การประมาณค่าจุดตรึงของการส่งแบบนอนเอ็คซแพนซีฟ

นางสาวเยาวลักษณ์ แซ่ซอร์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2544

ISBN 974-03-0251-3

ON THE APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE MAPPINGS

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic Year 2001

ISBN 974-03-0251-3

Thesis Title	On the Approximation of Fixed Points of Nonexpansive Mappings
By	Miss Yaowaluk Saesor
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Imchit Termwuttipong, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master 's Degree

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เยาวลักษณ์ แซ่ซอร์ : การประมาณค่าจุดตรึงของการส่งแบบนอนเอ็คซแพนซีฟ
(ON THE APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE MAPPINGS)
อ.ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ ดร. อิ่มจิตต์ เติมวุฒิพงษ์, 33 หน้า ISBN 974-03-0251-3

ให้ B เป็นปริภูมิบานาคซึ่งมีนอร์มที่หาอนุพันธ์แบบยูนิฟอร์มในเชิง Gâteaux ได้ และให้ T เป็นการส่งแบบนอนเอ็คซแพนซีฟบนเซตนูนปิด C ของ B ไปยัง C โดยที่ T มีจุดตรึง เราจะสร้างลำดับ (x_n)ใน C แบบเวียนเกิด โดยกำหนดให้

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \text{ สำหรับ } n = 0, 1, 2, 3, \ldots$$

โดยที่สำหรับแต่ละ n $\alpha_n \in [0,1]$ ในงานวิจัยนี้เราหาเงื่อนไขที่เพียงพอของลำดับ (α_n) ที่ทำให้ลำดับ(x_n) ลู่เข้าแบบเข้มสู่จุดตรึงของ T

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ปีการศึกษา **2544**	ลายมือชื่ออาจารย์ที่ปรึกษาร่วม -

4172401023 : MAJOR MATHEMATICS
KEYWORD : BANACH SPACE / UNIFORMLY GÂTEAUX DIFFERENTIABLE NORM / NONEXPANSIVE MAPPING / FIXED POINT

YAOWALUK SAESOR : ON THE APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE MAPPINGS. THESIS ADVISOR : ASSISTANT PROFESSOR IMCHIT TERMWUTTIPONG, Ph.D., 33 pp. ISBN 974-03-0251-3.

Let B be a Banach space whose norm is uniformly Gâteaux differentiable and T a nonexpansive mapping on a closed convex set C of B into itself with the fixed point property. Define a sequence (x_n) in C recursively by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \text{ for } n = 0, 1, 2, 3, \ldots,$$

where $\alpha_n \in [0,1]$ for each n and $x_0 \in C$ is arbitrary. In our investigation, we find a sufficient condition on (α_n) to assure that (x_n) converges strongly to a fixed point of T.

Department **Mathematics** Student's signature……………………………..
Field of Study **Mathematics** Advisor's signature…………………………….
Academic year **2001** Co-advisor's signature -

Acknowledgements

I am greatly indebted to Assistant Professor Dr. Imchit Termwuttipong, my thesis advisor, for her untired offering me some thoughtful and helpful advice in preparing and writing this thesis. I would like to thank Associate Professor Dr. Kritsana Neammanee and Dr. Wicharn Lewkeeratiyutkul, the chairman and member of my thesis committee for making my complete thesis. I wish to express my sincere thanks to Associate Professor Dr. Naoki Shioji for giving advice in some proof of his paper and this paper became the main thing to develop my theorem in this thesis. I would also like to thank all of my kind friends in the United States of America for sending some reference papers to me. I am also grateful to all teachers who have taught me for my knowledge and skills.

I would also like to express my sincere gratitude to the Development and Promotion of Science and Technology Talents Project (DPST) for providing me support throughout my graduate study.

A special word for appreciation also goes to my friends for their help as well as their friendship. In addition, I got support and endurance from my beloved family for the whole time. Finally, I would like to thank every one, who supported me, but I did not refer to the above-mentioned.

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CHAPTER I

INTRODUCTION

This thesis is concerned with the approximation of fixed points for nonexpansive self-mappings. The problem of this kind has widely been studied. By a **nonexpansive mapping** we mean a map T from a normed linear space X into itself such that

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \text{for every } x, y \in X.$$

A nonexpansive mapping T is said to have ***the fixed point property*** if the set $\{x : Tx = x\}$ is nonempty.

In 1965, Browder [1] established that a nonexpansive mapping from the unit ball of a real Hilbert space into itself always possesses at least one fixed point. In 1967, Browder [2] used the Banach Fixed Point Theorem to prove the following :

> Let H be a Hilbert space, and U the unit ball of H, i.e., $U = \{x \in H : \|x\| \leq 1\}$, and $T : U \to U$ a nonexpansive mapping with the fixed point property and y_k the unique element of U satisfying $y_k = kTy_k$ for $k \in (-1, 1)$. Then
>
> $$\lim_{k \to 1} y_k = y,$$
>
> where y is the unique fixed point of T with the smallest norm.

In 1967, Halpern [3] gave a simple iterative method for approximating fixed points of such map. In fact, he generated a sequence (x_n) by the recursive formula

$$x_{n+1} = \alpha_{n+1} T x_n, \qquad \text{for } n \in \mathbb{N}_0,$$

where T is the nonexpansive mapping and (α_n) is an appropriate sequence of real numbers. Also he introduced the definition of an ***acceptable sequence*** as follows : A sequence (α_n) is said to be ***acceptable*** if, for any Hilbert space H, any nonexpansive mapping $T : U \to U$ with the fixed point property, where U is the unit ball of H, and for any point $a \in U$, the sequence (z_n), defined by

$$z_0 = a, z_n = \alpha_n T z_{n-1}, \text{ for } n \in \mathbb{N},$$

converges to y, where y is the fixed point of T with the smallest norm.

Moreover, by using the above result of Browder, Halpern [3] gives necessary conditions and sufficient conditions on (α_n) to insure that the sequence of (x_n) converges to a fixed point of T. The statements are the followings :

The necessary conditions for the sequence (α_n) to be acceptable are

(i) $\alpha_n \in (-1, 1)$, (ii) $\lim_{n\to\infty} \alpha_n = 1$, and (iii) $\prod_{n=1}^{\infty} \alpha_n = 0$.

The sufficient conditions for (α_n) to be acceptable are

(i) $\alpha_n < 1$ for all $n \in \mathbb{N}$, (ii) α_n is an increasing sequence,

(iii) $\lim_{n\to\infty} \alpha_n = 1$ and

(iv) there exists an increasing sequence (k_n) of positive integers such that

$$(a) \lim_{n\to\infty} \frac{(1-\alpha_{n+k_n})}{(1-\alpha_n)} = 1 \quad \text{and} \quad (b) \lim_{n\to\infty} k_n(1-\alpha_n) = \infty.$$

Let C be a closed convex subset of a normed linear space X and T a nonexpansive mapping from C into itself, with the fixed point property.

Define a sequence (x_n) in C recursively by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) T x_n, \text{ for } n \in \mathbb{N}_0, \tag{1.1}$$

where $\alpha_n \in [0, 1]$ for each n and $x_0 \in C$ is arbitrary.

