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PLANETARY ORBIT DETERMINATION FROM OBSERVATIONS BY GUASS METHOD

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งานวิจัยนี้ได้ทำการคำนวณเพื่อกำหนดวงโคจรของดาวพฤหัสบดีและดาวอังคารโดยวิธีการแบบเกาส์ จากการสังเกตดาวเคราะห์ด้วยการถ่ายภาพจากโลกพิจารณาได้ด้วยวิธีการดีเพนเดนซ์ซึ่งชุดของข้อมูลการสังเกตประกอบด้วย เวลา มุมตามวิถีศูนย์สูตร และมุมห่างจากวิถีศูนย์สูตร ในวิธีการของเกาส์นั้นข้อมูลการสังเกตสามชุดได้ถูกประมวลผลด้วยวิธีการเชิงตัวเลขซึ่งใช้บัญญัติสากลที่มีเงื่อนไขบังคับสองแบบคือเงื่อนไขบังคับพลวัตและเงื่อนไขบังคับเรขาคณิตเพื่อคำนวณหาตำแหน่งและความเร็วของดาวพฤหัสบดีและดาวอังคาร ณ เวลานั้น อนึ่งค่าตำแหน่งและความเร็วนี้ได้ถูกแปลงให้อยู่ในรูปของค่าคงที่ของวงโคจรที่เรียกว่าหลักมูลคลาสสิคเพื่ออธิบายถึงลักษณะวงโคจรของดาวเคราะห์ในอวกาศ เมื่อเปรียบเทียบผลการคำนวณจากวิธีการแบบเกาส์กับผลที่ได้จากปฏิทินดาราศาสตร์พบว่ามีความสอดคล้องกันด้วยดี และโปรแกรมการกำหนดวงโคจรของดาวพฤหัสบดี และดาวอังคารที่ได้พัฒนาขึ้นในงานวิจัยนี้สามารถประยุกต์ใช้กำหนดวงโคจรของดาวเคราะห์อื่นและดาวหางได้

ภาควิชา ฟิสิกส์ ลายมือชื่อนิสิต...

สาขาวิชา ฟิสิกส์

ปีการศึกษา 2544 ลายมือชื่ออาจารย์ที่ปรึกษา..

4172399523: MAJOR PHYSICS
KEY WORD: CELESTIAL MECHANICS / ORBIT DETERMINATION / METHOD OF GAUSS

YUTTAKAN RATTHANACHAI: PLANETARY ORBIT DETERMINATION FROM OBSERVATIONS BY GUASS METHOD. THESIS ADVISOR: ASST. PROF. PIRAPAT SIRISOMBOONLARP D.Sc., 180 pp. ISBN 974-17-0110-1.

The orbital determination of planets Jupiter and Mars is performed using Gauss method. From the photographs of the planets taken on Earth, the observational data comprising of time, right ascension and declination is obtained via the method of dependence. In Gauss method, three sets of the observational data are then processed numerically using the universal formulations subject to dynamic and geometric constraints to obtain the position and velocity of Jupiter and Mars at a given epoch time. Moreover, these position and velocity components are transformed to six orbital constants known as the classical elements which describe the planetary orbit in space. Our results agree very well with the astronomical almanac. Our computer program developed in this thesis can also be used in the orbital determination of other planets and comets.

Department	Physics	Student's signature.......................................…….
Field of study	Physics	
Academic year	2001	Advisor's signature.......................................……

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CHAPTER 1

INTRODUCTION

The beginning of the history of celestial mechanics is the model stating that the Sun is the center of the sky, proposed by Poland astronomer Nicolas Copernicus [1]. He settled as a canon of Frauenburg where his famous book *De Revolutionibus Orbium Coelestium* was written. He was afraid of ridicule and allowed publication of his book only after his death. The way Copernicus and his students proposed the heliocentric idea was that they suggested this as a convenient way of computing planetary orbits rather than as a physical fact. Nevertheless, the Catholic Church still put Copernicus's work on the index of prohibited book in 1616. Note that Copernicus considered all planets on circular orbits around the Sun.

The next significant steps were made by the German astronomer Johannes Kepler who in addition to a holiocentric view introduced the idea of elliptic orbits of the planets. His three laws of planetary motion and the equation named after him, which connects the eccentric anomaly with the mean anomaly, are still in use today. Many of his results were based on the observations of the Dutch astronomer Tycho Brahe whom he followed as court mathematician in Prague after his professorship in Graz was terminated in 1600.

In 1687, Sir Isaac Newton published the important scienctifically titled "Philosophiae Naturalis Principia Mathematica". The book is the foundation of laws of motions in classical mechanics under gravitational influence. From such principles, we could determine position and velocity of the object at a given time by solving Newton's equations of motion.

In celestial mechanics the components of position and velocity provide a completely general description of orbital motion; however, their vector form does not clearly reveal the orbit's size, shape, and orientation in space. So we have to transform their components into the set of the parameters known as "classical elements".

In "orbit determination" the classical elements of a celestial body observed in the solar system are found from reduced observational data. In this research, we shall refer to the observed body as the planets and find a "preliminary orbit", approximation of orbit from a minimum of observations of their planets using data from three observations, each one comprising time, right ascension and declination.

An observer on another star would recognize the bodies in the solar system as moving in elliptic orbits about the Sun; but observations from the Earth are affected by the motion of the Earth. The observed geocentric will obviously not be on ellipse, and Figure 1.1, showing part of the path of comet Arend-Roland in 1956 [2], demonstrated how complicated to observed path will become. The position of the Earth in the solar system at any time is, of course, accurately known. If we could observe the distance of the comet, then there would be no difficulty in calculating its position in the solar system; unfortunately only its direction can be observed, and the calculation of its distance is one of the process of orbit determination. In astrodynamics more and different information may be available from observations. The processes of orbit determination can be modified (and simplified) to take advantage of this extra information.

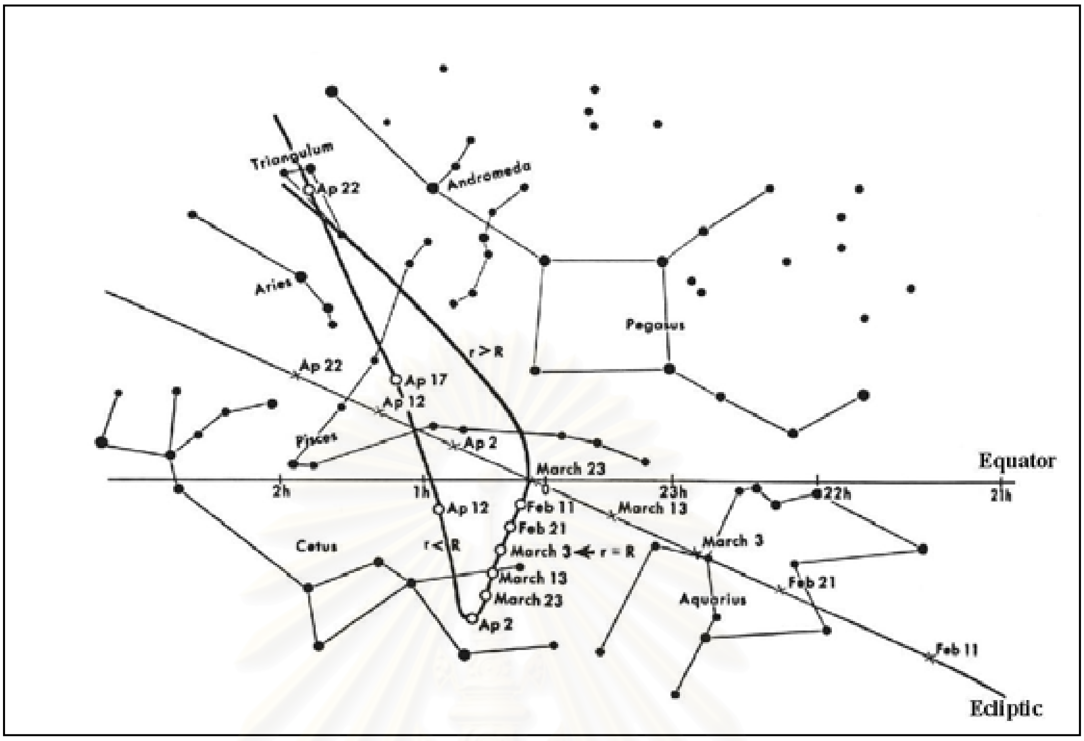


Figure 1.1 The chart is a central projection, so that all great circles appear as straight lines.[2]

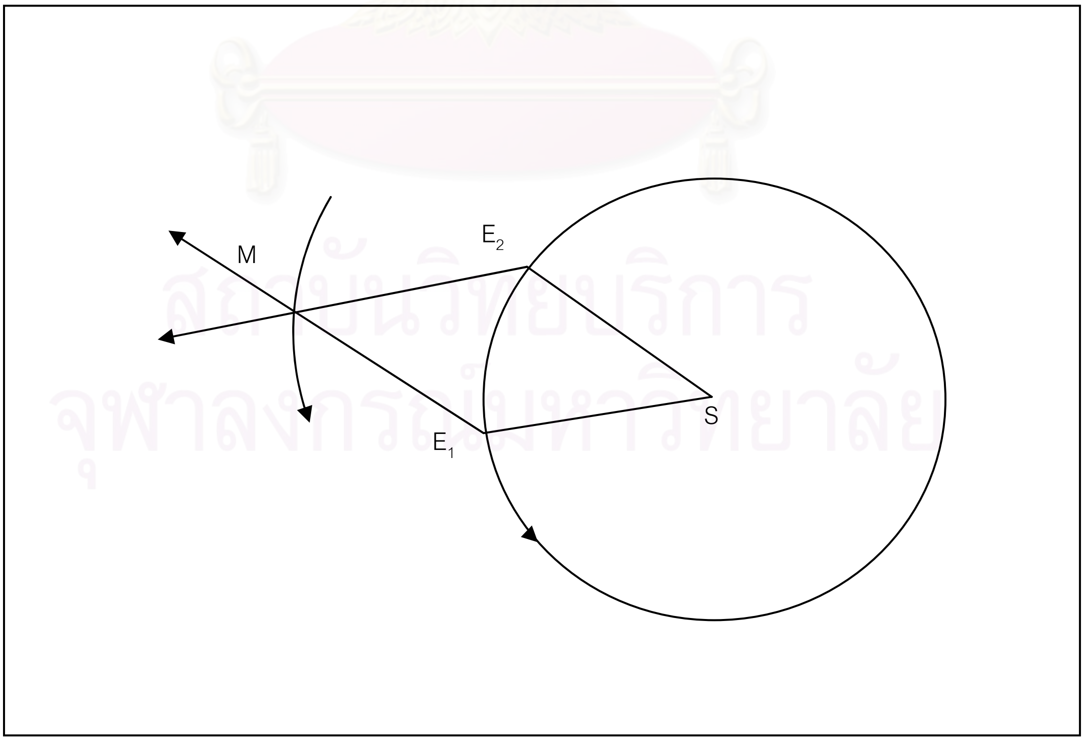


Figure 1.2 The heliocentric orbit of Earth and Mars.[2]

For interest we shall describe in principle the method used by Kepler to find the distance, and thence the orbit, of Mars. The sidereal period of Mars was accurately known, and Figure 1.2 shows the situation for two observations separated by one sidereal period, so that Mars has returned to the same position in the solar system. Since the sidereal period is 1.88 years, the Earth will have revolved through approximately 677°, so that the angle E_1SE_2 is known, as is the distance E_1E_2 (but only in term of the astronomical unit). Observations furnish the angles E_1E_2M and E_2E_1M, so that the triangle E_1E_2M can be solved to find the lengths of the sides, and ultimately the distance MS.

The price of simplicity of Kepler's method is that observations are needed over many revolutions, and this is a luxury that we cannot afford. The history of the discovery of Ceres will illustrate this [3,4,5]. Two centuries ago, no asteroids had yet made their appearance in astronomical catalogs of the solar system, which then included just the Sun, seven planets, and the mysteriously evanescent comets. The first sighting of an asteroid occurred on Jan. 1, 1801, when the monk Giuseppe Piazzi [6] noticed a faint, star like object not included in a star catalog that he was checking. Fascinated by astronomy, Piazzi had in 1790 established and equipped an observatory in Palermo on the island of Sicily. Taking advantage of a favorable climate for astronomical viewing, he launched a lengthy project dedicated to determining precisely the astronomical coordinates of several thousand stars.

Piazzi's observations of the mysterious object on successive nights revealed that it moved slowly against its starry backdrop, first drifting backward, then reversing direction and overtaking the background stars. Unsure whether the object was a comet or a planet, Piazzi watched it regularly until Feb. 11, when he fell ill. By the time he recovered a few days later, he was able to make only one more observation before the object advanced sufficiently close to the Sun to disappear in its glare. Piazzi named the tiny new planet Ceres.

Piazzi had already begun notifying colleagues in other parts of Europe of his discovery, but political turmoil in Italy delayed the mail. As a result, no one else had a chance to observe the object. Only one-tenth the brightness of Uranus and on the fringe of visibility in most telescopes of the time, this faint speck had no telltale planetary disk to make it easier to locate. To recover the object once it emerged from the sun's glare several months later, astronomers needed to know its orbit. Piazzi's observations, however, covered a period of just 41 days, during which time the object had moved through an arc of only 3 degrees across the sky. Any attempt to compute the orbit of such an inconspicuous object from this meager set of date appeared futile.

To Carl Friedrich Gauss [7], a 24-year-old mathematician who early in life had displayed a prodigious talent for mathematics and a remarkable facility for highly involved mental arithmetic, this problem presented an enticing challenge. Having completed his studies at the University of Gottingen, Gauss was living on a small allowance granted by his patron, the Duke of Brunswick. With a major mathematical work just published and little else to occupy his time during the latter part of 1801, Gauss brought his formidable powers to bear on celestial mechanics. Like a skillful mechanics, he systematically disassembled the creaky, ponderous engine that had long been used for determining approximate orbits and rebuilt it into an efficient, streamlined machine that could function reliably given even minimal data. Assuming that Piazzi's object circumnavigated the Sun on an elliptical course and using only three observations of its place in the sky to compute its preliminary orbit, Gauss calculated what its position would be when the time came to resume observations. In December, after three months of labor, he delivered his prediction to the Hungarian astronomer Franz Xaver von Zach, who had organized a self-proclaimed “celestial police” to track down a "missing" planet between the orbits of Mars and Jupiter. Any hope of locating Piazzi’s celestial mote after a lapse of nearly a year rested on the reliability of Gauss’s innovative methods and the accuracy of his calculations. On Dec. 7, von Zach relocated the object only a short distance away from where Gauss had predicted it would lie. Gauss became a celebrity. After Piazzi’s discovery, astronomers quickly found additional minor planets. To Gauss, the discovery of one asteroid after another furnished new opportunities for testing the efficiency and generality of his methods.

From the history of Ceres, as such, we find that Ceres is a faint object, and it was obviously important to predict when and where it could be observed again; this prediction could not be based on the leisurely study of several revolutions but had to depend on a small arc of one revolution supported by Gauss.

In 1984, however, Taff has issued a strange condemnation of Gauss’s method for determination of preliminary orbit [8]. When he can really offer as an alternative is the use of Laplace’s method in situation where he presupposes a wealth of observational data. The criticism of Gauss’s method enters on the grounds of the small radius of convergence of **f** and **g** series that leads to divergent iterating result and so makes the determination of orbits using 3 observations failed.

Using modern computational technology that plays central roles in astronomical computation after, however, in 1985 Marsden [9] published the article titled "Initial orbit determination: the pragmatist’s point of view" in the paper, he argued that the comment by Taff is immaterial. Technically, he used Gauss’s method to determine the trajectory of the amor-type minor planet 1982 DV and 1971 SL_1. Developing the values of **f** and **g** "function" from their closed expressions in terms of the so-called universal variable [10,11]. He could show that Taff's comment could be unwaranted.

From the achievement of Marsden, using Gauss's method, in the paper referred above, the author realizes advantages of Gauss's method and tries, in this research, to follow Marsden’s research appoarch. Importantly, the goal of the research is to give some ideas that hopefully can improve Thai astronomy wisdom partly because (as we know) none in Thailand tries to figure out the trajectory of celestial body by observation before. The main objective of the research is determining the trajectory of planets in the solar system by observational data obtaining from photographing with simple-widely-used instruments.

There are many reasons why we choose Mars and Jupiter as observed celestial bodies. First, they can be observed over night sometimes, making observing easier. Secondly, the position of Mars is suitable for observing. Thirdly, Jupiter is one of the brightest planets in the solar system. Finally, Jupiter is interesting since it is the heaviest planet in the sysytem as well.

The scope of the research is to figure out the solution of two-body equation of motion in terms of six numerical quantities, called classical elements. These elements can be obtained by transforming the position and velocity vectors at a given time. To transform those vectors we have to use celestial mechanics and analytical geometry with numerical computation techniques to find the position and velocity vector from

3-sets of observational data that passed a reduction technique in the astronomical photography. Moreover, two forms of constraints, geometric constraint and dynamic constraint are needed for numerical technique, especially in iteration process. The approach is a standard Gauss's method improved by Encke [12] and Merton [13].

CHAPTER 2

FUNDAMENTALS OF ORBITAL MOTION

Suppose we carefully follow the apparent motion of a celestial body as it travels across the heavens. Its track against the background of fixed stars might turn out to be an interesting curve such as that depicted in Figure 2.1. If we wish to determine the orbit of this celestial body, we must make accurate measurements of its position at a series of convenient times and use certain clever procedures to disentangle its orbital motion from the motion of our observing station on the surface of the moving Earth. Although such observational data contain a complex mixture of several independent motions, the orbit of the celestial body can, in principle, be determined by employing a theory of celestial mechanics developed from three general laws of motion and one law which accounts for the acceleration caused by gravity. Armed with these four fundamental principles, the orbits of planets can be computed using only elementary physics and simple calculus.

This chapter introduces the basic physics of celestial mechanics. The methods by which these principles are applied and the numerical techniques used to solve the orbital equations are taken up later in the thesis.

2.1 The Laws of Motion

Three laws of motion received explicit formulation by Isaac Newton in the Seventeenth Century. The importance of these principles to the development of celestial mechanics can hardly be overestimated. They may be stated as follows:

Law 1 *A body continues in a state of rest or uniform motion in a straight line unless compelled to change its state by forces impressed upon it.*

Law 2 *The acceleration of a body is directly proportional to the net force impressed upon the body, inversely proportional to the mass of the body, and in the same direction as the net force*.

Law 3 *When one body exerts a force on a second body, the second body exerts a force of equal magnitude, but opposite direction, upon the first body*.

2.2 The Law of Gravitation

Newton's law of universal gravitation is the fourth fundamental principle of orbital motion. It may be stated as follows:

Law 4 *Every particle of matter in the universe attracts every other particle of matter with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.*

The gravitational law is summarized by the following expression:

$$F = \frac{k^2 m_1 m_2}{r^2}, \qquad \textbf{(2.1)}$$

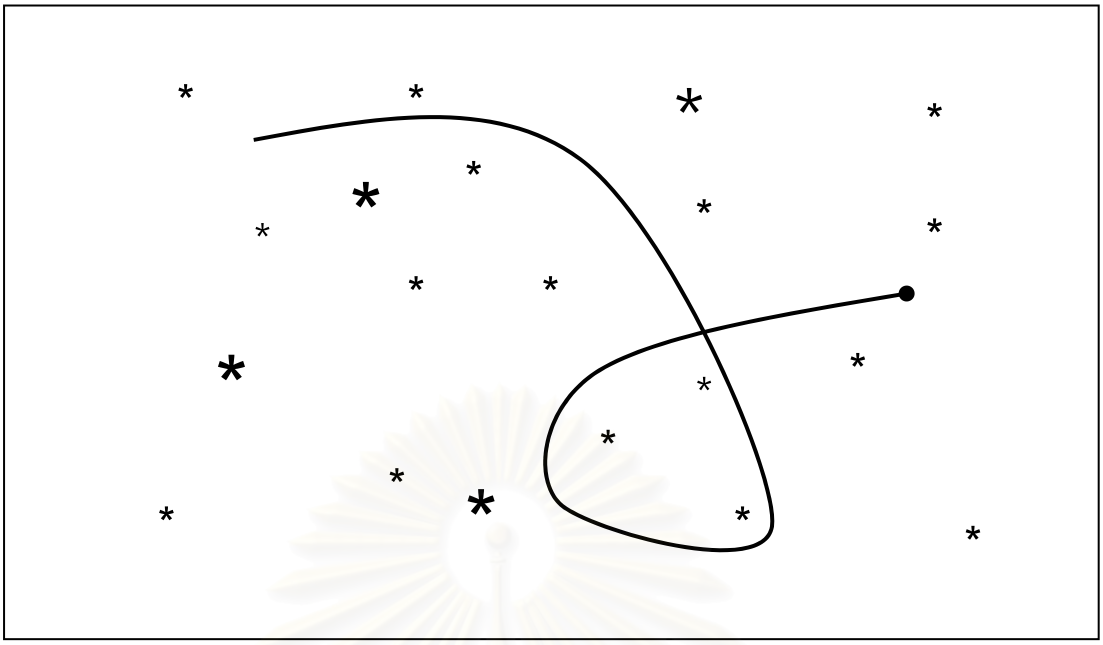


Figure 2.1 The apparent path of a celestial body [27].

where **F** is the magnitude of the force, $\mathbf{k^2}$ is the *gravitational constant*, $\mathbf{m_1}$ and $\mathbf{m_2}$ are the masses, and **r** is the distance. Although the law is stated for particles, Equation 2.1 can be applied to large accumulations of matter if we assume that the mass distribution of each body is spherically symmetric about its center of that all bodies are separated by distances which are very great in comparison to their sizes.

2.3 Equations of Motion

The orbital motion of a celestial body must be described by an equation which expresses its instantaneous acceleration in terms of all gravitational and non-gravitation forces. We shall reduce the problem considerably by dealing only with spherically symmetric gravitational force fields and ignoring all non-gravitational influences. Applying the fundamental principles already discussed in the context of these simplifying assumptions, the result is an equation of motion which can be used to compute the movement of a planet, taking into account the effects of any number of perturbing masses.

2.3.1 The Equation of Inertial Motion

Figure 2.2 depicts a mass $\mathbf{m_1}$ subject to the gravitational attractions of several other masses $\mathbf{m_2}$, $\mathbf{m_3}$, …., $\mathbf{m_N}$. The position vector $\vec{R}_1$ defines the location of $\mathbf{m_1}$ with respect to the center of $\mathbf{m_1}$. The magnitude of the force exerted on $\mathbf{m_1}$ by any other mass $\mathbf{m_q}$ is as follows:

$$F_q = \frac{k^2 m_1 m_q}{r_q^2}. \qquad \textbf{(2.2)}$$

The force vector radiating from $\mathbf{m_1}$ toward any of other bodies:

$$\vec{F}_q = \frac{k^2 m_1 m_q \vec{r}_q}{r_q^3}. \qquad \textbf{(2.3)}$$

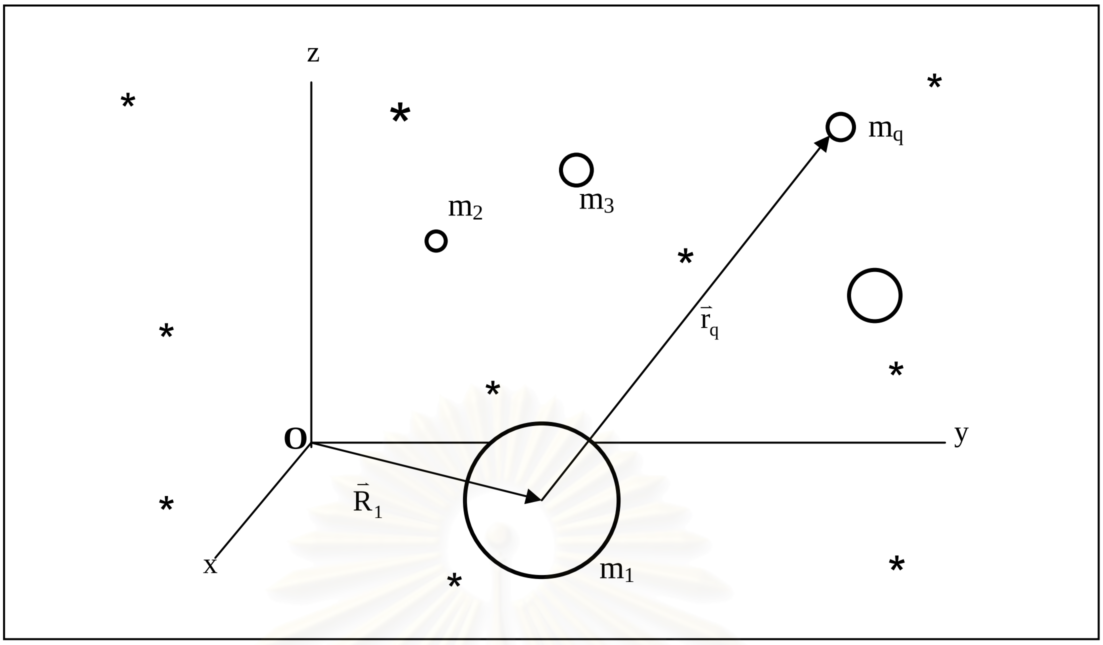


Figure 2.2 Mass $\mathbf{m_1}$ subject to the gravitational attractions of other bodies [27].

