frac{1}{3}} \qquad \text{(by 2. and Lemma 4.6(1))} \\ &\le 4(1+a)E|\widehat{V}_a(\rho)-V_a(\rho)|^3 + \frac{12}{n-1} \\ &\quad + 3\{E|\widehat{V}_a(\rho)-V_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta V_a + \Delta\widehat{V}_a|^3\}^{\frac{1}{3}} \\ &\quad + 24(1+a)\{E|\widehat{V}_a(\rho)-V_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta V_a|^3\}^{\frac{1}{3}} \\ &\le \frac{C(1+a)\delta}{n} + \frac{12}{n-1}. \end{aligned}$$

□

Lemma 4.8. *Let $a \ge 0$ be such that $\frac{\delta}{(1+a)^3} < 1$. Then there exist constants C_k such that*

$$|EV_a^k(\rho)| \le C_k + C_k(1+a)^{k-3}\delta + C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta$$

for $k = 1, 2, \ldots, 17$.

Proof. We shall prove by induction. The basis step follows from Lemma 4.7(*i*).

Let $k \in \{1, 2, \ldots, 16\}$, by Lemma 4.7, we may assume that $5 \leq k \leq 16$.

Assume that there exist constants C_m such that

$$|EV_a^m(\rho)| \leq C_m + C_m(1+a)^{m-3}\delta + C_m(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{m-5}(1+a)^{m-3}\delta \tag{4.29}$$

for $m = 1, 2, \ldots, k-1$.

Case 1. $n^{\frac{1}{3}}\delta^{\frac{2}{3}} > 1$.

It suffices to show that

$$|EV_a^k(\rho)| \leq C_k + C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta.$$

Since $n^{\frac{1}{3}}\delta^{\frac{2}{3}} > 1$, by (4.29), we have

$$|EV_a^m(\rho)| \leq C_m + C_m(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{m-5}(1+a)^{m-3}\delta \tag{4.30}$$

for $m = 1, 2, \ldots, k-1$.

By (4.25),

$$|EV_a^k(\rho)| \leq \frac{n-1}{4}|E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{k-1}(\rho) - V_a^{k-1}(\rho))| + |\frac{1}{n}\sum_{i,j} x_{a,ij}EV_a^{k-1}(\rho)|. \tag{4.31}$$

By (4.26), (4.30) and the fact that $\delta < (1+a)^3$, we have

$$\begin{aligned}
|\frac{1}{n}\sum_{i,j} x_{a,ij}EV_a^{k-1}(\rho)| &\leq \frac{C\delta}{(1+a)^2}(C_{k-1} + C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta) \\
&= \frac{C_k\delta}{(1+a)^2} + C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-6}\delta^2 \\
&\leq C_k(1+a)^{k-3}\delta + C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta \\
&\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta. \qquad (4.32)
\end{aligned}$$

Next, we consider

$$\begin{aligned}
&\frac{n-1}{4}\Big|E(\widehat{V}_a(\rho) - V_a(\rho))(\widehat{V}_a^{k-1}(\rho) - V_a^{k-1}(\rho))\Big| \\
&= \frac{n-1}{4}\Big|E(\widehat{V}_a(\rho) - V_a(\rho))^2\sum_{i=0}^{k-2}\widehat{V}_a^i(\rho)V_a^{(k-2)-i}(\rho)\Big| \\
&= \frac{n-1}{4}\Big|E(\widehat{V}_a(\rho) - V_a(\rho))^2\sum_{i=0}^{k-2}\Big(\Delta\widehat{V}_a + V_a(\tau)\Big)^i\Big(\Delta V_a + V_a(\tau)\Big)^{(k-2)-i}\Big| \\
&\leq C_k(n-1)E(\widehat{V}_a(\rho) - V_a(\rho))^2\sum_{i=0}^{k-2}\Big(|\Delta\widehat{V}_a|^i + |V_a(\tau)|^i\Big)\Big(|\Delta V_a|^{(k-2)-i} + |V_a(\tau)|^{(k-2)-i}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= C_k(n-1)E(\widehat{V}_a(\rho) - V_a(\rho))^2 \sum_{i=0}^{k-2} \Big(|\Delta\widehat{V}_a|^i |\Delta V_a|^{(k-2)-i} + |\Delta\widehat{V}_a|^i |V_a(\tau)|^{(k-2)-i} \\
&\qquad\qquad + |V_a(\tau)|^i |\Delta V_a|^{(k-2)-i} + |V_a(\tau)|^{k-2} \Big) \\
&= C_k(n-1)\Big\{ \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|\Delta\widehat{V}_a|^i |\Delta V_a|^{(k-2)-i}) \\
&\qquad + \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|\Delta\widehat{V}_a|^i |V_a(\tau)|^{(k-2)-i}) \\
&\qquad + \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|V_a(\tau)|^i |\Delta V_a|^{(k-2)-i}) \\
&\qquad + \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 |V_a(\tau)|^{k-2} \Big\} \\
&= C_k(n-1)\Big\{ \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|\Delta\widehat{V}_a|^i |\Delta V_a|^{(k-2)-i}) \\
&\qquad + \sum_{i=1}^{k-3} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|V_a(\tau)|^i |\Delta V_a|^{(k-2)-i}) \\
&\qquad + \sum_{i=1}^{k-3} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|V_a(\tau)|^i |\Delta\widehat{V}_a|^{(k-2)-i}) \\
&\qquad + \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 |V_a(\tau)|^{k-2} \Big\} \\
&:= B_1 + B_2 + B_3 + B_4 \qquad (4.33)
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &:= C_k(n-1) \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|\Delta\widehat{V}_a|^i |\Delta V_a|^{(k-2)-i}) \\
B_2 &:= C_k(n-1) \sum_{i=1}^{k-3} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|V_a(\tau)|^i |\Delta V_a|^{(k-2)-i}) \\
B_3 &:= C_k(n-1) \sum_{i=1}^{k-3} E(\widehat{V}_a(\rho) - V_a(\rho))^2 (|V_a(\tau)|^i |\Delta\widehat{V}_a|^{(k-2)-i}) \\
B_4 &:= C_k(n-1) \sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 |V_a(\tau)|^{k-2}.
\end{aligned}
$$

Since $\Delta V_a = V_a(\rho) - V_a(\tau)$ and $\Delta \widehat{V}_a = \widehat{V}_a(\rho) - V_a(\tau)$,

$$\begin{aligned}|\Delta V_a| &= |x_{a,\tau^{-1}(L)\tau(I)} + x_{a,\tau^{-1}(M)\tau(J)} + x_{a,IL} + x_{a,JM} \\ &\qquad - x_{a,I\tau(I)} - x_{a,J\tau(J)} - x_{a,\tau^{-1}(L)L} - x_{a,\tau^{-1}(M)M}| \\ &\leq 8(1+a) \qquad (4.34)\end{aligned}$$

and

$$\begin{aligned}|\Delta \widehat{V}_a| &= |x_{a,\tau^{-1}(L)\tau(I)} + x_{a,\tau^{-1}(M)\tau(J)} + x_{a,IM} + x_{a,JL} \\ &\qquad - x_{a,I\tau(I)} - x_{a,J\tau(J)} - x_{a,\tau^{-1}(L)L} - x_{a,\tau^{-1}(M)M}| \\ &\leq 8(1+a). \qquad (4.35)\end{aligned}$$

By (4.34) and (4.35), we have

$$B_1 \leq C_k(n-1)(1+a)^{k-3}E(\widehat{V}_a(\rho) - V_a(\rho))^2|\Delta V_a|$$

or

$$B_1 \leq C_k(n-1)(1+a)^{k-3}E(\widehat{V}_a(\rho) - V_a(\rho))^2|\Delta \widehat{V}_a|.$$

From these and the fact that

$$E(\widehat{V}_a(\rho) - V_a(\rho))^2|\Delta \widehat{V}_a| \leq \{E|\widehat{V}_a(\rho) - V_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta \widehat{V}_a|^3\}^{\frac{1}{3}} \leq \frac{C\delta}{n}$$

and

$$E(\widehat{V}_a(\rho) - V_a(\rho))^2|\Delta V_a| \leq \{E|\widehat{V}_a(\rho) - V_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta \widehat{V}_a|^3\}^{\frac{1}{3}} \leq \frac{C\delta}{n},$$

we have

$$B_1 \leq C_k(1+a)^{k-3}\delta \leq (n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta. \qquad (4.36)$$

By Lemma 4.6(i) and (4.30),

$$\begin{aligned}B_4 &= C_k(n-1)\sum_{i=0}^{k-2} E(\widehat{V}_a(\rho) - V_a(\rho))^2 E|V_a(\tau)|^{k-2} \\ &\leq C_k E|V_a(\tau)|^{k-2} \\ &\leq \begin{cases} C_k|EV_a(\tau)|^{k-2} & \text{if } k \text{ is even} \\ C_k(|EV_a(\tau)|^{k-1})^{\frac{k-2}{k-1}} & \text{if } k \text{ is odd} \end{cases} \\ &\leq \begin{cases} C_k(C_{k-2} + C_{k-2}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta) & \text{if } k \text{ is even} \\ C_k(C_{k-1} + C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta)^{\frac{k-2}{k-1}} & \text{if } k \text{ is odd} \end{cases}\end{aligned}$$

$$\le \begin{cases} C_k(C_{k-2} + C_{k-2}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta) & \text{if } k \text{ is even} \\ C_k(C_{k-1} + C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta) & \text{if } k \text{ is odd} \end{cases}$$

$$\le C_k + C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta. \tag{4.37}$$

To bound B_2, let

$$B_2 := \sum_{i=1}^{k-3} B_{2,i}$$

where $\quad B_{2,i} = C_k(n-1)E(\widehat{V}_a(\rho) - V_a(\rho))^2(|V_a(\tau)|^i|\Delta V_a|^{(k-2)-i})$.

Case 1.1 $\dfrac{2(k-2)}{i} > 3$.

