## CHAPTER I

## THE VELOCITY OF ELECTROMAGNETIC WAVES

The general forms of Maxwell's equations for electromagnetic waves in any medium are

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\left(\frac{\partial \overline{D}}{\partial t} + 4\pi\rho\overline{v}\right) \\ \nabla \times \overline{E} &= -\frac{1}{c}\frac{\partial \overline{B}}{\partial t} \\ \nabla \cdot \overline{D} &= 4\pi\rho \\ \nabla \cdot \overline{B} &= 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

where $\overline{H}$ is magnetic field strength in electromagnetic units (e.m.u)

$\overline{E}$ is electric field strength in electrostatic units (e.s.u)

$\overline{B}$ is magnetic flux density (e.m.u) ; $\overline{B} = \mu\overline{H}$, $\mu$ is called magnetic permeability.

$\overline{D}$ is electric flux density (e.s.u); $\overline{D} = \epsilon\overline{E}$, $\epsilon$ is called dielectric constant.

$\overline{V}$ is velocity of charge (c.g.s)

$\rho$ is the charge density (e.s.u)

c is a constant which equal to the ratio of the electromagnetic unit to the electrostatic unit of charge, from experiment $c = 3 \times 10^{10}$ c.g.s.

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Bars over symbols denote vectors.

When the medium is ether , we have $\epsilon = 1$, $\mu = 1$ and the system (1) becomes

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\left(\frac{\partial \overline{E}}{\partial t} + 4\pi\rho\overline{v}\right) \\ \nabla \times \overline{E} &= -\frac{1}{c}\frac{\partial \overline{H}}{\partial t} \\ \nabla \cdot \overline{E} &= 4\pi\rho \\ \nabla \cdot \overline{H} &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2)$$

These are the forms of the equations for electromagnetic waves in the atmosphere where $\epsilon$ and $\mu$ are both close to unity. We shall pay attention to these forms latter.

In the case of free ether or there is no charge, we have $\rho = 0$ , and the system (2) becomes

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\frac{\partial \overline{E}}{\partial t} \\ \nabla \times \overline{E} &= -\frac{1}{c}\frac{\partial \overline{H}}{\partial t} \\ \nabla \cdot \overline{E} &= 0 \\ \nabla \cdot \overline{H} &= 0 \end{aligned}\right\} \ldots\ldots\ldots (3)$$

The equations in (3) were first obtained by Maxwell.

From the first equation of (3), we have

$$\nabla \times (\nabla \times \overline{H}) = \frac{1}{c}\nabla \times \left(\frac{\partial \overline{E}}{\partial t}\right)$$

or

$$\nabla(\nabla \cdot \overline{H}) - \nabla^2 \overline{H} = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \overline{E})$$

By using the second and the fourth equations in (3), we then obtain

$$\nabla^2 \bar{H} = \frac{1}{c^2} \frac{\partial^2 \bar{H}}{\partial t^2} \qquad \text{.............} (4)$$

similarly we may show that

$$\nabla^2 \bar{E} = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} \qquad \text{.............} (5)$$

the equations (4) and (5) may be replaced by the joint equation

$$\nabla^2 U = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} \qquad \text{.............} (6)$$

where U represent any component of $\bar{H}$ and $\bar{E}$ in the directions of the axes of a cartessian system of co - ordinates. The equation (6) is the well -known equation of the wave motion. The solution may be written down by inspection. For try the form

$$U(x, y, z, t) = F(\alpha x + \beta y + \gamma z - ct) + G(\alpha x + \beta y + \gamma z + ct).$$

Substitution in the equation shows that this is a solution if the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$ hold, no matter what functions F and G may be. Note that the equation

$$\alpha x + \beta y + \gamma z = s$$

is the equation of a plane at a perpendicular distance s from the origin and the direction cosines of its normal are $(\alpha, \beta, \gamma)$ If the time t increases from t to t + $\Delta$t, s must change from s to s + c$\cdot\Delta$t in order that the function $F(\alpha x + \beta y + \gamma z - ct)$ remains constant. Thus u = F represents a plane wave moving away from the origin with a relocity c. In a similar manner U = G represents a plane wave approaching the origin. The general solution U = F + G of (6) represents the superposition of an advancing and a retreating plane wave.

To find the velocity of electromagnetic waves in a region where the ether contains the charges or where $\rho \neq 0$, we shall find the integrals of the system (2), from which we can deduce the velocity of the wave.

By the fourth equation of (2), we can put

$$\overline{H} = \nabla \times \overline{A} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where $\overline{A}$ is a vector function of position and time.

Substituting this in the second equation of (2), we have

$$\nabla \times \overline{E} = -\frac{1}{c} \nabla \times \left( \frac{\partial \overline{A}}{\partial t} \right)$$

or

$$\nabla \times \left( \overline{E} + \frac{1}{c} \frac{\partial \overline{A}}{\partial t} \right) = 0.$$

Hence, there exists a scalar function $\phi$ of position and time such that

$$\overline{E} = -\frac{1}{c} \frac{\partial \overline{A}}{\partial t} - \nabla \phi \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Substituting (7) and (8) in the first equation of (2), we obtain

$$\nabla \times (\nabla \times \overline{A}) = -\frac{1}{c^2} \frac{\partial^2 \overline{A}}{\partial t^2} - \frac{1}{c} \nabla \left( \frac{\partial \phi}{\partial t} \right) + \frac{1}{c} 4\pi \rho \overline{v}$$

or

$$\nabla^2 \overline{A} - \frac{1}{c^2} \frac{\partial^2 \overline{A}}{\partial t^2} = \left( \nabla \cdot \overline{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) - \frac{1}{c} 4\pi \rho \overline{v}.$$