In 1980, Reich [4] showed that if X is a uniformly smooth Banach space, i.e., the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{1.2}$$

is attained uniformly for $x, y \in S$, where $S = \{x \in X : \|x\| = 1\}$, and a sequence (α_n) in [0,1] satisfies

$$\alpha_n = \frac{1}{n^a}, \text{ where } 0 < a < 1,$$

then the sequence (x_n) defined as in (1.1) converges to a fixed point of T.

In 1992, Wittmann [5] proved that if X is a Hilbert space and a sequence (α_n) in $[0, 1]$ satisfies

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then the sequence (x_n) in C defined as in (1.1) converges to a fixed point of T.

In 1997, Shioji and Takahashi [6] obtained the same result as Wittmann's in the case when X is a Banach space whose norm is uniformly Gâteaux differentiable, i.e., for each $y \in S$, the limit (1.2) is attained uniformly for $x \in S$, where $S = \{x \in X : \|x\| = 1\}$.

In our work, we confine our study on Banach spaces whose norm is uniformly Gâteaux differentiable and we find conditions on (α_n) weaker than those of Shioji and Takahashi [6] that also give the same conclusion.

CHAPTER II

PRELIMINARIES

In this chapter, in order to make the subject more complete and understandable definitions, theorems and propositions whose results will be needed in our work will be stated, and examples of some definitions are also given.

Notation

Let $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{R}$ denote the set of all positive integers, the set of all nonnegative integers and the set of all real numbers, respectively. The set B stands for a real Banach space and B^* is its dual. For each x in B, $\|x\|$ and $\langle x \rangle$ denote the norm of x and the subspace of B spanned by x. If $x \in B$ and $f \in B^*$, the value of f at x is denoted by $\langle x, f \rangle$, and by $\|f\|$ we mean the norm of the linear functional f.

Hahn-Banach Theorem

2.1 Definition Let X be a real vector space and $p : X \to \mathbb{R}$.

(i) If for any x, y in X , $p(x + y) \leq p(x) + p(y)$ then p is said to be ***subadditive***.

(ii) If for any x in X and for any $\alpha > 0$, $p(\alpha x) = \alpha p(x)$ then p is said to be ***positively homogeneous***.

(iii) If p is subadditive and positively homogeneous then p is called a ***sublinear functional*** on X.

2.2 Theorem (*Hahn-Banach Theorem*) Let X be a real vector space, $p : X \to \mathbb{R}$ a sublinear functional on X and $f : M \to \mathbb{R}$ a linear functional on a subspace M of X satisfying

$$f(x) \leq p(x), \text{ for all } x \in M.$$

Then f has a linear extension $\widetilde{f}$ from M to X such that

$$\widetilde{f}(x) \leq p(x), \text{ for all } x \in X.$$

2.3 Definition Let (X, d) be a metric space and $f : X \to X$. If there is a positive real number k such that

$$d(f(x), f(y)) \leq kd(x, y), \quad \text{ for all } x, y \in X,$$

then f is said to be a ***contraction mapping*** if $k < 1$ and f is said to be a ***nonexpansive mapping*** if $k = 1$.

2.4 Theorem (*Banach Fixed Point Theorem*) Let (X, d) be a complete metric space and $f : X \to X$ a contraction mapping on X. Then f has precisely one fixed point.

Duality Mapping and Differentiability of the Norm in Banach Spaces

We will introduce the notion of a duality mapping of a Banach space and state some relation between the duality mapping and the differentiability of the norm of the Banach space. We present here the definitions of the terms which will be used in our work and some of their examples, and also state some important propositions concerning duality mapping and differentiability of the norm of the Banach space but the proofs of them will be omitted, since they can be found in Takahashi [7].

2.5 Definition Let B be a Banach space. For each $x \in B$, let

$$Jx = \{f \in B^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}.$$

The mapping J defined above is called the ***duality mapping*** of B. J can be considered as a multi-valued mapping from B into B^*.

The following proposition give some basic properties of the duality mapping. These properties play an important role in the proof of our main theorem.

2.6 Proposition Let B be a Banach space and let J be the duality mapping of B. Then for each $x \in B$,

(i) $Jx \neq \varnothing$,

(ii) for each $\alpha \in \mathbb{R}$, $J(\alpha x) = \alpha Jx$, and

(iii) for each $f \in Jx$, $\|x\|^2 - \|y\|^2 \leq 2\langle x - y, f \rangle$, for all $y \in B$.

The differentiability of the norm of the Banach space has an interesting relation to the property that the duality mapping of B is single-valued. Now we present the definitions of some types of differentiability on Banach spaces.

2.7 Definition For a Banach space B, let $S = \{x \in B : \|x\| = 1\}$.

(i) If the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2.1}$$

exists for each $x, y \in S$, then the norm of B is said to be ***Gâteaux differentiable*** and B is said to be ***smooth***.

(ii) If for each $y \in S$, the limit (2.1) is attained uniformly for $x \in S$, then the norm of B is said to be ***uniformly Gâteaux differentiable*** and B is said to have a ***uniformly Gâteaux differentiable norm***.

(iii) If for each $x \in S$, the limit (2.1) is attained uniformly for $y \in S$, then the norm of B is said to be ***Fréchet differentiable*** and B is said to have a ***Fréchet differentiable norm***.

(iv) If the limit (2.1) is attained uniformly for $x, y \in S$, then the norm of B is said to be ***uniformly Fréchet differentiable*** and B is said to be ***uniformly smooth***.

2.8 Remark In the definitions of differentiability of the norms of Banach spaces above, we can see that

(i) if the norm of B is uniformly Fréchet differentiable, then it is Fréchet differentiable, uniformly Gâteaux differentiable and Gâteaux differentiable.

(ii) if the norm of B is uniformly Gâteaux differentiable, then it is Gâteaux differentiable.

(iii) if the norm of B is Fréchet differentiable, then it is Gâteaux differentiable.

2.9 Example Every Hilbert space has a uniformly Fréchet differentiable norm.

Proof. Let H be a Hilbert space, $S := \{x \in H : \|x\| = 1\}$.

We shall show that the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $x, y \in S$ and equals to $\langle x, y \rangle$, the inner product of x and y.

Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$.

For $0 < |t| < \delta$, we have for each $x, y \in S$

$$\left|\frac{\|x+ty\|-\|x\|}{t} - \langle x,y\rangle\right| = \left|\frac{\sqrt{\langle x+ty, x+ty\rangle}-1}{t} - \langle x,y\rangle\right|$$

$$= \left|\frac{\sqrt{\langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle}-1}{t} - \langle x,y\rangle\right|$$

$$= \left|\frac{\sqrt{1+2t\langle x,y\rangle + t^2}-1}{t} - \langle x,y\rangle\right|$$

$$= \left|\frac{2\langle x,y\rangle + t}{\sqrt{1+2t\langle x,y\rangle + t^2}+1} - \langle x,y\rangle\right|.$$

Let $g(t) = \sqrt{1+2t\langle x,y\rangle + t^2}+1$. Then

$$\left|\frac{\|x+ty\|-\|x\|}{t} - \langle x,y\rangle\right| = \left|\frac{2\langle x,y\rangle + t}{g(t)} - \langle x,y\rangle\right|$$

$$\leq \left|\frac{2\langle x,y\rangle + t}{g(t)} - \frac{2\langle x,y\rangle + t}{2}\right|$$

$$+ \left|\frac{2\langle x,y\rangle + t}{2} - \frac{2\langle x,y\rangle}{2}\right|$$

$$= |2\langle x,y\rangle + t|\left|\frac{1}{g(t)} - \frac{1}{2}\right| + \frac{1}{2}|t|$$

$$\leq (2\|x\|\|y\| + |t|)\frac{|g(t)-2|}{2|g(t)|} + \frac{1}{2}|t|$$

$$= (2+|t|)\frac{|g(t)-2|}{2|g(t)|} + \frac{1}{2}|t|.$$

It is easy to see that $2-|t| \le \left|\sqrt{1+2t\dfrac{\langle x,y\rangle}{\|x\|\|y\|}+t^2}+1\right| \le 2+|t|$, since $\left|\dfrac{\langle x,y\rangle}{\|x\|\|y\|}\right| \le 1$.