By Lemma 4.6(i), (ii) and (4.30),

$$\begin{aligned} B_{2,i} &\le C_k(n-1)\big\{E|\widehat{V}_a(\rho) - V_a(\rho)|^{\frac{2(k-2)}{i}}|V_a(\tau)|^{k-2}\big\}^{\frac{i}{k-2}}\big\{E|\Delta V_a|^{k-2}\big\}^{\frac{(k-2)-i}{k-2}} \\ &\le C_k(n-1)\big\{E|\widehat{V}_a(\rho) - V_a(\rho)|^{\frac{2(k-2)}{i}}\big\}^{\frac{i}{k-2}}\big\{E|V_a(\tau)|^{k-2}\big\}^{\frac{i}{k-2}}\big\{E|\Delta V_a|^{k-2}\big\}^{\frac{(k-2)-i}{k-2}} \\ &\le C_k(n-1)\big\{(1+a)^{\frac{2(k-2)}{i}-3}E|\widehat{V}_a(\rho) - V_a(\rho)|^3\big\}^{\frac{i}{k-2}}\big\{E|V_a(\tau)|^{k-2}\big\}^{\frac{i}{k-2}} \\ &\qquad\qquad \times \big\{(1+a)^{k-5}E|\Delta V_a|^3\big\}^{\frac{(k-2)-i}{k-2}} \\ &\le C_k(n-1)\big\{(1+a)^{\frac{2(k-2)}{i}-3}\frac{\delta}{n}\big\}^{\frac{i}{k-2}}\big\{E|V_a(\tau)|^{k-2}\big\}^{\frac{i}{k-2}}\big\{(1+a)^{k-5}\frac{\delta}{n}\big\}^{\frac{(k-2)-i}{k-2}} \\ &= C_k(1+a)^{k-3-i}\delta\big\{E|V_a(\tau)|^{k-2}\big\}^{\frac{i}{k-2}}. \end{aligned}$$

Case 1.1.1 k is odd.

$$\begin{aligned} B_{2,i} &\le C_k(1+a)^{k-3-i}\delta\big\{E|V_a(\tau)|^{k-1}\big\}^{\frac{i}{k-1}} \\ &\le C_k(1+a)^{k-3-i}\delta(C_{k-1} + C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta)^{\frac{i}{k-1}}. \end{aligned}$$

If $C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta \le 1$, then

$$B_{2,i} \le C_k(1+a)^{k-3-i}\delta \le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta.$$

If $C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta > 1$, then

$$\begin{aligned} B_{2,i} &\le C_k(1+a)^{k-3-i}\delta((n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta)^{\frac{i}{k-1}} \\ &\le C_k(1+a)^{k-3-i}\delta((n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-1})^{\frac{i}{k-1}} \\ &\le C_k(1+a)^{k-3}\delta(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6} \\ &\le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta. \end{aligned}$$

Case 1.1.2 k is even.

$$\begin{aligned} B_{2,i} &\le C_k(1+a)^{k-3-i}\delta\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \\ &\le C_k(1+a)^{k-3-i}\delta(C_{k-2}+C_{k-2}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta)^{\frac{i}{k-2}}. \end{aligned}$$

If $C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta \le 1$, then

$$B_{2,i} \le C_k(1+a)^{k-3-i}\delta \le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta.$$

If $C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta > 1$, then

$$\begin{aligned} B_{2,i} &\le C_k(1+a)^{k-3-i}\delta((n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta)^{\frac{i}{k-2}} \\ &\le C_k(1+a)^{k-3-i}\delta((n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-2})^{\frac{i}{k-2}} \\ &\le C_k(1+a)^{k-3}\delta(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7} \\ &\le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta. \end{aligned}$$

Case 1.2 $\dfrac{2(k-2)}{i} \le 3$.

By Lemma 4.6(i), (ii) and (4.30),

$$\begin{aligned} B_{2,i} &\le C_k(n-1)\{E|\widehat{V}_a(\rho)-V_a(\rho)|^{\frac{2(k-2)}{i}}\}^{\frac{i}{k-2}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}}\{E|\Delta V_a|^{k-2}\}^{\frac{(k-2)-i}{k-2}} \\ &\le C_k(n-1)\{E|\widehat{V}_a(\rho)-V_a(\rho)|^{3}\}^{\frac{2}{3}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}}\{(1+a)^{k-5}E|\Delta V_a|^{3}\}^{\frac{(k-2)-i}{k-2}} \\ &\le C_k(n-1)\{\frac{\delta}{n}\}^{\frac{2}{3}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}}\{(1+a)^{k-5}\frac{\delta}{n}\}^{\frac{(k-2)-i}{k-2}} \\ &\le C_k n^{\frac{1}{3}-\frac{(k-2)-i}{k-2}}\delta^{\frac{2}{3}+\frac{(k-2)-i}{k-2}}(1+a)^{\frac{(k-5)(k-2-i)}{k-2}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \\ &= C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta^{3-\frac{3i}{k-2}}(1+a)^{\frac{(k-5)(k-2-i)}{k-2}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \\ &= C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta\delta^{2-\frac{3i}{k-2}}(1+a)^{\frac{(k-5)(k-2-i)}{k-2}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \\ &\le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{3(2-\frac{3i}{k-2})}(1+a)^{\frac{(k-5)(k-2-i)}{k-2}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \\ &= C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{\frac{k^2-k-4i-2-ki}{k-2}}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \\ &\le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \end{aligned}$$

where we have used the fact that $\dfrac{k^2-k-4i-2-ki}{k-2} \le k-3-i$ for $i \ge \dfrac{2(k-2)}{3}$ in the last inequality.

Case 1.2.1 k is odd.

$$\begin{aligned} B_{2,i} &\le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}\{E|V_a(\tau)|^{k-1}\}^{\frac{i}{k-1}} \\ &\le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}(C_{k-1}+C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta)^{\frac{i}{k-1}}. \end{aligned}$$

If $C_{k-2}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta \leq 1$, then

$$B_{2,i} \leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i} \leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}\delta(1+a)^{k-3}$$

where we have used the fact that $\dfrac{4+3i-2k}{k-2} \leq k-5$ for $i \leq k-3$ in the last inequality.

If $C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta > 1$, then

$$\begin{aligned} B_{2,i} &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}\{(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-4}\delta\}^{\frac{i}{k-1}} \\ &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}\{(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-6}(1+a)^{k-1}\}^{\frac{i}{k-1}} \\ &= C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}+\frac{(k-6)i}{k-1}}\delta(1+a)^{k-3} \\ &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}\delta(1+a)^{k-3} \end{aligned}$$

where we have used the fact that $\dfrac{4+3i-2k}{k-2}+\dfrac{(k-6)i}{k-1} \leq k-5$ for $\dfrac{2(k-3)}{3} \leq i \leq k-3$ in the last inequality.

Case 1.2.2 k is even.

$$\begin{aligned} B_{2,i} &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}\{E|V_a(\tau)|^{k-2}\}^{\frac{i}{k-2}} \\ &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}(C_{k-2}+C_{k-2}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta)^{\frac{i}{k-2}}. \end{aligned}$$

If $C_{k-2}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta \leq 1$, then

$$B_{2,i} \leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i} \leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}\delta(1+a)^{k-3}$$

where we have used the fact that $\dfrac{4+3i-2k}{k-2} \leq k-5$ for $i \leq k-3$ in the last inequality.

If $C_{k-1}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta > 1$, then

$$\begin{aligned} B_{2,i} &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-5}\delta\}^{\frac{i}{k-2}} \\ &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}}\delta(1+a)^{k-3-i}\{(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-7}(1+a)^{k-2}\}^{\frac{i}{k-2}} \\ &= C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{\frac{4+3i-2k}{k-2}+\frac{(k-7)i}{k-2}}\delta(1+a)^{k-3} \\ &\leq C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}\delta(1+a)^{k-3} \end{aligned}$$

where we have used the fact that $\dfrac{4+3i-2k}{k-2}+\dfrac{(k-7)i}{k-2} \leq k-5$ for $\dfrac{2(k-2)}{3} \leq i \leq k-3$ and $k \leq 17$ in the last inequality.

Therefore

$$B_{2,i} \le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}\delta(1+a)^{k-3}$$

for $i = 1, 2, \dots, k-3$ which implies

$$B_2 \le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}\delta(1+a)^{k-3}. \tag{4.38}$$

For B_3, we can use the same argument of B_2 to show that

$$B_3 \le C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}\delta(1+a)^{k-3}. \tag{4.39}$$

Therefore, by (4.31) – (4.33) and (4.36) – (4.39),

$$|EV_a^k\rho| \le C_k + C_k(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^{k-5}(1+a)^{k-3}\delta \ \text{ if } \ n^{\frac{1}{3}}\delta^{\frac{2}{3}} > 1.$$

Case 2. $n^{\frac{1}{3}}\delta^{\frac{2}{3}} \le 1$.

We can prove by the similar argument of case 1 by replacing $n^{\frac{1}{3}}\delta^{\frac{2}{3}}$ with 1. Therefore, the lemma holds. □

To prove the concentration inequality(Lemma 4.11), we construct the following system.

Let $\overline{I}, \overline{J}, \overline{L}, \overline{M}$ be defined as in Chapter 3 and

$$\begin{aligned}
c_a[(i,j),(l,m)] &:= |x_{a,il} + x_{a,jm} - x_{a,im} - x_{a,jl}| \min(|x_{a,il} + x_{a,jm} - x_{a,im} - x_{a,jl}|, \delta), \\
c'_a[(i,j),(l,m)] &:= c_a[(i,j),(l,m)] - Ec_a[(i,j),(\rho(i),\rho(j))], \\
C_a(\rho) &:= \sum_{\substack{i,j \\ i\neq j}} c'_a[(i,j),(\rho(i),\rho(j))], \\
C_a(\tau) &:= \sum_{\substack{i,j \\ i\neq j}} c'_a[(i,j),(\tau(i),\tau(j))]
\end{aligned}$$

and

$$\begin{aligned}
\widehat{C}_a(\rho) := C_a(\rho) &- c'_a[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] - c'_a[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))] \\
&+ c'_a[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + c'_a[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))].
\end{aligned}$$

It follows that $(\widehat{C}_a(\rho), C_a(\rho))$ is an exchangeable pairs.

Lemma 4.9. *Let $a \ge 0$ be such that $\dfrac{\delta}{(1+a)^3} < 1$. Then*

(i) $E(\widehat{C}_a(\rho) - C_a(\rho))^2 \le \dfrac{96\delta^2}{n-1}$,

(ii) $E|\widehat{C}_a(\rho) - C_a(\rho)|^3 \le \dfrac{C\delta^4}{n}$ *for some constant C,*

(iii) $EC_a^2(\rho) \le 16n\delta^2$.