We may impose the further condition

$$\nabla \cdot \overline{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0. \qquad \ldots\ldots\ldots\ldots\ldots (9)$$

Then we have

$$\nabla^2 \overline{A} - \frac{1}{c^2} \frac{\partial^2 \overline{A}}{\partial t^2} = -\frac{1}{c} 4\pi \rho \overline{v} \qquad \ldots\ldots\ldots\ldots\ldots (10)$$

Substituting from (8) in the third equation of (2), we have

$$-\frac{1}{c}\frac{\partial}{\partial t}\;(\nabla\cdot\overline{A})\;-\;\nabla\cdot(\nabla\phi)\;=\;4\pi\rho$$

or

$$-\frac{1}{c}\frac{\partial}{\partial t}\,(\nabla\cdot\overline{A})-\nabla^2\phi\;=\;4\pi\rho$$

Substituting for $\nabla\cdot\overline{A}$ from condition (9), we obtain

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\;-\nabla^2\phi\;=\;4\pi\rho$$

or

$$\nabla^2\phi\;-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\;=\;-4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots (11)$$

We thus have a new system of equations (9), (10) and (11) which is equivalent to the system (2). $\overline{H}$ and $\overline{E}$ ~~are~~ are found from $\overline{A}$ and $\phi$ by means of (7) and (8). $\phi$ is called the scalar potential, $\overline{A}$ the vector potential. We have now to integrate (10) and (11) subject to the condition (9) when $\rho$ and $\overline{v}$ are given as functions of position and time.

Consider the equation

$$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2}\;=\;-f \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

which has the form of each component in (10), and of (11). By Fourier's theorem we may represent f as a function of t in the following way :

$$f\;(t)\;=\;\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\;(\alpha)\;e^{i\beta(t-\alpha)}\;d\alpha d\beta \quad \ldots\ldots\ldots (13)$$

Similarly

$$W(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\alpha)\, e^{i\beta(t-\alpha)}\, d\alpha d\beta \qquad .......(14)$$

Hence, by (12)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\nabla^2 W(\alpha) + \frac{\beta^2}{c^2} W(\alpha) + f(\alpha)\right] e^{-\beta(t-\alpha)}\, d\alpha d\beta = 0$$

This equation is satisfied by

$$\nabla^2 W(\alpha) + \frac{\beta^2}{c^2} W(\alpha) = - f(\alpha) \qquad .............(15)$$

or

$$\nabla^2 W + k^2 W = - f \qquad .............(15')$$

where $k = \frac{\beta}{c}$.

We now require the integral of this equation. Such integral can be found by the following way : consider its reduced equation

$$\nabla^2 W + k^2 W = 0 \qquad ............. (16)$$

If W is depended on r only, and not on $\theta$ and $\phi$ , the equation (16) in spherical coordinates becomes

$$\frac{1}{r}\frac{d^2}{dr^2}(rW) + k^2 W = 0$$

This is satisfied by

$$W = \frac{1}{r} e^{\pm ikr}$$

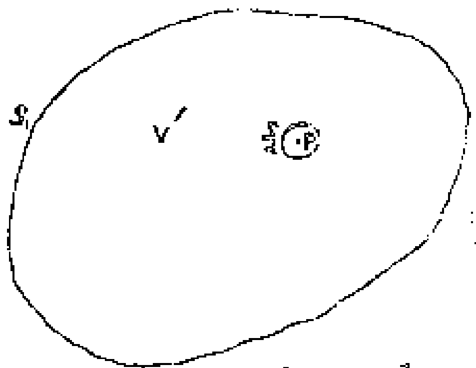
where r is the distance of the variable point from a fixed point P. This is true for any region not including P.

Now, consider Green's theorem :

$$\int_v \left[\Phi_1 \nabla^2 \Phi_2 - \Phi_2 \nabla^2 \Phi_1\right] dv = \int_s \left[\Phi_1 \frac{\partial \Phi_2}{\partial n} - \Phi_2 \frac{\partial \Phi_1}{\partial n}\right] ds$$

Put $\Phi_1 = \frac{1}{r} e^{-ikr}$ (one integral of (16) ), and $\Phi_2 = W$ (required integral of (15 ) ). Since the quantities involved in Green's formula must be finite, the region of integration must exclude P, at which $\Phi_1 = \frac{1}{r} e^{-ikr}$ is infinite.