Since $|t| \le 1$ and $\|x\| = 1 = \|y\|$, we have $\dfrac{1}{|g(t)|} \le \dfrac{1}{2-|t|}$.

We note that

$$\begin{aligned}
|g(t)-2| &= |\sqrt{1+2t\langle x,y\rangle+t^2}-1| \\
&\le |\sqrt{1+2t\langle x,y\rangle+t^2}-1||\sqrt{1+2t\langle x,y\rangle+t^2}+1| \\
&= |2t\langle x,y\rangle+t^2| \\
&\le 2|t|\|x\|\|y\|+|t|^2 = 2|t|+|t|^2 \\
&< 3|t|.
\end{aligned}$$

Since $\dfrac{2+|t|}{2-|t|} < 3$, we have

$$\begin{aligned}
\left|\frac{\|x+ty\|-\|x\|}{t}-\langle x,y\rangle\right| &\le (2+|t|)\frac{|g(t)-2|}{2|g(t)|}+\frac{1}{2}|t| \\
&< (2+|t|)\frac{3|t|}{2(2-|t|)}+\frac{1}{2}|t| \\
&\le \frac{3}{2}(3|t|)+\frac{1}{2}|t| \\
&= 5|t| \le \varepsilon.
\end{aligned}$$

Hence $\displaystyle\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} = \langle x,y\rangle$ uniformly for $x,y \in S$ and thus $\|\cdot\|$ is uniformly Fréchet differentiable. □

2.10 Proposition If B is a smooth Banach space, then the duality mapping J of B is single-valued.

2.11 Corollary Let B be a Banach space.

(i) If B has a uniformly Gâteaux differentiable norm, then the duality mapping J of B is single-valued.

(ii) If B has a Fréchet differentiable norm, then the duality mapping J of B is single-valued.

(iii) If B has a uniformly Fréchet differentiable norm, then the duality mapping J of B is single-valued.

2.12 Proposition If the norm of B is uniformly Gâteaux differentiable then the single-valued duality mapping J is (norm to weak star) uniformly continuous on each bounded subset of B.

Banach Limits

Since a Banach limit is an important tool of the proof in our main theorem, in this section, we give a definition and some remarks on Banach limits.

We denote by l^∞ the space of all real bounded functions on the nonnegative integers.

2.13 Definition Let μ be a bounded linear functional on l^∞ and $(x_n) \in l^\infty$. We write $\mu(x_n)$ instead of $\mu((x_n))$. If μ satisfies $\mu(x_n) = \mu(x_{n+1})$ for all $(x_n) \in l^\infty$ and $\mu(1) = 1$, then we call μ a ***Banach limit***.

2.14 Proposition There exists a Banach limit .

Proof. Let $q : l^\infty \to \mathbb{R}$ be defined by

$$q(x_n) = \limsup_{n\to\infty} \frac{x_0 + x_1 + \cdots + x_{n-1}}{n}, \text{ for } (x_n) \in l^\infty.$$

Let $c := \{x = (x_n) \in l^\infty : x \text{ is a convergent sequence}\}$. Then c is a subspace of l^∞.

If $(x_n) \in c$, then the map $l : c \to \mathbb{R}$ with $l(x_n) = \lim_{n\to\infty} x_n$ is defined, and l is a linear functional on c.

Moreover, it is known that $l(x_n) = \lim_{n\to\infty} \frac{x_0 + x_1 + \cdots + x_{n-1}}{n}$.

Clearly, q is a sublinear functional on l^∞.

By the *Hahn-Banach theorem*, there exists a linear functional $\mu : l^\infty \to \mathbb{R}$ such that $\mu(x_n) = l(x_n)$ on c and

$$\mu(x_n) \leq q(x_n) \text{ for all } (x_n) \in l^\infty.$$

Next, we will show that μ is a Banach limit.

Since it is obvious that $\mu(1) = l(1) = 1 = \|\mu\|$, it suffices to show that $\mu(x_{n+1}) = \mu(x_n)$, for all $(x_n) \in l^\infty$.

We note that

$$\begin{aligned} \mu(x_{n+1} - x_n) &= -\mu(-(x_{n+1} - x_n)) \geq -q[-(x_{n+1} - x_n)] \\ &= \liminf_{n\to\infty} \left(\frac{x_{n+1} - x_0}{n} \right) \geq 0, \end{aligned}$$

and

$$\mu(x_{n+1} - x_n) \leq q(x_{n+1} - x_n) = \limsup_{n\to\infty} \left(\frac{x_{n+1} - x_0}{n} \right) \leq 0.$$

Thus $\mu(x_{n+1}) = \mu(x_n)$. Hence μ is a Banach limit. $\square$

2.15 Proposition For every Banach limit μ,

$$\liminf_{n\to\infty} x_n \leq \ \mu(x_n) \leq \limsup_{n\to\infty} x_n, \text{ for } (x_n) \in l^\infty.$$

Proof. Let $x = (x_n)$ be arbitrary in l^∞. First, we shall show that $\mu(x_n) \geq 0$ if $x_n \geq 0$ for all $n \in \mathbb{N}_0$.

If $x_n = 0$, for all $n \in \mathbb{N}_0$, then the inequality is immediately obtained, since $\mu(0) = 0$.

Assume that there exists $n \in \mathbb{N}_0$ such that $x_n \neq 0$. Then $\|x\| > 0$.

Suppose that $\mu(x_n) < 0$.

Let $y = (y_n) \in l^\infty$ be such that for each $n \in \mathbb{N}_0$,

$$y_n = 1 - \frac{x_n}{\|x\|}.$$

Then $\|y\| \leq 1$ but $\mu(y) > 1$.

Hence $\|\mu\| = \sup\limits_{\substack{z \in l^\infty \\ \|z\| \leq 1}} |\mu(z)| > 1$, a contradiction.

Now we have $\mu(x_n) \geq 0$ if $x_n \geq 0$ for all $n \in \mathbb{N}_0$.

Next, we shall show that $\inf\limits_{n\in\mathbb{N}_0} x_n \leq \mu(x_n) \leq \sup\limits_{n\in\mathbb{N}_0} x_n$.

We shall show only $\inf\limits_{n\in\mathbb{N}_0} x_n \leq \mu(x_n)$.

For $\varepsilon > 0$, choose n_0 so that

$$\inf_{n\in\mathbb{N}_0} x_n \leq x_{n_0} < \inf_{n\in\mathbb{N}_0} x_n + \varepsilon,$$

Then $x_n + \varepsilon - x_{n_0} > 0$ for all $n \in \mathbb{N}_0$.