Thus, the net force $\vec{\mathbf{F}}$ acting on $\mathbf{m_1}$ is the vector sum

$$\vec{F} = \sum_{q=2}^{N} F_q \, . \tag{2.4}$$

According to Newton's second law, the inertial acceleration $\vec{\mathbf{A}}_1$ of the mass $\mathbf{m_1}$ is given by

$$\vec{A}_1 = \frac{\vec{F}}{m_1} \, .$$

Dividing $\vec{\mathbf{F}}$ by $\mathbf{m_1}$ is equivalent to dividing each term on the right side of Equation 2.4 by $\mathbf{m_1}$. Therefore, by Equation 2.3

$$\vec{A}_1 = \sum_{q=2}^{N} \frac{k^2 m_q \vec{r}_q}{r_q^3} \, , \tag{2.5}$$

which is the *equation of inertial motion* for mass $\mathbf{m_1}$ with respect to the Newtonian frame of reference.

2.3.2 The Equation of Relative Motion

Figure 2.3 illustrates the gravitational problem we mast solve in order to compute an orbit. Several masses $\mathbf{m_1}$, $\mathbf{m_2}$, …, $\mathbf{m_N}$ are moving under the influence of their mutual attractions. Vector $\vec{\mathbf{R}}_1$ and $\vec{\mathbf{R}}_2$ define the locations of $\mathbf{m_1}$ and $\mathbf{m_2}$ with respect to the inertial origin **O**, and $\vec{\mathbf{r}}_\mathbf{q}$ and $\vec{\mathbf{p}}_\mathbf{q}$ define the position of any other mass $\mathbf{m_q}$ relative to $\mathbf{m_1}$ and $\mathbf{m_2}$, respectively. According to the second law of motion, the forces of gravitational attraction will cause $\mathbf{m_1}$ and $\mathbf{m_2}$ to accelerate with respect to the inertial origin. The acceleration $\vec{\mathbf{a}}_2$ of $\mathbf{m_2}$ with respect to $\mathbf{m_1}$ can by found by differentiating the vector $\vec{\mathbf{r}}_2$ twice with respect to time. Obtained

$$\vec{a}_2 = \vec{A}_2 - \vec{A}_1 \, , \tag{2.6}$$

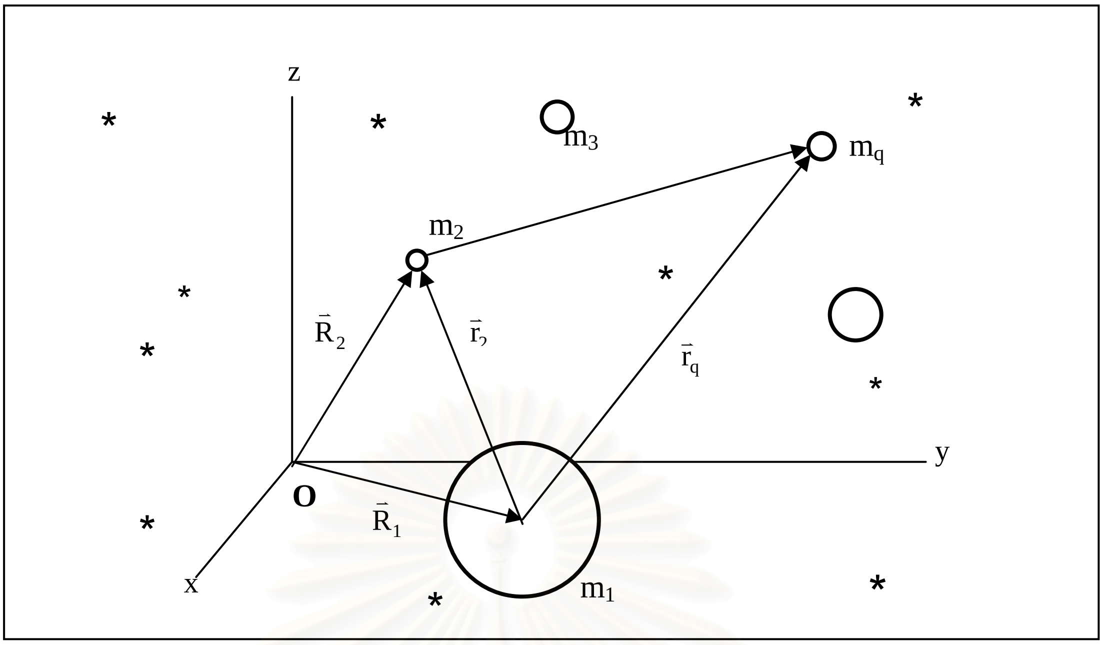


Figure 2.3 The many-body gravitational problem [27].

where $\vec{\mathbf{A}}_1$ and $\vec{\mathbf{A}}_2$ represent the inertial accelerations of $\mathbf{m_1}$ and $\mathbf{m_2}$, respectively. Now, let Equation 2.5 be rewritten as follows:

$$\vec{A}_1 = \frac{k^2 m_2 \vec{r}_2}{r_2^3} + \sum_{q=3}^{N} \frac{k^2 m_q \vec{r}_q}{r_q^3} . \qquad \textbf{(2.7)}$$

By analogy with Equation 2.7 we can also write an expression for $\vec{\mathbf{A}}_2$ in term of $\mathbf{m_1}$, $\vec{\mathbf{r}}_2$, $\mathbf{m_q}$, and $\vec{\mathbf{p}}_\mathbf{q}$. The result is

$$\vec{A}_2 = -\frac{k^2 m_1 \vec{r}_2}{r_2^3} + \sum_{q=3}^{N} \frac{k^2 m_q \vec{p}_q}{p_q^3} , \qquad \textbf{(2.8)}$$

where $\vec{p}_q = \vec{r}_q - \vec{r}_2$, $p_q = |\vec{p}_q|$, and first term on the right side is negative because the acceleration of $\mathbf{m_2}$ due to $\mathbf{m_1}$ is in a direction opposite to vector $\vec{\mathbf{r}}_2$. Finally, we obtain

$$\vec{a}_2 = -\frac{k^2(m_1 + m_2)\vec{r}_2}{r_2^3} + \sum_{q=3}^{N} k^2 m_q \left(\frac{\vec{p}_q}{p_q^3} - \frac{\vec{r}_q}{r_q^3} \right) . \qquad \textbf{(2.9)}$$

This is the *equation of relative motion* for mass $\mathbf{m_2}$ with respect to an origin at the center of mass $\mathbf{m_1}$.

2.4 Working Units and Constant: The Heliocentric System

We are concerned with applying Equation 2.9 to orbital problems in which the motion of interest is about the Sun, heliocentric motion. The motion of a body orbiting the Sun is referred to a rectangular coordinate system centered in the Sun. The fundamental defining constant of the heliocentric system of units is the *Gaussian gravitational constant* given exactly by

$$\mathbf{k} \equiv 0.0172029895 .$$

Length, mass, and time are expressed in *astronomical units* (**AU**), *solar masses* (**M**), and *days* (**day**), respectively. One day is defined to be 86400 seconds, and the astronomical unit is that length for which the Gaussian gravitational constant takes the value defined above when the units of measurements are astronomical units, solar masses, and days. The resulting value is approximately equal to the Earth's average distance from the Sun. This practice of holding **k** fixed while allowing the astronomical unit to vary insures that whenever better data for the masses of the Earth and Sun become available all calculations functionally dependent on **k** do not have to be repeated, but only scaled for the new value of the au.

2.5 The Working Equation of Motion

Consider again Equation 2.9 which describes the relative motion of $\mathbf{m_2}$ with respect to $\mathbf{m_1}$:

$$\vec{a}_2 = -\frac{k^2(m_1 + m_2)\vec{r}_2}{r_2^3} + \sum_{q=3}^{N} k^2 m_q \left(\frac{\vec{p}_q}{p_q^3} - \frac{\vec{r}_q}{r_q^3} \right). \qquad \textbf{(2.10)}$$

It is possible to write a simpler version of this equation if we consistently let $\mathbf{m_1}$ represent the central mass and $\mathbf{m_2}$ the mass of the orbiting body of primary interest. Then, $\mathbf{m_1}$ will be unity, so that we can define a *combined mass*

$$\mu \equiv 1 + m_2 \qquad \textbf{(2.11)}$$

and drop the subscripts on $\vec{\mathbf{a}}_2$ and $\vec{\mathbf{r}}_2$. When this accomplished, the summation indices can be adjusted to begin at **q** = 1 and end at a new value n which is equal only to the number of perturbing bodies. Thus, when all these changes are made, Equation 2.10 becomes

$$\vec{a} = -\frac{k^2 \mu \vec{r}}{r^3} + \sum_{q=1}^{n} k^2 m_q \left(\frac{\vec{p}_q}{p_q^3} - \frac{\vec{r}_q}{r_q^3} \right). \qquad \textbf{(2.12)}$$

Simplification can be carried one step farther by defining *modified time* $\boldsymbol{\tau}$ as follows [14]:

$$\tau \equiv k(t - t_0) \qquad \textbf{(2.13)}$$

so that

$$d\tau = kdt\ , \qquad \textbf{(2.14)}$$

where **k** is appropriate gravitational constant, **t** is given instant of time, and $\mathbf{t_0}$ is an arbitrarily initial time or *epoch*. If we use a dot to indicate differentiation with respect to modified time, then

$$\dot{\vec{r}} = \left(\frac{1}{k} \right) \vec{v} \qquad \textbf{(2.15)}$$

$$\ddot{\vec{r}} = \left(\frac{1}{k^2} \right) \vec{a}\ . \qquad \textbf{(2.16)}$$

It follows that if both sides of Equation 2.12 are multiplied by $1/\mathbf{k}^2$, we are able to write

$$\ddot{\vec{r}} = -\frac{\mu \vec{r}}{r^3} + \sum_{q=1}^{n} m_q \left(\frac{\vec{p}}{p_q^3} - \frac{\vec{r}}{r_q^3} \right). \qquad \textbf{(2.17)}$$

Equation 2.17 is the *working equation of motion*. If a celestial body's position and velocity are known for a given time, then, in principle, Equation 2.17 can be integrated to yield the body's position and velocity at some other time.

CHAPTER 3

TIME AND POSITION

Nothing in the heavens is at rest, and not one of its movements is precisely uniform. Consequently, the specification of position is tied closely to a measure of time, and the convenient inertial reference frame we have thus far assumed does not exist. Of course, the situation is not really very serious. We know from experience that a carefully chosen origin along with a celestial coordinate system defined for a particular epoch can be used to accurately define a position in space at a given instant of time. This is achieved by determining initial values for certain fundamental parameters of the coordinate system and measuring the rate at which these quantities change with time [15].

3.1 The Fundamental References

Consider the *celestial sphere* depicted in Figure 3.1. The nighttime sky created the strong impression that the stars are fixed to an enormous curved surface which appears to be equally distant in all directions regardless of our location on the Earth. Indeed, this imaginary sphere is so vast that any location within the solar system **S** can serve as the origin of its radius. The celestial sphere is banded by two fundamental reference circles which correspond to its intersections with the fundamental planes of the Earth's equator and orbit. These great circles are called the *celestial equator* and *ecliptic*, respectively. The two-dimensional starry surface of the celestial sphere provides the background upon which the reference circles are traced. This is accomplished by meticulous observations of the motions of the Sun and planets relative to a network of fundamental reference stars [16]. The ecliptic crosses the celestial equator at an inclination of approximately 23.5 degrees, forming an angle ε known as the *obliquity of the ecliptic*. The two points of intersection lie on a line passing through the center of the celestial sphere at **S**. One of these intersections has been defined as the fundamental direction. Originally named the *First Point of Aries* ϒ**,** it is often symbolized by the horns of a ram. This designation was applied in the second century B.C. when ϒ was in the constellation of Aries.

The physical significance of ϒ is further illustrated in Figure 3.2, where the Earth is shown moving into spring (for its northern hemisphere) as it approaches the point **E** where day and night are of equal length (equinox). To an observer on the Earth, the Sun's projection **S** against the inside surface of the celestial sphere will travel along the path of the ecliptic toward the celestial equator, crossing it from south to north at the moment the Earth passes through **E**. Thus, the fundamental reference direction is also called the *vernal* (*spring*) *equinox*. The corresponding point on the opposite side of the celestial sphere is known as the autumnal equinox. When the term equinox is used in this text, it will always refer to the fundamental direction ϒ.

3.2 Time Scales

Before we can define an inertial frame of reference for some particular epoch, we must have at least one accurate method for measuring the passage of long intervals of time. Two different scales used in this thesis: universal time and Julian date.

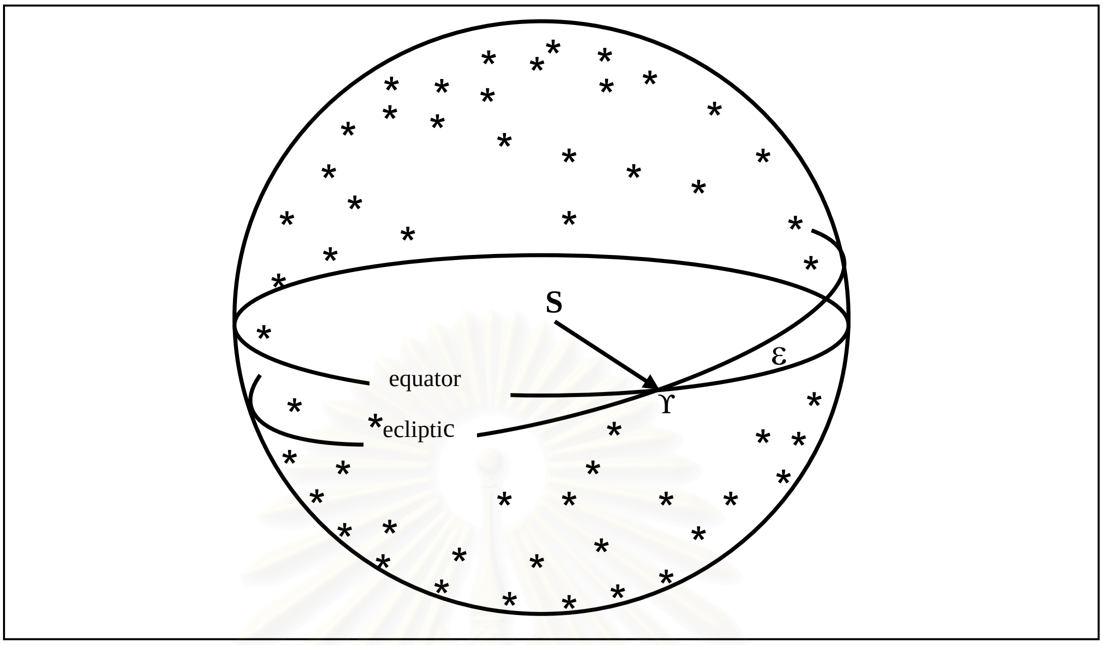


Figure 3.1 The celestial sphere [27].

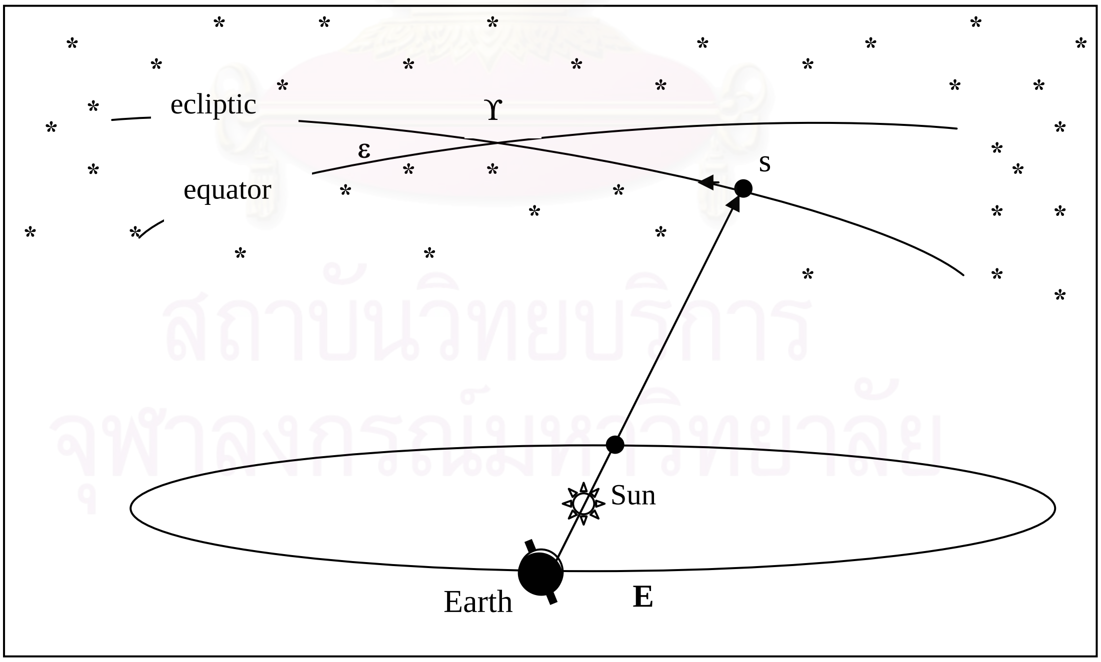


Figure 3.2 The physical significance of ϒ **[27].**

3.2.1 Universal Time

Universal time (UT) is a form of solar time which corresponds closely to the daily (diurnal) motion of the Sun across the sky as seen from a point on the Greenwich meridian of zero terrestrial longitude. It serves as the basis for worldwide civil timekeeping. Universal time is actually determined from observations of the diurnal motions of the stars and made to correspond to solar time by a formula which relates it to Greenwich mean sidereal time [17].

Universal time can be easily computed for any given local standard civil time (CT) by using the following relationship:

$$UT = CT + Z, \qquad \textbf{(3.1)}$$

where **Z** is the number of standard time zones which the locality is displaced to the west of the Greenwich meridian.

3.2.2 Julian date

A Concept of fundamental importance in the reckoning of time is that of the Julian Date (JD). This is nothing more than another arbitrary benchmark that is a continuing count of each day elapsed since some particular epoch. This epoch was arbitrarily selected as January 1, 4713 B. C. Thus, at given universal time,

$$JD = J_0 + \frac{UT}{24}, \qquad \textbf{(3.2)}$$

where UT is expressed in hours and $\mathbf{J_0}$ is the tabular value of the Julian date at 0^h UT. As an alternative to using the table, we have a formula which can be used in a subroutine to automatically compute $\mathbf{J_0}$ when the calendar date is given:

$$J_0 = 367\,Y - \left\langle \frac{7\left[Y + \langle (M+9)/12 \rangle\right]}{4} \right\rangle + \left\langle \frac{275M}{9} \right\rangle + D + 1721013.5, \qquad \textbf{(3.3)}$$

where the symbolism is defined as follows:

- $\langle \mathbf{x} \rangle$ represents a truncation function which extracts the integral part of **x**. For example, $\langle -7.32 \rangle = -7$ and $\langle 3.91 \rangle = 3$.
- **Y** is the year. It must be an integer in the range 1901 to 2099.
- **M** is the month. It must be an integer in the range 1 to 12.
- **D** is the day of the month. It must be an integer in the range 1 to 31.

3.3 Coordinate Systems

We now have the pieces necessary to assemble the coordinate systems used for the computation of orbits. Based on what has been said up to this point. It should come as no surprise that the major astronomical coordinates systems are based on either the celestial equator or ecliptic, and share the vernal equinox as their principal coordinate direction. Furthermore, in the case of spherical coordinate systems, it is customary for the angle in the fundamental plane to be measured positively toward the east from ϒ and to measure the perpendicular angle positively toward the north from the fundamental plane. Finally, the rectangular coordinate systems are all right-handed systems of three mutually perpendicular axes. The **+x**-axis is directed toward ϒ, the **xy**-plane lies in the fundamental plane, and **+z**-axis points northward.

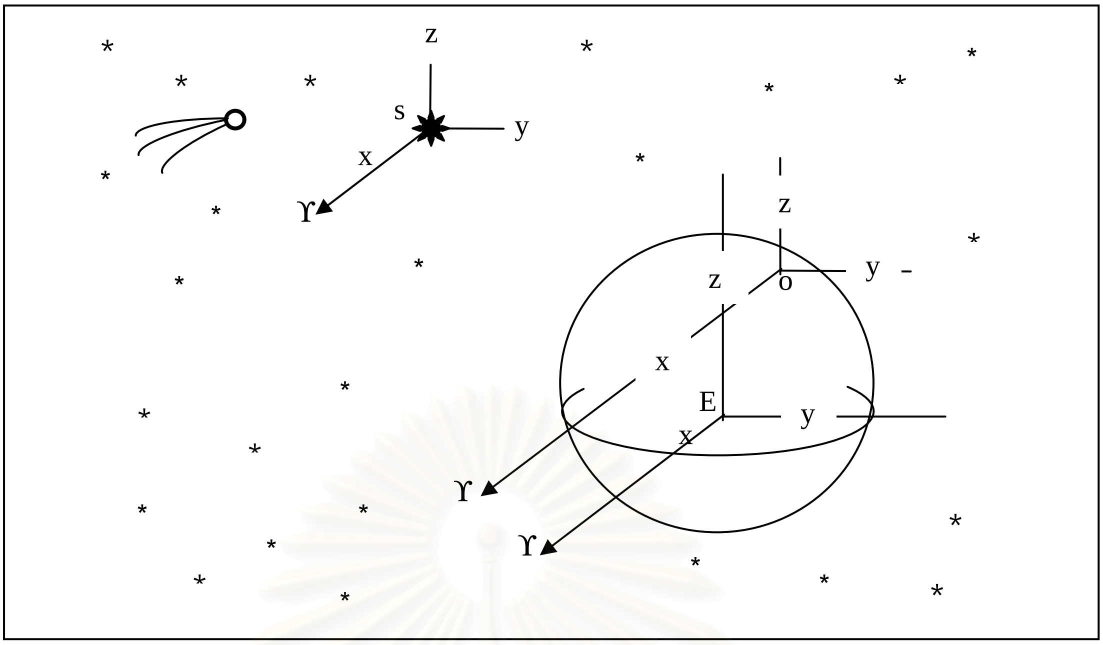


Figure 3.3 Parallel astronomical coordinate systems [27].

It is convenient to group the primary coordinate systems used for orbit determination into three general categories: *celestial equatorial*, *terrestrial equatorial* and *celestial ecliptic*. Within any one of these categories, the coordinate systems differ only by the location of their origins. As shown in Figure 3.3, the origin of a *geocentric* system is centered in the Earth at **E**, the origin of a *topocentric* system is at the observer **O**, and the origin of a *heliocentric* system is at **S** in the center of the Sun. it is important to realize that because ϒ is on the infinite celestial sphere, all axes pointing toward it are parallel. Therefore, although the coordinate systems in a particular category may be widely separated in space, their respective fundamental planes and axes are parallel. For the heliocentric orbit, the object moving around the Sun, two systems are now considered: celestial equatorial and celestial ecliptic systems.