We isolate P by describing a small sphere of radius $\epsilon$ with P as center




Substituting the above values of $\Phi_1$ and $\Phi_2$ in Green's formula, we have

$$\int_{V'} \frac{1}{r} e^{-ikr} \left( \nabla^2 W + k^2 W \right) d \;\; = \int_{s_1} \left[ \frac{1}{r} e^{-ikr} \frac{\partial W}{\partial n} - W \frac{\partial}{\partial n} \left( \frac{1}{r} e^{-ikr} \right) \right] ds_1$$

$$+ \int_{s_2} \left[ \frac{1}{\epsilon} e^{-ik\epsilon} \frac{\partial W}{\partial n} - W \frac{\partial}{\partial n} \left( \frac{1}{\epsilon} e^{-ik\epsilon} \right) \right] ds_2 \quad \ldots\ldots\ldots \; (17)$$

where $s_1$ is the outer surface, $s_2$ is the surface of the sphere, and $V'$ is the region between $s_1$ and $s_2$. On $s_2$, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial \epsilon}$, and $ds_2 = \epsilon^2 dw$ where $dw$ is the element of solid angle subtended by $ds_2$ at P. Hence the second integral on the right hand side of (17) becomes,

$$\oint \left[ -\frac{1}{\epsilon} e^{-ik\epsilon} \frac{\partial W}{\partial \epsilon} + W \frac{\partial}{\partial \epsilon} \left( \frac{1}{\epsilon} e^{-ik\epsilon} \right) \right] \epsilon^2 \, dw$$

or

$$\oint \left[ -e^{-ik\epsilon} \frac{\partial W}{\partial \epsilon} + W e^{-ik\epsilon} - Wik\epsilon\, e^{-ik\epsilon} \right] d\omega$$

Let $\epsilon \longrightarrow 0$, then $v' \longrightarrow v$, the first and the third terms in the above integral approach zero, $\oint We^{-ik\epsilon}\, d\omega \longrightarrow 4\pi W_p$ where $W_p$ is the value of W at the point P, and in the limit equation (17) becomes

$$4\pi W_p = -\int_v \frac{1}{r} e^{-ikr} (\nabla^2 W + k^2 W)\, dv$$
$$+ \int_{S_1} \left[ \frac{1}{r} e^{-ikr} \frac{\partial W}{\partial n} - W \frac{\partial}{\partial n} \left( \frac{1}{r} e^{-ikr} \right) \right] ds_1$$

Now, let $s_1$ be a sphere with P as center and whose radius approaches infinity. Then if Wr tends to a finite constant in such a way that $\frac{\partial W}{\partial n} \cdot r^2$ tends to a finite constant as $r \to \infty$, the above surface integral appraches zero as a limit. It follows that

$$W_p = -\frac{1}{4\pi} \int_v \frac{1}{r} e^{-ikr} (\nabla^2 W + k^2 W)\, dv .$$

According to equation (15′), this becomes

$$W_p = \frac{1}{4\pi} \int_v \frac{1}{r} e^{-ikr} f\, dv$$

This is a particular integral of (15′) and

$$W_p(\alpha) = \frac{1}{4\pi} \int_v \frac{1}{r} e^{-ikr} f(\alpha)\, dv$$

is a particular integral of (15). Substituting this expression in (14) and changing the order of integration we obtain

$$W_p(t) = \frac{1}{4\pi} \int_v \frac{1}{r} \left[ \frac{1}{2\pi} \int \int f(\alpha)\, e^{i\beta(t-\frac{r}{c}-\alpha)}\, d\alpha d\beta \right] dv .$$

But in view of (13), this equivalent to

$$W_p(t) = \frac{1}{4\pi} \int_v \frac{f(t - r/c)}{r}\, dv$$

Return to equations (7) and (8). Since $|\overline{H}| \cdot r^2$ and $|\overline{E}| \cdot r^2$ tend to a finite constant as $r \longrightarrow \infty$, $|\overline{A}| \cdot r$ and $\phi \cdot r$ tend to a finite constant. Thus, in view of the above result, the equations. (10) and (11) give

$$\overline{A}_p(t) = \frac{1}{c} \int_v \frac{[\rho\overline{v}]}{r} \, dv \qquad \ldots\ldots\ldots\ldots\ldots(18)$$

$$\phi_p(t) = \int_v \frac{[\rho]}{r} \, dv \qquad \ldots\ldots\ldots\ldots\ldots(19)$$

where the square bracket indicates that the value of the function is to be taken not at the instant t, but for the previous instant $t - \frac{r}{c}$. The functions of this nature are called retarded potentials. Further, the relation (9) is sactisfied by the above value of $\overline{A}$ and $\phi$. For

$$\nabla \cdot \overline{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{1}{c} \int_v \left[\nabla_p \cdot \frac{[\rho\overline{v}]}{r} + \frac{\partial}{\partial t}\frac{[\rho]}{r}\right] dv \qquad \ldots\ldots\ldots\ldots (20)$$

where $\nabla_p$ is the vector operator at the point P. We shall use x, y, z for the coordinates of P and $\xi, \eta, \zeta$ for those of Q. Then

$$\nabla_P = \overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}$$

$$\nabla_Q = \overline{i}\frac{\partial}{\partial \xi} + \overline{j}\frac{\partial}{\partial \eta} + \overline{k}\frac{\partial}{\partial \zeta}$$

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}\,.$$

By expansion, we have

$$\nabla_P \cdot \frac{[\rho\overline{v}]}{r} \equiv [\rho\overline{v}] \cdot \nabla_P \left(\frac{1}{r}\right) + \frac{1}{r}\nabla_P \cdot [\rho\overline{v}] \qquad \ldots\ldots\ldots\ldots(21)$$

Since $[\rho\overline{v}]$ is the value of $\rho\overline{v}$ at the point Q, $\nabla_P \cdot [\rho\overline{v}] = 0$.