Proof. (i) By the same argument of (3.8), we have

$$
\begin{aligned}
&E(\widehat{C}_a(\rho) - C_a(\rho))^2 \\
&\le 8Ec_a^2[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + 8Ec_a^2[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] \\
&= \frac{8}{n(n-1)(n(n-1)-1)} \sum_{\substack{i,j \\ i\neq j \\ (i,j)\neq}} \sum_{\substack{l,m \\ l\neq m \\ (l,m)}} Ec_a^2[(i,j),(\rho(l),\rho(m))] \\
&\quad + \frac{8}{n(n-1)} \sum_{\substack{i,j \\ i\neq j}} Ec_a^2[(i,j),(\rho(i),\rho(j))] \\
&= \frac{8}{n(n-1)(n(n-1)-1)} \sum_{\substack{i,j \\ i\neq j \\ (i,j)\neq}} \sum_{\substack{l,m \\ l\neq m \\ (l,m)}} c_a^2[(i,j),(l,m)] \\
&\quad + \frac{8}{n^2(n-1)^2} \sum_{\substack{i,j \\ i\neq j}} \sum_{\substack{l,m \\ l\neq m}} c_a^2[(i,j),(l,m)] \\
&\le \frac{16}{n^2(n-1)^2} \sum_{\substack{i,j \\ i\neq j}} \sum_{\substack{l,m \\ l\neq m}} c_a^2[(i,j),(l,m)] + \frac{8}{n^2(n-1)^2} \sum_{\substack{i,j \\ i\neq j}} \sum_{\substack{l,m \\ l\neq m}} c_a^2[(i,j),(l,m)] \\
&\le \frac{24\delta^2}{n^2(n-1)^2} \sum_{\substack{i,j \\ i\neq j}} \sum_{\substack{l,m \\ l\neq m}} |x_{a,il} + x_{a,jm} - x_{a,im} - x_{a,jl}|^2 \\
&= 24\delta^2 E(\widehat{V}_a(\rho) - V_a(\rho))^2 \\
&\le \frac{96\delta^2}{n-1}
\end{aligned}
$$

where we have used Lemma 4.6(i) in the last inequality.

(ii) By the same argument of (i) and Lemma 4.6(ii), we have

$$
\begin{aligned}
E|\widehat{C}_a(\rho) - C_a(\rho)|^3 &= E|c_a[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] + c_a[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))] \\
&\qquad - c_a[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] - c_a[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))]|^3 \\
&\le C\delta^3 E|\widehat{V}_a(\rho) - V_a(\rho)|^3 \\
&\le C\delta^3\Big(\frac{4\delta}{n}\Big) \\
&= \frac{C\delta^4}{n}.
\end{aligned}
$$

(iii) By the same argument of (3.9), we have

$$
E^\rho \widehat{C}_a(\rho) = \Big(1 - \frac{2}{n(n-1)-1}\Big) C_a(\rho). \tag{4.40}
$$

Applying the argument of (4.24) by using (4.40) and $\widehat{V}_a(\rho) = \widehat{C}_a(\rho)$, $V_a(\rho) = C_a(\rho)$ and $m = 2$, respectively. Then

$$EC_a^2(\rho) = \frac{n(n-1)-1}{4} E(\widehat{C}_a(\rho) - C_a(\rho))^2. \tag{4.41}$$

Therefore, (iii) follows from (i). □

Lemma 4.10. *Let $a \geq 0$ be such that $(1+a)\delta < 1$. Then*

$$E(\eta_a(\delta) - E\eta_a(\delta))^4 \leq C\delta^4$$

where $\eta_a(\delta) := \dfrac{n-1}{4} E^\rho\{|\widehat{V}_a(\rho) - V_a(\rho)| \min(|\widehat{V}_a(\rho) - V_a(\rho)|, \delta)\}$.

Proof. By the same argument of (3.11), we have

$$E(\eta_a(\delta) - E\eta_a(\delta))^4 = \frac{1}{256} EC_a^4(\rho),$$

so it suffices to show that

$$EC_a^4(\rho) \leq Cn^4\delta^4.$$

Let κ and Γ be defined as in Lemma 3.5. Then

$$|\Gamma| \leq Cn \tag{4.42}$$

and $c'_a[(i,j),(\rho(i),\rho(j))] = c'_a[(i,j),(\tau(i),\tau(j))]$ on $\kappa - \Gamma$. Hence

$$\Delta C_a = \sum_{(i,j)\in\Gamma} c'_a[(i,j),(\rho(i),\rho(j))] - \sum_{(i,j)\in\Gamma} c'_a[(i,j),(\tau(i),\tau(j))], \tag{4.43}$$

which implies that

$$\begin{aligned} E|\Delta C_a|^3 &= E\Big|\sum_{(i,j)\in\Gamma} c'_a[(i,j),(\rho(i),\rho(j))] - \sum_{(i,j)\in\Gamma} c'_a[(i,j),(\tau(i),\tau(j))]\Big|^3 \\ &= E\Big|\sum_{(i,j)\in\Gamma} c_a[(i,j),(\rho(i),\rho(j))] - \sum_{(i,j)\in\Gamma} c_a[(i,j),(\tau(i),\tau(j))]\Big|^3 \\ &\leq Cn^2 E \sum_{\substack{i,j \\ i\neq j}} |c_a[(i,j),(\rho(i),\rho(j))]|^3 \\ &\leq Cn^2\delta^3 \sum_{\substack{i,j \\ i\neq j}} E|x_{a,i\rho(i)} + x_{a,i\rho(j)} - x_{a,i\rho(j)} - x_{a,j\rho(i)}|^3 \\ &= Cn^3(n-1)\delta^3 E|\widehat{V}_a(\rho) - V_a(\rho)|^3 \\ &\leq Cn^3\delta^4. \qquad \text{(by Lemma 4.6}(ii)\text{)} \end{aligned} \tag{4.44}$$

We observe that

$$\begin{aligned} E|\Delta\widehat{C}_a|^3 &= E|\Delta C_a + \widehat{C}_a(\rho) - C_a(\rho)|^3 \\ &\le C\big(E|\Delta C_a|^3 + E|\widehat{C}_a(\rho) - C_a(\rho)|^3\big) \\ &\le Cn^3\delta^4. \end{aligned} \tag{4.45}$$

Using the same technique of (4.41), we have

$$EC_a^4(\rho) = \frac{n(n-1)-1}{4}E(\widehat{C}_a(\rho) - C_a(\rho))(\widehat{C}_a^3(\rho) - C_a^3(\rho)). \tag{4.46}$$

By Lemma 4.9, (4.44), (4.45) and the fact that

$$\begin{aligned} |\Delta\widehat{C}_a| &= |\widehat{C}_a(\rho) - C_a(\tau)| \\ &= |C_a(\rho) - C_a(\tau) - c_a'[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))] - c_a'[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))] \\ &\quad + c_a'[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))] + c_a'[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))]| \\ &\le |\Delta C_a| + |c_a[(\overline{I},\overline{J}),(\rho(\overline{I}),\rho(\overline{J}))]| + |c_a[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]| \\ &\quad + |c_a[(\overline{I},\overline{J}),(\rho(\overline{L}),\rho(\overline{M}))]| + |c_a[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{J}))]| \\ &\le |\Delta C_a| + 16(1+a)\delta \\ &\le 2|\Gamma(I,J)|(1+a)\delta + 16(1+a)\delta \qquad \text{(by (4.43))} \\ &\le Cn, \qquad \text{(by (4.42))} \end{aligned}$$

we have

$$\begin{aligned} &E(\widehat{C}_a(\rho) - C_a(\rho))(\widehat{C}_a^3(\rho) - C_a^3(\rho)) \\ &= \; E[(\widehat{C}_a(\rho) - C_a(\rho))^2\{(\widehat{C}_a(\rho) - C_a(\rho))^2 + 3\widehat{C}_a(\rho)C(\rho)\}] \\ &= \; E(\widehat{C}_a(\rho) - C_a(\rho))^4 + 3E(\widehat{C}_a(\rho) - C_a(\rho))^2C_a^2(\tau) \\ &\quad + 3E(\widehat{C}_a(\rho) - C_a(\rho))^2C_a(\tau)(\Delta C_a + \Delta\widehat{C}_a) \\ &\quad + 3E(\widehat{C}_a(\rho) - C_a(\rho))^2\Delta C_a\Delta\widehat{C}_a \\ &\le \; 16(1+a)\delta E|\widehat{C}_a(\rho) - C_a(\rho)|^3 + 3E(\widehat{C}_a(\rho) - C_a(\rho))^2EC_a^2(\tau) \\ &\quad + 3E(\widehat{C}_a(\rho) - C_a(\rho))^2|C_a(\tau)||\Delta C_a + \Delta\widehat{C}_a| \\ &\quad + CnE(\widehat{C}_a(\rho) - C_a(\rho))^2|\Delta C_a| \\ &\le \; CE|\widehat{C}_a(\rho) - C_a(\rho)|^3 + 3E(\widehat{C}_a(\rho) - C_a(\rho))^2EC_a^2(\tau) \\ &\quad + 3\{E|\widehat{C}_a(\rho) - C_a(\rho)|^3|C_a(\tau)|^{\frac{3}{2}}\}^{\frac{2}{3}}\{E|\Delta C_a + \Delta\widehat{C}_a|^3\}^{\frac{1}{3}} \\ &\quad + Cn\{E|\widehat{C}_a(\rho) - C_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta C_a|^3\}^{\frac{1}{3}} \end{aligned}$$

$$
\begin{aligned}
&= CE|\widehat{C}_a(\rho)-C_a(\rho)|^3+3E(\widehat{C}_a(\rho)-C_a(\rho))^2EC_a^2(\tau)\\
&\quad +3\{E|\widehat{C}_a(\rho)-C_a(\rho)|^3E|C_a(\tau)|^{\frac{3}{2}}\}^{\frac{2}{3}}\{E|\Delta C_a+\Delta\widehat{C}_a|^3\}^{\frac{1}{3}}\\
&\quad +Cn\{E|\widehat{C}_a(\rho)-C_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta C_a|^3\}^{\frac{1}{3}}\\
&\le CE|\widehat{C}_a(\rho)-C_a(\rho)|^3+3E(\widehat{C}_a(\rho)-C_a(\rho))^2EC_a^2(\tau)\\
&\quad +3\{E|\widehat{C}_a(\rho)-C_a(\rho)|^3\}^{\frac{2}{3}}\{EC_a^2(\tau)\}^{\frac{1}{2}}\{E|\Delta C_a+\Delta\widehat{C}_a|^3\}^{\frac{1}{3}}\\
&\quad +Cn\{E|\widehat{C}_a(\rho)-C_a(\rho)|^3\}^{\frac{2}{3}}\{E|\Delta C_a|^3\}^{\frac{1}{3}}\\
&\le \frac{C\delta^4}{n}+3\Big(\frac{96\delta^2}{n-1}\Big)(16n\delta^2)+3\Big(\frac{C\delta^4}{n}\Big)^{\frac{2}{3}}(16n\delta^2)^{\frac{1}{2}}(Cn^3\delta^4)^{\frac{1}{3}}\\
&\quad +Cn\Big(\frac{C\delta^4}{n}\Big)^{\frac{2}{3}}(Cn^3\delta^4)^{\frac{1}{3}}\\
&\le \frac{C\delta^4}{n}+C\delta^4+Cn\delta^5+Cn^2\delta^4\\
&\le Cn^2\delta^4.
\end{aligned}
$$

Therefore

$$EC_a^4(\rho)\le Cn^4\delta^4.$$

This complete the proof.

□

4.2 Concentration Inequality

In this section, we give a concentration inequality lemma which we will use to prove the main theorem in the next section.