3.3.1 Celestial Equatorial Systems

Figure 3.4 illustrates the characteristic features of celestial equatorial systems. The position of a given point **P** in space is specified by three spherical coordinates: the *radial distance* **r** from the origin **C**, the angle of *right ascension* α, and the angle of *declination* δ. Right ascension (**RA**) is measured eastward from ϒ around the celestial equator in units of time or degrees. Declination (**DEC**) is always measured in degrees and ranges from 0° at the equator to +90° at the north celestial pole (**NCP**) or -90° at the south celestial pole (**SCP**). Given the spherical coordinates of a point **P**, its rectangular coordinates can be found from the following:

$$x = r \cos \delta \cos \alpha \qquad \textbf{(3.4)}$$

$$y = r \cos \delta \sin \alpha \qquad \textbf{(3.5)}$$

$$z = r \sin \delta \,. \qquad \textbf{(3.6)}$$

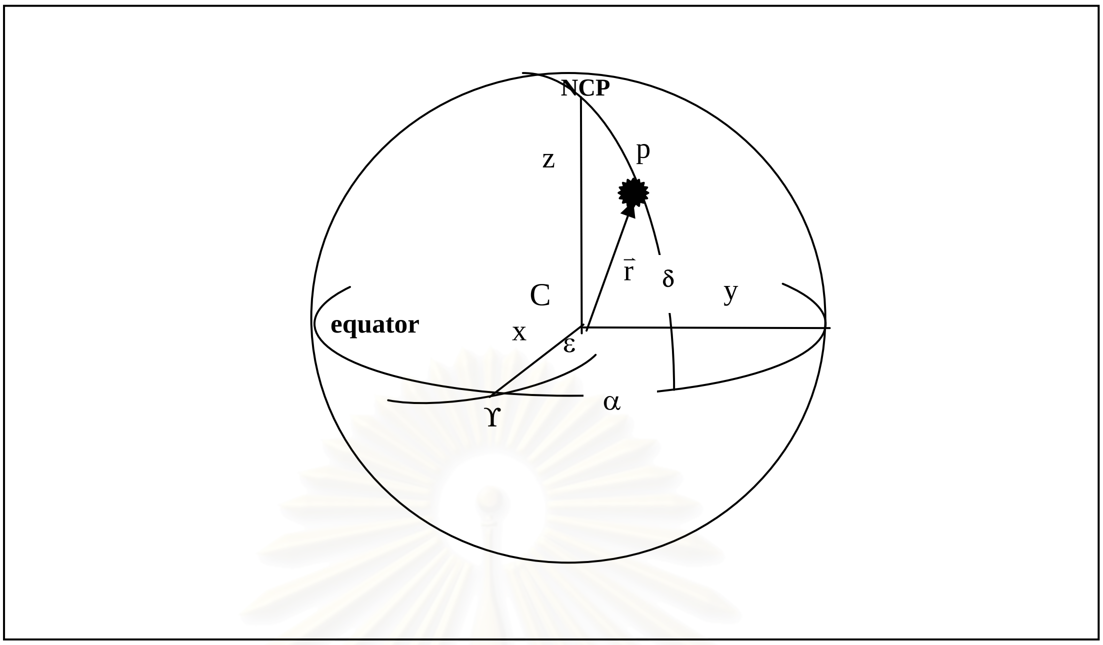


Figure 3.4 The celestial equatorial coordinate system [27].

The process of orbit determination will also require converting equatorial rectangular coordinates to equatorial spherical coordinates. Thus, the declination can be found from

$$\sin\delta = \frac{z}{r}\,. \tag{3.7}$$

Then, for the right ascension,

$$\cos\delta = \sqrt{1-\sin^2\delta} \tag{3.8}$$

$$\cos\alpha = \frac{x}{r\cos\delta} \tag{3.9}$$

$$\sin\alpha = \frac{y}{r\cos\delta}\,. \tag{3.10}$$

3.3.2 Celestial Ecliptic Systems

The spherical and rectangular celestial ecliptic systems are shown in Figure 3.5. The fundamental circle and plane are those defined by the ecliptic, and the origin **C** is usually centered in the Sun. The position of a point **P** in space is defined by its *radial distance* **r**, *ecliptic longitude* λ, and *ecliptic latitude* β. The angle λ is measured eastward from ϒ around the ecliptic from 0° to 360°. The angle β is measured from the ecliptic plane to +90° at the north ecliptic pole (**NEP**) or –90° at the south ecliptic pole (**SEP**). Given these spherical coordinates, the rectangular coordinates are computed as follows:

$$\tilde{x} = r\cos\beta\cos\lambda \tag{3.11}$$

$$\tilde{y} = r\cos\beta\sin\lambda \tag{3.12}$$

$$\tilde{z} = r\sin\beta\,. \tag{3.13}$$

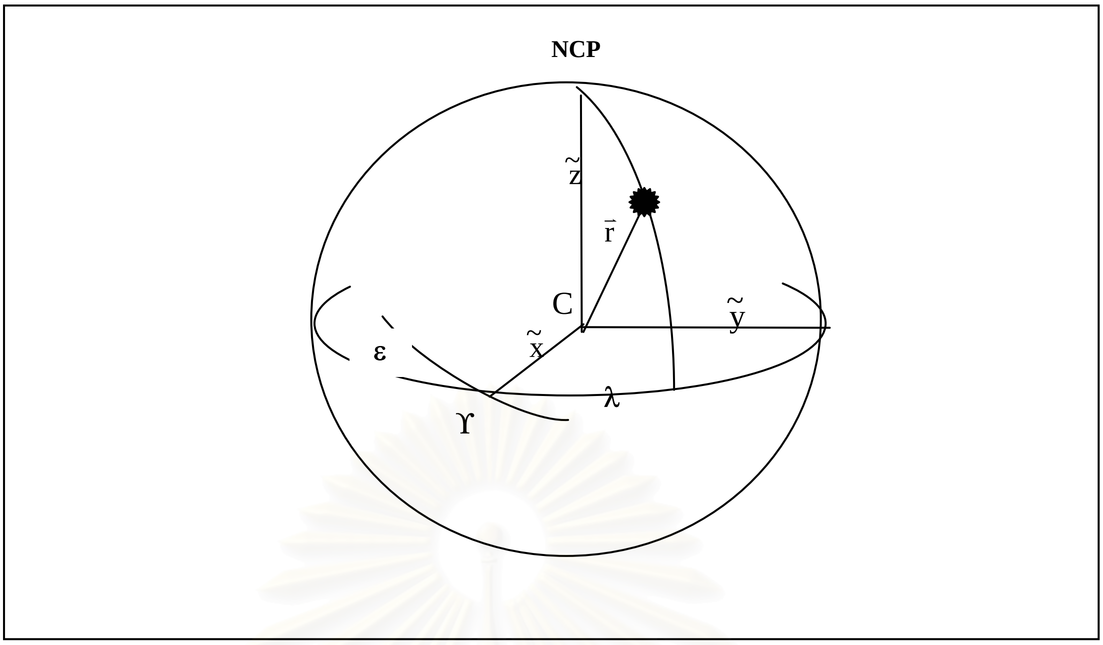


Figure 3.5 The celestial ecliptic coordinate system [27].

3.4 Ecliptic-Equatorial Transformations

It is often convenient and sometimes necessary to transform rectangular coordinates between the ecliptic and equatorial frames of reference. As both frames share the same principal direction for their **x**-axes, the transformation is simply a rotation about the **x**-axis through an angle equal to the obliquity of the ecliptic ε. Because of the effects of general precession, the numerical value of ε is not constant. However, the obliquity of the ecliptic for date **t**, with respect to the equator of date **t**, can be computed from

$$\varepsilon = 23^{\circ}0439291 - 0^{\circ}.0130042\ T - 0^{\circ}.00000016\ T^2, \qquad \textbf{(3.14)}$$

where

$$T = \frac{t - 2000.0}{100} = \frac{JD - 2451545.0}{36525}. \qquad \textbf{(3.15)}$$

Using the appropriate value of ε from Equation 3.14, the transformation of equatorial rectangular coordinates at epoch **t** to ecliptic rectangular coordinates at the same epoch is given by

$$\tilde{x} = x \qquad \textbf{(3.16)}$$
$$\tilde{y} = x \sin\varepsilon + y \cos\varepsilon \qquad \textbf{(3.17)}$$
$$\tilde{z} = z \cos\varepsilon - y \sin\varepsilon. \qquad \textbf{(3.18)}$$

The reverse transformation from ecliptic rectangular coordinates to equatorial rectangular coordinates is given by

$$x = \tilde{x} \qquad \textbf{(3.19)}$$
$$y = \tilde{y} \cos\varepsilon - \tilde{z} \sin\varepsilon \qquad \textbf{(3.20)}$$
$$z = \tilde{y} \sin\varepsilon + \tilde{z} \cos\varepsilon. \qquad \textbf{(3.21)}$$

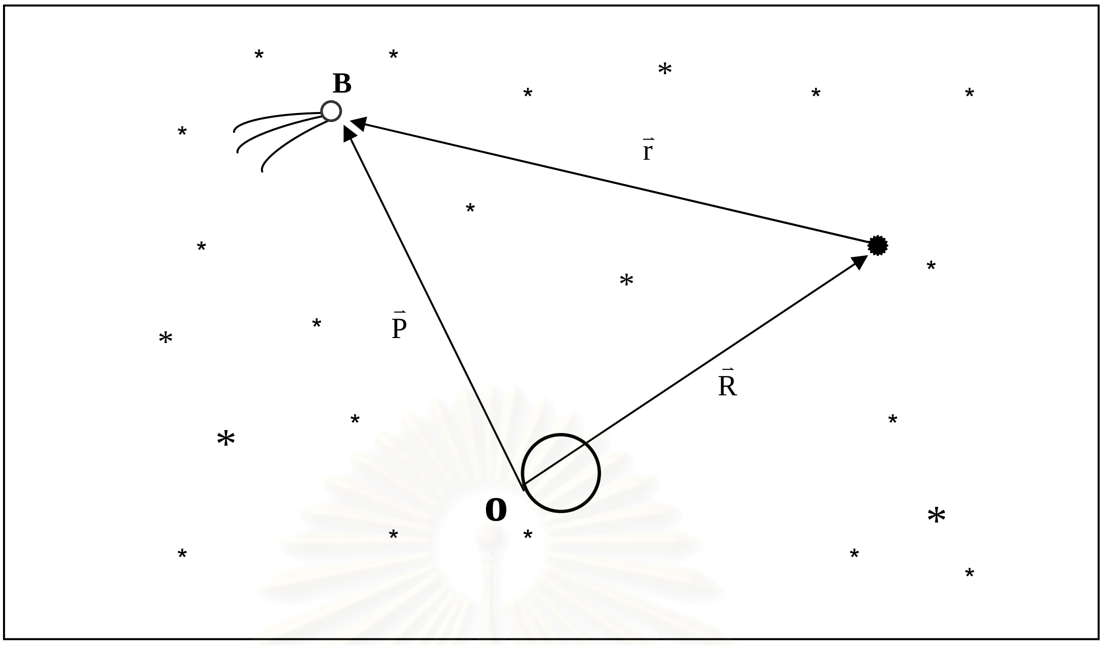


Figure 3.6 The fundamental vector triangle [27].

3.5 The Fundamental Vector Triangle

Consider the general vector relationship illustrated in Figure 3.6, where $\vec{\mathbf{R}}$, $\vec{\mathbf{r}}$, and $\vec{p}$ define the relative positions of the observer **O**, center of force **C**, and object **B**. If it is assumed that all vectors are referred to the same inertial coordinate system, then

$$\vec{p} = \vec{r} + \vec{R} \,. \qquad \textbf{(3.22)}$$

Equation 3.22 represents the *fundamental vector triangle* of orbit computation. If we let

$$\vec{r} = \{x,y,z\} \qquad \textbf{(3.23)}$$

$$\vec{R} = \{X,Y,Z\} \,, \qquad \textbf{(3.24)}$$

then the topocentric position vector to **B** can be expressed as

$$\vec{p} = \{x + X, y + Y, z + Z\} \,. \qquad \textbf{(3.25)}$$

Let it now be assumed, as almost always the case, that the object's topocentric position is measured in the celestial equatorial coordinate system. Then, by Equations 3.4 through 3.6, we can write

$$\frac{x + X}{p} = \cos\delta\cos\alpha \qquad \textbf{(3.26)}$$

$$\frac{y + Y}{p} = \cos\delta\sin\alpha \qquad \textbf{(3.27)}$$

$$\frac{z + Z}{p} = \sin\delta \,. \qquad \textbf{(3.28)}$$

Where $p = |\vec{p}|$ is the range of the object from the observer. The ratios on the left sides of these equations are knows as *direction cosines*. Since we have divided each component of $\vec{p}$ by p, we have created a unit vector

$$\vec{L} = \frac{\vec{p}}{p} \qquad \textbf{(3.29)}$$

pointing toward the position of the object on the celestial sphere. Thus,

$$\vec{L} = \{\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta\}\,. \qquad \textbf{(3.30)}$$

So, we obtain

$$p\vec{L} = \vec{r} + \vec{R} \qquad \textbf{(3.31)}$$

or, equivalently,

$$\vec{r} = p\vec{L} - \vec{R}\,. \qquad \textbf{(3.32)}$$

Equations 3.31 and 3.32 are the most convenient forms of the equation of fundamental vector triangle.

3.6 Reduction of Astronomical Coordinates: Planetary Aberration

Before two or more sets of position data are compared, their coordinates should be based on a common inertial frame of reference. The choice of common reference frame is largely a matter of convenience according to the nature of the available date and the problem to be solved. For a physical standpoint, consider the general type of the correction for the motions of celestial body and the observer with respect to the inertial reference frame. It is the reduction for the aberration of light, which causes the observed direction toward the celestial body to depend on the motions of both the body and the observer during the time interval required for light to travel from the body to the observer.

Because the velocity of light is finite, the *apparent* direction toward a moving celestial body as viewed by a moving observer is not the same as the *geometric direction* toward the object a the same instant of time. This displacement from the geometric position results from two separate effects. The first, caused by the motion of the celestial body independent of the motion of the observer, is known as the *correction for light-time*. The second, caused by the motion of the observer independent of the motion of the celestial body, is called *stellar aberration* because it typically affects observations of the fixed stars. The total effect due to light-time and stellar aberration is called *planetary aberration*.

Consider the situation depicted in Figure 3.7, where a celestial body **B** and observer **O** are shown in the geometric positions they occupy at a time **t** when **O** observes **B**. let **B′** be the position of the celestial body at a previous instant $\mathbf{t_c}$, when the light left the body to reach **O** at the observation time. If $\boldsymbol{\delta t}$ is the *light-time*, or interval required for light to travel from **B′** to **O**, then

$$t_c = t - \delta t\,. \qquad \textbf{(3.33)}$$

Furthermore, if **c** is *speed of light* and **p** is distance from **B′** to **O**, we have

$$\delta t = \frac{p}{c}\,, \qquad \textbf{(3.34)}$$

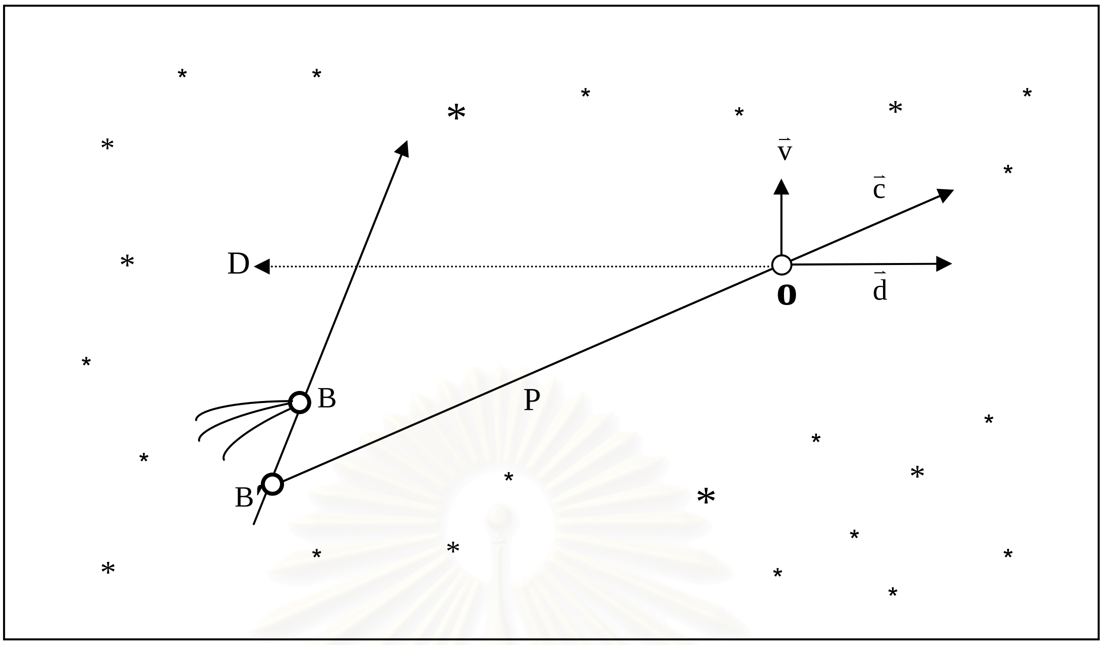


Figure 3.7 The aberration of light [27].

and

$$t_c = t - \frac{p}{c}, \qquad \textbf{(3.35)}$$

where, in heliocentric working units,

$$c = 173.1446 \ \text{AU/Day}. \qquad \textbf{(3.36)}$$

If it can be assumed that the observer is at rest, the apparent direction of the celestial body will be toward the point **B′** which the body occupied at time $\mathbf{t_c}$. Therefore, if the light-time and the body’s motion are known, its geometric position at $\mathbf{t_c}$ can also be found.

In situations where the observer cannot be assumed to be at rest, stellar aberration caused the problem to be more complex. When the light which left the celestial body at time $\mathbf{t_c}$ arrives at the observer at time **t**, it will not appear to be coming from **B′** but from its direction relative to the moving observer. According to classical physics, the observed direction of the celestial body will be opposite that of the vector $\vec{\mathbf{d}}$, which is difference between the velocity of light $\vec{\mathbf{c}}$ and the velocity of the observer $\vec{\mathbf{v}}$. In other words,

$$\vec{d} = \vec{c} - \vec{v}. \qquad \textbf{(3.37)}$$

Therefore, the apparent position of the celestial body will be some point **D** on the celestial sphere which is displaced from **B′** toward the direction of the observer’s motion.

CHAPTER 4

THE TWO – BODY PROBLEM

We have at our disposal a celestial frame of reference. Now we shall apply that equation to a specific type of orbital problem which is great practical significance. The opportunity presents it because our interest in heliocentric orbit restricts the application of the general equation to orbits dominated by the mutual gravitational attraction of two celestial bodies. In this context, theory and experience have shown that a *two-body* orbit can be computed using a simple mathematical model which ignores all perturbations and considers only the attraction between the orbiting and central masses.

4.1 The Two-Body Equation of Motion

Two-body motion is nothing more than a special case of the many-body motion modeled by Equation 2.17, namely,

$$\ddot{\vec{r}} = -\frac{\mu\vec{r}}{r^3} + \sum_{q=1}^{n} m_q\left(\frac{\vec{p}_q}{p_q^3} - \frac{\vec{r}_q}{r_q^3}\right). \qquad \textbf{(4.1)}$$

The two-body equation follows immediately from the above when we make the simplifying assumption that all term involving masses $\mathbf{m_q}$ can be neglected when computing a first approximation of the orbit. Thus,

$$\ddot{\vec{r}} = -\frac{\mu\vec{r}}{r^3} \qquad \textbf{(4.2)}$$

is the *two-body equation of motion*. As shown in Figure 4.1, the acceleration $\ddot{\vec{\mathbf{r}}}$ will always point directly toward the origin **O** at the center of the central body **C** because that is the direction of the net force acting on boy **B**. As a consequence of this, there is no tendency for **B** to move out of the plane formed by $\vec{\mathbf{r}}$, $\dot{\vec{\mathbf{r}}}$, and **O**. Therefore, *a two-body orbit is always confined to a plane which passes through the center of the central body.*

4.2 The Orbital and Radial Rates

Before taking up the solution of the two-body equation of motion, we must digress to discuss the potentially confusing relationship between the rate of motion along the orbital path and the rate of change of the magnitude of the radius vector $\vec{\mathbf{r}}$. As illustrated in Figure 4.2, a body **B** has velocity $\dot{\vec{\mathbf{r}}}$ which is tangent to the orbit at $\vec{\mathbf{r}}$. Consider the magnitude of $\dot{\vec{\mathbf{r}}}$ is the orbital speed $\mathbf{v}$. Thus,

$$v = |\dot{\vec{r}}|\,. \qquad \textbf{(4.3)}$$

Letting $\dot{\mathbf{r}}$ represent the rate of change of the scalar **r** with respect to modified time, then

$$\dot{r} = \frac{dr}{d\tau}, \qquad \textbf{(4.4)}$$

where $\mathbf{r} = |\vec{\mathbf{r}}|$ and $\mathbf{dr}$ is the incremental change in **r** during an infinitesimal interval of modified time $\mathbf{d\tau}$. Equation 4.4 is a scalar relationship which expresses only the rate at which the *distance* between bodies **B** and **C** changes with time.

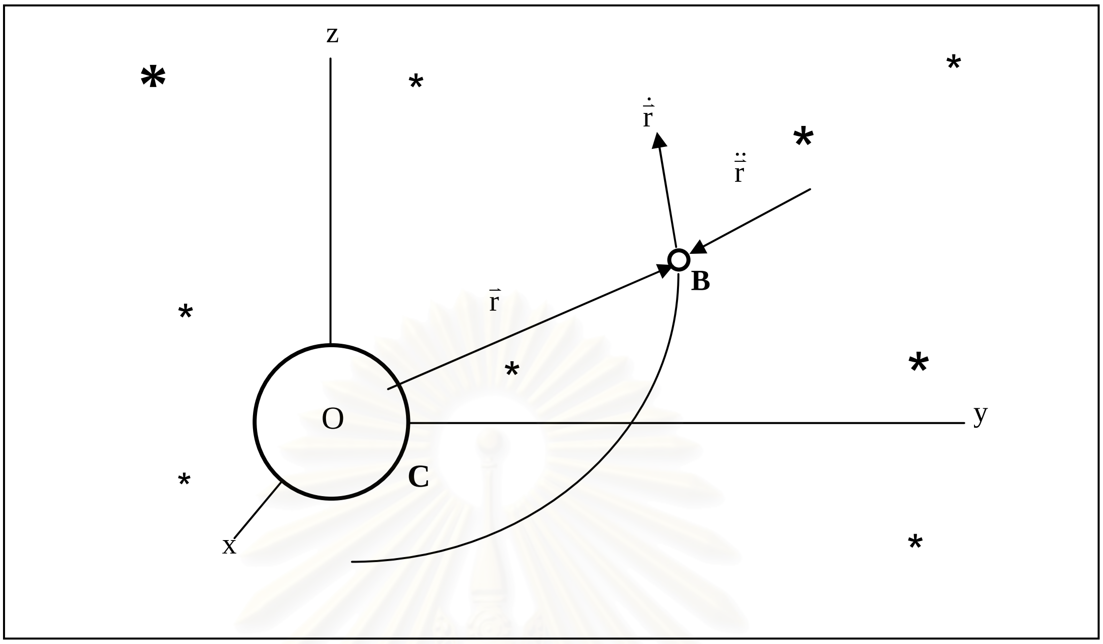


Figure 4.1 Two-body orbital motion [27].