And $\nabla_P \left(\frac{1}{r}\right) = -\nabla_Q \left(\frac{1}{r}\right)$. Hence, we have

$$\nabla_P \cdot \frac{[\rho \bar{v}]}{r} = - [\rho \bar{v}] \cdot \nabla_Q \left(\frac{1}{r}\right)$$

Applying the identity (21) to the right hand side of the last equation, we obtain

$$\nabla_P \cdot \frac{[\rho \bar{v}]}{r} = -\nabla_Q \cdot \frac{[\rho \bar{v}]}{r} + \frac{1}{r}\nabla_Q \cdot [\rho \bar{v}] \quad \ldots\ldots\ldots\ldots\ldots (22)$$

For the last integral of (20), we have

$$\frac{\partial}{\partial t} \cdot \frac{[\rho]}{r} = \frac{1}{r}\frac{\partial[\rho]}{\partial t} = \frac{1}{r}\frac{\partial[\rho]}{\partial t'} \quad \ldots\ldots\ldots\ldots\ldots (23)$$

where $t' = t - r/c$.

Substituting the values from (22) and (23) in (20), we have

$$\nabla_P \cdot \bar{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{1}{c}\int_v \left[-\nabla_Q \cdot \frac{[\rho \bar{v}]}{r} + \frac{1}{r}\nabla_Q \cdot [\rho \bar{v}] + \frac{1}{r}\frac{\partial[\rho]}{\partial t'}\right] dv \ldots . (24)$$

Now, since $\nabla \cdot (\rho \bar{v}) + \frac{\partial \rho}{\partial t} = 0$, the relation (24) becomes

$$\nabla_P \cdot \bar{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = -\frac{1}{c}\int_v \left(\nabla_Q \cdot \frac{[\rho \bar{v}]}{r}\right) dv$$

By applying divergence theorem, we have

$$\nabla_P \cdot \bar{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = -\frac{1}{c}\int_s \frac{[\rho \bar{v}]}{r} \cdot d\bar{s} \quad \ldots\ldots\ldots (25)$$

where s is the boundary surface of region v. We may transform the surface integral into integral taken over a surface at infinity, at which $\rho \bar{v}$ is assumed (on physical grounds) to vanish. Hence

$$\nabla_P \cdot \bar{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0.$$

We now have the relations (18) and (19) as the solutions of the wave equations (10) and (11) from which the values of $\overline{H}$ and $\overline{E}$ can be found by the relations (7) and (8).

Equation (19) has the following meaning : The value of the potential at the point P and the instant t, depends upon the value of $\rho$ (charge density) at the point Q, not at this instant, but at the previous instant $t - \frac{r}{c}$, where r = PQ, that is, at a time earlier by the interval required for its influence to move from Q to P. Equations (18) and (19) therefore express the fact that electromagnetic disturbances are propagated in ether with a velocity c.

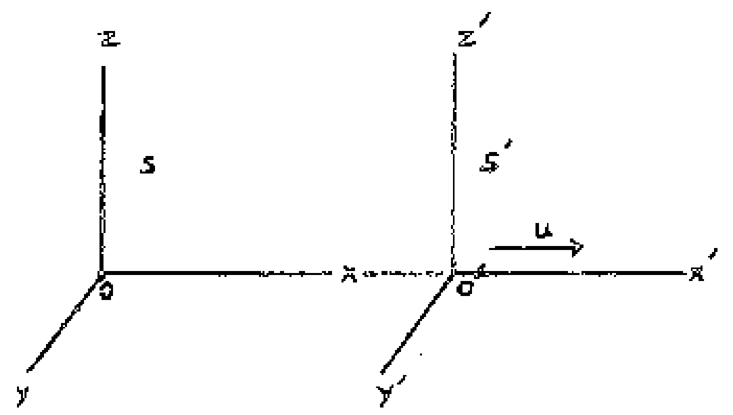
The main object of what we have done is to deduce the velocity of electromagnetic waves from Maxwell's equations, and we have found that in ether it is equal to c, or $3 \times 10^{10}$ cm./sec. This value agrees with the observed velocity of light in ether. From this and a variety of other reasons, light is identified as consistency of such electromagnetic waves.

We have to realise that the experiments on which the electromagnetic equations are based are performed by an observer at rest relative to the medium.. What we have called the velocity of electromagnetic waves in ether is actually the velocity deduced by such an observer. The problem then arises as to what value would be obtained by another observer in motion relative to the first.

# CHAPTER II

## TRANSFORMATION OF MAXWELL'S EQUATIONS

## BY THE GALILEAN TRANSFORMATION .




Consider two observers : An observer O is at rest at the origin of the system of coordinates s, and he describes any event by a set of numbers (x, y, z, t). Another observer O' is at rest at the origin of the system s', and he therefore describes any event by the set of numbers (x', y', z', t'). The system s' moves with uniform velocity u in the positive x - direction with respect to the system s. When $t = t' = 0$, the origins of the two systems coincide. And their axes are always parallel.

The observers O and O' observe the same electromagnetic waves. Suppose that the observations of O for the electric field make $\overline{E} = (Ex, Ey, Ez)$ and for the magnetic field make $\overline{H} = (Hx, Hy, Hz)$, while the observations of O' for the same electromagnetic field make $\overline{E}' = (E'_x, E'_y, E'_z)$ and $\overline{H}' = (H'_x, H'_y, H'_z)$.