By $\mu(x_n) \geq 0$ if $x_n \geq 0$ and $\mu(1) = 1$, we have

$$\inf_{n\in\mathbb{N}_0} x_n \leq \mu(x_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\inf\limits_{n\in\mathbb{N}_0} x_n \leq \mu(x_n)$.

Similarly, $\sup\limits_{n\in\mathbb{N}_0} x_n \geq \mu(x_n)$.

Hence, we have for each $k \in \mathbb{N}_0$,

$$\inf_{n\geq k} x_n \leq \mu(x_k) \leq \sup_{n\geq k} x_n$$

By $\mu(x_n) = \mu(x_{n+1})$, for all $n \in \mathbb{N}_0$, we then obtain that $\liminf\limits_{n\to\infty} x_n \leq \mu(x_n) \leq \limsup\limits_{n\to\infty} x_n$. □

2.16 Proposition For every Banach limit μ, if (a_n) and (b_n) in l^∞ are such that $a_n \geq b_n$ for every $n \in \mathbb{N}_0$, then $\mu(a_n) \geq \mu(b_n)$.

Proof. Let (a_n) and (b_n) be sequences in l^∞ such that $a_n \geq b_n$ for all $n \in \mathbb{N}_0$.

Then for each $n \in \mathbb{N}_0, a_n - b_n \geq 0$.

By Proposition 2.15, we have $\mu(a_n) - \mu(b_n) = \mu(a_n - b_n) \geq 0$

Hence $\mu(a_n) \geq \mu(b_n)$. □

CHAPTER III

APPROXIMATED SEQUENCES FOR NONEXPANSIVE MAPPING

In this chapter, we will prove our main result which can be stated as follows :

Let B be a Banach space whose norm is uniformly Gâteaux differentiable, C a closed convex subset of B. Let $T : C \to C$ be a nonexpansive mapping, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \text{for all } x, y \in C.$$

Let (α_n) be any sequence in $[0,1]$ satisfying

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} (1-\alpha_{n+2})|\alpha_{n+1} - \alpha_n| < \infty. \tag{3.1}$$

Let $x_0 \in C$ be arbitrary. Define (x_n) in C by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad \text{for } n \in \mathbb{N}_0. \tag{3.2}$$

For each $t \in (0,1)$, define z_t to be the unique element in C such that

$$z_t = tx_0 + (1-t)Tz_t. \tag{3.3}$$

If T has a fixed point and $\lim_{t\to 0} \|z_t - z\| = 0$ for some fixed point z of T then $\lim_{n\to\infty} \|x_n - z\| = 0$.

The following lemma guarantees the existence of z_t in (3.3), for each $t \in (0,1)$.

3.1 Lemma Let $(X, \|\cdot\|)$ be a Banach space and C a closed convex subset of X. If $T : C \to C$ is a nonexpansive mapping, then for any $x_0 \in C$ and for any $t \in (0,1)$, the mapping $g_t : C \to C$ defined by $g_t(x) = tx_0 + (1-t)Tx$, for each $x \in C$, is a contraction. Hence by the Banach fixed point theorem g_t has a unique fixed point.

Proof. The conclusion that g_t is a contraction is immediately obtained from the assumption that T is nonexpansive and $0 < 1 - t < 1$. □

From the above lemma, we have that for each $x_0 \in C$, for each $t \in (0,1)$, there is a unique z_t in C such that

$$z_t = tx_0 + (1-t)Tz_t.$$

3.2 Definition Let $T : X \to X$. A subset A of X is said to be ***T-invariant*** if $Tx \in A$ for every $x \in A$.

It is noticed that if C has some additional condition, then there will be a fixed point z of T such that $\lim_{t\to 0} \|z_t - z\| = 0$. This result is shown in Takahashi and Ueda [8]. The statement is as follows :

3.3 Lemma Let B be a Banach space whose norm is uniformly Gâteaux differentiable, let C be a weakly compact, convex subset of B and let $T : C \to C$ be a nonexpansive mapping with fixed point property. Let $x \in C$ and for each $0 < t < 1$, let z_t be the unique element of C which satisfies $z_t = tx + (1-t)Tz_t$. Assume that each nonempty, T-invariant, closed convex subset of C contains a fixed point of T. Then there is a fixed point z of T such that $\lim_{t\to 0} \|z_t - z\| = 0$.

The next two Lemmas state key properties of a Banach limit which are used in the proof of our main theorem. The idea of the proofs are based on those of Shioji and Takahashi [6]. We present here the proofs in detail.

3.4 Lemma Let $a \in \mathbb{R}$ and let $(a_n) \in l^\infty$. Then $\mu(a_n) \leq a$ for all Banach limits μ if and only if for each $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \varepsilon, \quad \text{for all } p \geq p_0 \text{ and } n \in \mathbb{N}_0.$$

Proof. To prove the necessity, assume that $\mu(a_n) \leq a$ for all Banach limits μ.

Define $q : l^\infty \to \mathbb{R}$ by

$$q((x_n)) = \limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{p} \sum_{i=n}^{n+p-1} x_i, \qquad \text{for all } (x_n) \in l^\infty.$$

Then q is a sublinear functional on l^∞.

Clearly, $q\big|_{\langle(a_n)\rangle}$ is a linear functional on $\langle(a_n)\rangle$, the subspace of B generated by (a_n).

By the *Hahn-Banach Theorem*, there exists a linear functional $\mu : l^\infty \to \mathbb{R}$ such that $\mu(x_n) \leq q(x_n)$ for all $(x_n) \in l^\infty$ and $\mu\big|_{\langle(a_n)\rangle} = q\big|_{\langle(a_n)\rangle}$.

Claim that μ is a Banach limit.

Since $\mu(x_n) \leq q(x_n)$ on l^∞, $-q(-x_n) \leq -\mu(-x_n) = \mu(x_n) \leq q(x_n)$.

Thus for all $(x_n) \in l^\infty$,

$$\liminf_{p\to\infty} \inf_{n\in\mathbb{N}} \frac{1}{p} \sum_{i=n}^{n+p-1} x_i \leq \mu(x_n) \leq \limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{p} \sum_{i=n}^{n+p-1} x_i.$$

So

$$\mu(x_{n+1}-x_n) \geq \liminf_{p\to\infty}\ \inf_{n\in\mathbb{N}} \frac{1}{p}\sum_{i=n}^{n+p-1}(x_{i+1}-x_i)$$
$$= \liminf_{p\to\infty}\ \inf_{n\in\mathbb{N}} \frac{1}{p}(x_{n+p}-x_n) \geq 0$$

Similarly,

$$\mu(x_{n+1}-x_n) \leq \limsup_{p\to\infty}\ \sup_{n\in\mathbb{N}} \frac{1}{p}(x_{n+p}-x_n) \leq 0.$$

This implies that $\mu(x_{n+1}) = \mu(x_n)$, for all $(x_n) \in l^\infty$.

It is clear that $\mu(1) = 1$ and $|\mu(x_n)| \leq 1$, for any $(x_n) \in l^\infty$ and $\|x_n\| = 1$.