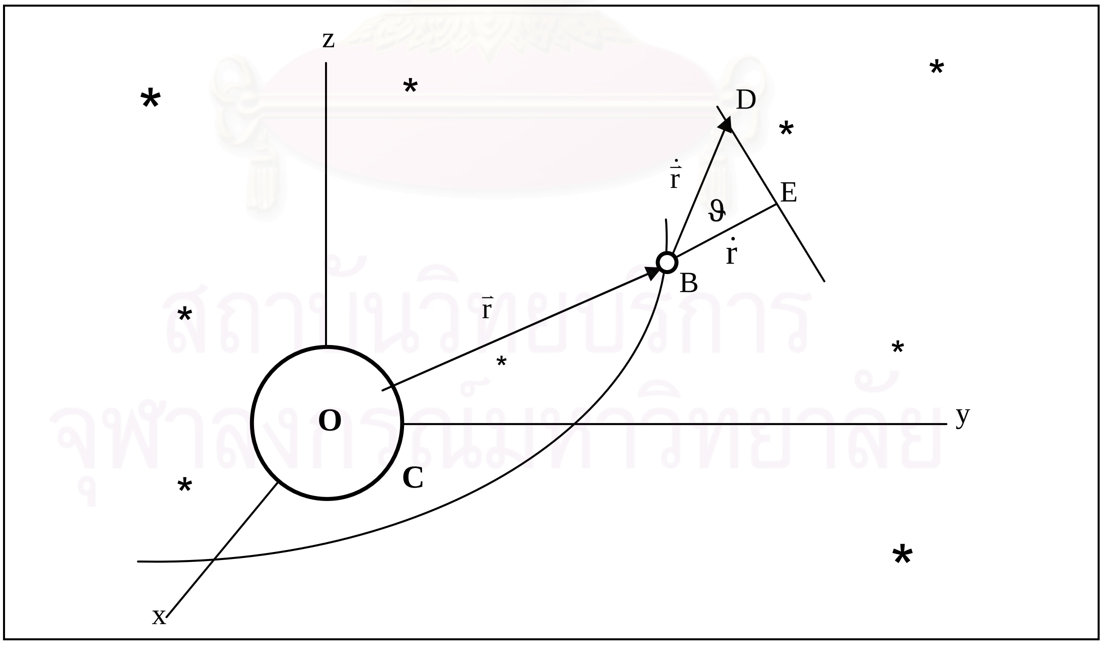


Figure 4.2 The orbital and radial rates [27].

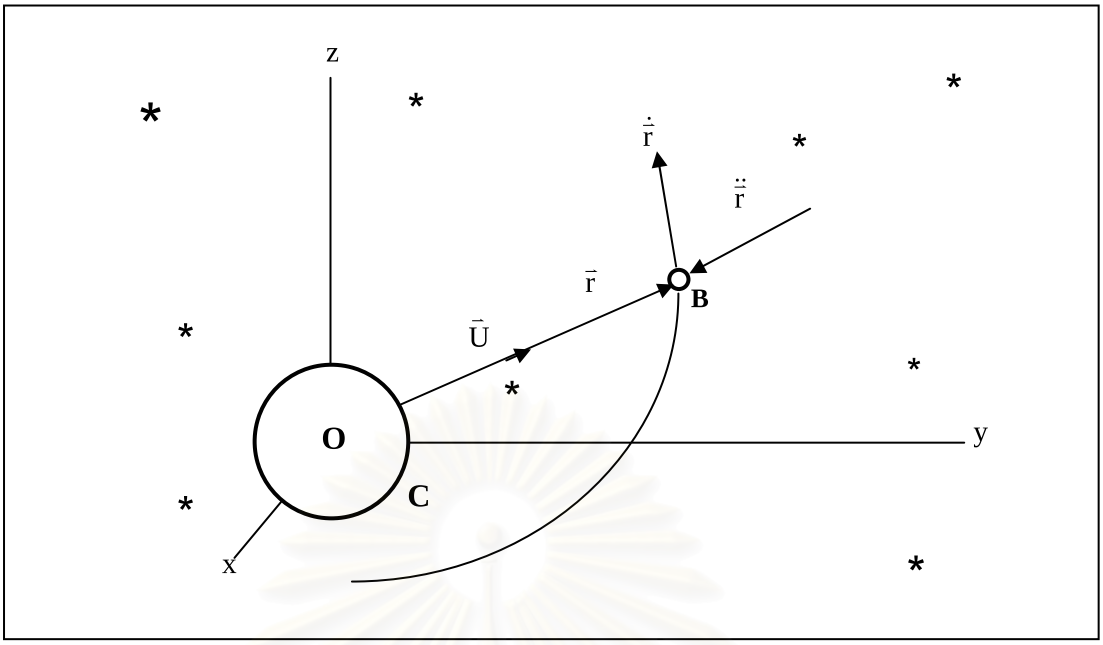


Figure 4.3 The two-body problem [27].

In the right triangle **BDE** of Figure 4.2, ϑ represents the angle between two sides whose lengths are in the ratio $\dot{\mathbf{r}}/\mathbf{v}$. Thus,

$$\dot{r} = v\cos\vartheta . \tag{4.5}$$

By the definition of the dot produce to give

$$\vec{r}\cdot\dot{\vec{r}} = rv\cos\vartheta . \tag{4.6}$$

4.3 The Law of Two-Body Motion

When the techniques of integral calculus are used to solve the Newtonian two-body differential equation of motion, the results confirm the law derived empirically by Johannes Kepler [18,19,20]:

1. *The orbits of the planets are ellipses, with the Sun at one focus*.
2. *The line joining a planet to the Sun sweeps out equal areas in equal times*.
3. *The square of a planet's period is proportional to the cube of its mean distance from the Sun*.

Moreover, the Newtonian formulation expresses these original principles in a general fashion, and an important relationship is shown to exist between the speed and position of a celestial body in any two-body orbit

4.3.1 The Conic Section Law

Let the fundamental equation of two-body motion be written as follows [14]:

$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\vec{U}, \tag{4.7}$$

where

$$\vec{U} = \frac{\vec{r}}{r}, \tag{4.8}$$

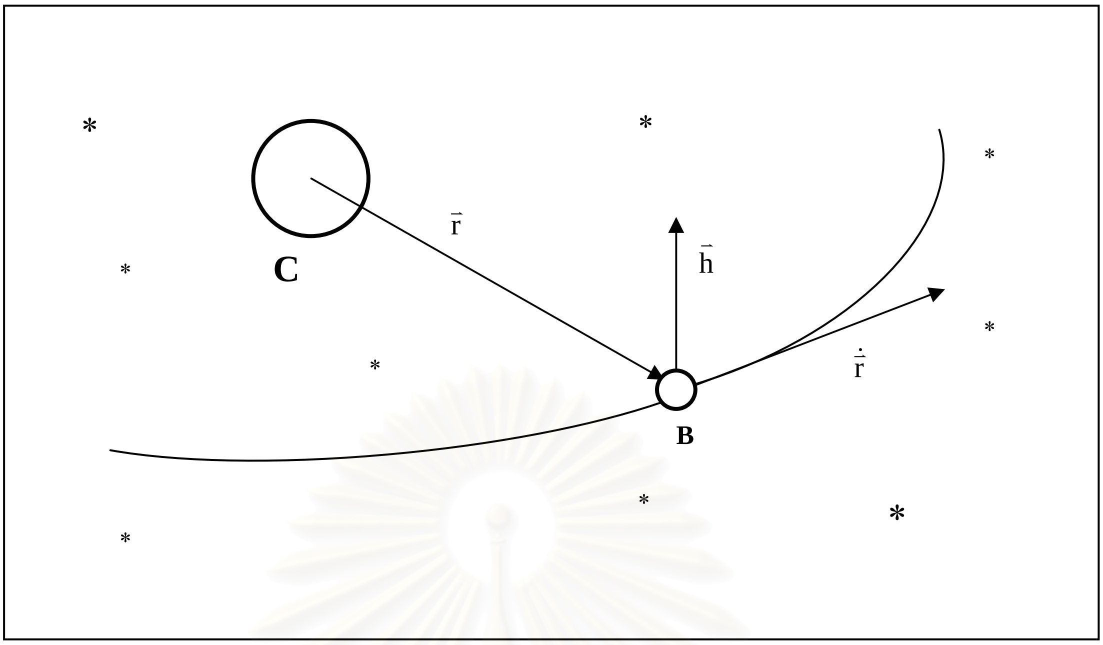


Figure 4.4 The angular momentum vector $\vec{h}$ [27].

is a unit vector pointing in the direction of the radius vector as shown in Figure 4.3.

Taking the cross product of Equation 4.7 with $\vec{\mathbf{r}}$,
we obtain

$$\vec{r} \times \ddot{\vec{r}} = -\frac{\mu}{r^2}\left(\vec{r} \times \vec{U}\right),$$

so that

$$\vec{r} \times \ddot{\vec{r}} = 0 \tag{4.9}$$

because $\vec{\mathbf{r}}$ and $\vec{\mathbf{U}}$ are parallel, causing their cross product to be the *null vector* $\vec{0}$. Now consider the following differentiation with respect to modified time:

$$\frac{d}{d\tau}\left(\vec{r} \times \dot{\vec{r}}\right) = 0 \tag{4.10}$$

since $\vec{\mathbf{r}} \times \ddot{\vec{\mathbf{r}}} = 0$ by Equation 4.9, and $\dot{\vec{\mathbf{r}}} \times \dot{\vec{\mathbf{r}}} = 0$ for the same reason. If we integrate Equation 4.10 to reverse the differentiation process, we obtain

$$\vec{r} \times \dot{\vec{r}} = \vec{h}, \tag{4.11}$$

where $\vec{\mathbf{h}}$ is a vector constant of integration which is equal to the *angular momentum per unit mass* of the two-body system.

The physical significance of this important result is illustrated in Figure 4.4. The fact that $\vec{\mathbf{h}}$ is constant implies that its magnitude and direction in inertial space never change. Therefore, vector $\vec{\mathbf{r}}$ and $\dot{\vec{\mathbf{r}}}$ always lie in a fixed paned which passes through the central body **C**. The direction of $\vec{\mathbf{h}}$ establishes the orientation of the orbital plane with respect to the inertial rectangular coordinates system. When the **z** component of $\vec{\mathbf{h}}$ is positive, the celestial body **B** is moving in a counter-clockwise direction as viewed from the positive **z**-axis, and the motion is called *direct*.

When the **z** component of $\vec{\mathbf{h}}$ is negative, the orbital motion is clockwise as seen from the positive **z**-axis, and the motion is called *retrograde*.

The angular momentum vector can be used to transform the fundamental equation of motion into an expression which can be easily integrated.

Returning to Equation 4.7 and taking the cross product of that expression with $\vec{\mathbf{h}}$

$$\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{r^2}\left[\vec{U} \times (\vec{r} \times \dot{\vec{r}})\right], \qquad \textbf{(4.12)}$$

because $\vec{\mathbf{h}} = \vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}$ by definition. Now, apply a vector identity to Equation 4.12, the result is

$$\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{r^2}\left[(\vec{U} \cdot \dot{\vec{r}})\vec{r} - (\vec{U} \cdot \vec{r})\dot{\vec{r}}\right].$$

Replacing $\vec{\mathbf{U}}$ by its definition to simplify the result yields

$$\ddot{\vec{r}} \times \vec{h} = \frac{\mu}{r^2}\left[r\dot{\vec{r}} - \dot{r}\vec{r}\right]. \qquad \textbf{(4.13)}$$

Before we can integrate Equation 4.13, we must pause to show that the expression on each side of the equal sign represents a perfect differential. Consider the following:

$$\frac{d\vec{U}}{d\tau} = \frac{d}{d\tau}\left(\frac{\vec{r}}{r}\right). \qquad \textbf{(4.14)}$$

Carrying out the differentiation on the right side of Equation 4.14 produces

$$\frac{d}{d\tau}\left(\frac{\vec{r}}{r}\right) = \frac{r\dot{\vec{r}} - \dot{r}\vec{r}}{r^2}. \qquad \textbf{(4.15)}$$

Consider also the following:

$$\frac{d}{d\tau}\left(\dot{\vec{r}} \times \vec{h}\right) = (\ddot{\vec{r}} \times \vec{h}) + (\dot{\vec{r}} \times \dot{\vec{h}}). \qquad \textbf{(4.16)}$$

However, by Equations 4.10 and 4.11, $\vec{\mathbf{h}}$ is a constant vector so that

$$\dot{\vec{h}} = 0. \qquad \textbf{(4.17)}$$

Therefore, Equation 4.16 becomes simply

$$\frac{d}{d\tau}\left(\dot{\vec{r}} \times \vec{h}\right) = (\ddot{\vec{r}} \times \vec{h}). \qquad \textbf{(4.18)}$$

Returning to Equation 4.13, we see that its right and left ides can be replaced by Equations 4.15 and 4.18, respectively. Upon making those substitutions, we have

$$\frac{d}{d\tau}\left(\dot{\vec{r}} \times \vec{h}\right) = \mu\frac{d}{d\tau}\left(\frac{\vec{r}}{r}\right). \qquad \textbf{(4.19)}$$

Equation 4.19 can be immediately integrated to eliminate the differentiation. The result can be written as follows:

$$\dot{\vec{r}} \times \vec{h} = \mu\left(\frac{\vec{r}}{r} + \vec{e}\right), \qquad \textbf{(4.20)}$$

where $\vec{\mathbf{e}}$ is an arbitrary vector constant of integration.

If we take the dot product of Equation 4.20 with $\vec{\mathbf{r}}$, the result can be manipulated to obtain a key relationship. First,

$$(\dot{\vec{r}} \times \vec{h}) \cdot \vec{r} = \mu\left(\frac{\vec{r} \cdot \vec{r}}{r} + \vec{e} \cdot \vec{r}\right). \qquad \textbf{(4.21)}$$

Using the vector identity, Equation 4.21 can be rewritten to obtain

$$(\vec{r} \times \dot{\vec{r}}) \cdot \vec{h} = \mu(r + \vec{e} \cdot \vec{r}), \qquad \textbf{(4.22)}$$

which, according to Equation 4.11, becomes

$$h^2 = \mu(r + \vec{e} \cdot \vec{r}). \qquad \textbf{(4.23)}$$

Now, using the definition of the dot product produces

$$\vec{e} \cdot \vec{r} = er \cos \upsilon, \qquad \textbf{(4.24)}$$

where $\mathbf{e} = |\vec{\mathbf{e}}|$ and $\boldsymbol{\upsilon}$ is angle between $\vec{\mathbf{e}}$ and $\vec{\mathbf{r}}$. Therefore,

$$h^2 = \mu r(1 + e \cos \upsilon), \qquad \textbf{(4.25)}$$

which can also be written

$$r = \frac{h^2/\mu}{1 + e \cos \upsilon}. \qquad \textbf{(4.26)}$$

The geometric significance of Equation 4.26 can be seen by comparing it to the general equation of a conic section written in polar coordinates

$$r = \frac{\wp}{1 + e \cos \upsilon}, \qquad \textbf{(4.27)}$$

where the origin is at a focus, the polar angle $\boldsymbol{\upsilon}$ is the angle between the radius vector and the point on the conic nearest the focus, and

$$\wp = \frac{h^2}{\mu}. \qquad \textbf{(4.28)}$$

The conclusion is that a two-body orbit is always a conic section which lies in a fixed plane which passes through the central body at the focus. This is a generalized statement of Kepler’s first law.

The elements commonly used to describe conic sections are illustrated for the case of an ellipse in Figure 4.5, where $\vec{\mathbf{r}}$ is the radius vector and $\vec{\mathbf{e}}$ defines the direction of perifocus. The angle $\boldsymbol{\upsilon}$ is called the *true anomaly*, the quantity $\wp$ is the *semiparameter*, the constant **e** is the *eccentricity*, the length **q** is the *perifocal distance*, and the length **a** is the *semimajor axis*.

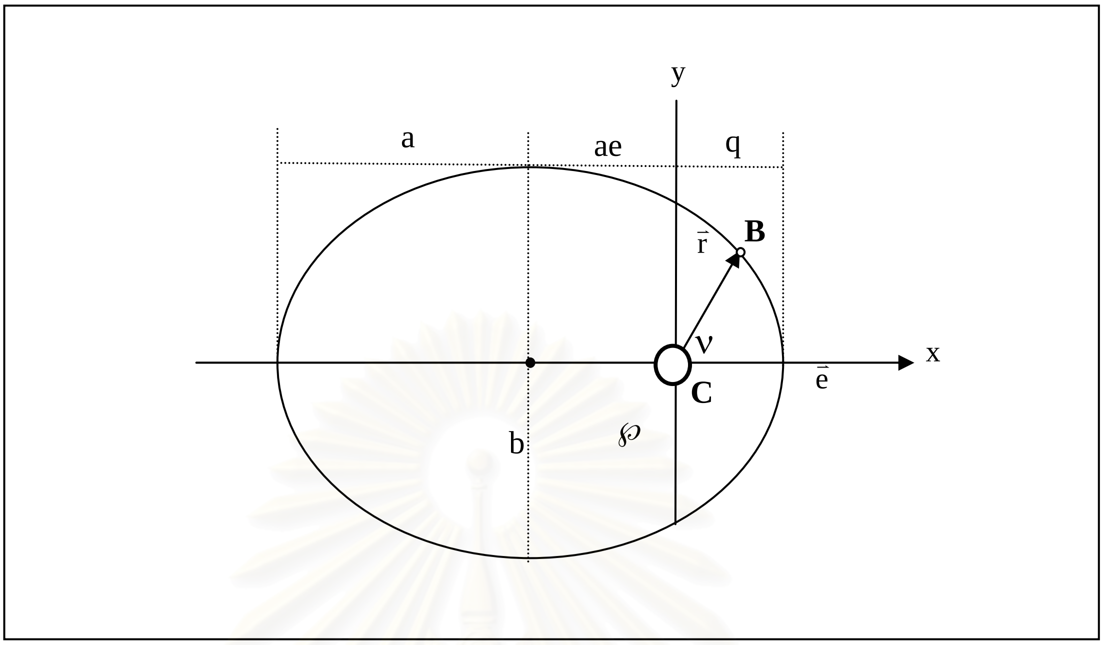


Figure 4.5 The descriptive elements of a conic [27].

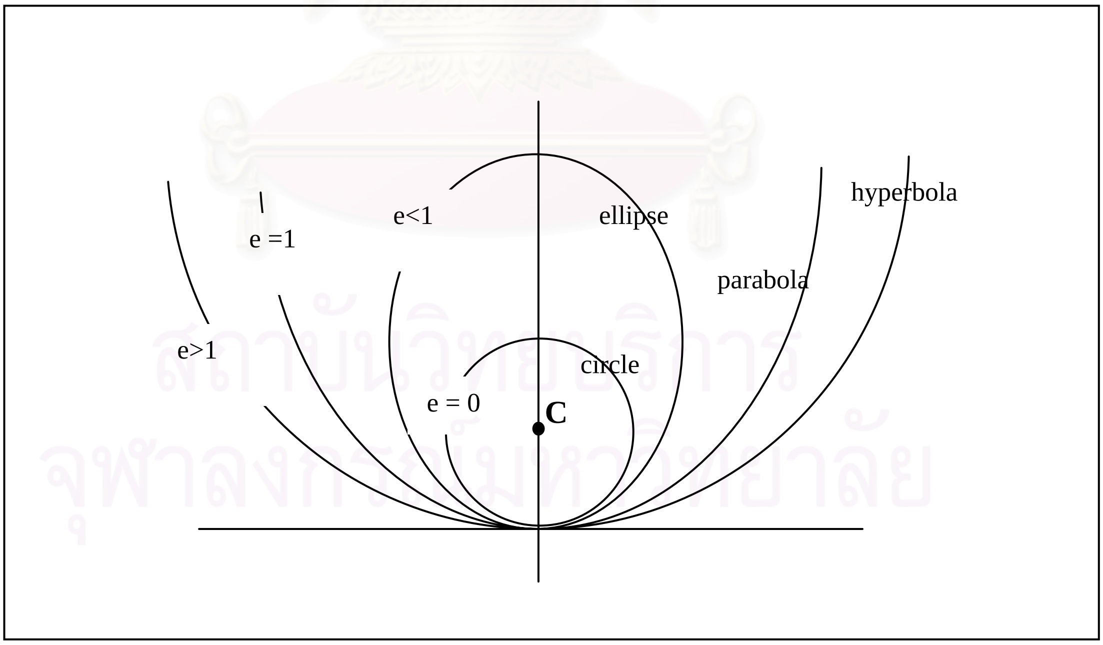


Figure 4.6 Four conic sections [27].

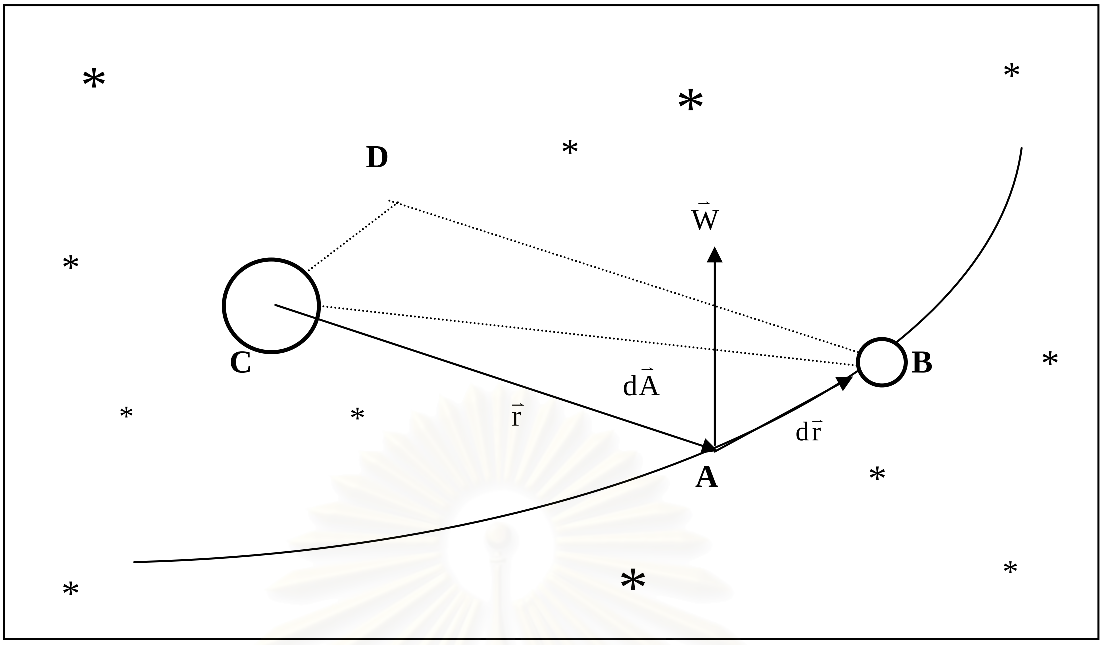


Figure 4.7 The vector area $\vec{W}dA$ [27].

These elements are related as follows:

$$\wp = q(1 + e) \quad \textbf{(4.29)}$$

$$q = a(1 - e) \quad \textbf{(4.30)}$$

$$\wp = a(1 - e^2)\,. \quad \textbf{(4.31)}$$

Additionally, the *semiminor axis* **b** is related to the above by

$$b = a\sqrt{1 - e^2} \quad \textbf{(4.32)}$$

$$b = \sqrt{\wp a}\,. \quad \textbf{(4.33)}$$

The value of the eccentricity determines the specific type of conic section represented by Equation 4.26. Four possibilities are depicted in Figure 4.6, and Equation 4.20 provides the useful vector relationship,

$$e = \frac{\dot{\vec{r}} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}, \quad \textbf{(4.34)}$$

which permits us to find the magnitude $\mathbf{e} = |\,\vec{\mathbf{e}}\,|$ and the direction of the perifocus in inertial space when $\vec{\mathbf{r}}$ and $\dot{\vec{\mathbf{r}}}$ are known at any point on the orbit.

4.3.2 The Law of Areas

Consider the vector cross product shown below in light of the geometric relationships depicted in Figure 4.7:

$$\vec{W}dA = \frac{1}{2}(\vec{r} \times d\vec{r})\,, \quad \textbf{(4.35)}$$

where dA is a very narrow triangular area swept out by the radius vector $\vec{\mathbf{r}}$ during an infinitesimal modified time interval $d\tau$, $d\vec{r}$ is the incremental change in $\vec{r}$ during the interval, and $\vec{W}$ is a unit vector normal to the orbital plane. The numerical constant 1/2 accounts fact that the cross product is equivalent to the vector area of the entire parallelogram **ABCD**. The unit normal $\vec{W}$ defines the direction associated with the vector area.

