Chapter I

Background

1.1 Notation

As is usual for a mathematics thesis, we use many symbols here. Most of them are notation for new objects, and will be defined in later chapter. However, there is some special notation for well-known objects that we will use, which we list here.

$\mathbb{N}$ = set of all nonnegative integers.

$\mathfrak{P}(A)$ = power set of the set A.

$A \to B$ = the set of all functions from the set A to the set B.

1.2 Universe construction

In Chapter 2 we will define type systems which include function and tuple types. To simplify our definition of the interpretation of such type systems we introduce the universe constructed from a set.

Definition 1.2.1 *Let A be a set. The* **universe constructed from** *A, denoted by $\mathcal{U}_A$, is defined as follows. We start by defining a sequence $\{A_k\}$ of sets by induction. Let $A_0 = A$. If A_k is defined for some $k \in \mathbb{N}$, then we define $A_{k+1} = \mathfrak{P}(A_k) \cup A_k$. Finally, we let*

$$\mathcal{U}_A = \bigcup_{k=0}^{\infty} A_k.$$

In set theory, see [1], the order pair (a, b) is typically defined to be $\{\{a\}, \{a, b\}\}$. From this definition, we have the following lemma.

Lemma 1.2.2 *If $X_1, X_2 \subseteq A_k$ for some $k \in \mathbb{N}$, then $X_1 \times X_2 \subseteq A_{k+2}$.*

Proof: Let $(x, y) \in X_1 \times X_2$. By definition, $(x, y) = \{\{x\}, \{x, y\}\}$. Since X_1 and X_2 are subsets of A_k, $\{x\}$ and $\{x, y\}$ are also subsets of A_k. Impliy that $\{x\}$ and $\{x, y\}$ are in A_{k+1}. Hence, $\{\{x\}, \{x, y\}\} \subseteq A_{k+1}$. It follows that $\{\{x\}, \{x, y\}\} \in A_{k+2}$. That is, $(x, y) \in A_{k+2}$. So we have $X_1 \times X_2 \subseteq A_{k+2}$.

One simple way to define the arbitrary finite Cartesian product $X_1 \times X_2 \times \cdots \times X_m$ is as $((\cdots(X_1 \times X_2) \times \cdots) \times X_m)$. Using this definition it is easy to prove the following corollary by induction.

Corollary 1.2.3 *If $X_1, X_2, ..., X_m \subseteq A_k$ for some $k \in \mathbb{N}$, then $(X_1 \times X_2 \times \cdots \times X_m) \subseteq A_{k+2(m-1)}$.*

Proposition 1.2.4 *If $X_1, X_2 \subseteq A_k$ for some $k \in \mathbb{N}$, then $X_1 \to X_2 \subseteq A_{k+3}$.*

Proof: Let $f \in X_1 \to X_2$. By definition and Lemma 1.2.2 $f \subseteq X_1 \times X_2 \subseteq A_{k+2}$. Then $f \in\subseteq A_{k+3}$. Hence $X_1 \to X_2 \subseteq A_{k+3}$.

1.3 Propositional logic

We will use propositional logic to help define our set of logical axioms in chapter 2, so we will give a regorous definition of propositional formulas here. Let $\mathbb{V}^p = \{P_1, P_2, P_3, ...\}$ be the set of all propositional variables. A formula in propositional logic is constructed from $\mathbb{V}^p$ using logical operators as follows.

Definition 1.3.1 *The set of all* **propositional formulas**, *denoted by $\mathbb{F}^p$, is defined by induction as follows. Let $\mathbb{F}^p_0 = \mathbb{V}^p$. Let $k \in \mathbb{N}$, and assume $\mathbb{F}^p_k$ is already defined. Define*

$$\mathbb{F}^p_{k+1} = \mathbb{F}^p_k \cup \{\neg\Phi \mid \Phi \in \mathbb{F}^p_k\} \cup \{\Phi \wedge \Psi \mid \Phi, \Psi \in \mathbb{F}^p_k\}$$

Finally, define

$$\mathbb{F}^p = \bigcup_{k=0}^{\infty} \mathbb{F}^p_k$$

If we want to know whether a formula $\Phi \in \mathbb{F}^p$ is true or false, we need to assign a truth value to each propositional variable that occurs in Φ. We can do this by using a **truth assignment**, which is a functions from $\mathbb{V}^p$ to $\{T, F\}$. The **interpretation of a propositional formula Φ** under a truth assignment α, denoted by $I^{\mathfrak{P}}(\Phi, \alpha)$, is defined inductively as follows.

Definition 1.3.2 *Let $\Phi \in \mathbb{F}^p$ and let α be a truth assignment. If $\Phi \in \mathbb{F}^p_0$, then define $I^{\mathfrak{P}}(\Phi, \alpha) = \alpha(\Phi)$. Assume that $\Phi \in \mathbb{F}^p_k$ for some $k > 0$, and that for all $\ell < k$ and all $\Psi \in \mathbb{F}^p_\ell$, $I^{\mathfrak{P}}(\Psi, \alpha)$ is defined. There are two cases that we must consider.*

Case $\Phi \equiv \neg\Psi$ for some $\Psi \in \mathbb{F}^p_\ell, \ell < k$. Define

$$I^{\mathfrak{P}}(\Phi, \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{P}}(\Psi, \alpha) = F \\ F & \text{otherwise.} \end{cases}$$

Case $\Phi \equiv (\Psi \wedge \Theta)$ for some $\Psi, \Theta \in \mathbb{F}^p_\ell, \ell < k$. Define

$$I^{\mathfrak{P}}(\Phi, \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{P}}(\Psi, \alpha) = T \text{ and } I^{\mathfrak{P}}(\Theta, \alpha) = T \\ F & \text{otherwise.} \end{cases}$$

If $I^{\mathfrak{P}}(\Phi, \alpha) = T$ for all truth assignments α, then we call Φ a **tautology**.