Hence $\|\mu\| = \sup_{\substack{(x_n)\in l^\infty \\ \|x_n\|=1}} |\mu(x_n)| = 1 = \mu(1)$. Therefore we have the claim.

Since $q\big|_{\langle(a_n)\rangle} = \mu\big|_{\langle(a_n)\rangle}$ and μ is a Banach limit, then $q(a_n) \leq a$.

Then for each $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \varepsilon, \quad \text{for all } p \geq p_0 \text{ and } n \in \mathbb{N}_0.$$

To prove the sufficiency, we assume that for each $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ which satisfies

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \varepsilon, \tag{3.4}$$

for all $p \geq p_0$ and for all $n \in \mathbb{N}_0$.

Let μ be a Banach limit and $\varepsilon > 0$ be given.

By the hypothesis, there exists $p_0 \in \mathbb{N}$ which satisfies (3.4), for all $p \geq p_0$ and for all $n \in \mathbb{N}_0$.

Since μ is a Banach limit,

$$\mu(a_{n+p_0-1}) = \cdots = \mu(a_{n+2}) = \mu(a_{n+1}) = \mu(a_n).$$

Then by Proposition 2.16 and the properties of Banach limit,

$$\begin{aligned}
\mu(a_n) &= \frac{1}{p_0}(p_0\mu(a_n)) \\
&= \frac{1}{p_0}[\mu(a_n) + \mu(a_{n+1}) + \cdots + \mu(a_{n+p_0-1})] \\
&= \mu\left(\frac{a_n + a_{n+1} + \cdots + a_{n+p_0-1}}{p_0}\right) \\
&\leq \mu(a+\varepsilon) \\
&= (a+\varepsilon)\mu(1) \\
&= a+\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\mu(a_n) \leq a$. □

3.5 Lemma Let a be a real number and $(a_n) \in l^\infty$ be such that $\mu(a_n) \leq a$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then $\limsup_{n\to\infty} a_n \leq a$.

Proof. Let $\varepsilon > 0$ be given. Since $\mu(a_n) \leq a$ for all Banach limit μ, by Lemma 3.4, there exists $p \in \mathbb{N}$ such that

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}_0$.

If $p = 1$, it is obvious that $\limsup_{n\to\infty} a_n \leq a$.

Assume that $p \geq 2$.

Since $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, there exists $n_0 \in \mathbb{N}_0$ such that

$$a_{n+1} - a_n < \frac{\varepsilon}{(p-1)}, \text{ for all } n \geq n_0.$$

Let $n \geq n_0 + p$. Then for each $i \in \{0, 1, \ldots, p-1\}$, we have

$$\begin{aligned} a_n &= a_{n-i} + (a_{n-i+1} - a_{n-i}) + (a_{n-i+2} - a_{n-i+1}) + \cdots + (a_n - a_{n-1}) \\ &\leq a_{n-i} + \frac{i\varepsilon}{(p-1)}. \end{aligned}$$

So we get

$$pa_n \leq (a_n + a_{n-1} + a_{n-2} + \cdots + a_{n-p+1}) + (0 + 1 + 2 + \cdots + (p-1))\frac{\varepsilon}{(p-1)}$$

Thus

$$\begin{aligned} a_n &\leq \frac{a_n + a_{n-1} + a_{n-2} + \cdots + a_{n-p+1}}{p} + \frac{1}{p}.\frac{p(p-1)}{2}.\frac{\varepsilon}{(p-1)} \\ &\leq a + \varepsilon. \end{aligned}$$

Hence $\limsup_{n\to\infty} a_n \leq a + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\limsup_{n\to\infty} a_n \leq a$. □

In case that the Banach space B has a uniformly Gêteaux differentiable norm we have a nice result on the convergence of nets in each bounded subset of B. This is because in such a Banach space the duality mapping J is single-valued and norm to weak star uniformly continuous on each bounded subset of B. We state precisely the following lemma :

3.6 Lemma Let B be a Banach space such that the duality mapping J is single-valued and norm to weak star uniformly continuous on each bounded subset of B. For each $t \in (0, 1)$, let z_t be a point in B. If $z \in B$ is such that $\lim_{t\to 0} z_t = z$, then for any x, y in B

$$\lim_{t\to 0}\langle x - z_t, J(y - z_t)\rangle = \langle x - z, J(y - z)\rangle.$$

Proof. First we show that $\lim_{t\to 0}\langle x-y, J(y-z_t)\rangle = \langle x-y, J(y-z)\rangle$.

Without loss of generality, we may assume that $x \neq y$. Therefore $\|x-y\| > 0$.

Let $\varepsilon > 0$ be given. Since J is uniformly continuous on each bounded subset of B, it is continuous at $y-z$.

There exists a $\delta > 0$ such that for every $v \in B, \|v-(y-z)\| < \delta$ implies

$$|\langle \omega, Jv - J(y-z)\rangle| < \frac{\varepsilon}{\|x-y\|} \text{ for every } \omega \in B \text{ with } \|\omega\| \leq 1.$$

Since $\lim_{t\to 0} z_t = z$, $\lim_{t\to 0} y - z_t = y - z$.

Let γ be a positive real number such that $|t| < \gamma$, implies $\|(y-z_t)-(y-z)\| < \delta$.

Then for $t \in (0,1)$ with $|t| < \gamma, \|(y-z_t)-(y-z)\| < \delta$ and hence

$$\left|\left\langle \frac{x-y}{\|x-y\|}, J(y-z_t) - J(y-z)\right\rangle\right| < \frac{\varepsilon}{\|x-y\|}.$$

That is if $t \in (0,1)$ is such that $|t| < \gamma$, then

$$|\langle x-y, J(y-z_t) - J(y-z)\rangle| < \varepsilon.$$

Therefore $\lim_{t\to 0}\langle x-y, J(y-z_t)\rangle = \langle x-y, J(y-z)\rangle$.

Next, we shall show that

$$\lim_{t\to 0}\langle x-z_t, J(y-z_t)\rangle = \langle x-z, J(y-z)\rangle.$$

Since $\|\cdot\|^2$ is continuous and $\lim_{t\to 0} z_t = z$, $\lim_{t\to 0}\|y-z_t\|^2 = \|y-z\|^2$.

We have for each $t \in (0,1)$,

$$
\begin{aligned}
&|\langle x - z_t, J(y - z_t)\rangle - \langle x - z, J(y - z)\rangle| \\
&= |\langle x - y + y - z_t, J(y - z_t)\rangle - \langle x - y + y - z, J(y - z)\rangle| \\
&= |\langle x - y, J(y - z_t)\rangle + \|y - z_t\|^2 - \|y - z\|^2 - \langle x - y, J(y - z)\rangle| \\
&\leq |\langle x - y, J(y - z_t)\rangle - \langle x - y, J(y - z)\rangle| + |\ \|y - z_t\|^2 - \|y - z\|^2|.
\end{aligned}
$$

Then
$$
\begin{aligned}
0 &\leq \lim_{t\to 0} |\langle x - z_t, J(y - z_t)\rangle - \langle x - z, J(y - z)\rangle| \\
&\leq \lim_{t\to 0} |\langle x - y, J(y - z_t)\rangle - \langle x - y, J(y - z)\rangle| \\
&\quad + \lim_{t\to 0} |\ \|y - z_t\|^2 - \|y - z\|^2| = 0.
\end{aligned}
$$

That is $\lim_{t\to 0}\langle x - z_t, J(y - z_t)\rangle = \langle x - z, J(y - z)\rangle$. □

The following Lemma obtains some idea from Wittmann [5].