Dividing both sides of Equation 4.35 by the interval $d\tau$, during which the triangular area is swept out, we obtain

$$\vec{W}\frac{dA}{d\tau} = \frac{1}{2}(\vec{r} \times \dot{\vec{r}}),$$

which, by Equation 4.11, is equivalent to

$$\vec{W}\frac{dA}{d\tau} = \frac{1}{2}\vec{h}. \tag{4.36}$$

Taking the magnitude of Equation 4.36, the result is

$$\frac{dA}{d\tau} = \frac{h}{2}. \tag{4.37}$$

Therefore, the rate at which the radius vector sweeps out area is constant. In other word, the radius vector generates equal areas in equal times, which is Kepler's second law.

4.3.3 The Harmonic Law

Let Equation 4.37 be rearranged and the modified time interval replaced by the corresponding normal time interval kdt as follows:

$$2(dA) = hk(dt). \tag{4.38}$$

If we assume the orbit to be an ellipse, integrating the area swept out by the radius vector over a time interval of exactly one orbital period will yield the following expression:

$$2(\pi ab) = hkP, \tag{4.39}$$

where the term in parentheses is the area of an ellipse, and **P** is orbital period. Recalling that, by Equations 4.28 and 4.33, we can write

$$P^2 = \left[\frac{1}{\mu}\left(\frac{2\pi}{k}\right)^2\right]a^3, \tag{4.40}$$

which is the generalized statement of Kepler's third law [21].

4.3.4 The Vis-viva Law

An extremely important relationship between orbital speed and position can be derived through another integration of the two-body equation of motion

$$\ddot{\vec{r}} = -\frac{\mu\vec{r}}{r^3}.$$

If we take dot product of this equation with $2\dot{\vec{r}}$ we can obtain

$$2(\ddot{\vec{r}} \cdot \dot{\vec{r}}) = 2\left(-\frac{\mu\dot{r}}{r^2}\right). \tag{4.41}$$

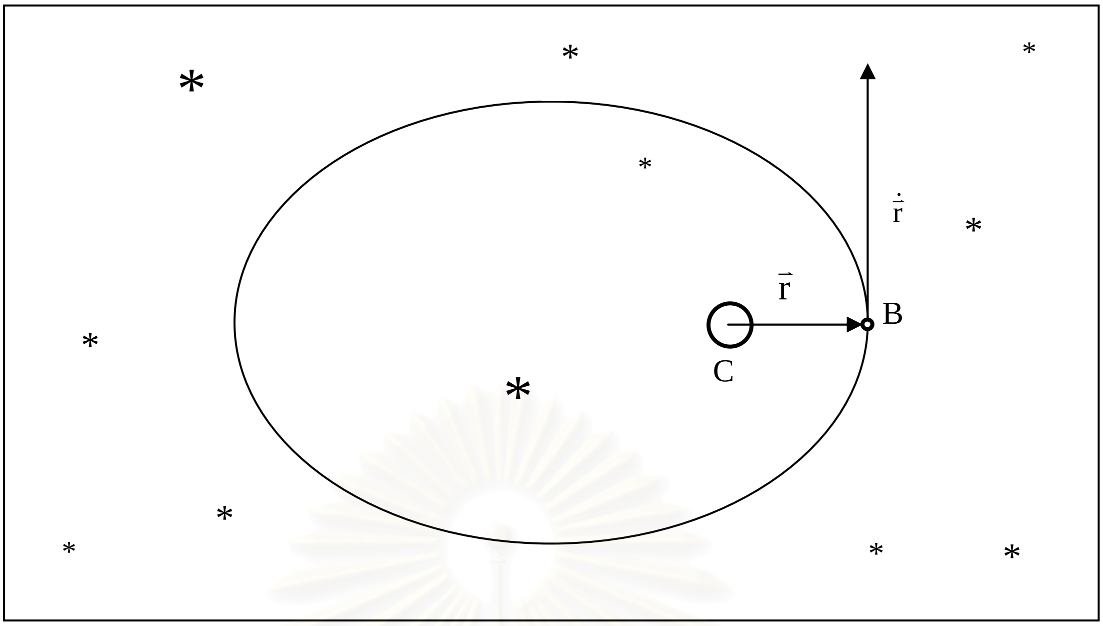


Figure 4.8 An orbiting body at perifocus [27].

Now consider the following two derivatives:

$$\frac{d}{d\tau}\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right) = (\dot{\vec{r}}\cdot\ddot{\vec{r}}) + (\ddot{\vec{r}}\cdot\dot{\vec{r}}) = 2(\ddot{\vec{r}}\cdot\dot{\vec{r}}) \quad \textbf{(4.42)}$$

and

$$\frac{d}{d\tau}\left(\frac{\mu}{r}\right) = -\frac{\mu\dot{r}}{r^2}. \quad \textbf{(4.43)}$$

Therefore, substituting Equations 4.42 and 4.43 into 4.41 yields

$$\frac{d}{d\tau}(\dot{\vec{r}}\cdot\dot{\vec{r}}) = 2\frac{d}{d\tau}\left(\frac{\mu}{r}\right). \quad \textbf{(4.44)}$$

Integrating Equation 4.44 will reverse the differentiation and produce an arbitrary constant η. Thus,

$$\dot{\vec{r}}\cdot\dot{\vec{r}} = \frac{2\mu}{r} + \eta,$$

which is equivalent to

$$v^2 = \frac{2\mu}{r} + \eta. \quad \textbf{(4.45)}$$

The constant η can be evaluated by imposing the conditions which exist when the orbiting body is at perifocus, as shown in Figure 4.8. In this situation, $\mathbf{r} = \mathbf{q}$, so that

$$\eta = v^2 - \frac{2\mu}{q}. \quad \textbf{(4.46)}$$

Furthermore, because $\vec{\mathbf{r}}$ and $\dot{\vec{\mathbf{r}}}$ are perpendicular at the perifocus, we may use the definition of the vector cross product to obtain the simple relationship

$$h = |\vec{r} \times \dot{\vec{r}}| = rv \sin 90^\circ = qv(\text{at peri focus}) . \quad \mathbf{(4.47)}$$

Squaring Equation 4.47 and substituting Equations 4.28 and 4.29, obtained

$$v^2 = \frac{\mu(1+e)}{q} . \quad \mathbf{(4.48)}$$

Using the above expression for $\mathbf{v}^2$ and the fact that $\mathbf{q = a(1 - e)}$, Equation 4.46 yieds the following expression for η :

$$\eta = -\frac{\mu}{a} . \quad \mathbf{(4.49)}$$

Finally, employing Equation 4.49 to replace η in Equation 4.45, the result is

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) . \quad \mathbf{(4.50)}$$

Equation 4.50 is called the *vis-viva equation*. This relation is particularly useful as a means of determining the semimajor axis when the position and velocity vectors are known for any point on the orbit. Thus,

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu} . \quad \mathbf{(4.51)}$$

4.4 Two-Body Motion by Numerical Integration: The f and g Series

When a celestial body's position $\vec{\mathbf{r}}_0$ and velocity $\dot{\vec{\mathbf{r}}}_0$ are given for some epoch time $\mathbf{t_0}$, then its acceleration $\ddot{\vec{\mathbf{r}}}_0$ is also known since

$$\ddot{\vec{r}}_0 = -\frac{\mu \vec{r}_0}{r_0^3} .$$

Knowing the position, velocity, and acceleration at some epoch makes it possible to extrapolate the orbiting body's trajectory over a short interval of time to yield a new position and velocity. Furthermore, the new position permits the calculation of a new acceleration, and the whole process can be repeated. Therefore, in principle, it is possible to calculate the body's motion over an extended period or time by means of a sequence of relatively short steps if its initial position and velocity are known for a given epoch. This process is called *numerical integration*, which is a broad designation for a variety of clever procedures that are among the most powerful tools of celestial mechanics [2,22,23].

Consider the free-fall motion illustrated in Figure 4.9, where body **B** has position $\vec{\mathbf{r}}_0$ and velocity $\dot{\vec{\mathbf{r}}}_0$ at an arbitrary epoch $\mathbf{t_0}$. If we want to compute the position of **B** at some other time **t**, we must have an expression for $\vec{\mathbf{r}}_0$ which will satisfy the initial conditions at $\mathbf{t_0}$ and equation of motion

$$\ddot{\vec{r}} = -\frac{\mu \vec{r}}{r^3} \quad \mathbf{(4.52)}$$

for any given time before or after $\mathbf{t_0}$.

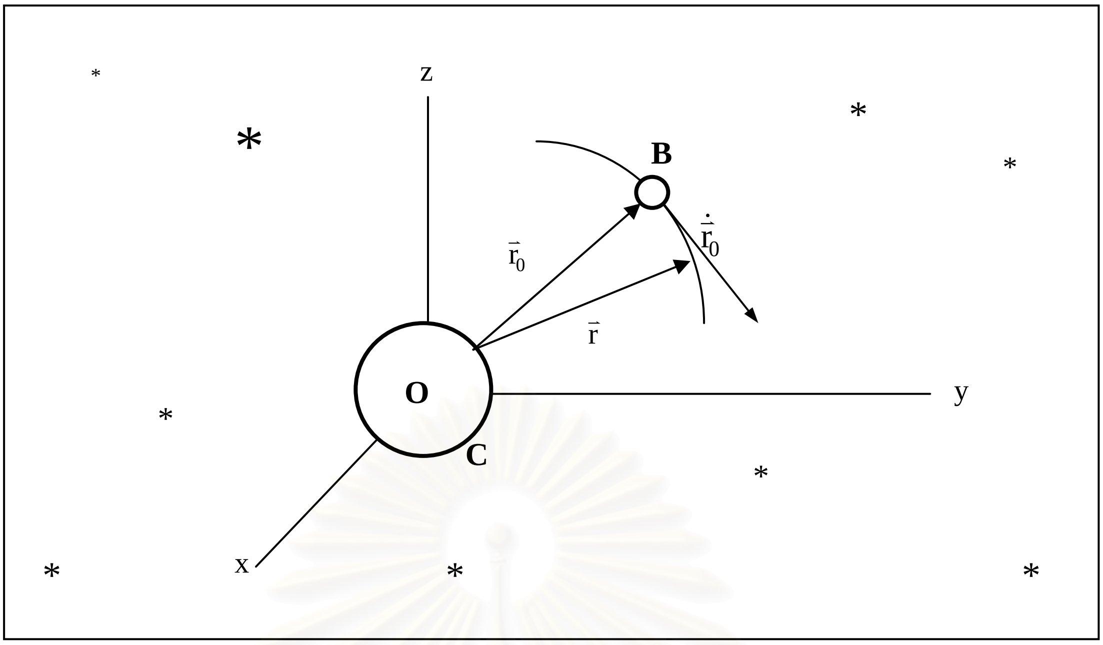


Figure 4.9 The two-body free-fall trajectory [27].

It can be shown that the solution of Equation 4.52 may be written as an infinite power series expansion in modified time about the position $\vec{\mathbf{r}}_0$ [22,24]. Therefore,

$$r = C_0 + C_1\tau + C_2\tau^2 + C_3\tau^3 + C_4\tau^4 + ...,$$

where $\boldsymbol{\tau} = \mathbf{k}(\mathbf{t} - \mathbf{t}_0)$ is the modified time interval, and the coefficients are vector constants which must be determined by applying the constraints imposed by the initial conditions. If this infinite series is repeatedly differentiated with respect to modified time, we obtain

$$\begin{aligned} r &= C_0 + C_1\tau + C_2\tau^2 + C_3\tau^3 + C_4\tau^4 + ... \\ \dot{\vec{r}} &= C_1 + 2C_2\tau + 3C_3\tau^2 + 4C_4\tau^3 + ... \\ \ddot{\vec{r}} &= 2C_2 + 6C_3\tau + 12C_4\tau^2 + ... \\ \dddot{\vec{r}} &= 6C_3 + 24C_4\tau + ... \\ &. \\ &. \end{aligned} \tag{4.53}$$

Evaluating these equations at $\mathbf{t_0}$ when $\boldsymbol{\tau} = 0$ and replace the vector coefficients in Equation 4.53, the result is

$$\vec{r} = \vec{r}_0 + \dot{\vec{r}}_0\tau + \ddot{\vec{r}}_0\frac{\tau^2}{2!} + \dddot{\vec{r}}_0\frac{\tau^3}{3!} + \ddddot{\vec{r}}_0\frac{\tau^4}{4!} + ... \tag{4.54}$$

Notice that the first three terms are similar to the more familiar scalar equation for free fall near the surface of the Earth. Equation 4.54 is a very useful model for two-body orbital motion because the higher derivatives can be found by successively differentiating Equation 4.52 and evaluating the results at the epoch $\mathbf{t_0}$.

Rather than attacking the problem head-on, it is more convenient to differentiate Equation 4.54 by means of auxiliary quantities. Let **u**, **z**, and **q** (not the perifocal distance) be defined as follows [14,24]:

$$u = \frac{\mu}{r^3} \qquad \textbf{(4.55)}$$

$$z = \frac{\vec{r} \cdot \dot{\vec{r}}}{r^2} \qquad \textbf{(4.56)}$$

$$q = \frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{r^2} - u\,, \qquad \textbf{(4.57)}$$

and let Equation 4.53 be rewritten in the simpler form

$$\ddot{\vec{r}} = -u\vec{r}\,. \qquad \textbf{(4.58)}$$

Now, Equation 4.58 is the expression we want to differentiate; however, for reasons which will be apparent shortly, we shall delay its differentiation until after we have developed expressions for the first derivatives of **u**, **z**, and **q**. Commencing with **u**, we have

$$\dot{u} = -3u(\vec{r} \cdot \dot{\vec{r}})r^{-2}\,,$$

which becomes

$$\dot{u} = -3uz\,. \qquad \textbf{(4.59)}$$

Next, for quantity **z**,

$$\dot{z} = \left[(\dot{\vec{r}} \cdot \dot{\vec{r}}) + (\vec{r} \cdot \ddot{\vec{r}})\right]r^{-2} + (\vec{r} \cdot \dot{\vec{r}})(-2r^{-3}\dot{r})\,.$$

Using Equations 4.57 and 4.58, the result is

$$\dot{z} = q - 2z^2\,. \qquad \textbf{(4.60)}$$

Finally, by means of a similar process for **q**, we have

$$\dot{q} = 2(\dot{\vec{r}} \cdot \ddot{\vec{r}})r^{-2} - 2(\dot{\vec{r}} \cdot \dot{\vec{r}})r\dot{r}r^{-4} - \dot{u}\,,$$

and Equations 4.58 and 4.59 can be used to replace their corresponding identities. Now, according to Equations 4.56 and 4.57 to give

$$\dot{q} = -(uz + 2zq)\,. \qquad \textbf{(4.61)}$$

The derivatives $\dot{u}$, $\dot{z}$, and $\dot{q}$, are now given in terms of the original quantities **u**, **z**, and **q**. At last we are ready to complete the series solution by finding expressions for the derivatives required by Equation 4.54. To do this, we differentiate

$$\ddot{\vec{r}} = -u\vec{r} \qquad \textbf{(4.62)}$$

repeatedly, replacing $\dot{u}$, $\dot{z}$, $\dot{q}$, and $\ddot{\vec{r}}$ by their corresponding values in terms of **u**, **z**, **q**, and $-u\vec{r}$ whenever they appear during the differentiation process. Thus,

$$\dddot{\vec{r}} = 3uz\vec{r} - u\dot{\vec{r}}$$

$$\ddddot{\vec{r}} = (3uq - 15uz^2 + u^2)\vec{r} + 6uz\dot{\vec{r}}$$

$$\vdots$$

Evaluating all derivatives at $\mathbf{t_0}$ and substituting the resulting expressions into the power series of Equation 4.54 yields

$$\begin{aligned}\vec{r} &= \vec{r}_0 + \dot{\vec{r}}_0\tau + (-u)\vec{r}_0\frac{\tau^2}{2} + (3uz\vec{r}_0 - u\dot{\vec{r}}_0)\frac{\tau^3}{6} \\ &\quad + \left[(3uq - 15uz^2 + u^2)\vec{r}_0 + 6uz\dot{\vec{r}}_0\right]\frac{\tau^4}{24} + \ldots,\end{aligned} \qquad \textbf{(4.63)}$$

where, $\mathbf{t} = \mathbf{t_0}$, The solution of the two-body equation of motion can be now written in a very convenient form by rearranging and collecting the terms of Equation 4.63 into coefficients of $\vec{\mathbf{r}}_0$ and $\dot{\vec{\mathbf{r}}}_0$. The result is

$$\vec{r} = f\vec{r}_0 + g\dot{\vec{r}}_0, \qquad \textbf{(4.64)}$$

where **f** and **g** series are given by

$$f = 1 - \frac{u}{2}\tau^2 + \frac{uz}{2}\tau^3 + \frac{3uq - 15uz^2 + u^2}{24}\tau^4 + \ldots \qquad \textbf{(4.65)}$$

$$g = \tau - \frac{u}{6}\tau^3 + \frac{uz}{4}\tau^4 + \ldots \qquad \textbf{(4.66)}$$

Finally, an expression for velocity is obtained by differentiating the equation for position. Thus,

$$\dot{\vec{r}} = \dot{f}\vec{r}_0 + \dot{g}\dot{\vec{r}}_0, \qquad \textbf{(4.67)}$$

CHAPTER 5

ORBIT GEOMETRY

The solution of the two-body problem can be characterized by six numerical quantities which are related to the arbitrary constants resulting from the integration of the differential equation

$$\ddot{\vec{r}} = -\frac{\mu\vec{r}}{r^3} .$$

These fundamental parameters are called *orbital element*, and, when they are known, the orbiting body's motion can be computed. The set of elements we shall use often consists of the six scalar components of position and velocity evaluated at a given instant of time. However, although this set uniquely defines the size and orientation of the orbit in space, there are a number of other convenient sets from which to choose.

This chapter continues the discussion of the two-body problem by introducing an orbit-plane coordinates system which facilitates the derivation of elliptic which can be used to change the form of the orbital elements.

5.1 General Relationships

Figure 5.1 depicts the two-body orbit of a celestial body **B** about a dynamical center **C** located at the origin of an inertial rectangular coordinate system. The $\bar{\mathbf{x}}\bar{\mathbf{y}}$-plane coincides with the orbit plane, and the $\bar{\mathbf{x}}$-axis is aligned with the orbit's semimajor axis. The vector $\vec{\mathbf{v}}$ is the velocity of the celestial body at a given instant when its radius vector $\vec{\mathbf{r}}$ is displaced from the $\bar{\mathbf{x}}$-axis by the angle of the true anomaly υ . Recalling Equation 4.11, we can write

$$\vec{h} = \vec{r} \times \vec{v} , \tag{5.1}$$

where, in the orbit-plane coordinate system,

$$\begin{aligned} \vec{h} &= \{0,0,h\} \\ \vec{r} &= \{\bar{x},\bar{y},0\} \ . \\ \vec{v} &= \{\dot{\bar{x}},\dot{\bar{y}},0\} \end{aligned} \tag{5.2}$$

From Equation 5.1, we have

$$h = \bar{x}\dot{\bar{y}} - \bar{y}\dot{\bar{x}} . \tag{5.3}$$

5.1.1 Angular Momentum and Angular Speed

Equation 5.3 can be used to derive a useful relationship between **h** and the angular speed $\dot{\upsilon}$. From the geometry of Figure 5.1, we can write

$$\bar{x} = r\cos\upsilon \tag{5.4}$$

$$\bar{y} = r\sin\upsilon . \tag{5.5}$$

Differentiating these two equations with respect to modified time produces

$$\dot{\bar{x}} = \dot{r}\cos\upsilon - r\dot{\upsilon}\sin\upsilon \tag{5.6}$$

$$\dot{\bar{y}} = \dot{r}\sin\upsilon + r\dot{\upsilon}\cos\upsilon . \tag{5.7}$$

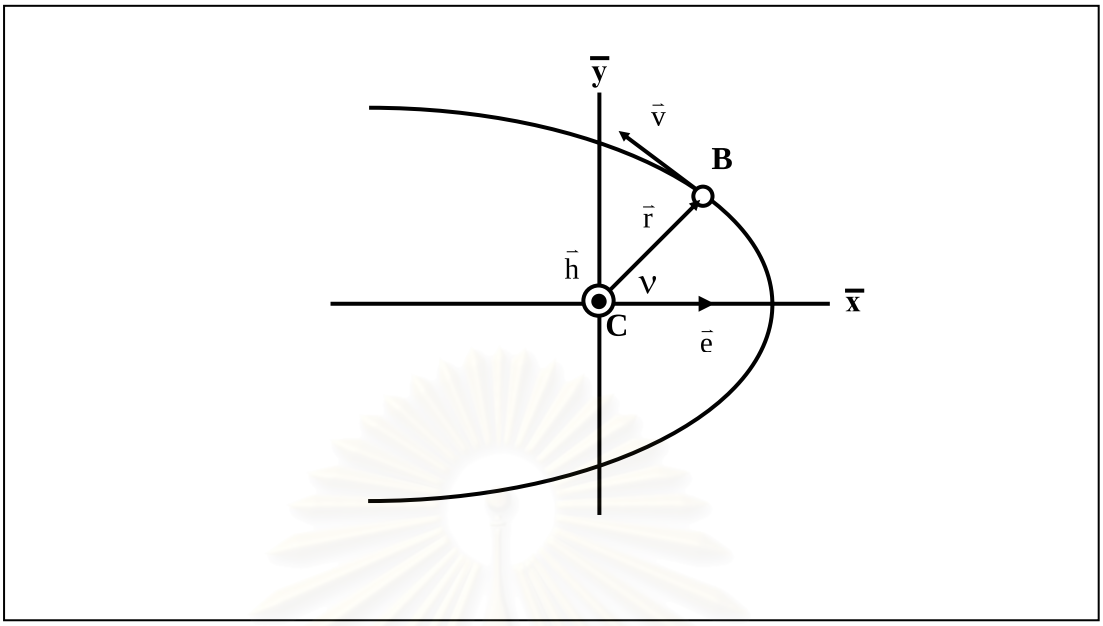


Figure 5.1 The orbit-plane coordinate system [27].