The relations between $\overline{E}$ and $\overline{H}$ in the system s are

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\left(\frac{\partial \overline{E}}{\partial t} + 4\pi\rho\overline{v}\right) \\ \nabla \times \overline{E} &= -\frac{1}{c}\frac{\partial \overline{H}}{\partial t} \\ \nabla \cdot \overline{E} &= 4\pi\rho \\ \nabla \cdot \overline{H} &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (1)$$

If we put $\overline{\Psi} = \overline{E} + i\overline{H}$ where $i = \sqrt{-1}$, then the system (1) becomes.

$$\left.\begin{aligned} \nabla \times \overline{\Psi} &= \frac{i}{c}\left(\frac{\partial \overline{\Psi}}{\partial t} + 4\pi\rho\overline{v}\right) \\ \nabla \cdot \overline{\Psi} &= 4\pi\rho \end{aligned}\right\} \cdots\cdots\cdots\cdots (1')$$

Are they invariant under the Galilean transformation ? If they are, the form of the equations after transformation should be

$$\begin{aligned} \nabla' \times \overline{H}' &= \frac{1}{c}\left(\frac{\partial \overline{E}'}{\partial t'} + 4\pi\rho\overline{v}'\right) \\ \nabla' \times \overline{E}' &= -\frac{1}{c}\frac{\partial \overline{H}'}{\partial t'} \\ \nabla' \cdot \overline{E}' &= 4\pi\rho \\ \nabla' \cdot \overline{H}' &= 0 \end{aligned}$$

or

$$\begin{aligned} \nabla' \times \overline{\Psi}' &= \frac{i}{c}\left(\frac{\partial \overline{\Psi}'}{\partial t'} + 4\pi\rho\overline{v}'\right) \\ \nabla' \cdot \overline{\Psi}' &= 4\pi\rho \end{aligned}$$

where $\nabla' = \bar{i}\frac{\partial}{\partial x'} + \bar{j}\frac{\partial}{\partial y'} + \bar{k}\frac{\partial}{\partial z'}$.

Equations (1 ) may be written as

$$\begin{bmatrix} 0 + \frac{\partial \Psi_z}{\partial y} - \frac{\partial \Psi_y}{\partial z} - \frac{i}{c}\frac{\partial \Psi_x}{\partial t} \\ -\frac{\partial \Psi_z}{\partial x} + 0 + \frac{\partial \Psi_x}{\partial z} - \frac{i}{c}\frac{\partial \Psi_y}{\partial t} \\ \frac{\partial \Psi_y}{\partial x} - \frac{\partial \Psi_x}{\partial y} + 0 - \frac{i}{c}\frac{\partial \Psi_z}{\partial t} \\ \frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y} + \frac{\partial \Psi_z}{\partial z} + 0 \end{bmatrix} = \begin{bmatrix} \frac{i}{c} 4\pi \rho v_x \\ \frac{i}{c} 4\pi \rho v_y \\ \frac{i}{c} 4\pi \rho v_z \\ 4\pi\rho \end{bmatrix} \quad \ldots\ldots (1'')$$

Since

$$\begin{bmatrix} \frac{\partial \Psi_\alpha}{\partial x} \\ \frac{\partial \Psi_\alpha}{\partial y} \\ \frac{\partial \Psi_\alpha}{\partial z} \\ \frac{\partial \Psi_\alpha}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} & \frac{\partial t'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} & \frac{\partial t'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} & \frac{\partial t'}{\partial z} \\ \frac{\partial x'}{\partial t} & \frac{\partial y'}{\partial t} & \frac{\partial z'}{\partial t} & \frac{\partial t'}{\partial t} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \Psi_\alpha}{\partial x'} \\ \frac{\partial \Psi_\alpha}{\partial y'} \\ \frac{\partial \Psi_\alpha}{\partial z'} \\ \frac{\partial \Psi_\alpha}{\partial t'} \end{bmatrix}$$

where $\alpha$ can be replaced by x, y or z,

the Galilean transformation : $x' = x - ut$ , $y' = y$ , $z' = z$ ,

$t' = t$, gives

$$\begin{bmatrix} \frac{\partial \Psi_\alpha}{\partial x} \\ \frac{\partial \Psi_\alpha}{\partial y} \\ \frac{\partial \Psi_\alpha}{\partial z} \\ \frac{\partial \Psi_\alpha}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -u & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \Psi_\alpha}{\partial x'} \\ \frac{\partial \Psi_\alpha}{\partial y'} \\ \frac{\partial \Psi_\alpha}{\partial z'} \\ \frac{\partial \Psi_\alpha}{\partial t'} \end{bmatrix}$$

or

$$\left.\begin{aligned}
\frac{\partial \Psi_\alpha}{\partial x} &= \frac{\partial \Psi_\alpha}{\partial x'} \\
\frac{\partial \Psi_\alpha}{\partial y} &= \frac{\partial \Psi_\alpha}{\partial y'} \\
\frac{\partial \Psi_\alpha}{\partial z} &= \frac{\partial \Psi_\alpha}{\partial z'} \\
\frac{\partial \Psi_\alpha}{\partial t} &= - u \frac{\partial \Psi_\alpha}{\partial x'} + \frac{\partial \Psi_\alpha}{\partial t'}
\end{aligned}\right\} \cdots\cdots\cdots\cdots (2)$$

Sutstibution the values from (2) in the first and fourth equations of $(1'')$ , we have

$$0 + \frac{\partial \Psi_z}{\partial y'} - \frac{\partial \Psi_y}{\partial z'} - \frac{1}{c} \left( -u \frac{\partial \Psi_x}{\partial x'} + \frac{\partial \Psi_x}{\partial t'} \right) = \frac{1}{c} 4\pi\rho v_x \quad \cdots\cdots (3)$$

$$\frac{\partial \Psi_x}{\partial x'} + \frac{\partial \Psi_y}{\partial y'} + \frac{\partial \Psi_z}{\partial z'} + 0 = 4\pi\rho \quad \cdots\cdots (4)$$