Lemma 4.11. (Concentration inequality) *Assume that* $(1+z)^5\delta<1$. *Then for every* $z\ge 0$,

$$
\begin{aligned}
&\max\Big\{P\big(z-|\Delta(\rho)|\le V_z(\rho)\le z,|\Delta(\rho)|\le\frac{z}{3}\big),P\big(z\le V_z(\rho)\le z+|\Delta(\rho)|,|\Delta(\rho)|\le\frac{z}{3}\big)\Big\}\\
&\le\frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}
\end{aligned}
$$

for some constant C.

Proof. We shall show only the case that

$$
\begin{aligned}
&P\big(z-|\Delta(\rho)|\le V_z(\rho)\le z,|\Delta(\rho)|\le\frac{z}{3}\big)\\
&\le\frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}
\end{aligned}
$$

For the case

$$\begin{aligned}&P\big(z \le V_z(\rho) \le z + |\Delta(\rho)|, |\Delta(\rho)| \le \frac{z}{3}\big)\\ &\le \frac{C}{1+z^3}\Big\{\delta + \delta^2 + \delta_2 + \delta\delta_2 + \delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}} + \delta_2(n^{\frac{1}{2}}\delta) + \delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}} + \delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\},\end{aligned}$$

we can prove by the same argument.

For any $\gamma \in \mathbb{R}$, we define $f_\gamma : \mathbb{R} \to \mathbb{R}$ by

$$f_\gamma(w) = \begin{cases} 0 & \text{for } \ w \le z - |\gamma| - \delta \\ (1+|w|^3)(w - z + |\gamma| + \delta) & \text{for } \ z - |\gamma| - \delta \le w \le z + \delta \\ (1+|w|^3)(|\gamma| + 2\delta) & \text{for } \ w > z + \delta. \end{cases}$$

First, we note that

$$\min(x, y) \ge x - \frac{x^2}{4y} \qquad \text{for } \ x \ge 0, y > 0,$$

so that, by Lemma 4.6(i), (ii) and the fact that $\delta < 1$, we have

$$\begin{aligned}E(\eta_z(\delta)) &= \frac{n-1}{4}E\{|\widehat{V}_z(\rho) - V_z(\rho)|\min(|\widehat{V}_z(\rho) - V_z(\rho)|, \delta)\}\\ &\ge \frac{n-1}{4}E\Big\{|\widehat{V}_z(\rho) - V_z(\rho)|\Big(|\widehat{V}_z(\rho) - V_z(\rho)| - \frac{|\widehat{V}_z(\rho) - V_z(\rho)|^2}{4\delta}\Big)\Big\}\\ &= \frac{n-1}{4}\{E|\widehat{V}_z(\rho) - V_z(\rho)|^2 - \frac{1}{4\delta}E|\widehat{V}_z(\rho) - V_z(\rho)|^3\}\\ &\ge \frac{n-1}{4}\{\frac{4}{n-1} - \frac{17\delta}{16(n-1)(1+a)} - \frac{1}{4\delta}(\frac{4\delta}{n})\}\\ &= 1 - \frac{17\delta}{64(1+a)} - \frac{n-1}{4n}\\ &\ge 1 - \frac{17\delta}{64} - \frac{1}{4}\\ &\ge \frac{1}{4}. \end{aligned} \tag{4.47}$$

Let

$$M(t) = \frac{n-1}{4}(\widehat{V}_z(\rho) - V_z(\rho))\{\mathit{1}(0 \le t \le \widehat{V}_z(\rho) - V_z(\rho)) - \mathit{1}(\widehat{V}_z(\rho) - V_z(\rho) \le t < 0)\}.$$

From the facts that

$$\begin{aligned}&M(t) \ge 0 \ \text{ for all } \ t,\\ &f'_{\Delta(\rho)}(w) \ge 1 + |w|^3 \ \text{ for } \ z - |\Delta(\rho)| - \delta \le w \le z + \delta \ \text{ and}\\ &f'_{\Delta(\rho)}(w) \ge 0 \ \text{ for all } \ w,\end{aligned}$$

and (4.47), we have

$$
\begin{aligned}
&\frac{n-1}{4}E\Big\{\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)\Big\}\\
&=\frac{n-1}{4}E\Big\{\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\int_0^{\widehat{V}_z(\rho)-V_z(\rho)} f'_{\Delta(\rho)}\big(V_z(\rho)+t\big)dt\Big\}\\
&=E\Big\{\int_{-\infty}^{\infty} f'_{\Delta(\rho)}\big(V_z(\rho)+t\big)M(t)dt\Big\}\\
&\geq E\Big\{(1+|V_z(\rho)|^3)\mathit{1}(z-|\Delta(\rho)|\leq V_z(\rho)\leq z)\int_{|t|\leq\delta}M(t)dt\Big\}\\
&\geq E\Big\{(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\int_{|t|\leq\delta}M(t)dt\Big\}\\
&=\frac{n-1}{4}E\Big\{(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\\
&\qquad\qquad\times|\widehat{V}_z(\rho)-V_z(\rho)|\min(|\widehat{V}_z(\rho)-V_z(\rho)|,\delta)\Big\}\\
&=\frac{n-1}{4}E\Big\{(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\\
&\qquad\qquad\times E^{\rho}|\widehat{V}_z(\rho)-V_z(\rho)|\min(|\widehat{V}_z(\rho)-V_z(\rho)|,\delta)\Big\}\\
&=E\Big\{(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\eta_z(\delta)\Big\}\\
&=E\Big\{(E\eta_z(\delta)+\{\eta_z(\delta)-E\eta_z(\delta)\})(1+|V_z(\rho)|^3)\\
&\qquad\quad\times\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\Big\}\\
&\geq E\eta_z(\delta)E(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\\
&\quad-\Big|E\{\eta_z(\delta)-E\eta_z(\delta)\}(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\Big|\\
&\geq\frac{1}{4}\Big(1+(\frac{2z}{3})^3\Big)P\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\\
&\quad-\Big|E\{\eta_z(\delta)-E\eta_z(\delta)\}(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\Big|. \qquad (4.48)
\end{aligned}
$$

Next, we will bound the second term on the right hand side of (4.48).

By the Cauchy-Schwarz inequality, Lemma 4.8 and Lemma 4.10, we have

$$
\begin{aligned}
&E\Big|\{\eta_z(\delta)-E\eta_z(\delta)\}(1+|V_z(\rho)|^3)\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\Big|\\
&=E\Big\{\Big(\frac{1}{2}\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)\Big)\Big(2(1+|V_z(\rho)|^3)|\eta_z(\delta)-E\eta_z(\delta)|\Big)\Big\}\\
&\leq\frac{1}{8}E\mathit{1}\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)+2E(1+|V_z(\rho)|^3)^2|\eta_z(\delta)-E\eta_z(\delta)|^2\\
&\leq\frac{1}{8}P\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z,|\Delta(\rho)|\leq\frac{z}{3}\big)+2\{E(1+|V_z(\rho)|^3)^4\}^{\frac{1}{2}}(E|\eta_z(\delta)-E\eta_z(\delta)|^4)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \frac{1}{8}\Big(1+\Big(\frac{2z}{3}\Big)^3\Big)P\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z, |\Delta(\rho)|\leq \frac{z}{3}\big)$$
$$+ C(1+E|V_z(\rho)|^{12})(E|\eta_z(\delta)-E\eta_z(\delta)|^4)^{\frac{1}{2}}$$
$$\leq \frac{1}{8}\Big(1+\Big(\frac{2z}{3}\Big)^3\Big)P\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z, |\Delta(\rho)|\leq \frac{z}{3}\big)$$
$$+ C(1+(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^7(1+z)^9\delta+(1+z)^9\delta)\delta^2$$
$$\leq \frac{1}{8}\Big(1+\Big(\frac{2z}{3}\Big)^3\Big)P\big(z-|\Delta(\rho)|\leq V_z(\rho)\leq z, |\Delta(\rho)|\leq \frac{z}{3}\big)+C(1+(n^{\frac{1}{3}}\delta^{\frac{2}{3}})^7)\delta \qquad (4.49)$$

By (4.48) and (4.49), we have

$$P(z-|\Delta(\rho)|\leq V_z(\rho)\leq z, |\Delta(\rho)|\leq \frac{z}{3})\leq \frac{C}{1+\Big(\frac{2z}{3}\Big)^3}\Big\{B+\delta+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}\Big\}, \qquad (4.50)$$

where $B=\frac{n-1}{4}E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$.
By Lemma 4.4 and the exchangeability of $(\widehat{V}_z(\rho), V_z(\rho))$ and $(\widehat{\Delta}(\rho), \Delta(\rho))$, we have

$$0 = E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))+f_{\widehat{\Delta}(\rho)}(V_z(\rho))\big)$$
$$= E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big[2f_{\Delta(\rho)}(V_z(\rho))+\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$+\big(f_{\widehat{\Delta}(\rho)}(V_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)\big]$$
$$= 2E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)f_{\Delta(\rho)}(V_z(\rho))$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\widehat{\Delta}(\rho)}(V_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$= 2E\big(E^\rho\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big)f_{\Delta(\rho)}(V_z(\rho))$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\widehat{\Delta}(\rho)}(V_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$= 2E\big(-\frac{2}{n-1}V_z(\rho)+\frac{2}{n(n-1)}\sum_{i,j}x_{z,ij}\big)f_{\Delta(\rho)}(V_z(\rho)) \qquad \text{(by Lemma 4.4)}$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\widehat{\Delta}(\rho)}(V_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$= -\frac{4}{n-1}Ef_{\Delta(\rho)}(V_z(\rho))\big(V_z(\rho)-\frac{1}{n}\sum_{i,j}x_{z,ij}\big)$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\Delta(\rho)}(\widehat{V}_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big)$$
$$+E\big(\widehat{V}_z(\rho)-V_z(\rho)\big)\big(f_{\widehat{\Delta}(\rho)}(V_z(\rho))-f_{\Delta(\rho)}(V_z(\rho))\big).$$

Hence

$$B = B_1 + B_2 \tag{4.51}$$

where

$$B_1 = Ef_{\Delta(\rho)}(V_z(\rho))\big(V_z(\rho) - \frac{1}{n}\sum_{i,j} x_{z,ij}\big) \quad \text{and}$$

$$B_2 = -\frac{n-1}{4}E\big(\widehat{V}_z(\rho) - V_z(\rho)\big)\big(f_{\widehat{\Delta}(\rho)}(V_z(\rho)) - f_{\Delta(\rho)}(V_z(\rho))\big).$$