Substituting Equations 5.4 through 5.7 into Equation 5.3 and utilizing the trigonometric identity [25,26], we obtain

$$h^2 = r^2 \dot{\upsilon} . \qquad \textbf{(5.8)}$$

5.1.2 Radial Speed and True Anomaly

We can now use Equation 5.8 to derive an expression for the radial speed $\dot{\mathbf{r}}$. Recalling the general equation of a conic, we know that

$$\wp = r(1 + e \cos \upsilon) , \qquad \textbf{(5.9)}$$

where the semiparameter $\wp = h^2/\mu$. Differentiating Equation 5.9 with respect to modified time produces

$$\dot{r}(1 + e \cos \upsilon) - re\dot{\upsilon} \sin \upsilon = 0 .$$

Multiplying by **r** and make use of Equations 5.8 and 5.9, we obtain

$$\dot{r}\wp - he \sin \upsilon = 0 ,$$

$$\dot{r} = \sqrt{\frac{\mu}{\wp}} e \sin \upsilon . \qquad \textbf{(5.10)}$$

5.2 Relationship between Geometry and Time: Elliptic Formulation

When Equation 5.3 is applied to the elliptic orbit, it is possible to derive mathematical relationships between position in the orbit plane and time elapsed from a given epoch. We shall refer to this expression collectively as Kepler equation [14,27,28,29]. Consider the geometric construction of Figure 5.2, where an auxiliary circle centered at **K** circumscribes the actual ellipse of motion about the dynamical center **C**. As the celestial body **B** moves along the ellipse,

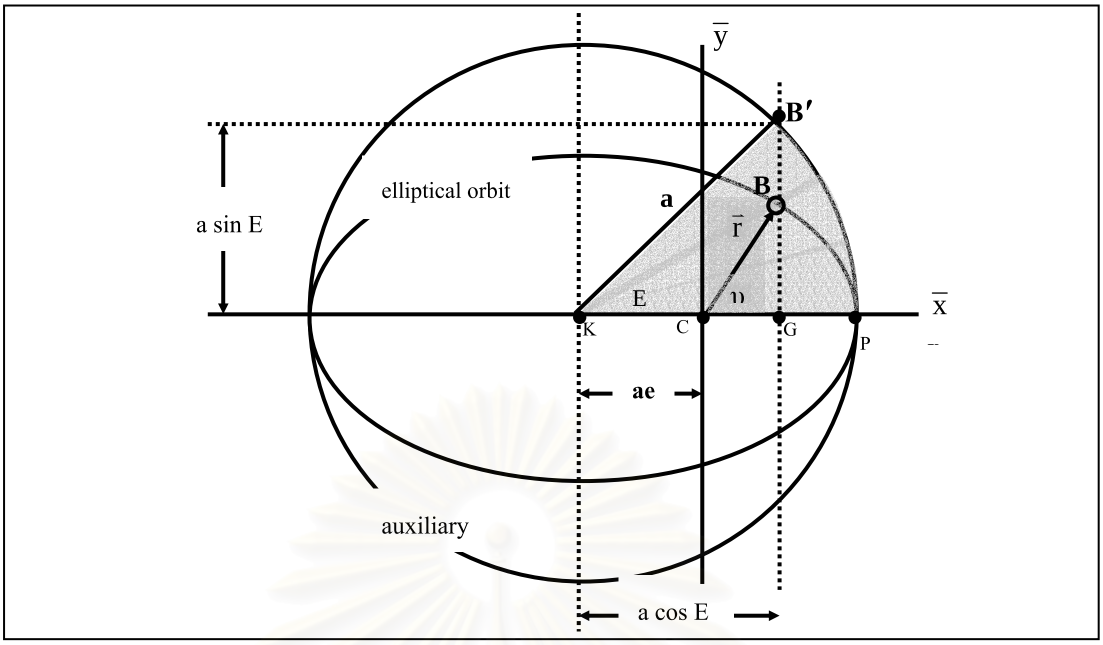


Figure 5.2 Elliptic formulation [27].

it is followed by a point **B′** defined by the projection of **B** in the **y**-direction upon the circle. The angle **E**, which is proportional to the shaded area, is called the *eccentric anomaly* and is measured in the orbital plane from the $\bar{\mathbf{x}}$-axis to the line **KB′**. The distance from **B′** to **K** is always equal to **a**, the length of the semimajor axis of the ellipse, and distance between **K** and **C** is equal to **ae**, where **e** is the eccentricity of the ellipse.

The advantage of the auxiliary circle is that by expressing $\bar{x}$ and $\bar{y}$ in terms of the eccentric anomaly, instead of the true anomaly, Equation 5.3 can be reduced to a simple form which is easily integrated.

By carefully examining Figure 5.2, we see that the $\bar{x}$-coordinate of the celestial body is related to the eccentric anomaly by

$$\bar{x} = a(\cos E - e) \ . \qquad \textbf{(5.11)}$$

Also, the general conic equation allows us to write

$$\wp = r + e(r \cos \upsilon) \ . \qquad \textbf{(5.12)}$$

Now, by comparing Equations 5.4 and 5.11, we find that

$$r \cos \upsilon = a \cos E - ae \ . \qquad \textbf{(5.13)}$$

Thus, substituting Equation 5.13 into Equation 5.12 produces

$$\wp = r + ae \cos E - ae^2 \, . \qquad \textbf{(5.14)}$$

Since

$$\wp = a(1 - e^2) \, .$$

Equation 5.14 can be rearranged to yield an expression for **r**:

$$r = a(1 - e \cos E) \, . \qquad \textbf{(5.15)}$$

We derive the equation for $\bar{y}$ by substituting the expressions for $\bar{x}$ and **r** into general relationship

$$r^2 = \bar{x}^2 + \bar{y}^2 .$$

The substitutions produce

$$\bar{y}^2 = a^2(1 - e^2)(1 - \cos^2 E) .$$

So that,

$$\bar{y} = a\sqrt{1 - e^2}\,\sin E . \qquad \textbf{(5.16)}$$

Expression for components of the velocity vector are found by differentiating the equations for $\bar{x}$ and $\bar{y}$. Therefore,

$$\dot{\bar{x}} = -a\dot{E}\sin E , \qquad \textbf{(5.17)}$$

$$\dot{\bar{y}} = a\sqrt{1 - e^2}\,\dot{E}\cos E . \qquad \textbf{(5.18)}$$

Now, if we substitute Equations 5.11, 5.16, 5.17, and 5.18 into Equation 5.3, then

$$h = a^2\sqrt{1 - e^2}(\cos^2 E - e\cos E + \sin^2 E)\dot{E} ,$$

which simplifies to

$$\sqrt{\frac{\mu}{a^3}} = (1 - e\cos E)\dot{E} \qquad \textbf{(5.19)}$$

when we apply the trigonometric identity and make use of the fact that

$$h = \sqrt{\mu\wp} = \sqrt{\mu a(1 - e^2)} .$$

Finally, Equation 5.19 becomes

$$\sqrt{\frac{\mu}{a^3}}d\tau = (1 - e\cos E)dE .$$

This equation is easily integrated to produce

$$\sqrt{\frac{\mu}{a^3}}\tau = E - e\sin E ,$$

where the arbitrary constant of integration is zero because we define the modified time interval τ to be zero when the eccentric anomaly **E** is zero. Letting **T** represent the *time of perifocal passage*, then the celestial body's position at any time **t** can be written

$$n(t - T) = E - e\sin E ,$$

where **n**, known as the *mean motion*, is given by

$$n = k\sqrt{\frac{\mu}{a^3}} . \qquad \textbf{(5.20)}$$

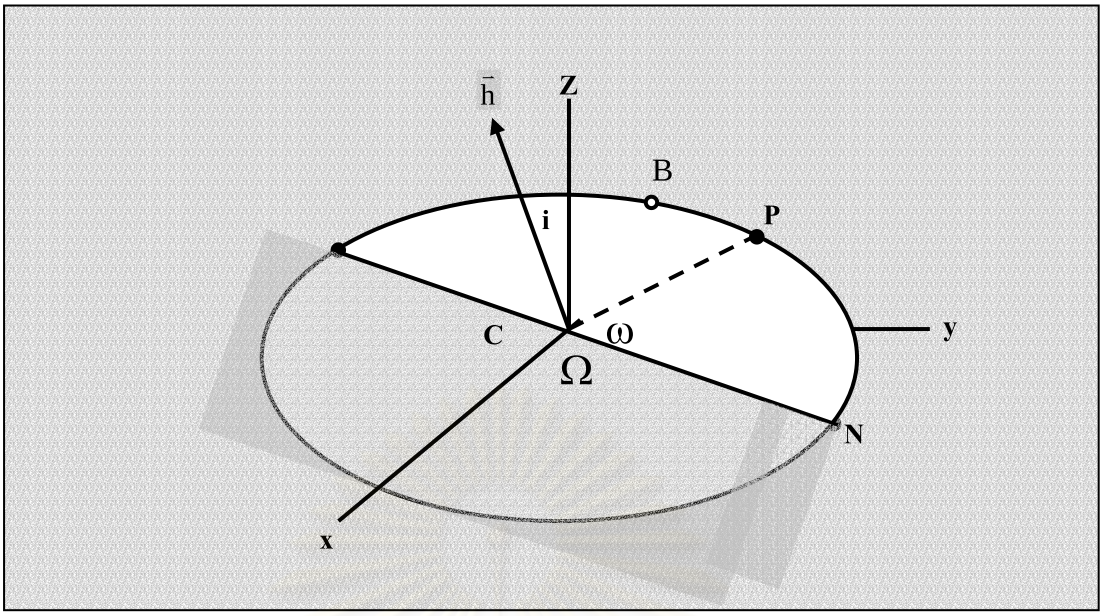


Figure 5.3 Classical geometric elements [27].

It is also convenient to define a quantity **M**, called the *mean anomaly*, such that

$$M = n(t - T)\,, \tag{5.21}$$

so that we can finally write

$$M = E - e\sin E\,, \tag{5.22}$$

which is *Kepler's equation*. Although developed for elliptic orbits, Equation 5.22 also holds for the case of circular motion when **e** = 0

5.3 The Classical Elements from Position and Velocity

The components of $\vec{r}$ and $\vec{v}$ provide a completely general description of orbital motion; however, their vector form does not clearly reveal the orbit's size, shape, and orientation in space. Since it is often helpful to have a geometric perspective such as that depicted in Figure 5.3, we will develop a procedure which will transform $\vec{r}$ and $\vec{v}$ into the following set of parameters known as *classical elements* [27,28,29,30]:

a **semimajor axis** The conic parameter which is used to define the size of an elliptic orbit.

e **eccentricity** The conic parameter which define the shape of an orbit.

i **inclination** The angle between the +**z**-axis and angular momentum vector $\vec{h}$, which is perpendicular to the orbit plane, measured from 0° to 180°. If **i** < 90°, the orbital motion is counterclockwise when viewed from the north side of the fundamental plane (direct motion). If **i** > 90°, the orbital motion is clockwise when viewed from the north side of the fundamental plane (retrograde motion).

Ω **longitude of the ascending node** The angle in the fundamental plane between the +**x**-axis and a line from the dynamical center **C** to the point **N** where the celestial body crosses through the fundamental plane from south to north (ascending node), measured counterclockwise from 0° to 360° as viewed from the north side of the fundamental plane. If **i** = 0, then Ω is undefined.

ω **argument of the perifocus** The angle in the orbital plane, between the line of the ascending node and a line from the dynamical center **C** to the perifocus **P**, measured from 0° to 360° in the direction of the celestial body's motion. If **e** = 0 or **i** = 0, then ω is undefined.

n **mean motion** A mathematical quantity whose value is the constant angular speed which would be required for the celestial body to complete its orbit in one period.

M **mean anomaly** A mathematical quantity whose value relates the position of the celestial body in the orbit to the elapsed time by means of the Kepler equations. The mean anomaly changes at a uniform rate equal to the mean motion **n**.

T **time of perifocal passage** The moment when the celestial body passes the perifocus **P**. This quantity can also be used to relate position along the orbit to the elapsed time by means of the Kepler equation. If **e** = 0, then T is undefined.

5.3.1 Three Fundamental Vectors

We begin process of determining the classical elements by forming the fundamental vector $\vec{e}$, $\vec{h}$, and $\vec{N}$ illustrated in Figure 5.4. The origin of the inertial rectangular coordinate system is at the dynamical center **C**, and $\vec{I}$, $\vec{J}$, and $\vec{K}$ are unit vectors parallel to the **x**, **y**, and **z**-axes, respectively. The coordinate system is aligned so that +**x**-axis points toward the vernal equinox, and the **xy**-plane coincides with the fundamental plane of the celestial coordinate system. The fundamental plane will correspond to the equatorial plane if the orbit is geocentric or to ecliptic plane if the orbit is heliocentric. In the latter case, the vectors $\vec{r}$ and $\vec{v}$ must be referred to the ecliptic coordinate system by using the equatorial-to-ecliptic transformation described in Section 3.4. Thus, we have for any given time **t**

$$\begin{aligned} r &= |\vec{r}| \\ v^2 &= \vec{v}\cdot\vec{v} \\ r\dot{r} &= \vec{r}\cdot\vec{v}\,. \end{aligned} \qquad \textbf{(5.23)}$$

Therefore, according to Equation 4.34

$$\vec{e} = \frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r},$$

because $\vec{h} = \vec{r}\times\vec{v}$, utilizing the vector identity, the *eccentricity vector* is

$$\vec{e} = \left(\frac{v^2}{\mu} - \frac{1}{r}\right)\vec{r} - \left(\frac{r\dot{r}}{\mu}\right)\vec{v}\,. \qquad \textbf{(5.24)}$$

The *angular momentum vector* is computed by means of its cross product definition

$$\vec{h} = \vec{r}\times\vec{v}\,, \qquad \textbf{(5.25)}$$

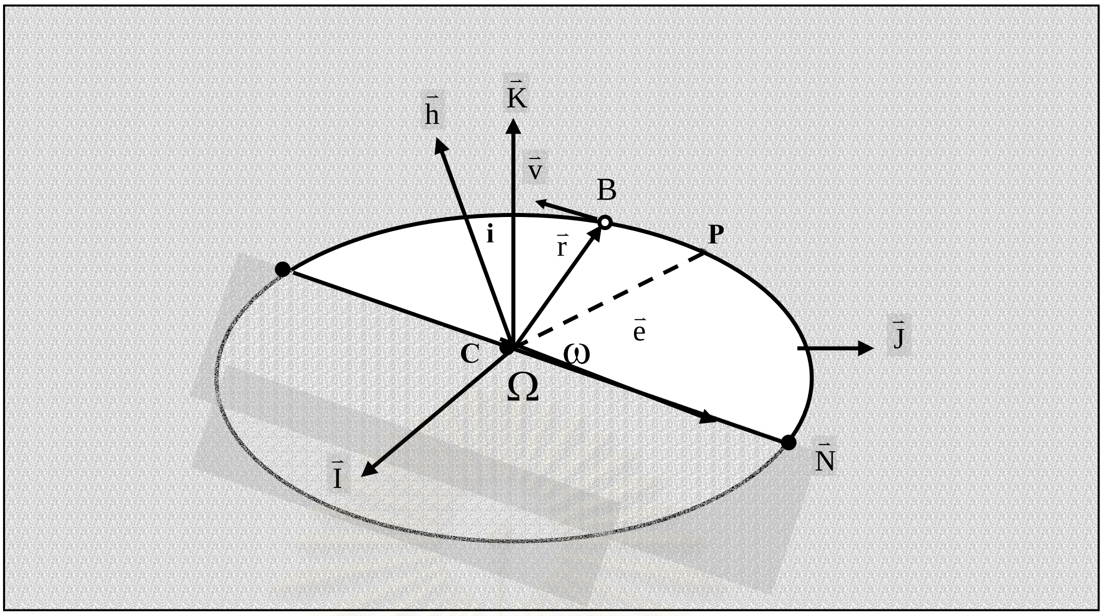


Figure 5.4 The fundamental vectors $\vec{e}$, $\vec{h}$, and $\vec{N}$ [27].

where

$$
\begin{aligned}
h_x &= y\dot{z} - z\dot{y} \\
h_y &= z\dot{x} - x\dot{z} \\
h_z &= x\dot{y} - y\dot{x}\,.
\end{aligned}
\qquad \textbf{(5.26)}
$$

Finally, the *ascending node vector* $\vec{\mathbf{N}}$ is obtained from the cross product of vector $\vec{\mathbf{K}}$ and $\vec{\mathbf{h}}$. Accordingly,

$$\vec{N} = \vec{K} \times \vec{h}\,, \qquad \textbf{(5.27)}$$

where

$$
\begin{aligned}
\vec{K} &= \{0,0,1\} \\
\vec{h} &= \{h_x, h_y, h_z\},
\end{aligned}
$$

$$
\begin{aligned}
N_x &= -h_y \\
N_y &= +h_z \\
N_z &= 0\,.
\end{aligned}
\qquad \textbf{(5.28)}
$$

5.3.2 The Conic Parameters

The conic parameter **a**, **e**, and **q** easily found by using relationships already discussed. According to the vis-viva equation, we have

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}\,, \qquad \textbf{(5.29)}$$

so the semimajor axis is determined. The eccentricity is obtained form Equation 5.24. Thus,

$$e = |\vec{e}|\,. \qquad \textbf{(5.30)}$$

We find the perifocal distance by employing Equation 5.25 to compute

$$h = |\vec{h}|\,, \tag{5.31}$$

so that, by Equation 4.28, we have

$$\wp = \frac{h^2}{\mu}\,, \tag{5.32}$$

and Equation 4.29 yields

$$q = \frac{\wp}{1+e}\,. \tag{5.33}$$

5.3.3 The Orientation Angles

Returning to Figure 5.4, we see that orientation angles **i**, $\boldsymbol{\Omega}$, $\boldsymbol{\omega}$ can all be determined from various dot products between fundamental vectors $\vec{\mathbf{e}}$, $\vec{\mathbf{h}}$, $\vec{\mathbf{N}}$ and unit vectors $\vec{\mathbf{I}}$, $\vec{\mathbf{J}}$, $\vec{\mathbf{K}}$. Accordingly, the *angle of inclination* is computed from the dot product

$$\vec{K}\cdot\vec{h} = |\vec{K}||\vec{h}|\cos i\,, \tag{5.34}$$

so Equation 5.34 simplifies to

$$\cos i = \frac{h_z}{h}\,, \tag{5.35}$$

and the inclination is determined. In the case of the longitude of the ascending node, we begin with the dot product

$$\vec{I}\cdot\vec{N} = |\vec{I}||\vec{N}|\cos\Omega\,, \tag{5.36}$$

where

$$\begin{aligned}\vec{I} &= \{1,0,0\}\\ \vec{N} &= \{N_x,N_y,N_z\}\end{aligned}\,.$$

So Equation 5.36 reduces to

$$\cos\Omega = \frac{N_x}{N}\,, \tag{5.37}$$

where $\boldsymbol{\Omega}$ > 180° if $\mathbf{N_y}$ < 0. Therefore, the longitude of the ascending node is determined.

Finally, the *argument of the perifocus* is obtained from the dot product

$$\vec{N}\cdot\vec{e} = |N||e| \cos\omega\,, \tag{5.38}$$

where

$$\begin{aligned}\vec{N} &= \{N_x,N_y,N_z\}\\ \vec{e} &= \{e_x,e_y,e_z\}\,.\end{aligned}$$

Equation 5.38 can be rewritten as

$$\cos\omega = \frac{\vec{N}\cdot\vec{e}}{Ne}\,, \tag{5.39}$$

where ω > 180° if $\mathbf{e_z}$ < 0. Thus, the argument of the perifocus is determined. This element is frequently replaced by the *longitude of the perifocus*,

$$\varpi = \Omega + \omega \,,$$
$$\omega = \varpi - \Omega \,.$$

5.3.4 The Mean Anomaly

According to Equations 5.4 and 5.5, the celestial body's position with reference so the orbit-plane coordinate system is given by

$$\bar{x} = r\cos\upsilon \qquad \textbf{(5.40)}$$
$$\bar{y} = r\sin\upsilon \,. \qquad \textbf{(5.41)}$$

Rearranging the conic equation to obtain

$$r\cos\upsilon = \frac{\wp - r}{e}$$

permits us to write Equation 5.40 in the form

$$\bar{x} = \frac{\wp - r}{e} \,. \qquad \textbf{(5.42)}$$

Now, recalling Equation 5.10, we have

$$\dot{r} = \sqrt{\frac{\mu}{\wp}} e\sin\upsilon \,,$$

so that, multiplying through by **r**, we get

$$r\sin\upsilon = \frac{r\dot{r}}{e}\sqrt{\frac{\wp}{\mu}} \,.$$

Substituting the above expression in Equation 5.41, the result is

$$\bar{y} = \frac{r\dot{r}}{e}\sqrt{\frac{\wp}{\mu}} \,. \qquad \textbf{(5.43)}$$

Now, if the eccentricity determined by Equation 5.30 is less than unity, the mean anomaly must be computed using the elliptic equations of motion. We begin with Equations 5.11 and 5.16, namely

$$\bar{x} = a(\cos E - e) \qquad \textbf{(5.44)}$$
$$\bar{y} = b\sin E \,, \qquad \textbf{(5.45)}$$

where, for convenience, we have let

$$b = a\sqrt{1 - e^2} \,. \qquad \textbf{(5.46)}$$

Since $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are known quantities given by Equations 5.42 and 5.43, we can write

$$\cos E = \frac{\bar{x}}{a} + e \qquad \textbf{(5.47)}$$

$$\sin E = \frac{\bar{y}}{b} \,, \qquad \textbf{(5.48)}$$

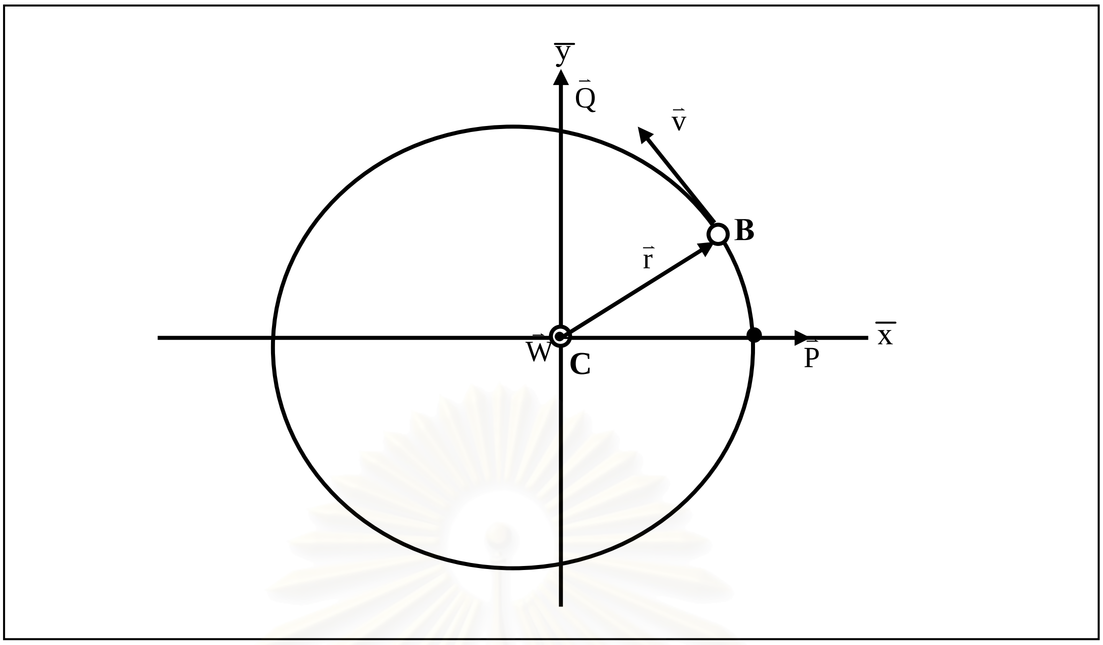


Figure 5.5 The unit vector $\vec{P}$, $\vec{Q}$ and $\vec{W}$ [27].

which determine the eccentric anomaly without ambiguity. The elliptic mean anomaly and mean motion can now be computed in radian measure from

$$M = E - e\sin E, \tag{5.49}$$

$$n = k\sqrt{\frac{\mu}{a^3}}. \tag{5.50}$$

If desired, the period of the elliptic orbit can be found from Equation 4.40.

5.4 Position and Velocity from the Classical Elements

Occasionally, situations arise where it is necessary to convert a set of classical elements into the elements of position and velocity at a given epoch time. This transformation is facilitated by defining a set of mutually perpendicular unit vectors $\vec{\mathbf{P}}, \vec{\mathbf{Q}}$, and $\vec{\mathbf{W}}$ in the orbit-plane coordinate system. As shown in Figure 5.5 $\vec{\mathbf{P}}$ is directed along the $\bar{\mathbf{x}}$-axis toward the perifocus, $\vec{\mathbf{Q}}$ lies along the $\bar{\mathbf{y}}$-axis, and $\vec{\mathbf{W}}$ is perpendicular to the orbit plane. Thus,

$$\vec{W} = \vec{P} \times \vec{Q}. \tag{5.51}$$

therefore, as celestial body **B** orbits dynamical center **C**, its position and velocity may be described in terms of the unit vectors as follows:

$$\begin{aligned} \vec{r} &= \bar{x}\vec{P} + \bar{y}\vec{Q} \\ \vec{v} &= \dot{\bar{x}}\vec{P} + \dot{\bar{y}}\vec{Q}. \end{aligned} \tag{5.52}$$

Now, if we express all the quantities on the right sides of these equations in terms of the classical elements, the vector elements $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ can be computed. We being by transforming the scalar components using expression which are appropriate to the conic section describes the orbit.