Eliminating $\frac{\partial \Psi_x}{\partial x'}$ from (3) and (4) , we obtain

$$0 + \frac{\partial}{\partial y'} \left( \Psi_z - \frac{1}{c} u \Psi_y \right) - \frac{\partial}{\partial z'} \left( \Psi_y + \frac{1}{c} u \Psi_z \right) - \frac{1}{c} \frac{\partial \Psi_x}{\partial t'}$$

$$= \frac{1}{c} 4\pi\rho (v_x - u)$$

This has the same form as the first equation in $(1'')$ if we put



$$\left.\begin{aligned}
\Psi'_x &= \Psi_x \\
\Psi'_y &= \Psi_y + \frac{1}{c} u \Psi_z \\
\Psi'_z &= \Psi_z - \frac{1}{c} u \Psi_y \\
v'_x &= v_x - u
\end{aligned}\right\} \cdots\cdots\cdots\cdots (5)$$

Now, from $y' = y$ , $z' = z$ , $t' = t$ , we have the relations

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{dt} = v_y$$

$$v'_z = \frac{dz'}{dt'} = \frac{dz}{dt} = v_z$$

From these and (5) , we have the complete system of transformation equations.

$$\left.\begin{aligned} \Psi'_x &= \Psi_x \\ \Psi'_y &= \Psi_y + \frac{1}{c} u \Psi_z \\ \Psi'_z &= \Psi_z - \frac{1}{c} u \Psi_y \\ v'_x &= v_x - u \\ v'_y &= v_y \\ v'_z &= v_z \end{aligned}\right\} \quad \ldots\ldots\ldots (6)$$

Again, substituting the values from (2) in the second equation of (1''), we obtain

$$-\frac{\partial \Psi_z}{\partial x'} + 0 + \frac{\partial \Psi_x}{\partial z'} - \frac{1}{c}\left(-u \frac{\partial \Psi_y}{\partial x'} + \frac{\partial \Psi_y}{\partial t'}\right) = 4\pi\rho v_y$$

or

$$-\frac{\partial}{\partial x'}\left(\Psi_z - \frac{1}{c} u \Psi_y\right) + 0 - \frac{\partial \Psi_x}{\partial z'} - \frac{1}{c}\frac{\partial \Psi_y}{\partial t} = 4\pi\rho v_y$$

If we put :

$$\left.\begin{aligned} \Psi'_x &= \Psi_x \\ \Psi'_y &= \Psi_y \\ \Psi'_z &= \Psi_z - \frac{1}{c} u \Psi_y \\ v'_y &= v_y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7)$$

Then the second equation of $(1^{\prime})$ is invariant under the transformation. Note that the transformation of the y - component of $\overline{\Psi'}$ in (7) is different from that in (6), and the other components are unaltered.

Again, if we substitute the values from (2) in the third equation of $(1^{\prime})$ , we have

$$\frac{\partial \Psi_y}{\partial x'} - \frac{\partial \Psi_x}{\partial y'} + 0 - \frac{1}{c} \left(-u \frac{\partial \Psi_z}{\partial x'} + \frac{\partial \Psi_z}{\partial t'}\right) = 4\pi \rho v_z$$

or

$$\frac{\partial}{\partial x'} \left( \Psi_y + \frac{1}{c} u \Psi_z\right) - \frac{\partial \Psi_z}{\partial y'} + 0 - \frac{1}{c} \frac{\partial \Psi_z}{\partial t'} = 4\pi \rho v_z .$$

We obtain a new transformation for z - component of $\overline{\Psi'}$ , ie.

$$\Psi_z' = \Psi_z \qquad \ldots\ldots\ldots\ldots\ldots (8)$$

While the other components are the same as in (6)

The difference of the results (6) , (7) and (8) , imply that Maxwell's equations are not invariant under the Galilean transformation. That is , we cannot have the relations

$$\nabla' \times \overline{\Psi}' = \frac{1}{c} \left( \frac{\partial \overline{\Psi}'}{\partial t'} + 4\pi\rho \overline{v}' \right)$$

$$\nabla' \cdot \overline{\Psi}' = 4\pi \rho$$

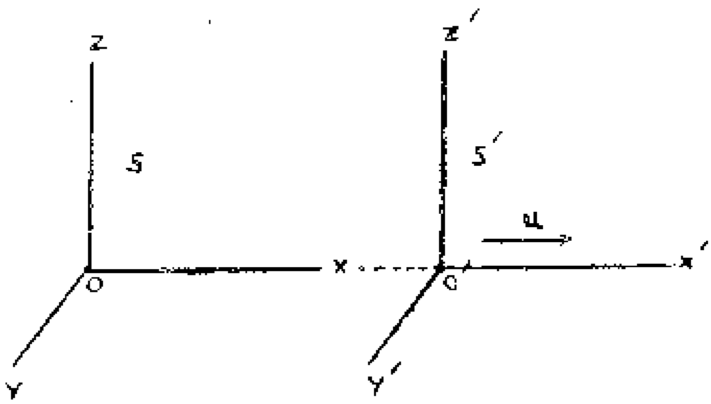
under the Galilean transformation. It follows then that the velocity of propagation of electromagnetic waves deduced by the observer $O'$ (in the moving system ) is not equal c. This disagrees with the results of Michelson and Morley experiment which indicated that the velocity of light (which are electro-magnetic waves) is always constant and equal to c in all systems (at rest or moving). These show the inapplicability of the Galilean transformation.