By the definition of $f_{\Delta(\rho)}$, (4.26), Lemma 4.7 and Lemma 4.8,

$$\begin{aligned}
|B_1| &\le E\big|f_{\Delta(\rho)}(V_z(\rho))\big(V_z(\rho) - \frac{1}{n}\sum_{i,j} x_{z,ij}\big)\big| \\
&\le E\Big\{\big|(1+|V_z(\rho)|^3)(|\Delta(\rho)| + 2\delta)\big|\big(\big|V_z(\rho)\big| + \big|\frac{1}{n}\sum_{i,j} x_{z,ij}\big|\big)\Big\} \\
&= E\Big\{|\Delta(\rho)||V_z(\rho)| + |\Delta(\rho)|\big|\frac{1}{n}\sum_{i,j} x_{z,ij}\big| + 2\delta|V_z(\rho)| + 2\delta\big|\frac{1}{n}\sum_{i,j} x_{z,ij}\big| \\
&\quad + |\Delta(\rho)||V_z(\rho)|^4 + |\Delta(\rho)|\big|\frac{1}{n}\sum_{i,j} x_{z,ij}\big||V_z(\rho)|^3 + 2\delta|V_z(\rho)|^4 \\
&\quad + 2\delta\big|\frac{1}{n}\sum_{i,j} x_{z,ij}\big||V_z(\rho)|^3\Big\} \\
&\le C\Big\{E|\Delta(\rho)||V_z(\rho)| + \frac{\delta}{(1+z)^2}E|\Delta(\rho)| + \delta E|V_z(\rho)| + \frac{\delta^2}{(1+z)^2} \\
&\quad + E|\Delta(\rho)||V_z(\rho)|^4 + \frac{\delta}{(1+z)^2}E|\Delta(\rho)||V_z(\rho)|^3 + \delta E|V_z(\rho)|^4 \\
&\quad + \frac{\delta^2}{(1+z)^2}E|V_z(\rho)|^3\Big\} \\
&\le C\Big\{\big\{E|\Delta(\rho)|^2\big\}^{\frac{1}{2}}\big\{E|V_z(\rho)|^2\big\}^{\frac{1}{2}} + \frac{\delta}{(1+z)^2}\big\{E|\Delta(\rho)|^2\big\}^{\frac{1}{2}} + \delta\big\{E|V_z(\rho)|^2\big\}^{\frac{1}{2}} \\
&\quad + \frac{\delta^2}{(1+z)^2} + \big\{E|\Delta(\rho)|^2\big\}^{\frac{1}{2}}\big\{E|V_z(\rho)|^8\big\}^{\frac{1}{2}} + \frac{\delta}{(1+z)^2}\big\{E|\Delta(\rho)|^2\big\}^{\frac{1}{2}}\big\{E|V_z(\rho)|^6\big\}^{\frac{1}{2}} \\
&\quad + \delta E|V_z(\rho)|^4 + \frac{\delta^2}{(1+z)^2}\big\{E|V_z(\rho)|^4\big\}^{\frac{3}{4}}\Big\} \\
&\le C\Big\{\delta_2 + \frac{\delta\delta_2}{(1+z)^2} + \delta + \frac{\delta^2}{(1+z)^2} + \delta_2\big\{1 + (n^{\frac{1}{3}}\delta^{\frac{2}{3}})^3(1+z)^5\delta + (1+z)^5\delta\big\}^{\frac{1}{2}} \\
&\quad + \frac{\delta\delta_2}{(1+z)^2}\big\{1 + (n^{\frac{1}{3}}\delta^{\frac{2}{3}})(1+z)^3\delta + (1+z)^3\delta\big\}^{\frac{1}{2}} \\
&\quad + \delta(1 + (1+z)\delta) + \frac{\delta^2}{(1+z)^2}\big\{1 + (1+z)\delta\big\}^{\frac{3}{4}}\Big\}
\end{aligned}$$

$$\le C\Big\{\delta_2 + \delta\delta_2 + \delta + \delta^2 + \delta_2\big\{1 + (n^{\frac{1}{3}}\delta^{\frac{2}{3}})^3\big\}^{\frac{1}{2}} + \delta\delta_2\big\{1 + (n^{\frac{1}{3}}\delta^{\frac{2}{3}})\big\}^{\frac{1}{2}}\Big\}$$
$$\le C\Big\{\delta + \delta^2 + \delta_2 + \delta\delta_2 + \delta_2(n^{\frac{1}{2}}\delta) + \delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}. \tag{4.52}$$

As to B_2, we note that

$$|f_{\widehat{\Delta}(\rho)}(w) - f_{\Delta(\rho)}(w)| \le (1+|w|^3)||\widehat{\Delta}(\rho)| - |\Delta(\rho)|| \le (1+|w|^3)|\widehat{\Delta}(\rho) - \Delta(\rho)|,$$

(see [3]).

Hence, by the Hölder inequality, (4.34), Lemma 4.5, Lemma 4.6 and Lemma 4.8,

$$\begin{aligned}
|B_2| &\le \frac{n-1}{4}E\Big\{|\widehat{V}_z(\rho) - V_z(\rho)||f_{\widehat{\Delta}(\rho)}(V_z(\rho)) - f_{\Delta(\rho)}(V_z(\rho))|\Big\} \\
&\le \frac{n-1}{4}E\Big\{|\widehat{V}_z(\rho) - V_z(\rho)|(1+|V_z(\rho)|^3)|\widehat{\Delta}(\rho) - \Delta(\rho)|\Big\} \\
&\le \frac{n-1}{4}\Big\{E|\widehat{V}_z(\rho) - V_z(\rho)|^2(1+|V_z(\rho)|^3)^2\Big\}^{\frac{1}{2}}\Big\{E|\widehat{\Delta}(\rho) - \Delta(\rho)|^2\Big\}^{\frac{1}{2}} \\
&\le \frac{n-1}{4}\Big\{E|\widehat{V}_z(\rho) - V_z(\rho)|^2(1+(|V_z(\tau)| + |\Delta V_z|)^3)^2\Big\}^{\frac{1}{2}}\Big\{E|\widehat{\Delta}(\rho) - \Delta(\rho)|^2\Big\}^{\frac{1}{2}} \\
&\le C(n-1)\Big\{E|\widehat{V}_z(\rho) - V_z(\rho)|^2(1+|V_z(\tau)|^3 + |\Delta V_z|^3)^2\Big\}^{\frac{1}{2}}\Big\{E|\widehat{\Delta}(\rho) - \Delta(\rho)|^2\Big\}^{\frac{1}{2}} \\
&\le C(n-1)\Big\{E|\widehat{V}_z(\rho) - V_z(\rho)|^2 + E|\widehat{V}_z(\rho) - V_z(\rho)|^2 E|V_z(\tau)|^6 \\
&\qquad\qquad + E|\widehat{V}_z(\rho) - V_z(\rho)|^2|\Delta V_z|^6)\Big\}^{\frac{1}{2}}\Big\{E|\widehat{\Delta}(\rho) - \Delta(\rho)|^2\Big\}^{\frac{1}{2}} \\
&\le C(n-1)\Big\{\frac{C}{n-1} + \frac{C}{n-1}E|V_z(\tau)|^6 \\
&\qquad\qquad + \big\{E|\widehat{V}_z(\rho) - V_z(\rho)|^3\big\}^{\frac{2}{3}}\big\{E|\Delta V_z|^{18}\big\}^{\frac{1}{3}}\Big\}^{\frac{1}{2}}\{\frac{\delta_2^2}{n}\}^{\frac{1}{2}} \\
&\le Cn^{\frac{1}{2}}\delta_2\Big\{\frac{C}{n-1} + \frac{C}{n-1}(1 + n^{\frac{1}{3}}\delta^{\frac{2}{3}}(1+a)^3\delta + (1+a)^3\delta) \\
&\qquad + \big\{\frac{\delta}{n}\big\}^{\frac{2}{3}}\big\{(1+a)^{15}E|\Delta V_z|^3\big\}^{\frac{1}{3}}\Big\}^{\frac{1}{2}} \\
&\le Cn^{\frac{1}{2}}\delta_2\Big\{\frac{C}{n-1} + \frac{C}{n-1}(1 + n^{\frac{1}{3}}\delta^{\frac{2}{3}}) + \big\{\frac{\delta}{n}\big\}^{\frac{2}{3}}\big\{(1+a)^{15}\frac{\delta}{n}\big\}^{\frac{1}{3}}\Big\}^{\frac{1}{2}} \\
&\le Cn^{\frac{1}{2}}\delta_2\Big\{\frac{C}{n-1} + \frac{C}{n-1}(1 + n^{\frac{1}{3}}\delta^{\frac{2}{3}}) + \frac{(1+a)^5\delta}{n}\Big\}^{\frac{1}{2}} \\
&\le C\{\delta_2 + \delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\}.
\end{aligned} \tag{4.53}$$

Hence by (4.50)-(4.53),

$$\begin{aligned}
&P(z-|\Delta(\rho)|\le V_z(\rho)\le z, |\Delta(\rho)|\le \frac{z}{3})\\
&\le \frac{C}{1+\left(\frac{2z}{3}\right)^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}\\
&\le \frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}
\end{aligned}$$

□

4.3 Proof of the Main Result

It suffices to consider $z \ge 0$ as we can apply the result to $-W$ when $z<0$. Let $z\ge 0$.

Case 1. $0\le z<1$.

By Theorem 4.1, we have

$$\Big|P\Big(\frac{W-\mu}{\sigma}\le z\Big)-P(V(\pi)\le z)\Big|\le C(\delta+\delta^2+\delta_2)\le \frac{C}{1+z^3}(\delta+\delta^2+\delta_2).$$

Case 2. $z\ge 1$.

If $\frac{\delta}{(1+z)^3}\ge 1$, then

$$\Big|P\Big(\frac{W-\mu}{\sigma}\le z\Big)-P(V(\pi)\le z)\Big|\le 1\le \frac{C\delta}{(1+z)^3}\le \frac{C\delta}{1+z^3}.$$

Assume that $\frac{\delta}{(1+z)^3}<1$.