5.4.1 The Scalar Components of Elliptic Motion

In the case of elliptic motion, we assume the following set of classical elements:

$$\{a, e, M, i, \Omega, \omega\},$$

where **e** < 1. The mean and eccentric anomalies are related by the elliptic Kepler equation, namely

$$M = E = e \sin E, \tag{5.53}$$

which can be written

$$f = E - e \sin E - M. \tag{5.54}$$

Differentiating **f** with respect to **E**, we obtain

$$\frac{df}{dE} = 1 - e \cos E. \tag{5.55}$$

If we choose **E** = **M** as a first approximation for the eccentric anomaly, Equations 5.54 and 5.55 can be solved for an accurate value of **E** by successive iterations using the Newton-Raphson method. This procedure is straightforward because a given value of the mean anomaly determines a unique value of the eccentric anomaly. When a satisfactory value of **E** has been found, we use Equation 5.15 to compute

$$r = a(1 - e \cos E) \tag{5.56}$$

and rewrite Equation 5.19 to obtain

$$\sqrt{\frac{\mu}{a}} = a(1 - e \cos E)\dot{E}. \tag{5.57}$$

Substituting Equation 5.56 into 5.57, the result can be arranged to yield

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}}. \tag{5.58}$$

We now use Equations 5.11, 5.16, 5.17, and 5.18 to compute the scalar components of position and velocity:

$$\begin{aligned} \bar{x} &= a(\cos E - e) \\ \bar{y} &= b \sin E \\ \dot{\bar{x}} &= -a\dot{E} \sin E \\ \dot{\bar{y}} &= b\dot{E} \cos E, \end{aligned} \tag{5.59}$$

where $b = a\sqrt{1 - e^2}$.

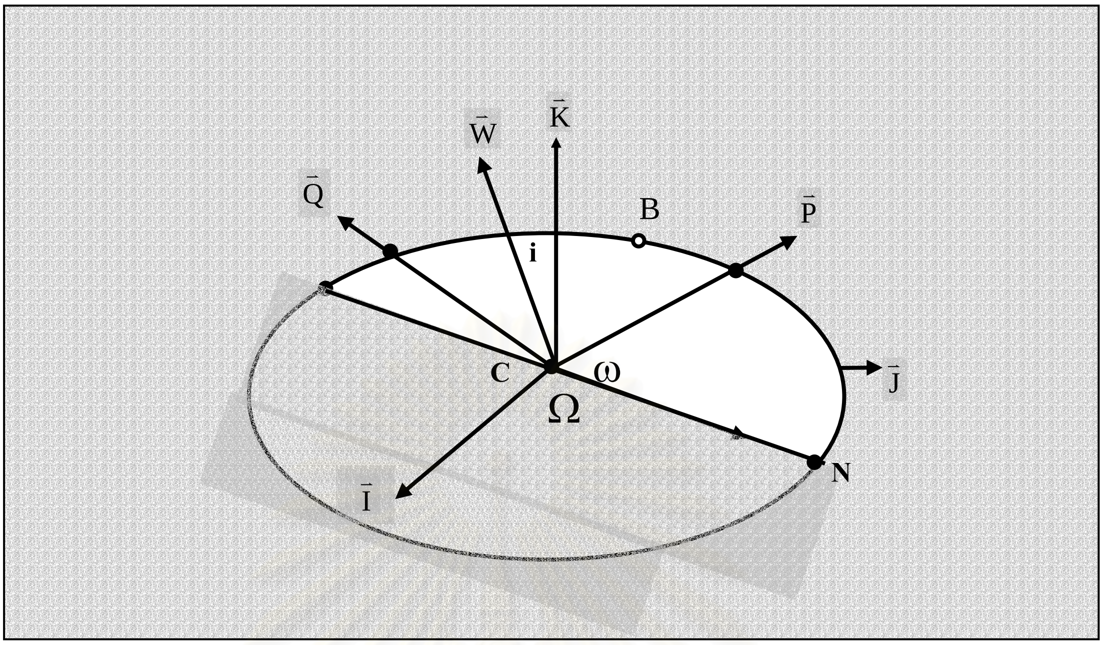


Figure 5.6 The rotation angles Ω , i, and ω [27].

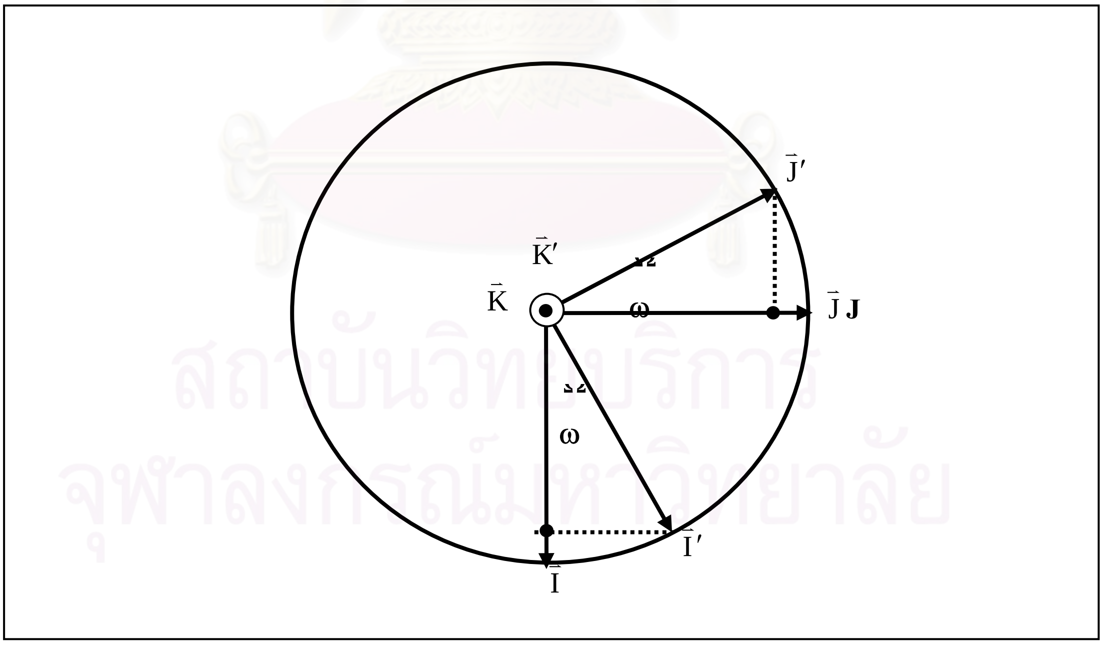


Figure 5.7 Rotation about $\vec{K}$ through angle Ω [27].

5.4.2 The Unit Vector Components of Motion

Now that we have developed equations which express the scalar components of position and velocity in terms of the classical elements [31], all that remains is to accomplish this for the vector components of $\vec{P}$ and $\vec{Q}$. It will then be possible to use Equations 5.52 to compute the vector elements $\vec{r}$ and $\vec{v}$ when the classical elements are given. Consider the geometric relationships depicted in Figure 5.6. The $\{\vec{P},\vec{Q},\vec{W}\}$ unit vector system can be obtained by successive rotations of the $\{\vec{I},\vec{J},\vec{K}\}$ unit vector system through angles $\mathbf{\Omega}$, **i**, and ω . We proceed as follows:

Step 1 Rotate the $\{\vec{I},\vec{J},\vec{K}\}$ system about $\vec{K}$ through the angle $\mathbf{\Omega}$, as shown in Figure 5.7. The result is an $\{\vec{\mathbf{I}}',\vec{\mathbf{J}}',\vec{\mathbf{K}}'\}$ system, where

$$\begin{aligned} \vec{I}' &= +\vec{I}\cos\Omega + \vec{J}\sin\Omega \\ \vec{J}' &= -\vec{I}\sin\Omega + \vec{J}\cos\Omega \\ \vec{K}' &= +\vec{K}. \end{aligned} \qquad \textbf{(5.60)}$$

Step 2 Rotate the $\{\vec{\mathbf{I}}',\vec{\mathbf{J}}',\vec{\mathbf{K}}'\}$ system about $\vec{\mathbf{I}}'$ through the angle **i**, as shown in Figure 5.8. The result is an $\{\vec{\mathbf{I}}'',\vec{\mathbf{J}}'',\vec{\mathbf{K}}''\}$ system, where

$$\begin{aligned} \vec{I}'' &= +\vec{I}' \\ \vec{J}'' &= +\vec{J}'\cos i + \vec{K}'\sin i \\ \vec{K}'' &= -\vec{J}'\sin i + \vec{K}'\cos i\,. \end{aligned} \qquad \textbf{(5.61)}$$

Step 3 Rotate the $\{\vec{\mathbf{I}}'',\vec{\mathbf{J}}'',\vec{\mathbf{K}}''\}$ system about $\vec{\mathbf{K}}''$ through the angle ω , as shown in Figure 5.9. The result is the $\{\vec{\mathbf{P}},\vec{\mathbf{Q}},\vec{\mathbf{W}}\}$ system, where

$$\begin{aligned} \vec{P} &= \vec{I}''\cos\omega + \vec{J}''\sin\omega \\ \vec{Q} &= -\vec{I}''\sin\omega + \vec{J}''\cos\omega \\ \vec{W} &= +\vec{K}''\,. \end{aligned} \qquad \textbf{(5.62)}$$

Step 4 Substitute Equations 5.60 into Equations 5.61 to obtain the following relationships:

$$\begin{aligned} \vec{I}'' &= +\vec{I}\cos\Omega + \vec{J}\sin\Omega \\ \vec{J}'' &= -\vec{I}(\sin\Omega\cos i) + \vec{J}(\cos\Omega\cos i) + \vec{K}\sin i \\ \vec{K}'' &= +\vec{I}(\sin\Omega\sin i) - \vec{J}(\cos\Omega\sin i) + \vec{K}\cos i\,. \end{aligned} \qquad \textbf{(5.63)}$$

Step 5 Substitute Equation 5.63 into Equation 5.62, factor, and rearrange to obtain the final forms of the unit vectors:

$$\begin{aligned} \vec{P} = \ & \vec{I}(+\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i) + \\ & \vec{J}(+\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i) + \\ & \vec{K}(+\sin\omega\sin i) \end{aligned} \qquad \textbf{(5.64)}$$

$$\begin{aligned} \vec{Q} = \ & \vec{I}(-\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i) + \\ & \vec{J}(-\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i) + \\ & \vec{K}(+\cos\omega\sin i) \end{aligned} \qquad \textbf{(5.65)}$$

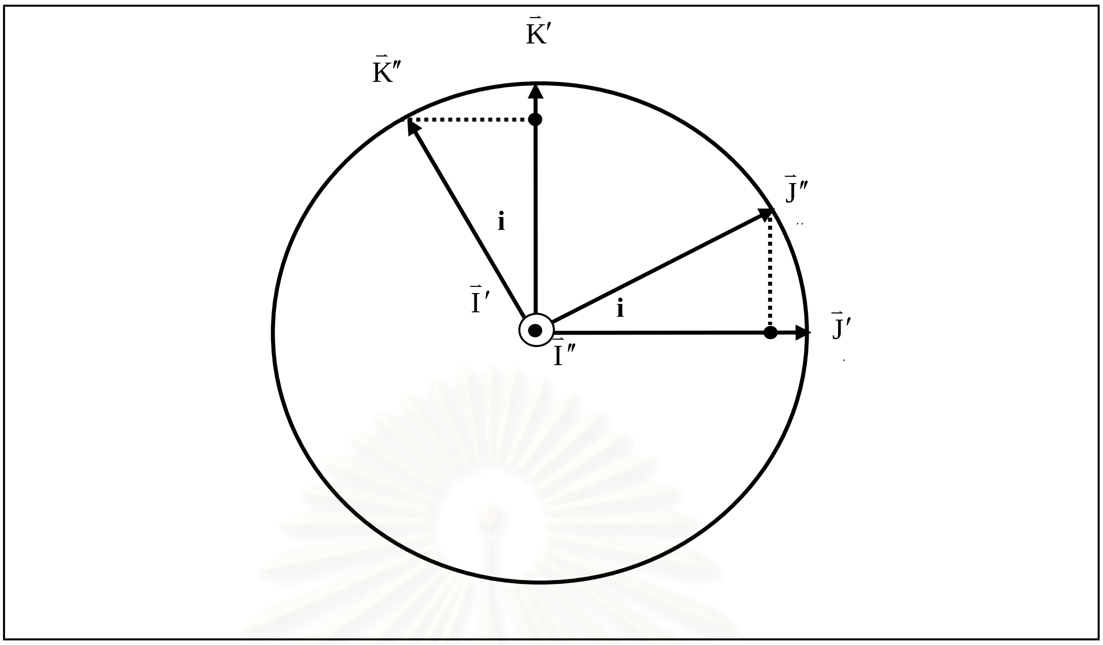


Figure 5.8 Rotation about $\vec{I}'$ through angle i [27].

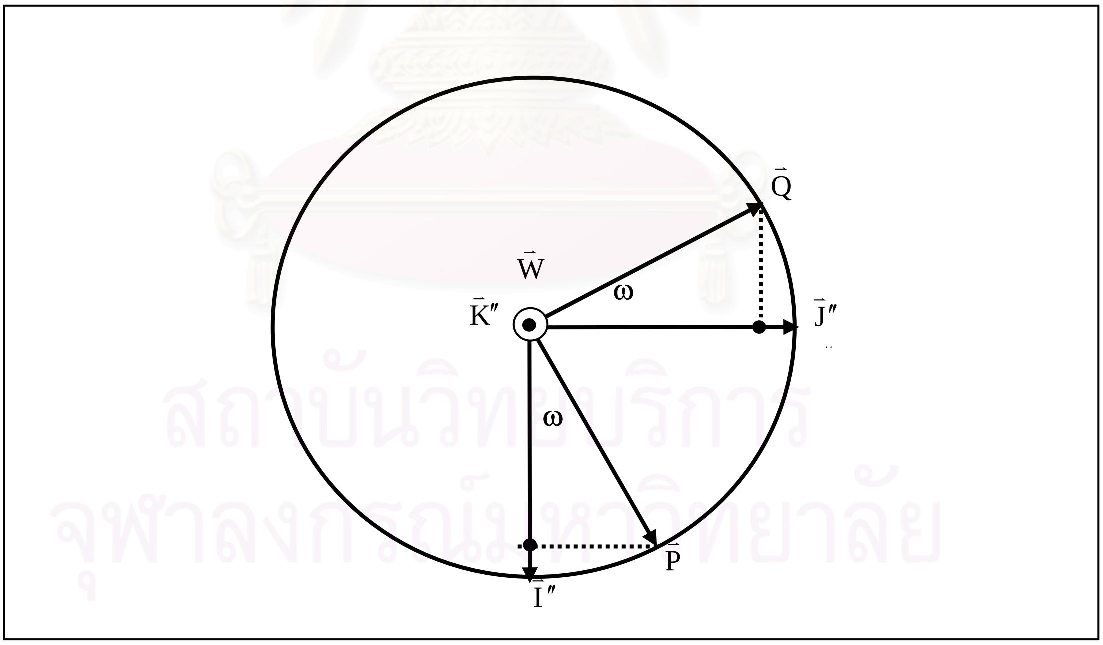


Figure 5.9 Rotation about $\vec{K}''$ through angle ω [27].

$$\vec{W} = \vec{I}(\sin\Omega\sin i) + \vec{J}(-\cos\Omega\sin i) + \vec{K}(+\cos i)\ . \quad \textbf{(5.66)}$$

Equations 5.64, 5.65, and 5.66 express the **x**, **y**, and **z** components of the unit vectors $\vec{\mathbf{P}}$, $\vec{\mathbf{Q}}$, and $\vec{\mathbf{W}}$ in terms of the classical elements. Now, in the case of the heliocentric orbits, **i**, $\mathbf{\Omega}$, and ω are measured with respect to the ecliptic coordinate system, and the resulting values of $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ must be reduced to the equator by the ecliptic-to-equatorial transformation given in Section 3.4.

CHAPTER 6

EPHEMERIS GENERATION

In the previous chapter we derived the elliptic equation of motion which relate position in the orbital plane to the time elapsed from the moment of perifocal passage:

$$M = E - e \sin E$$

However, since the expression is not the most convenient form for our application, we shall modify this equation to obtain the relationship which express motion in terms of time elapsed from an arbitrary epoch. This is done with a view toward using the modified forms with closed **f** and **g** expressions and as a starting point for a universal formulation which is equally applicable to conic sections.

6.1 The Differenced Kepler Equations: Elliptic Formulation

Consider the case of elliptic motion where, at some given epoch time $\mathbf{t_0}$, there exist corresponding values of the eccentric anomaly $\mathbf{E_0}$ and mean anomaly $\mathbf{M_0}$. Then we can write

$$M_0 = E_0 - e \sin E_0 . \qquad \textbf{(6.1)}$$

Subtracting Equation 6.1 from the more general form yields

$$M - M_0 = E - E_0 - e \sin E + e \sin E_0 , \qquad \textbf{(6.2)}$$

where, according to their definitions,

$$M - M_0 = n(t - t_0)$$

$$n = k\sqrt{\frac{\mu}{a^3}} . \qquad \textbf{(6.3)}$$

If we write the **sin E** term in the following form

$$\sin E = \sin (E - E_0 + E_0) ,$$

then we can apply a trigonometric identity to obtain

$$\sin E = \sin (E - E_0) \cos E_0 + \cos (E - E_0) \sin E_0 . \qquad \textbf{(6.4)}$$

Thus, substituting Equation 6.4 into Equation 6.2, the result is

$$\begin{aligned} M - M_0 = {} & (E - E_0) - (e \cos E_0) \sin (E - E_0) - \\ & (e \sin E_0) \cos (E - E_0) + (e \sin E_0) \end{aligned} . \qquad \textbf{(6.5)}$$

Equation 6.5 can be written in a more practical form by equation the three trigonometric terms in parentheses to quantities which are easy to compute when the position and velocity elements are known for the epoch. Equations 5.15 and 5.58 provide the following relationships:

$$r = a(1 - e \cos E) \qquad \textbf{(6.6)}$$

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}} . \qquad \textbf{(6.7)}$$

If we rearrange Equation 6.6, we can write

$$e \cos E = 1 - \frac{r}{a}. \tag{6.8}$$

Therefore, at the epoch time $\mathbf{t_0}$ we also have

$$e \cos E_0 = 1 - \frac{r_0}{a}. \tag{6.9}$$

Differentiating Equation 6.8 with respect to modified time and substituting Equation 6.7 for $\dot{E}$ produces

$$e \sin E = \frac{r\dot{r}}{\sqrt{\mu}}\sqrt{\frac{1}{a}}. \tag{6.10}$$

Define

$$D = \frac{r\dot{r}}{\sqrt{\mu}}. \tag{6.11}$$

Thus, at epoch $\mathbf{t_0}$ Equation 6.10 become

$$e \sin E_0 = D_0\sqrt{\frac{1}{a}}. \tag{6.12}$$

Now, if we define

$$\begin{aligned} C_0 &= e \cos E_0 \\ S_0 &= e \sin E_0 , \end{aligned} \tag{6.13}$$

then, by Equations 6.9 and 6.12, we have

$$C_0 = 1 - \frac{r_0}{a} \tag{6.14}$$

$$S_0 = D_0\sqrt{\frac{1}{a}}. \tag{6.15}$$

Finally, if we let

$$\begin{aligned} W &= M - M_0 \\ G &= E - E_0 , \end{aligned} \tag{6.16}$$

then Equation 6.5 can be written in the simpler form

$$W = G - C_0 \sin G - S_0 \cos G + S_0 . \tag{6.17}$$

6.2 The Closed f and g Expressions

Closed expressions for the **f** and **g** series can be developed for the elliptic motion. Since these close forms do not suffer from series truncation error, they maintain their accuracy when the computed positions are separated by long intervals of time [14].

Consider the situation shown in Figure 6.1, where the position and motion of celestial body **B** are referred to the orbit-plane coordinate system at an epoch time $\mathbf{t_0}$.

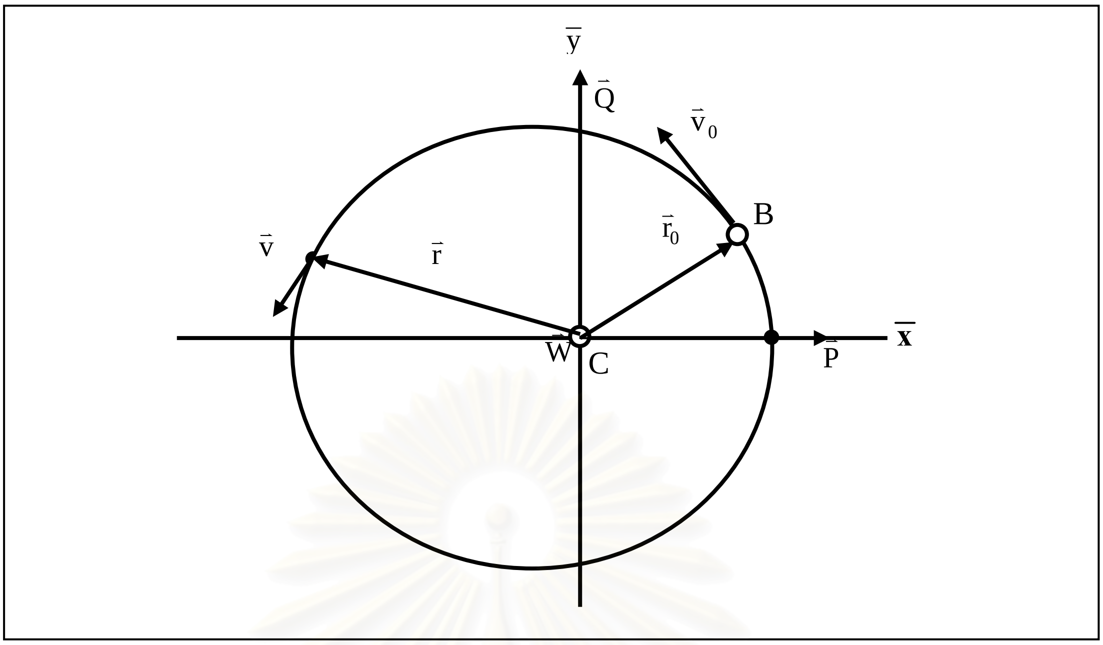


Figure 6.1 Orbital motion during the time interval t-t_0 [27].

Using $\vec{r}_0$ and $\vec{v}_0$ as the orbital elements, the position and velocity of **B** at some other time **t** is described by the expressions

$$\vec{r} = f\vec{r}_0 + g\vec{v}_0 \tag{6.18}$$

$$\vec{v} = \dot{f}\vec{r}_0 + \dot{g}\vec{v}_0 . \tag{6.19}$$

Equation 6.18 can be solved for **f** by taking its vector cross product with $\vec{v}_0$.