CHAPTER III

# TRANSFORMATION OF MAXWELL'S EQUATIONS

# BY THE LORENTZ TRANSFORMATION

The transformation that we shall next consider is the Lorentz transformation.




The Lorentz transformation from the system s to the system s is given by the equations

$$x' = \beta\,(x - ut)$$

$$y' = y$$

$$z' = z$$

$$t' = \beta(t - ux/c^2)$$

where $\beta = 1\Big/\sqrt{1 - u^2/c^2}$ .

Since

$$\begin{pmatrix} \frac{\partial \Psi_\alpha}{\partial x} \\ \frac{\partial \Psi_\alpha}{\partial y} \\ \frac{\partial \Psi_\alpha}{\partial z} \\ \frac{\partial \Psi_\alpha}{\partial t} \end{pmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} & \frac{\partial t'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} & \frac{\partial t'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} & \frac{\partial t'}{\partial z} \\ \frac{\partial x'}{\partial t} & \frac{\partial y'}{\partial t} & \frac{\partial z'}{\partial t} & \frac{\partial t'}{\partial t} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \Psi_\alpha}{\partial x'} \\ \frac{\partial \Psi_\alpha}{\partial y'} \\ \frac{\partial \Psi_\alpha}{\partial z'} \\ \frac{\partial \Psi_\alpha}{\partial t'} \end{bmatrix}$$

the Lorentz transformation gives

$$\begin{pmatrix} \frac{\partial \Psi_\alpha}{\partial x} \\ \frac{\partial \Psi_\alpha}{\partial y} \\ \frac{\partial \Psi_\alpha}{\partial z} \\ \frac{\partial \Psi_\alpha}{\partial t} \end{pmatrix} = \begin{pmatrix} \beta & 0 & 0 & -\beta \frac{u}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta u & 0 & 0 & \beta \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \Psi_\alpha}{\partial x'} \\ \frac{\partial \Psi_\alpha}{\partial y'} \\ \frac{\partial \Psi_\alpha}{\partial z'} \\ \frac{\partial \Psi_\alpha}{\partial t'} \end{pmatrix}.$$

or

$$\left.\begin{aligned} \frac{\partial \Psi_\alpha}{\partial x} &= \beta \frac{\partial \Psi_\alpha}{\partial x'} - \beta \frac{u}{c^2} \frac{\partial \Psi_\alpha}{\partial t'} \\ \frac{\partial \Psi_\alpha}{\partial y} &= \frac{\partial \Psi_\alpha}{\partial y'} \\ \frac{\partial \Psi_\alpha}{\partial z} &= \frac{\partial \Psi_\alpha}{\partial z'} \\ \frac{\partial \Psi_\alpha}{\partial t} &= -\beta u \frac{\partial \Psi_\alpha}{\partial x'} + \beta \frac{\partial \Psi_\alpha}{\partial t'} \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

where $\alpha$ can be replaced by x, y or z.

Substituting the values from (1) in the first and the fourth equations of the system ($1''$) , chap. II, we have

$$0 + \frac{\partial \Psi_z}{\partial y'} - \frac{\partial \Psi_y}{\partial z'} - \frac{1}{c} \left( - \beta u \frac{\partial \Psi_\alpha}{\partial x'} + \beta \frac{\partial \Psi_x}{\partial t'} \right) = \frac{1}{c} 4\pi \rho v_x$$

$$\left(\beta \frac{\partial \Psi_x}{\partial x'} - \beta \frac{u}{c^2} \frac{\partial \Psi_x}{\partial t'}\right) + \frac{\partial \Psi_x}{\partial y'} + \frac{\partial \Psi_z}{\partial z} + 0 = 4\pi\rho$$

Eliminating $\frac{\partial \Psi_x}{\partial x'}$ from these, we have

$$0 + \frac{\partial}{\partial y'}\left( \Psi_z - \frac{i}{c} u \Psi_y \right) - \frac{\partial}{\partial z'}\left( \Psi_y + \frac{i}{c} u \Psi_z \right) - \frac{i}{c} \frac{\partial}{\partial t'}\left(\beta - \beta \frac{u^2}{c^2}\right) \Psi_x$$

$$= \frac{i}{c} 4\pi \rho \, (v_x - u)$$

or

$$0 + \frac{\partial}{\partial y'}\left( \Psi_z - \frac{i}{c} u \Psi_y \right) - \frac{\partial}{\partial z'}\left( \Psi_y + \frac{i}{c} u \Psi_z \right) - \frac{i}{c} \frac{\partial}{\partial t'}\left( \frac{1}{\beta} \Psi_x \right)$$

$$= \frac{i}{c} 4\pi \rho \, (v_x - u).$$

This has the same form as the first equation of $(1'')$, chap. II, if we put

$$\left.\begin{aligned} \Psi_x' &= \frac{1}{\beta} \Psi_x \\ \Psi_y' &= \Psi_y + \frac{i}{c} u \Psi_z \\ \Psi_z' &= \Psi_z - \frac{i}{c} u \Psi_y \\ v_x' &= v_x - u \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2)$$

Since, under the Lorentz transformation, the time t is different from t, while y and z are unaltered, we have

$$v_y' = \frac{dy'}{dt'} \neq \frac{dy}{dt} = v_y$$

$$v_z' = \frac{dz'}{dt'} \neq \frac{dz}{dt} = v_z$$

We shall find the transformation of y and z components of $\overline{v}$ by the following procedure.