3.7 Lemma Let B be a Banach space and C a closed convex subset of B. Let $T : C \to C$ be a nonexpansive mapping such that the set F of all fixed points of T is nonempty. Let (α_n) be a sequence in $[0,1]$ satisfying (3.1), i.e.

$$
\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} (1 - \alpha_{n+2})|\alpha_{n+1} - \alpha_n| < \infty.
$$

Let $x_0 \in C$ and define (x_n) as in (3.2), i.e.

$$
x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \text{for } n \in \mathbb{N}_0.
$$

Then the sequence (x_n) and (Tx_n) are bounded.

Proof. In proving this lemma, we have two cases to consider :

Case 1. $0 \in F$.

For each $n \in \mathbb{N}_0$, $\|Tx_n\| = \|Tx_n - 0\| = \|Tx_n - T0\| \leq \|x_n - 0\| = \|x_n\|$.

Let $n \in \mathbb{N}_0$ be such that $\|x_n\| \leq \|x_0\|$. Thus

$$\begin{aligned}\|x_{n+1}\| = \|\alpha_n x_0 + (1-\alpha_n)Tx_n\| &\leq \alpha_n\|x_0\| + (1-\alpha_n)\|x_n\| \\ &\leq \alpha_n\|x_0\| + (1-\alpha_n)\|x_0\| = \|x_0\|.\end{aligned}$$

Then $\|x_n\| \leq \|x_0\|$ for all $n \in \mathbb{N}_0$. Hence (x_n) and (Tx_n) are bounded.

<u>*Case 2*</u>. $0 \notin F$.

Let p be a point in F. Let $\widetilde{C} = \{x - p : x \in C\}$ and define $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ by

$$\widetilde{T}(x) = T(x+p) - p, \text{ for all } x \in \widetilde{C}.$$

Then $\|\widetilde{T}x - \widetilde{T}y\| = \|(T(x+p)-p) - (T(y+p)-p)\|$

$\leq \|(x+p) - (y+p)\| = \|x - y\|$, for all $x, y \in \widetilde{C}$.

Then $\widetilde{T}$ is nonexpansive and $0 \in F(\widetilde{T})$, where $F(\widetilde{T})$ is the set of all fixed points of $\widetilde{T}$.

Set $\tilde{x}_0 = x_0 - p$ and for each $n \in \mathbb{N}_0$, let $\tilde{x}_{n+1} = \alpha_n\tilde{x}_0 + (1-\alpha_n)\widetilde{T}\tilde{x}_n$.

From *Case 1,* we have $(\tilde{x}_n)$ and $(\widetilde{T}\tilde{x}_n)$ are bounded.

We claim that $\tilde{x}_n = x_n - p$, for all $n \in \mathbb{N}_0$.

We already have $\tilde{x}_0 = x_0 - p$.

Let $n \in \mathbb{N}_0$ be such that $\tilde{x}_n = x_n - p$. Then

$$\begin{aligned}\tilde{x}_{n+1} &= \alpha_n\tilde{x}_0 + (1-\alpha_n)\widetilde{T}\tilde{x}_n = \alpha_n(x_0 - p) + (1-\alpha_n)\widetilde{T}(x_n - p) \\ &= \alpha_n(x_0 - p) + (1-\alpha_n)\{T[(x_n - p) + p] - p\} \\ &= \alpha_n x_0 - \alpha_n p + (1-\alpha_n)[T(x_n) - p] \\ &= \alpha_n x_0 + (1-\alpha_n)Tx_n - p \\ &= x_{n+1} - p.\end{aligned}$$

Thus $\tilde{x}_n = x_n - p$, for all $n \in \mathbb{N}_0$, and we have the claim.

So we have $\|x_n\| = \|\tilde{x}_n + p\| \leq \|\tilde{x}_n\| + \|p\|$ and

$$\|Tx_n\| = \|T(\tilde{x}_n + p)\| = \|\widetilde{T}\tilde{x}_n + p\| \leq \|\widetilde{T}\tilde{x}_n\| + \|p\|.$$

Hence our requirement is proved. $\square$

The following two Lemmas are based on the idea of the work of Shioji and Takahashi [6].

The first lemma states some asymptotic behavior of (x_n), defined in Lemma 3.7.

3.8 Lemma With the same hypothesis as in Lemma 3.7, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Proof. By Lemma 3.7, we obtain that (x_n) and (Tx_n) are bounded.

Let $M := sup\{\|Tx_n\| : n \in \mathbb{N}_0\}$. Then for each $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|[\alpha_n x_0 + (1-\alpha_n)Tx_n] - [\alpha_{n-1}x_0 + (1-\alpha_{n-1})Tx_{n-1}]\| \\
&= \|(\alpha_n - \alpha_{n-1})x_0 + (Tx_n - Tx_{n-1}) \\
&\qquad - \alpha_n(Tx_n - Tx_{n-1}) + (\alpha_{n-1} - \alpha_n)Tx_{n-1}\| \\
&= \|(\alpha_n - \alpha_{n-1})(x_0 - Tx_{n-1}) + (1-\alpha_n)(Tx_n - Tx_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|(\|x_0\| + M) + (1-\alpha_n)\|x_n - x_{n-1}\|.
\end{aligned}$$

Since for each $k \in \mathbb{N}_0, x_{k+1} = \alpha_k x_0 + (1-\alpha_k)Tx_k$, we have for each $m, n \in \mathbb{N}_0$,

$$
\begin{aligned}
&\|x_{n+m+1} - x_{n+m}\| \\
&\le |\alpha_{n+m} - \alpha_{n+m-1}|(\|x_0\| + M) + (1-\alpha_{n+m})\|x_{n+m} - x_{n+m-1}\| \\
&\le \left[|\alpha_{n+m} - \alpha_{n+m-1}| + \sum_{k=m}^{n+m-2} (1-\alpha_{k+2})|\alpha_{k+1} - \alpha_k| \right] (\|x_0\| + M) \\
&\quad + \prod_{k=m}^{n+m-1} (1-\alpha_{k+1})\|x_{m+1} - x_m\| \\
&\le \left[|\alpha_{n+m} - \alpha_{n+m-1}| + \sum_{k=m}^{n+m-2} (1-\alpha_{k+2})|\alpha_{k+1} - \alpha_k| \right] (\|x_0\| + M) \\
&\quad + \exp\left(-\sum_{k=m}^{n+m-1} \alpha_{k+1} \right) \|x_{m+1} - x_m\|. \qquad (3.5)
\end{aligned}
$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have for each $m \in \mathbb{N}_0$,

$\lim_{n\to\infty} |\alpha_{n+m} - \alpha_{n+m-1}| = 0$ and $\lim_{n\to\infty} \exp\left(-\sum_{k=m}^{n+m-1} \alpha_{k+1} \right) = 0.$

Therefore for each $m \in \mathbb{N}_0$, we have from (3.5) that

$$
\begin{aligned}
&\limsup_{n\to\infty} \|x_{n+1} - x_n\| \\
&= \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \\
&\le \limsup_{n\to\infty} \left\{ \left[|\alpha_{n+m} - \alpha_{n+m-1}| + \sum_{k=m}^{n+m-2} (1-\alpha_{k+2})|\alpha_{k+1} - \alpha_k| \right] (\|x_0\| + M) \right. \\
&\qquad \left. + \exp\left(-\sum_{k=m}^{n+m-1} \alpha_{k+1} \right) \|x_{m+1} - x_m\| \right\} \\
&= (\|x_0\| + M) \sum_{k=m}^{\infty} (1-\alpha_{k+2})|\alpha_{k+1} - \alpha_k|. \qquad (3.6)
\end{aligned}
$$

Let $\varepsilon > 0$ be arbitrary. We can choose m large enough that

$$\sum_{k=m}^{\infty} (1-\alpha_{k+2})|\alpha_{k+1} - \alpha_k| < \frac{\varepsilon}{(\|x_0\| + M)} \qquad \text{and then}$$

$\limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| < \varepsilon$. This implies that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

□

The next lemma is an essential result needed in proving our theorem.