We observe that

$$\begin{aligned}
-P(z-|\Delta(\pi)|\le V(\pi)\le z) &\le -P(V(\pi)\le z)+P(V(\pi)\le z-|\Delta(\pi)|)\\
&= -P(V(\pi)\le z)+P\Big(\frac{W-\mu}{\sigma}-\Delta(\pi)\le z-|\Delta(\pi)|\Big)\\
&\le P\Big(\frac{W-\mu}{\sigma}\le z\Big)-P(V(\pi)\le z)\\
&\le P\Big(\frac{W-\mu}{\sigma}-\Delta(\pi)\le z+|\Delta(\pi)|\Big)-P(V(\pi)\le z)\\
&\le P(z\le V(\pi)\le z+|\Delta(\pi)|) \qquad (4.54)
\end{aligned}$$

and

$$
\begin{aligned}
&P(z - |\Delta(\pi)| \le V(\pi) \le z) \\
&= P(z - |\Delta(\pi)| \le V(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}) + P(z - |\Delta(\pi)| \le V(\pi) \le z, |\Delta(\pi)| > \frac{z}{3}) \\
&\le P(z - |\Delta(\pi)| \le V(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}) + P(|\Delta(\pi)| > \frac{z}{3}) \\
&= P(z - |\Delta(\pi)| \le V(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}, \max_{1\le i\le n} |x_{i\pi(i)}| \le 1 + z) \\
&\quad + P(z - |\Delta(\pi)| \le V(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}, \max_{1\le i\le n} |x_{i\pi(i)}| > 1 + z) + P(|\Delta(\pi)| > \frac{z}{3}) \\
&\le P(z - |\Delta(\pi)| \le V_z(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}) + P(V(\pi) \ge \frac{2z}{3}, \max_{1\le i\le n} |x_{i\pi(i)}| > 1 + z) \\
&\quad + P(|\Delta(\pi)| > \frac{z}{3}) \\
&\le P(z - |\Delta(\pi)| \le V_z(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}) + \sum_i P(V(\pi) \ge \frac{2z}{3}, |x_{i\pi(i)}| > 1 + z) \\
&\quad + P(|\Delta(\pi)| > \frac{z}{3}) \\
&= P(z - |\Delta(\pi)| \le V_z(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}) + \sum_i P(|x_{i\pi(i)}| > 1 + z) + P(|\Delta(\pi)| > \frac{z}{3}) \\
&= P(z - |\Delta(\pi)| \le V_z(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}) + \sum_i \frac{E|x_{i\pi(i)}|^3}{(1+z)^3} + \frac{E|\Delta(\pi)|^2}{(\frac{z}{3})^2} \\
&= P(z - |\Delta(\pi)| \le V_z(\pi) \le z, |\Delta(\pi)| \le \frac{z}{3}) + \frac{C\delta}{1+z^3} + \frac{C\delta_2^2}{1+z^2}. \qquad (4.55)
\end{aligned}
$$

Next, we give the argument for

$$P(z - |\Delta(\pi)| \le V_z(\pi) \le z, |\Delta(\rho)| \le \frac{z}{3}).$$

Case 2.1 $(1+z)^5\delta \ge 1$.

$$
\begin{aligned}
&P(z - |\Delta(\pi)| \le V_z(\pi) \le z, |\Delta(\rho)| \le \frac{z}{3}) \\
&\le P(V_z(\pi) \ge \frac{2z}{3}) \\
&\le \frac{E|V_z(\pi)|^8}{(\frac{2z}{3})^8} \\
&\le \frac{CE|V_z(\pi)|^8}{(1+z)^8} \\
&\le \frac{C(1 + (1+z)^5\delta + (n^{\frac{1}{2}}\delta)^2(1+z)^5\delta)}{(1+z)^8} \\
&\le \frac{C}{(1+z)^8} + \frac{C((1+z)^5\delta + (n^{\frac{1}{2}}\delta)^2(1+z)^5\delta)}{(1+z)^8}
\end{aligned}
$$

$$\leq \frac{C(1+z)^5\delta}{(1+z)^8} + \frac{C(n^{\frac{1}{2}}\delta)^2(1+z)^5\delta}{(1+z)^8}$$
$$\leq \frac{C\delta}{(1+z)^3} + \frac{C(n\delta^2)\delta}{(1+z)^3}$$
$$\leq \frac{C(\delta+\delta(n^{\frac{1}{2}}\delta)^2)}{1+z^3}.$$

Case 2.2 $(1+z)^5\delta < 1$.

By Lemma 4.11, we see that

$$P(z-|\Delta(\rho)| \leq V_z(\rho) \leq z, |\Delta(\rho)| \leq \frac{z}{3})$$
$$\leq \frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}.$$

Therefore by (4.54) and (4.55), we have

$$P\Big(\frac{W-\mu}{\sigma} \leq z\Big) - P(V(\pi) \leq z)$$
$$\geq -P(z-|\Delta(\pi)| \leq V_z(\pi) \leq z, |\Delta(\pi)| \leq \frac{z}{3}) - \frac{C\delta}{1+z^3} - \frac{C\delta_2^2}{1+z^2}$$
$$\geq -\frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}$$
$$-\frac{C\delta}{1+z^3} - \frac{C\delta_2^2}{1+z^2}$$
$$= -\frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}$$
$$-\frac{C\delta_2^2}{1+z^2}.$$

Similarly, we can show that

$$P\Big(\frac{W-\mu}{\sigma} \leq z\Big) - P(V(\pi) \leq z)$$
$$\leq \frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac{1}{2}}\delta)^2+\delta(n^{\frac{1}{2}}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac{1}{2}}\delta)+\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}+\delta\delta_2(n^{\frac{1}{2}}\delta)^{\frac{1}{3}}\Big\}$$
$$+\frac{C\delta_2^2}{1+z^2}.$$

Therefore (4.2) holds.

To prove (4.3), we observe that

$$V(\pi) = \sum_i x_{i\pi(i)} \quad \text{and} \quad \sup_{i,j}|x_{ij}| \leq \frac{K}{\sqrt{n}}$$

so by Theorem 3.2, we have

$$|P(V(\pi) \le z) - \Phi(z)| \le \frac{C}{(1+z^3)\sqrt{n}}. \tag{4.56}$$

Hence (4.3) follows from (4.2), (4.56) and the fact that

$$\left|P\Big(\frac{W-\mu}{\sigma} \le z\Big) - \Phi(z)\right| \le \left|P\Big(\frac{W-\mu}{\sigma} \le z\Big) - P(V(\pi) \le z)\right| + |P(V(\pi) \le z) - \Phi(z)|.$$

□

4.4 General Array

We now consider the more general statistic

$$X := X(\pi) := \sum_{i,j} C(i,j;\pi(i),\pi(j))$$

studied by Zhao *et al.*([30]), where C is an arbitrary 4-dimensional array and π is a random permutation of $\{1,2,\ldots,n\}$.

Define

$$\begin{aligned}
C(+,j;l,m) &:= \sum_{\substack{i \\ i\neq j}} C(i,j;l,m) \\
C(+,+;l,m) &:= \sum_{\substack{i,j \\ i\neq j}} C(i,j;l,m) \\
C(i,+;+,m) &:= \sum_{\substack{j \\ j\neq i}} \sum_{\substack{l \\ l\neq m}} C(i,j;l,m) \\
C((++);l,m) &:= \sum_{i} C(i,i;l,m) \\
C((++);(++)) &:= \sum_{i} \sum_{l} C(i,i;l,l).
\end{aligned}$$

With this notation, we set

$$
\begin{aligned}
&\widetilde{C}(i,j;l,m)\\
&:= C(i,j;l,m)\\
&- \frac{n-1}{n(n-2)}\big\{C(i,+;l,m)+C(+,j;l,m)+C(i,j;l,+)+C(i,j;+,m)\big\}\\
&- \frac{1}{n(n-2)}\big\{C(+,i;l,m)+C(j,+;l,m)+C(i,j;+,l)+C(i,j;m,+)\big\}\\
&- \frac{1}{(n-1)(n-2)}\big\{C(+,+;l,m)+C(i,j;+,+)\big\}\\
&+ \big(\frac{n-1}{n(n-2)}\big)^2\big\{C(i,+;l,+)+C(i,+;+,m)+C(+,j;l,+)+C(+,j;+,m)\big\}\\
&+ \frac{n-1}{n^2(n-2)^2}\big\{C(+,i;l,+)+C(+,i;+,m)+C(j,+;l,+)+C(j,+;+,m)\\
&\qquad\qquad +C(i,+;+,l)+C(i,+;m,+)+C(+,j;+,l)+C(+,j;m,+)\big\}\\
&+ \frac{1}{n^2(n-2)^2}\big\{C(+,i;+,l)+C(+,i;m,+)+C(j,+;+,l)+C(j,+;m,+)\big\}\\
&- \frac{1}{n(n-2)^2}\big\{C(i,+;+,+)+C(+,j;+,+)+C(+,+;l,+)+C(+,+;+,m)\big\}\\
&- \frac{1}{n(n-1)(n-2)^2}\big\{C(+,i;+,+)+C(j,+;+,+)+C(+,+;+,l)+C(+,+;m,+)\big\}\\
&+ \frac{1}{(n-1)^2(n-2)}\big\{C(+,+;+,+)\big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&2(n-2)c^*(i,l)\\
&:= \frac{n-1}{n(n-2)}\big\{C(i,+;l,+)+C(+,i;+,l)\big\}\\
&+ \frac{1}{n(n-2)}\big\{C(i,+;+,l)+C(+,i;l,+)\big\}\\
&- \frac{1}{n(n-2)}\big\{C(i,+;+,+)+C(+,i;+,+)+C(+,+;l,+)+C(+,+;+,l)\big\}\\
&+ \frac{2}{n^2(n-2)}C(+,+;+,+)\\
&+ C(i,i;l,l) - n^{-1}\big\{C(i,i;(++))+C((++);l,l)\big\} + n^{-2}C((++);(++)).
\end{aligned}
$$

Let

$$\mu := \frac{1}{n(n-1)}C(+,+;+,+) + \frac{1}{n}C((++);(++))$$
$$C_{12} := \frac{1}{n^2}\sum_i\sum_l\{c^*(i,l)\}^2,$$
$$C_{22} := \frac{1}{n^2(n-1)^2}\sum_{\substack{i,j\\ i\neq j}}\sum_{\substack{l,m\\ l\neq m}}\{\widetilde{C}(i,j;l,m)\}^2,$$
$$C_{13} := \frac{1}{n^2}\sum_i\sum_l|c^*(i,l)|^3.$$

With these definitions, it was shown in Barbour and Chen([2]) that

$$\sum_i c^*(i,l) = \sum_l c^*(i,l) = 0, \tag{4.57}$$
$$\widetilde{C}(+,j;l,m) = \widetilde{C}(i,+;l,m) = \widetilde{C}(i,j;+,m) = \widetilde{C}(i,j;l,+) = 0$$

and

$$X = \mu + \widetilde{V}(\pi) + \widetilde{\Delta}(\pi),$$
$$E\widetilde{V}(\pi) = E\widetilde{\Delta}(\pi) = E\widetilde{V}(\pi)\widetilde{\Delta}(\pi) = 0,$$
$$\sigma^2 := \operatorname{Var}\widetilde{V}(\pi) = \frac{4n^2(n-2)^2}{n-1}C_{12},$$
$$\operatorname{Var}\widetilde{\Delta}(\pi) \le \frac{2n(n-1)^2}{n-3}C_{22},$$

where $\widetilde{V}(\pi) := 2(n-2)\sum_i c^*(i,\pi(i))$ and $\widetilde{\Delta}(\pi) := \sum_{\substack{i,j\\ i\neq j}}\widetilde{C}(i,j;\pi(i),\pi(j)).$

This decomposition is the analogue of that given in (4.1).