Substituting the values from (1) in the second equation of $(1'')$, chap II, we have

$$- \left(\beta \frac{\partial \Psi_z}{\partial x'} - \beta \frac{u}{c^2} \frac{\partial \Psi_z}{\partial t'}\right) + 0 + \frac{\partial \Psi_x}{\partial z'} - \frac{1}{c}\left(-\beta u \frac{\partial \Psi_y}{\partial x'} + \beta \frac{\partial \Psi_y}{\partial t'}\right)$$

$$= \frac{1}{c} 4\pi\rho v_y$$

or

$$- \frac{\partial}{\partial x'}\left(\Psi_z - \frac{1}{c} u \Psi_y\right) + 0 + \frac{\partial}{\partial z'}\left(\frac{1}{\beta}\Psi_x\right) - \frac{1}{c}\frac{\partial}{\partial t'}\left(\Psi_y + \frac{1}{c} u \Psi_z\right)$$

$$= \frac{1}{c} 4\pi\rho \left(\frac{1}{\beta} v_y\right)$$

Now if we put

$$\left.\begin{aligned} \Psi_x' &= \frac{1}{\beta}\Psi_x \\ \Psi_y' &= \Psi_y + \frac{1}{c} u \Psi_z \\ \Psi_z' &= \Psi_z - \frac{1}{c} u \Psi_y \\ v_y' &= \frac{1}{\beta} v_y \end{aligned}\right\} \cdots\cdots\cdots\cdots (3)$$

we have an equation which has the same form as the second equation of $(1'')$ , chap. II. the relations (3) are those that make the form of the second equation of $(1'')$ , chap II, invariant. We note that the transformation of the components of $\overline{\Psi}$ in (3) is the same as in (2).

Again , if we substitute the values from (1) in the third equation of $(1'')$ , chap.II, we have

$$\left(\beta \frac{\partial \Psi_y}{\partial x'} - \beta \frac{u}{c^2} \frac{\partial \Psi_y}{\partial t'}\right) - \frac{\partial \Psi_x}{\partial y'} + 0 - \frac{1}{c}\left(-\beta u \frac{\partial \Psi_z}{\partial x'} + \beta \frac{\partial \Psi_z}{\partial t'}\right)$$

$$= 4\pi\rho v_z$$

or

$$\frac{\partial}{\partial x'}\left(\Psi_y + \frac{1}{c} u \Psi_z\right) - \frac{\partial}{\partial y'}\left(\frac{1}{\beta}\Psi_x\right) + 0 - \frac{1}{c}\frac{\partial}{\partial t'}\left(\Psi_z - \frac{1}{c} u \Psi_y\right)$$

$$= 4\pi\rho\left(\frac{1}{\beta} v_z\right)$$

This indicates that the third equation of $(1'')$ , chap.II is in-variant if we put

$$\left.\begin{aligned} \Psi'_x &= \frac{1}{\beta}\Psi_x \\ \Psi'_y &= \Psi_y + \frac{1}{c}u\Psi_z \\ \Psi'_z &= \Psi_z - \frac{1}{c}u\Psi_y \\ v'_z &= \frac{1}{\beta}v_z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Also note that the transformation of the components of $\overline{\Psi}$ in (4) is the same as in (2) and (3).

From (2), (3) and (4), we obtain the transformation that make Maxwell's equations invariant, as follows

$$\begin{aligned} \Psi'_x &= \frac{1}{\beta}\Psi_x \\ \Psi'_y &= \Psi_y + \frac{1}{c}u\Psi_z \\ \Psi'_z &= \Psi_z - \frac{1}{c}u\Psi_y \\ v'_x &= v_x - u \\ v'_y &= \frac{1}{\beta}v_y \\ v'_z &= \frac{1}{\beta}v_z . \end{aligned}$$

These are deduced from the Lorentz transformation.

The treatment just given has shown that Maxwell's equations have the same form in both systems (at rest and moving) if the space and time coordinates transform according to the Lorentz transformation, that is in the system $S'$, we have the relations.

$$\nabla' \times \overline{\Psi}' = \frac{1}{c}\left( \frac{\partial \overline{\Psi}'}{\partial t'} + 4\pi\rho\overline{v}' \right)$$
$$\nabla' \cdot \overline{\Psi}' = 4\pi\rho$$

for charged ether as a medium. In a similar manner, we can show that Maxwell's equations for free ether (containing no charge) are invariant under the Lorentz transformation but are not invariant under the Galilean transformation, that is, for free ether we have the relations:

$$\nabla' \times \overline{\Psi}' = \frac{1}{c}\frac{\partial \overline{\Psi}'}{\partial t'}$$
$$\nabla' \cdot \overline{\Psi}' = 0$$

Because of the invariance of Maxwell's equations under the Lorentz transformation, it follows that the velocity of electromagnetic waves deduced by the observers O and O' must be the same (equal to c. ) which agrees with the result of the Michelson and Morley experiment (mentioned in chapter II). This shows the correctness of the Lorentz transformation.