3.9 Lemma Let B be a Banach space such that the duality mapping J is single-valued and norm to weak star uniformly continuous on each bounded subset of B. Let C be a closed convex subset of B and x_0 be arbitrary in C. Let $T, (\alpha_n)$ and (x_n) be as in the hypothesis of Lemma 3.7, and for each $t \in (0,1)$ defined z_t as in (3.3), i.e.,

$$z_t = tx_0 + (1-t)Tz_t.$$

Then

$$\limsup_{n\to\infty}\langle x_0 - z, J(x_n - z)\rangle \leq 0.$$

Proof. Let μ be a Banach limit and $t \in (0,1)$.

By the boundedness of (x_n), we have $\{\|x_n - z_t\| : n \in \mathbb{N}_0\}$ is bounded. It implies that $\{\|x_n - z_t\|^2 : n \in \mathbb{N}_0\}$ is also bounded. Then for each $n \in \mathbb{N}_0$,

$$\begin{aligned}
\|x_n - Tz_t\|^2 &= \|\alpha_{n-1}x_0 + (1-\alpha_{n-1})Tx_{n-1} - Tz_t\|^2 \\
&\leq (\alpha_{n-1}\|x_0 - Tx_{n-1}\| + \|Tx_{n-1} - Tz_t\|)^2 \\
&\leq (\alpha_{n-1}(\|x_0\| + M) + \|x_{n-1} - z_t\|)^2 \\
&\leq \alpha_{n-1}(\|x_0\| + M)^2 + 2\alpha_{n-1}(\|x_0\| + M)\|x_{n-1} - z_t\| \\
&\quad + \|x_{n-1} - z_t\|^2
\end{aligned}$$

From the assumption that $\lim\limits_{n\to\infty}\alpha_n = 0$, we have

$$\|x_n - Tz_t\|^2 \leq \|x_{n-1} - z_t\|^2.$$

Since μ is a Banach limit, from Proposition 2.16, we obtain

$$\mu(\|x_n - Tz_t\|^2) \leq \mu(\|x_n - z_t\|^2). \tag{3.7}$$

Since $(1-t)(x_n - Tz_t) = (x_n - z_t) - t(x_n - x_0)$, we have that

$$(1-t)^2\|x_n - Tz_t\|^2 = \|(x_n - z_t) - t(x_n - x_0)\|^2 \tag{3.8}$$

By Proposition 2.6(iii) and J is single-valued we have for each $y \in B$,

$$\|x_n - z_t\|^2 - \|y\|^2 \leq 2\langle x_n - z_t - y, J(x_n - z_t)\rangle.$$

Or for each $y \in B$,

$$\|y\|^2 \geq \|x_n - z_t\|^2 - 2\langle x_n - z_t - y, J(x_n - z_t)\rangle. \tag{3.9}$$

Hence from (3.8) and (3.9) with $y = (x_n - z_t) - t(x_n - x_0)$ we have

$$\begin{aligned}(1-t)^2\|x_n - Tz_t\|^2 &\geq \|x_n - z_t\|^2 - 2\langle t(x_n - x_0), J(x_n - z_t)\rangle \\ &= \|x_n - z_t\|^2 - 2t\langle x_n - z_t - x_0 + z_t, J(x_n - z_t)\rangle \\ &= \|x_n - z_t\|^2 - 2t\langle x_n - z_t, J(x_n - z_t)\rangle \\ &\quad + 2t\langle x_0 - z_t, J(x_n - z_t)\rangle \\ &= \|x_n - z_t\|^2 - 2t\|x_n - z_t\|^2 + 2t\langle x_0 - z_t, J(x_n - z_t)\rangle \\ &= (1-2t)\|x_n - z_t\|^2 + 2t\langle x_0 - z_t, J(x_n - z_t)\rangle.\end{aligned}$$

By Proposition 2.16 and (3.7) we get

$$(1-t)^2\mu(\|x_n - z_t\|^2) \geq (1-2t)\mu(\|x_n - z_t\|^2) + 2t\,\mu(\langle x_0 - z_t, J(x_n - z_t)\rangle).$$

Hence

$$\frac{t}{2}\,\mu(\|x_n - z_t\|^2) \geq \mu(\langle x_0 - z_t, J(x_n - z_t)\rangle). \tag{3.10}$$

Since μ and $\|\cdot\|^2$ are continuous, we have

$$\begin{aligned}\limsup_{t\to 0}\ \frac{t}{2}\,\mu(\|x_n - z_t\|^2) &= \limsup_{t\to 0}\ \frac{t}{2}\ \limsup_{t\to 0}\mu(\|x_n - z_t\|^2) \\ &= 0 \cdot \mu(\limsup_{t\to 0}\|x_n - z_t\|^2) \\ &= 0 \cdot \mu(\|x_n - z\|^2) = 0. \qquad (3.11)\end{aligned}$$

By Lemma 3.6 we have for each $n \in \mathbb{N}_0$,

$$\lim_{t\to 0}\langle x_0 - z_t, J(x_n - z_t)\rangle = \langle x_0 - z, J(x_n - z)\rangle. \qquad (3.12)$$

From μ is continuous and (3.12), we obtain

$$\begin{aligned}\limsup_{t\to 0}\mu(\langle x_0 - z_t, J(x_n - z_t)\rangle) &= \mu(\limsup_{t\to 0}\langle x_0 - z_t, J(x_n - z_t)\rangle) \\ &= \mu(\langle x_0 - z, J(x_n - z)\rangle). \qquad (3.13)\end{aligned}$$

By (3.10), (3.11) and (3.13), we have

$$0 \geq \mu(\langle x_0 - z, J(x_n - z)\rangle).$$

We claim that $\lim_{n\to\infty} |\langle x_0 - z, J(x_{n+1} - z)\rangle - \langle x_0 - z, J(x_n - z)\rangle| = 0.$

Let $\varepsilon > 0$ be given.

Since J is norm to weak star uniformly continuous on each bounded subset of B and J is norm to weak star uniformly continuous on $\{x_n - z : n \in \mathbb{N}_0\}$.