For each $i,j = 1,2,\ldots,n$ and $a \ge 0$, let

$$c_{ij} := \frac{2(n-2)}{\sigma}c^*(i,j) \text{ and } c_{a,ij} := c_{ij}\mathit{1}(|c_{ij}| \le 1+a).$$

By (4.57), we have

$$\sum_i c_{ij} = 0 \text{ for each } j = 1,2,\ldots,n \quad \text{and}$$
$$\sum_j c_{ij} = 0 \text{ for each } i = 1,2,\ldots,n,$$

In 2005, Barbour and Chen([2]) gave a uniform Berry-Esseen bound for the normal approximation to X. The following is their result.

Theorem 4.12. *There exists a constant C such that*

$$\sup_{z\in\mathbb{R}}\left|P\Big(\frac{X-\mu}{\sigma}\le z\Big)-P(V(\pi)\le z)\right|\le C(\delta+\delta^2+\delta_2)$$

and

$$\sup_{z\in\mathbb{R}}\left|P\Big(\frac{X-\mu}{\sigma}\le z\Big)-\Phi(z)\right|\le C(\delta+\delta^2+\delta_2)$$

where $V(\pi):=\dfrac{\widetilde{V}(\pi)}{\sigma}$, $\delta=128\delta_1,\delta_1=n^4\sigma^{-3}C_{13}$ *and* $\delta_2^2:=E\Delta^2(\pi)\le\dfrac{2n(n-1)^2C_{22}}{(n-3)\sigma^2}$.

The argument is exactly as for Lemma 4.11 and its proof remains true in this context, with the new definitions fo $V(\pi),\delta_1$ and δ_2. This leads to the following theorem.

Theorem 4.13. *There exists a constant C such that for every real number z,*

$$\begin{aligned}&\left|P\Big(\frac{X-\mu}{\sigma}\le z\Big)-P(V(\pi)\le z)\right|\\&\le\frac{C}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac12}\delta)^2+\delta(n^{\frac12}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac12}\delta)+\delta_2(n^{\frac12}\delta)^{\frac13}+\delta\delta_2(n^{\frac12}\delta)^{\frac13}\Big\}\\&\quad+\frac{C\delta_2^2}{1+z^2}.\end{aligned}$$

In particular, if $\dfrac{2(n-2)}{\sigma}\sup_{i,j}|c^*(i,j)|\le\dfrac{K}{\sqrt{n}}$ *for some positive real number K, then*

$$\begin{aligned}&\left|P\Big(\frac{X-\mu}{\sigma}\le z\Big)-\Phi(z)\right|\\&\le\frac{CK}{1+z^3}\Big\{\delta+\delta^2+\delta_2+\delta\delta_2+\delta(n^{\frac12}\delta)^2+\delta(n^{\frac12}\delta)^{\frac{14}{3}}+\delta_2(n^{\frac12}\delta)+\delta_2(n^{\frac12}\delta)^{\frac13}\\&\quad+\delta\delta_2(n^{\frac12}\delta)^{\frac13}+\frac{1}{\sqrt{n}}\Big\}+\frac{C\delta_2^2}{1+z^2}.\end{aligned}$$

CHAPTER V
A NON-UNIFORM BOUND IN NORMAL APPROXIMATION FOR INDEPENDENT BOUNDED RANDOM VARIABLES

5.1 Main Results

Let $X_1, X_2, \ldots, X_n$ be independent and not necessary identically distributed random variables with zero means and finite variances.

Define

$$W = \sum_{i=1}^{n} X_i$$

and assume that $\mathrm{Var}(W) = 1$.

In this chapter, we give a non-uniform bound in normal approximation for W by using Stein's method without use of the concentration inequality. This method is simple and give a shaper result than the previous work.

Chen and Shao([11]) gave a non-uniform Berry-Esseen bound without assuming the existence of third moments. Their argument is based on a concentration inequality approach of Stein's method. In a special case, when random variables are bounded, they simplify the proof of uniform bound by not using the concentration inequality (see [12]). Their theorem is as follows.

Theorem 5.1. *Assume that $|X_i| \leq \delta$ for $i = 1, 2, \ldots, n$. Then*

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq 3.3\delta.$$

In this chapter, we use the idea of Chen and Shao([12]) to find a non-uniform Berry-Esseen bound for independent bounded random variables without use of the concentration inequality approach. The followings are our main results.

Theorem 5.2. *Assume that* $|X_i| \leq \delta$ *for* $i = 1, 2, \ldots, n$. *Then there exists a constant* C_δ *which depend on* δ *such that for every real number* z,

$$|P(W \leq z) - \Phi(z)| \leq C_\delta e^{-\frac{|z|}{2}} \delta$$

where $C_\delta = 4.45 + 2.21 e^{2\delta + (\delta^{-2}(e^{2\delta} - 1 - 2\delta))}$.

Theorem 5.3. *Assume that* $|X_i| \leq \delta$ *for* $i = 1, 2, \ldots, n$. *Then there exists a constant* C *which does not depend on* δ *such that for every real number* z,

$$|P(W \leq z) - \Phi(z)| \leq \frac{C\delta}{1 + |z|^3}.$$

Observe from Theorem 5.2 that if $\delta \to 0$ we have $C_\delta \to 20.78$.

To illustrate our results, we give an example of non-uniform bound in normal approximation of the binary expansion of a random integer.

Let $n \geq 2$ and X be a random variable uniformly distributed over $\{0, 1, \ldots, n-1\}$.

Let k be such that $2^{k-1} < n \leq 2^k$. Write the binary expansion of X

$$X = \sum_{i=1}^{k} X_i 2^{k-i}$$

and let $S = X_1 + X_2 + \cdots + X_k$ be the number of ones in the binary expansion of X.

When $n = 2^k$, the distribution of S is the binomial distribution for k trials with mean $\frac{k}{2}$ and variance $\frac{k}{4}$. Let $Y_i = \frac{X_i - (1/2)}{\sqrt{k/4}}$ for $i = 1, 2, \ldots, k$. Then $|Y_i| \leq \frac{1}{\sqrt{k}}$ and $\frac{S - (k/2)}{\sqrt{k/4}} = \sum_{i=1}^{k} Y_i$ and hence, by Theorem 5.2 and Theorem 5.3, for every real number z,

$$\left| P\Big(\frac{S - (k/2)}{\sqrt{k/4}} \leq z\Big) - \Phi(z) \right| \leq \frac{C_k e^{-\frac{|z|}{2}}}{k^{1/2}}$$

and

$$\left| P\Big(\frac{S - (k/2)}{\sqrt{k/4}} \leq z\Big) - \Phi(z) \right| \leq \frac{C}{(1 + |z|^3) k^{1/2}}$$

where $C_k = 4.45 + 2.21 e^{\frac{2}{\sqrt{k}} + (k(e^{\frac{2}{\sqrt{k}}} - 1 - \frac{2}{\sqrt{k}}))}$.

Note that if $n \neq 2^k$, then $X_1, X_2, \ldots, X_k$ are not independent, so we cannot apply Theorem 5.2 and Theorem 5.3.

5.2 Proof of Main Results

For each $i = 1, 2, \ldots, n$, assume that $|X_i| \leq \delta$ and let

$$W^{(i)} = W - X_i.$$

To prove the main result, we need the following proposition and lemmas.

Proposition 5.4. ([12]) *Let* $g(w) = (wf_z(w))'$. *Then*

$$g(w) \leq \begin{cases} 4(1+z^2)e^{\frac{z^2}{8}}(1-\Phi(z)) & \text{if } w \leq \frac{z}{2} \\ 4(1+z^2)e^{\frac{z^2}{2}}(1-\Phi(z)) & \text{if } w > \frac{z}{2}. \end{cases}$$

Lemma 5.5. (Bennet-Hoeffing inequality, [12]) *Let* $\eta_1, \eta_2, \ldots, \eta_n$ *be independent random variables satisfying* $E\eta_i \leq 0$, $\eta_i \leq a$ *for* $1 \leq i \leq n$ *and* $\sum_{i=1}^{n} E\eta_i^2 \leq B_n^2$. *Put* $S_n = \sum_{i=1}^{n} \eta_i$. *Then*

$$Ee^{tS_n} \leq \exp(a^{-2}(e^{ta} - 1 - ta)B_n^2)$$

for $t > 0$.

Lemma 5.6. *For* $s, t \leq \delta$ *and* $z \geq 2$, *we have*

$$\begin{aligned} &|E(W^{(i)} + t)f_z(W^{(i)} + t) - E(W^{(i)} + s)f_z(W^{(i)} + s)| \\ &\leq (2.43 + 1.47e^{2\delta + (\delta^{-2}(e^{2\delta} - 1 - 2\delta))})e^{-\frac{z}{2}}(|s| + |t|). \end{aligned}$$

Proof. Since $f(x) = \frac{1+z^2}{z}e^{-\frac{3z^2}{8}+\frac{z}{2}}$ is a decreasing function on $[2, \infty)$, so $f(2) = 1.52$ is the maximum value of f on $[2, \infty)$, which implies

$$\frac{1+z^2}{z}e^{-\frac{3z^2}{8}} \leq 1.52e^{-\frac{z}{2}} \tag{5.1}$$

for $z \geq 2$.

Similarly, we can show that

$$\frac{1+z^2}{z}e^{-z} \leq 0.92e^{-\frac{z}{2}} \tag{5.2}$$

for $z \geq 2$. Let $g(w) = (wf_z(w))'$. By Proposition 5.4, (5.1), (5.2), Lemma 5.5 and the fact that

$$1 - \Phi(w) \leq \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}w} \quad \text{for } \ w > 0, \tag{5.3}$$

(see [23], eq.25, p.23), we have for any $u \le \delta$,

$$\begin{aligned} Eg(W^{(i)}+u) &= Eg(W^{(i)}+u)1(W^{(i)}+u \le \frac{z}{2}) + Eg(W^{(i)}+u)1(W^{(i)}+u > \frac{z}{2}) \\ &\le 4(1+z^2)e^{\frac{z^2}{8}}(1-\Phi(z)) + 4(1+z^2)e^{\frac{z^2}{2}}(1-\Phi(z))P(W^{(i)}+u > \frac{z}{2}) \\ &\le \frac{4}{\sqrt{2\pi}}\frac{1+z^2}{z}e^{\frac{z^2}{8}}e^{-\frac{z^2}{2}} + \frac{4}{\sqrt{2\pi}}\frac{1+z^2}{z}e^{\frac{z^2}{2}}e^{-\frac{z^2}{2}}P(W^{(i)}+u > \frac{z}{2}) \\ &\le \frac{4}{\sqrt{2\pi}}\frac{1+z^2}{z}e^{-\frac{3z^2}{8}} + \frac{4}{\sqrt{2\pi}}\frac{1+z^2}{z}e^{-z+2u}Ee^{2W^{(i)}} \\ &\le 2.43e^{-\frac{z}{2}} + 1.47e^{-\frac{z}{2}+2u}Ee^{2W^{(i)}} \\ &\le 2.43e^{-\frac{z}{2}} + 1.47e^{-\frac{z}{2}+2u}e^{(\delta^{-2}(e^{2\delta}-1-2\delta))} \\ &\le 2.43e^{-\frac{z}{2}} + 1.47e^{-\frac{z}{2}+2\delta+(\delta^{-2}(e^{2\delta}-1-2\delta))}, \end{aligned}$$

which give

$$\begin{aligned} &|E(W^{(i)}+t)f_z(W^{(i)}+t) - E(W^{(i)}+s)f_z(W^{(i)}+s)| \\ &= |\int_s^t Eg(W^{(i)}+u)du| \\ &\le (2.43 + 1.47e^{2\delta+(\delta^{-2}(e^{2\delta}-1-2\delta))})e^{-\frac{z}{2}}(|s|+|t|). \end{aligned}$$

□

5.2.1 Proof of Theorem 5.2.

It suffices to consider $z \ge 0$ as we can apply the result to $-W$ when $z < 0$.

If $0 \le z < 2$, by Theorem 5.1 we have

$$|P(W \le z) - \Phi(z)| \le 3.3\delta \le 3.3(2.72)e^{-1}\delta \le 8.98e^{-\frac{z}{2}}\delta \le C_\delta e^{-\frac{z}{2}}\delta$$

where we have used the fact that $2.72e^{-1} > 1$ in the second inequality.

Assume that $z \ge 2$.

Case 1. $z \le 2\delta$.

Since $\delta \ge 1$, so $3.3e^{\delta} < 2.21e^{2\delta}$. Therefore, by Theorem 5.1, we have

$$|P(W \le z) - \Phi(z)| \le 3.3\delta \le 3.3e^{\delta}e^{-\frac{z}{2}}\delta \le 2.21e^{2\delta}e^{-\frac{z}{2}}\delta \le C_\delta e^{-\frac{z}{2}}\delta.$$

Case 2. $z > 2\delta$.

Let $K_i(t) = EX_i[1(0 \le t \le X_i) - 1(X_i \le t < 0)]$.

Hence $\sum_{i=1}^{n}\int_{-\infty}^{\infty}K_i(t)dt=\sum_{i=1}^{n}EX_i^2=1.$
By the fact that $|X_i|\le\delta$, we have $K_i(t)=0$ for $|t|>\delta$. This implies

$$\begin{aligned}\sum_{i=1}^{n}\int_{\infty}^{\infty}P(W^{(i)}+t\le z)K_i(t)dt &= \sum_{i=1}^{n}\int_{|t|\le\delta}P(W-X_i+t\le z)K_i(t)dt\\ &\ge \sum_{i=1}^{n}\int_{|t|\le\delta}P(W\le z-2\delta)K_i(t)dt\\ &= P(W\le z-2\delta). \qquad (5.4)\end{aligned}$$

Write $f=f_z$. Then

$$\begin{aligned}EWf(W) &= \sum_{i=1}^{n}EX_if(W)\\ &= \sum_{i=1}^{n}EX_i(f(W)-f(W^{(i)})\\ &= \sum_{i=1}^{n}EX_i\int_0^{X_i}f'(W^{(i)}+t)dt\\ &= \sum_{i=1}^{n}E\int_{-\infty}^{\infty}f'(W^{(i)}+t)X_i(\mathit{1}(0\le t\le X_i)-\mathit{1}(X_i\le t<0))dt\\ &= \sum_{i=1}^{n}E\int_{-\infty}^{\infty}f'(W^{(i)}+t)K_i(t)dt\\ &= \sum_{i=1}^{n}E\int_{-\infty}^{\infty}\big\{(W^{(i)}+t)f(W^{(i)}+t)+\mathit{1}(W^{(i)}+t\le z)-\Phi(z)\big\}K_i(t)dt\\ &= \sum_{i=1}^{n}E\int_{-\infty}^{\infty}(W^{(i)}+t)f(W^{(i)}+t)K_i(t)dt\\ &\quad+\sum_{i=1}^{n}\int_{-\infty}^{\infty}P(W^{(i)}+t\le z)K_i(t)dt-\Phi(z)\end{aligned}$$

where we have used (2.1) in the sixth equality.

By Lemma 5.6, we have

$$\sum_{i=1}^{n}\int_{-\infty}^{\infty}P(W^{(i)}+t\le z)K_i(t)dt-\Phi(z)$$

$$= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \big\{Wf(W) - (W^{(i)}+t)f(W^{(i)}+t)\big\} K_i(t)dt$$
$$\le \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \big|Wf(W) - (W^{(i)}+t)f(W^{(i)}+t)\big| K_i(t)dt$$
$$= \sum_{i=1}^{n} E \int_{|t|\le\delta} \big|(W^{(i)}+X_i)f(W^{(i)}+X_i) - (W^{(i)}+t)f(W^{(i)}+t)\big| K_i(t)dt$$
$$= \sum_{i=1}^{n} E \int_{|t|\le\delta} \big|E^{X_i}\{(W^{(i)}+X_i)f(W^{(i)}+X_i)\} - E(W^{(i)}+t)f(W^{(i)}+t)\big| K_i(t)dt$$
$$\le \widetilde{C}_\delta e^{-\frac{z}{2}} \sum_{i=1}^{n} E \int_{|t|\le\delta} (|X_i|+|t|)K_i(t)dt$$
$$= \widetilde{C}_\delta e^{-\frac{z}{2}} \sum_{i=1}^{n} \big\{E|X_i|EX_i^2 + 0.5E|X_i|^3\big\}$$
$$\le 1.5\widetilde{C}_\delta e^{-\frac{z}{2}}\delta \tag{5.5}$$

where $\widetilde{C}_\delta = 2.43 + 1.47e^{2\delta+(\delta^{-2}(e^{2\delta}-1-2\delta))}$.
Since $z > 2\delta$, we have $\dfrac{4\delta^2-4z\delta}{2} < 0$. Hence

$$\begin{aligned}
\Phi(z) - \Phi(z-2\delta) &= \frac{1}{\sqrt{2\pi}} \int_{z-2\delta}^{z} e^{-\frac{t^2}{2}} dt \\
&\le \frac{2\delta}{\sqrt{2\pi}} e^{-\frac{(z-2\delta)^2}{2}} \\
&= \frac{2\delta}{\sqrt{2\pi}} e^{-\frac{z^2}{2}-\frac{4\delta^2-4z\delta}{2}} \\
&\le \frac{2\delta}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \\
&\le 0.8e^{-\frac{z}{2}}\delta.
\end{aligned} \tag{5.6}$$

By (5.4) – (5.6), we have

$$\begin{aligned}
&P(W \le z-2\delta) - \Phi(z-2\delta) \\
&\le \sum_{i=1}^{n} \int_{-\infty}^{\infty} P(W^{(i)}+t \le z)K_i(t)dt - \Phi(z-2\delta) \\
&= \Phi(z) - \Phi(z-2\delta) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} P(W^{(i)}+t \le z)K_i(t)dt - \Phi(z) \\
&\le 0.8e^{-\frac{z}{2}}\delta + 1.5\widetilde{C}_\delta e^{-\frac{z}{2}}\delta \\
&= C_\delta e^{-\frac{z}{2}}\delta
\end{aligned}$$

where $C_\delta = 4.45 + 2.21e^{2\delta+(\delta^{-2}(e^{2\delta}-1-2\delta))}$.
Hence

$$P(W \le z) - \Phi(z) \le C_\delta e^{-\frac{z+2\delta}{2}}\delta \le C_\delta e^{-\frac{z}{2}}\delta.$$

Similarly, we can show that $P(W \le z) - \Phi(z) \ge -C_\delta e^{-\frac{z}{2}}\delta$.
Hence $|P(W \le z) - \Phi(z)| \le C_\delta e^{-\frac{z}{2}}\delta$. □

5.2.2 Proof of Theorem 5.3.

In view of Theorem 5.1, we may without loss of generality, assume $z \ge 2$.

Case 1. $(1+z)\delta \ge 1$.
Let

$$\begin{aligned} Y_i &= X_i 1(|X_i| \le z) - EX_i 1(|X_i| \le z), \\ T &= \sum_{i=1}^n Y_i, \\ r &= \sum_{i=1}^n EX_i 1(|X_i| > z). \end{aligned}$$

Thus $W = T - r$ when $\max\limits_{1\le j\le n} |X_j| \le z$.
Hence

$$\begin{aligned} P(W \ge z) &= P(T \ge z + r) + P(\max_{1\le j\le n} |X_j| > z) \\ &\le P(T \ge z + r) + \sum_{i=1}^n \frac{E|X_i|^3}{z^3} \\ &\le P(T \ge z + r) + \frac{\delta}{z^3} \\ &\le P(T \ge z + r) + \frac{C\delta}{1+z^3}. \end{aligned} \tag{5.7}$$

By the Rosenthal inequality, we have

$$\begin{aligned} E|T|^4 &\le C\{(E|T|^2)^2 + \sum_{i=1}^n E|X_i|^4 1(|X_i| \le z)\} \\ &\le C(1+z\delta) \end{aligned}$$

and hence

$$
\begin{aligned}
P(T \geq z+r) &\leq \frac{ET^4}{(z+r)^4} \\
&\leq \frac{ET^4}{(z-1)^4} \\
&\leq \frac{CET^4}{z^4} \\
&\leq \frac{C(1+z\delta)}{z^4} \\
&\leq \frac{C}{z^4} + \frac{\delta}{z^3} \\
&\leq \frac{C}{1+z^4} + \frac{C\delta}{1+z^3} \\
&\leq \frac{C}{(1+z)(1+z^3)} + \frac{C\delta}{1+z^3} \\
&\leq \frac{C\delta}{1+z^3} \qquad (5.8)
\end{aligned}
$$

where we have used the fact that $|r| \leq \sum_{i=1}^{n} EX_i^2 = 1$ in the last inequality.

By (5.7) and (5.8), we have

$$P(W \geq z) \leq \frac{C\delta}{1+z^3}.$$

Hence

$$
\begin{aligned}
|P(W \leq z) - \Phi(z)| &\leq P(W \geq z) + 1 - \Phi(z) \\
&\leq \frac{C\delta}{1+z^3} + \frac{C}{(1+z)^4} \\
&\leq \frac{C\delta}{1+z^3}.
\end{aligned}
$$

Case 2. $(1+z)\delta < 1$.

It follows from Theorem 5.2 and the fact that $\delta < \frac{1}{1+z} \leq \frac{1}{3}$.

This prove the theorem. □

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