Then there exists a $\delta > 0$, such that for every $n \in \mathbb{N}_0$ if $\|(x_{n+1} - z) - (x_n - z)\| < \delta$ then

$$|\langle x_0 - z, J(x_{n+1} - z)\rangle - \langle x_0 - z, J(x_n - z)\rangle| < \varepsilon.$$

From Lemma 3.8, there exists an $N \in \mathbb{N}_0$ such that for any $n \geq N, \|x_{n+1} - x_n\| < \delta$. Hence we have the claim.

By Lemma 3.5, we get

$$\limsup_{n\to\infty}\langle x_0 - z, J(x_n - z)\rangle \leq 0.$$

□

This Theorem is developed from some proof of Shioji and Takahashi's work [6] under our sufficient conditions.

3.10 Theorem Let B be a Banach space whose norm is uniformly Gâteaux differentiable and T a nonexpansive mapping on a closed convex subset C of B such that the set F of all fixed points of T is nonempty. Let (α_n) be a sequence in $[0,1]$ which satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} (1-\alpha_{n+2})|\alpha_{n+1}-\alpha_n| < \infty.$$

Let $x_0 \in C$ and define a sequence (x_n) in C recursively by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad \text{for } n \in \mathbb{N}_0.$$

For each $t \in (0,1)$, let z_t be the unique element in C such that

$$z_t = tx_0 + (1-t)Tz_t.$$

Assume that there is a fixed point z of T such that $\lim_{t\to 0} \|z_t - z\| = 0$. Then $\lim_{n\to\infty} \|x_n - z\| = 0$.

Proof. By Proposition 2.6(iii) we have for each $f \in J(x_{n+1}-z)$,

$$\|x_{n+1}-z\|^2 - \|(x_{n+1}-z) - \alpha_n(x_0-z)\|^2 \le 2\langle \alpha_n(x_0-z), f\rangle.$$

Since the duality mapping is single-value, we have

$$\|(x_{n+1}-z) - \alpha_n(x_0-z)\|^2 \ge \|x_{n+1}-z\|^2 - 2\alpha_n\langle x_0-z, J(x_{n+1}-z)\rangle. \tag{3.14}$$

From the identity $(1-\alpha_n)(Tx_n - z) = (x_{n+1}-z) - \alpha_n(x_0-z)$, we have from (3.14) for each $n \in \mathbb{N}_0$

$$(1-\alpha_n)^2 \|(Tx_n - z)\|^2 \ge \|x_{n+1}-z\|^2 - 2\alpha_n\langle x_0-z, J(x_{n+1}-z)\rangle.$$

Thus for each $n \in \mathbb{N}_0$,

$$\|x_{n+1} - z\|^2 \leq (1-\alpha_n)^2\|Tx_n - Tz\|^2 + 2\alpha_n\langle x_0 - z, J(x_{n+1} - z)\rangle$$
$$\leq (1-\alpha_n)\|x_n - z\|^2 + 2(1-(1-\alpha_n))\langle x_0 - z, J(x_{n+1} - z)\rangle. \tag{3.15}$$

Let $\varepsilon > 0$ be given. Since $\limsup\limits_{n\to\infty}\langle x_0 - z, J(x_n - z)\rangle \leq 0$, there exists an $m \in \mathbb{N}_0$ such that for $n \geq m$,

$$\langle x_0 - z, J(x_n - z)\rangle \leq \frac{\varepsilon}{2}.$$

Thus for $n \in \mathbb{N}_0$, (3.15) implies

$$\begin{aligned}\|x_{n+m} - z\|^2 &\leq (1-\alpha_{n+m-1})\|x_{n+m-1} - z\|^2 + (1-(1-\alpha_{n+m-1}))\varepsilon \\ &\leq \prod_{k=m}^{n+m-1}(1-\alpha_k)\|x_m - z\|^2 + \left(1 - \prod_{k=m}^{n+m-1}(1-\alpha_k)\right)\varepsilon \\ &\leq \exp\left(-\sum_{k=m}^{n+m-1}\alpha_k\right)\|x_m - z\|^2 + \varepsilon. \end{aligned} \tag{3.16}$$

Since $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, we obtain from (3.16) that

$$\limsup_{n\to\infty}\|x_n - z\|^2 = \limsup_{n\to\infty}\|x_{n+m} - z\|^2 \leq \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, $\limsup\limits_{n\to\infty}\|x_n - z\|^2 = 0$ and this implies $\lim\limits_{n\to\infty}\|x_n - z\|^2 = 0$ and thus

$$\lim_{n\to\infty}\|x_n - z\| = 0.$$

□

3.11 Corollary Let B be a Banach space whose norm is uniformly Gâteaux differentiable, C a weakly compact convex subset of B, and x_0 be any point in C. If $T : C \to C$ is a nonexpansive mapping such that every nonempty, T-invariant, closed convex subset of C contains a fixed point of T, and (α_n) is a sequence in $[0,1]$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} (1-\alpha_{n+2})|\alpha_{n+1} - \alpha_n| < \infty,$$

then the sequence (x_n) in C defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad \text{for } n \in \mathbb{N}_0,$$

converges strongly to a fixed point z of T.

Proof. The result is obtained immediately from Lemma 3.3 and Theorem 3.10. □

From the proof of Theorem 3.10 we note that the condition that the norm of B is uniformly Gâteaux differentiable is used only to make sure that the duality mapping of B is single-valued and norm to weak star uniformly continuous on each bounded subset of B, so we have the immediate corollary.

3.12 Corollary Let B be a Banach space such that the duality mapping J is single-valued and norm to weak star uniformly continuous on every bounded subset of B. Let T be a nonexpansive mapping on a closed convex subset C of B such that the set F of all fixed points of T is nonempty. Let (α_n) be a sequence in $[0,1]$ which satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} (1-\alpha_{n+2})|\alpha_{n+1} - \alpha_n| < \infty.$$

Let $x_0 \in C$ and define a sequence (x_n) in C recursively by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad \text{for } n \in \mathbb{N}_0.$$

For each $t \in (0,1)$, let z_t be the unique element in C such that

$$z_t = tx_0 + (1-t)Tz_t.$$

Assume that there is a fixed point z of T such that $\lim_{t\to 0} \|z_t - z\| = 0$. Then $\lim_{n\to\infty} \|x_n - z\| = 0$.

Proof. The result is obtained from the proof of Theorem 3.10. □

The assumption that the sequence (α_n) in [0,1] satisfying

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty}(1-\alpha_{n+2})|\alpha_{n+1}-\alpha_n| < \infty$$

can be obtained if (α_n) is a decreasing sequence in [0,1] with $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. Hence we have the following corollary.

3.13 Corollary Let B be a Banach space such that the duality mapping J is single-valued and norm to weak star uniformly continuous on every bounded subset of B. Let T be a nonexpansive mapping on a closed convex subset C of B such that the set F of all fixed points of T is nonempty. Let (α_n) be a decreasing sequence in $[0,1]$ which satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $x_0 \in C$ and define a sequence (x_n) in C recursively by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad \text{for } n \in \mathbb{N}_0.$$

For each $t \in (0,1)$, let z_t be the unique element in C such that

$$z_t = tx_0 + (1-t)Tz_t.$$

Assume that there is a fixed point z of T such that $\lim_{t\to 0} \|z_t - z\| = 0$. Then $\lim_{n\to\infty} \|x_n - z\| = 0$.

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Vita

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