Thus,

$$\vec{r} \times \vec{v}_0 = f(\vec{r}_0 \times \vec{v}_0) + g(\vec{v}_0 \times \vec{v}_0) ,$$

which becomes

$$\vec{r} \times \vec{v}_0 = f\vec{h} \tag{6.20}$$

because crossing any vector with itself yields the null vector, and

$$\vec{r}_0 \times \vec{v}_0 = \vec{h}$$

by definition. Now, if $\vec{W}$ is the unit vector perpendicular to the orbit-plane and **h** is the magnitude of the angular momentum vector, then Equation 6.20 can be written

$$(\bar{x}\dot{\bar{y}}_0 - \bar{y}\dot{\bar{x}}_0)\vec{W} = fh\vec{W} .$$

Equation the scalar coefficients of $\vec{W}$, we have the general equation

$$f = \frac{\bar{x}\dot{\bar{y}}_0 - \bar{y}\dot{\bar{x}}_0}{h} . \tag{6.21}$$

The expression for **g** is derived by crossing Equation 6.18 with $\vec{r}_0$. So that by following a process similar to that used to derive the expression for **f**, we obtain the general form

$$g = \frac{\bar{x}_0 \bar{y} - \bar{y}_0 \bar{x}}{h} . \tag{6.22}$$

According to the formulation of orbital motion described in Sections 5.2 and 5.4.1, we have the following relationships:

$$\bar{x} = a(\cos E - e) \tag{6.23}$$

$$\bar{y} = b \sin E \tag{6.24}$$

$$\dot{\bar{x}} = -a\dot{E} \sin E \tag{6.25}$$

$$\dot{\bar{y}} = b\dot{E} \cos E , \tag{6.26}$$

where

$$b = a\sqrt{1 - e^2} \tag{6.27}$$

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}} \tag{6.28}$$

$$r = a(1 - e \cos E) . \tag{6.29}$$

If we substitute Equations 6.23 through 6.26, evaluated at **t** and **t_0**, Into Equations 6.21 and 6.22, the resulting expressions for **f** and **g** can be written

$$\begin{aligned} f &= \frac{ab\dot{E}_0}{h}\left[(\cos E - e)\cos E_0 + \sin E \sin E_0\right] \\ g &= \frac{ab}{h}\left[(\cos E_0 - e)\sin E - \sin E_0(\cos E - e)\right], \end{aligned} \tag{6.30}$$

where, according to Equations 4.28 and 4.31

$$h = \sqrt{\mu a(1 - e^2)} . \tag{6.31}$$

So that,

$$\begin{aligned} f &= \frac{a}{r_0}\left[(\cos E - e)\cos E_0 + \sin E \sin E_0\right] \\ g &= \sqrt{\frac{a^3}{\mu}}\left[(\cos E_0 - e)\sin E - \sin E_0(\cos E - e)\right]. \end{aligned} \tag{6.32}$$

Employing the trigonometric identities [25], we can replace $\sin E$ and $\cos E$ in Equation 6.32 by the expressions

$$\begin{aligned} \sin(G + E_0) &= \sin G \cos E_0 + \cos G \sin E_0 \\ \cos(G + E_0) &= \cos G \cos E_0 - \sin G \sin E_0 , \end{aligned} \tag{6.33}$$

where

$$G = E - E_0 . \tag{6.34}$$

When this substitution is made, Equations 6.32 become

$$f = \frac{a}{r_0}(\cos G - e\cos E_0)$$
$$g = \sqrt{\frac{a^3}{\mu}}\left[\sin G(1 - e\cos E_0) + (1-\cos G)(e\sin E_0)\right]. \quad \textbf{(6.35)}$$

Recalling Equations 6.10 and 6.12, Equations 6.35 can be further reduced to yield

$$f = 1 - \frac{a}{r_0}(1-\cos G)$$
$$g = \sqrt{\frac{1}{\mu}}\left[r_0(\sqrt{a}\,\sin G) + D_0 a(1-\cos G)\right]. \quad \textbf{(6.36)}$$

If we define

$$C = a(1-\cos G)$$
$$S = \sqrt{a}\,\sin G\,, \quad \textbf{(6.37)}$$

then the closed **f** and **g** expressions can be written in the compact form

$$f = 1 - \frac{C}{r_0}$$
$$g = \sqrt{\frac{1}{\mu}}(r_0 S + D_0 C)\,. \quad \textbf{(6.38)}$$

Before proceeding to develop closed expressions for the derivatives of **f** and **g**, we must derive an equation for the magnitude of the radius vector at time identity from Equation 6.33, we can write

$$r = a\left[1 - \cos G(e\cos E_0) + \sin G(e\sin E_0)\right]. \quad \textbf{(6.39)}$$

Substituting Equations 6.10 and 6.12 for the appropriate terms above, the result can be arranged to obtain

$$r = a(1-\cos G) + r_0(\cos G) + D_0(\sqrt{a}\,\sin G)\,. \quad \textbf{(6.40)}$$

Finally, if Equations 6.37 are used to replace the terms in parentheses above, Equation 6.40 can be simplified to

$$r = r_0 + C\left(1 - \frac{r_0}{a}\right) + D_0 S\,. \quad \textbf{(6.41)}$$

The closed expressions for $\dot{f}$ and $\dot{g}$, required by Equation 6.19, are derived by differentiating Equations 6.38 with respect to modified time. Thus, we obtain

$$\dot{f} = -\frac{\dot{C}}{r_0}$$
$$\dot{g} = \sqrt{\frac{1}{\mu}}(r_0\dot{S} + D_0\dot{C}). \quad \textbf{(6.42)}$$

The next step is to find relationships which express $\dot{C}$ and $\dot{S}$ in term of quantities which can be computed from the vector elements. Beginning with the definition

$$G = E - E_0 ,$$

differentiation yields

$$\dot{G} = \dot{E} ,$$

since $\mathbf{E_0}$ is a constant. Therefore, according to Equation 6.28,

$$\dot{G} = \frac{1}{r}\sqrt{\frac{\mu}{a}} . \qquad \textbf{(6.43)}$$

Turning to the definition for **C**, we have

$$\dot{C} = a\dot{G}\sin G ,$$

which, according to Equation 6.43, can be written

$$\dot{C} = \frac{\sqrt{\mu}}{r}\sqrt{a}\,\sin G . \qquad \textbf{(6.44)}$$

Making use of Equations 6.37, we obtain

$$\dot{C} = \frac{\sqrt{\mu}}{r} S . \qquad \textbf{(6.45)}$$

Following a similar process for **S**, we have

$$\dot{S} = \sqrt{a}\dot{G}\cos G$$

which, by Equation 6.43, is

$$\dot{S} = \frac{\sqrt{\mu}}{r}\cos G . \qquad \textbf{(6.46)}$$

Again, employing Equations 6.37, we find that

$$\dot{S} = \frac{\sqrt{\mu}}{r}\left(1 - \frac{C}{a}\right). \qquad \textbf{(6.47)}$$

Now, if Equations 6.45 and 6.47 are substituted for the appropriate terms in expressions for $\dot{\mathbf{f}}$ and $\dot{\mathbf{g}}$, the results can be written

$$\begin{aligned} \dot{f} &= -\frac{\sqrt{\mu}}{rr_0} S \\ \dot{g} &= \frac{1}{r}\left\{\left[r_0 + C\left(1 - \frac{r_0}{a}\right) + D_0 S\right] - C\right\} . \end{aligned} \qquad \textbf{(6.48)}$$

Finally, if we replace the term in square brackets by Equation 6.41, the expression for $\dot{\mathbf{g}}$ reduces to the simple relationship

$$\dot{g} = 1 - \frac{C}{r} . \qquad \textbf{(6.49)}$$

In order to use the closed **f** and **g** expressions to compute elliptic motion over the time interval **t - t$_0$**, we must first solve the differenced Kepler equation for **G** by applying the Newton-Raphson method to the function

$$f(G) = G - C_0 \sin G - S_0 \cos G + S_0 - W \qquad \textbf{(6.50)}$$

and its derivative

$$\frac{df(G)}{dG} = 1 - C_0 \cos G + S_0 \sin G . \qquad \textbf{(6.51)}$$

Once **G** has been used to calculate **C** and **S**, the value of **f**, **g**, $\dot{\mathbf{f}}$, and $\dot{\mathbf{g}}$ are determined by their respective closed expressions, and the position and velocity at **t** are computed from Equations 6.18 and 6.19.

6.3 The Universal Formulation

The closed **f** and **g** expressions are very convenient for computing orbital motion when the value of the eccentricity clearly indicated that the orbit is an ellipse. Unfortunately, the closed elliptic formulations begin to yield inaccurate results as the eccentricity approaches unity in the ambiguous case of a nearly parabolic orbit. Therefore, it is often advantageous to avoid the need to switch formulas when changing from the conic section to another. Fortunately, general expressions have been developed which retain their accuracy for all values of the eccentricity; however, in order to obtain this universal formulation, we shall abandon the closed functions derived in the previous sections and use series expansions [2,28,32].

6.3.1 The Coefficients C, S, and U

According to Equations 6.37, the coefficients **C** and **S** are given by the trigonometric relationships

$$C = a(1 - \cos G) \qquad \textbf{(6.52)}$$

$$S = \sqrt{a} \sin G . \qquad \textbf{(6.53)}$$

These equations can be converted into series expansions by utilizing the following general identities [25]:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \qquad \textbf{(6.54)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \qquad \textbf{(6.55)}$$

Let

$$B_n = \frac{1}{n!} , \qquad \textbf{(6.56)}$$

then Equations 6.52 and 6.53 can be written

$$C = a(B_2 G^2 - B_4 G^4 + B_6 G^6 - B_8 G^8 + ...) \qquad \textbf{(6.57)}$$

$$S = \sqrt{a}(G - B_3 G^3 + B_5 G^5 - B_7 G^7 + ...) . \qquad \textbf{(6.58)}$$

We now make a crucial change of variable which will cause the value of the semimajor axis to appear only in the denominator so that the terms containing $1/a$ will vanish when the semimajor axis becomes infinite. Let

$$X = \sqrt{a}G, \tag{6.59}$$

substituting Equation 6.59 for **G** in Equations 6.57 and 6.58, the result is

$$C = B_2 X^2 - B_4 \frac{X^4}{a} + B_6 \frac{X^6}{a^2} + ... \tag{6.60}$$

$$S = X - B_3 \frac{X^3}{a} + B_5 \frac{X^5}{a^2} - .... \tag{6.61}$$

If we define a new coefficient **U** such that

$$U = B_3 X^3 - B_5 X^5 + ... \tag{6.62}$$

then we may also write

$$S = X - \frac{U}{a}. \tag{6.63}$$

The derivatives of **C** and **U** with respect to **X** will be required for the solution of the universal Kepler equation. When we take advantage of the values of the **B**-coefficients, the expressions for the derivatives can be simplified to yield

$$\frac{dC}{dX} = X - B_3 \frac{X^3}{a} + ... \tag{6.64}$$

$$\frac{dU}{dX} = B_2 X^2 - B_4 \frac{X^4}{a} + ... \tag{6.65}$$

Therefore, comparing the above expressions with Equations 6.60 and 6.61, we can finally write

$$\frac{dC}{dX} = S \tag{6.66}$$

$$\frac{dU}{dX} = C. \tag{6.67}$$

6.3.2 The Equations of Motion

Now that we have series expansion in **X** for the coefficients **C**, **S**, and **U**, we can use them to produce a universal formulation of the **f** and **g** equations of motion. We begin by combining Equations 6.3, 6.16, and 6.17 to produce the following expression for Kepler's equation:

$$n(t - t_0) = G - C_0 \sin G - S_0 \cos G + S_0, \tag{6.68}$$

where

$$n = k\sqrt{\frac{\mu}{a^3}}. \tag{6.69}$$

If we redefine **W** to be

$$W = k\sqrt{\mu}(t - t_0)\ , \tag{6.70}$$

then Equation 6.68 can be written

$$W = a\sqrt{a}G - aC_0\sqrt{a}\ \sin G + \sqrt{a}S_0 a(1 - \cos\ G)\ . \tag{6.71}$$

Substituting Equations 6.52 and 6.53 for the appropriate terms in Equation 6.71, we obtain

$$W = a\sqrt{a}G - aC_0 S + \sqrt{a}S_0 C\ . \tag{6.72}$$

Implementing the same change to the variable **X** introduced previously, the result is

$$W = aX - aC_0 S + \sqrt{a}S_0 C\ . \tag{6.73}$$

Now, when Equations 6.14, 6.15, and 6.63 are substituted into the above, the result can be arranged to yield

$$W = r_0 X + C_0 U + D_0 C\ , \tag{6.74}$$

which is the *universal Kepler's equation*.

The remaining universal equations of motion follow immediately by using the series expansions for **C** and **S** in the **f** and **g** formulation. Thus,

$$f = 1 - \frac{C}{r_0} \tag{6.75}$$

$$g = \sqrt{\frac{1}{\mu}}(r_0 S + D_0 C) \tag{6.76}$$

$$r = r_0 + C_0 C + D_0 S \tag{6.77}$$

$$\dot{f} = -\frac{\sqrt{\mu}}{r r_0} S \tag{6.78}$$

$$\dot{g} = 1 - \frac{C}{r}\ , \tag{6.79}$$

where

$$r_0 = |\vec{r}_0| \tag{6.80}$$

$$D_0 = \frac{\vec{r}_0 \cdot \vec{v}_0}{\sqrt{\mu}} \tag{6.81}$$

$$\frac{1}{a} = \frac{2}{r_0} - \frac{\vec{v}_0 \cdot \vec{v}_0}{\mu} \tag{6.82}$$

$$C_0 = 1 - \frac{r_0}{a}\ . \tag{6.83}$$

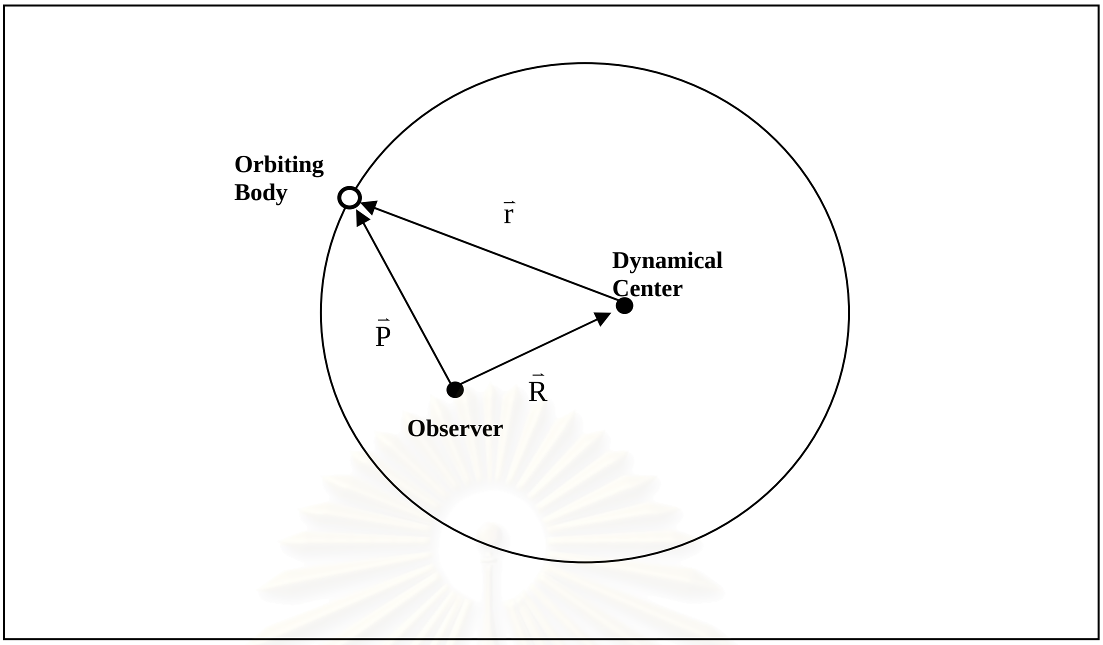


Figure 6.2 The fundamental vector triangle [27].

In order to use the universal **f** and **g** expressions to compute orbital motion over the time interval **t - t₀**, we must first solve the universal Kepler equation for **X** by applying the Newton-Raphson method to the function

$$f(X) = r_0 X + C_0 U + D_0 C - W \quad \textbf{(6.84)}$$

and its derivative

$$\frac{df(X)}{dX} = r_0 + C_0 \frac{dU}{dX} + D_0 \frac{dC}{dX}, \quad \textbf{(6.85)}$$

which can be written as simply

$$\frac{df(X)}{dX} = r_0 + C_0 C + D_0 S \quad \textbf{(6.86)}$$

when Equations 6.66 and 6.67 are substituted for the derivatives of **C** and **U**. Once **X** has been used to calculate **C** and **S**, the values of f, g, $\dot{f}$, and $\dot{g}$ are determined from their universal expressions, and the position and velocity at time **t** are computed from Equations 6.18 and 6.19 [2,27].

6.4 The Ephemeris

Given a celestial body's position and velocity at a particular epoch time, it is not difficult to compute a sequence of right ascension and declination coordinates at a series of other convenient times. Consider the fundamental vector equation:

$$\vec{p} = \vec{r} + \vec{R}. \quad \textbf{(6.87)}$$

As illustrated in Figure 6.2, $\vec{\mathbf{r}}$ is the radius vector from the dynamical center of motion to the orbiting celestial body, $\vec{\mathbf{R}}$ is the vector from the observer to the dynamical center, and $\vec{\mathbf{p}}$ is the vector which defines the position of the orbiting body with respect to the observer. Now, for any given time **t**, the vector $\vec{\mathbf{r}}$ can be computed from the orbital elements by numerical integration, the closed **f** and **g** series, or a universal formulation. Furthermore, we may assume that the vector $\vec{\mathbf{R}}$ is also known because the daily geocentric rectangular coordinates of the Sun tabulated in the *Astronomical Almanac* can be sued for heliocentric orbits. Therefore, $\vec{\mathbf{p}}$ can be determined and used to compute the unit vector $\vec{\mathbf{L}}$ from

$$\vec{L} = \frac{\vec{p}}{|\vec{p}|} . \qquad \textbf{(6.88)}$$

Finally, since

$$\vec{L} = \{\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta\}, \qquad \textbf{(6.89)}$$

the scalar components of $\vec{\mathbf{L}}$ are

$$L_x = \cos\delta\cos\alpha \qquad \textbf{(6.90)}$$

$$L_y = \cos\delta\sin\alpha \qquad \textbf{(6.91)}$$

$$L_z = \sin\delta , \qquad \textbf{(6.92)}$$

and since $|\vec{\mathbf{L}}| = 1$, Equations 3.7 through 3.10 reduce to

$$\sin\delta = L_z \qquad \textbf{(6.93)}$$

$$\cos\delta = \sqrt{1 - L_z^2} \qquad \textbf{(6.94)}$$

$$\cos\alpha = \frac{L_x}{\cos\delta} \qquad \textbf{(6.95)}$$

$$\sin\alpha = \frac{L_y}{\cos\delta} , \qquad \textbf{(6.96)}$$

which permit $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ to be found for time **t**.

In the case of geocentric orbits, we can usually assume that the effects of light-time are negligible. Therefore, the vector $\vec{\mathbf{p}}$ given by Equation 6.87 can used immediately to compute $\vec{\mathbf{L}}$ by Equation 6.88. However, when a heliocentric orbit is being computed, the effects of light-time are normally taken into account because the light which reaches the observer at a given time **t** had the observer to come from the direction of a slightly different orbital position $\vec{\mathbf{r}}_\mathbf{c}$. If we now let $\mathbf{p} = |\vec{\mathbf{p}}|$ represent the distance which the light must travel between the point where it leaves the celestial body and the point where it reaches the observer, then

$$t_c = t_0 - \frac{p}{c} , \qquad \textbf{(6.97)}$$

where **c** = 173.1446 AU/day, so that

$$\frac{1}{c} = 0.005775519 \quad \text{Day/AU} . \qquad \textbf{(6.98)}$$

In practice, since $\vec{\mathbf{r}}_\mathbf{c}$ is initially unknown, an approximate value of **p** is first calculated from Equation 6.87 and used in Equation 6.97 to find $\mathbf{t_c}$. The vector $\vec{\mathbf{r}}_\mathbf{c}$ is then computed for time $\mathbf{t_c}$ and used to find an improved value for $\vec{\mathbf{p}}$ from

$$\vec{p} = \vec{r}_c + \vec{R} \, . \qquad \textbf{(6.99)}$$

Therefore, we now have as before

$$\vec{L} = \frac{\vec{p}}{|\vec{p}|} \, , \qquad \textbf{(6.100)}$$

and Equations 6.90 through 6.96 yield $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ at time **t**.

CHAPTER 7

ASTRONOMICAL PHOTOGRAPHY

This chapter considers the photographic method of determining a star's position. This can, of course, only be a relative position, for it is necessary to assume that the celestial coordinates of some a star on the photographic plate or film are known a priori. The measurements of photographic astrometry must be underpinned by a satisfactory framework of reference stars. In the last section, we shall determine the star's position by comparison stars, the method of dependences, with applicable to obtain the preliminary orbit data in the method of Gauss.

7.1 The Tangent Plane

The central geometrical problem to be considered in astronomical photography is the mapping of the celestial sphere on to the plane surface of a photographic plate or film. To a high degree of approximation, this may be regarded as the simple central projection, shown in Figure 7.1. The diagram illustrates the essential features of an astronomical refractor when used for photography. In Figure 7.1 the celestial sphere, with **C** as center, is drawn. The tangent plane at **A** is drawn, this plane is at right angles to the radius **CA** and is therefore parallel to the photographic plate. It is to be remembered that **A** is the point on the celestial sphere towards which the optical axis of the telescope is directed. Produce **CB** to meet the tangent plane in **D**; then **D** will be called the projection of **B** on the tangent plane. The projection of any other point on the celestial sphere can be constructed in a similar manner by joining the center **C** to the point under consideration and producing the radius to meet the tangent plane. Consider a star **L** whose projection on the tangent plane is **N** and whose image on the photographic plate is at **M**. if $\phi = \mathbf{O\hat{C}M} = \mathbf{A\hat{C}L}$, we have [33,34]

$$\tan\phi = \frac{\mathrm{OM}}{\mathrm{OC}} = \frac{\mathrm{AN}}{\mathrm{AC}}. \qquad \mathbf{(7.1)}$$

It follows generally that the system of stellar images on the plate is similar to the system of the projections on the tangent plane, one system however being on a different linear scale from the other. Let **AR′**, **AS′** be the positive directions of rectangular axes in the tangent plane; let **OR**, **OS** be parallel to **AR′**, **AS′** respectively, in the plane of the plate, their positive directions being opposite to those of **AR′** and **AS′**. Let ξ', η' be the coordinates of the projection of a star on tangent plane and ξ, η the coordinates of the image on the plate; then by the principle of similarity, we have

$$\frac{\xi'}{\mathrm{AC}} = \frac{\xi}{\mathrm{OC}} \qquad \mathbf{(7.2)}$$

and

$$\frac{\eta'}{\mathrm{AC}} = \frac{\eta}{\mathrm{OC}}. \qquad \mathbf{(7.3)}$$

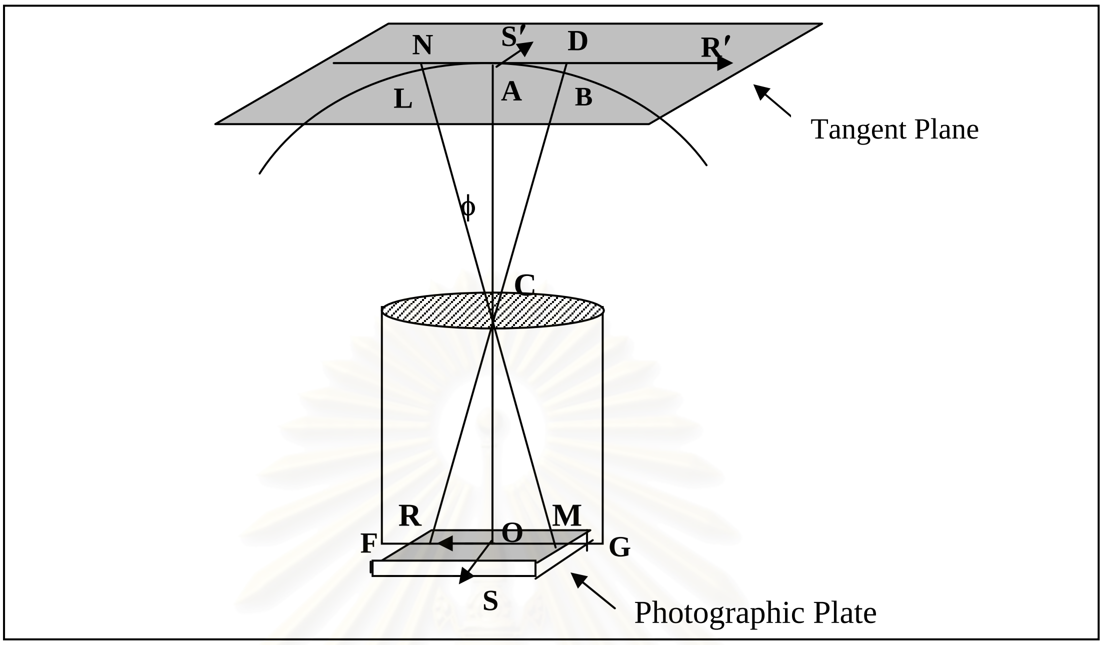


Figure 7.1 Central projection [33].

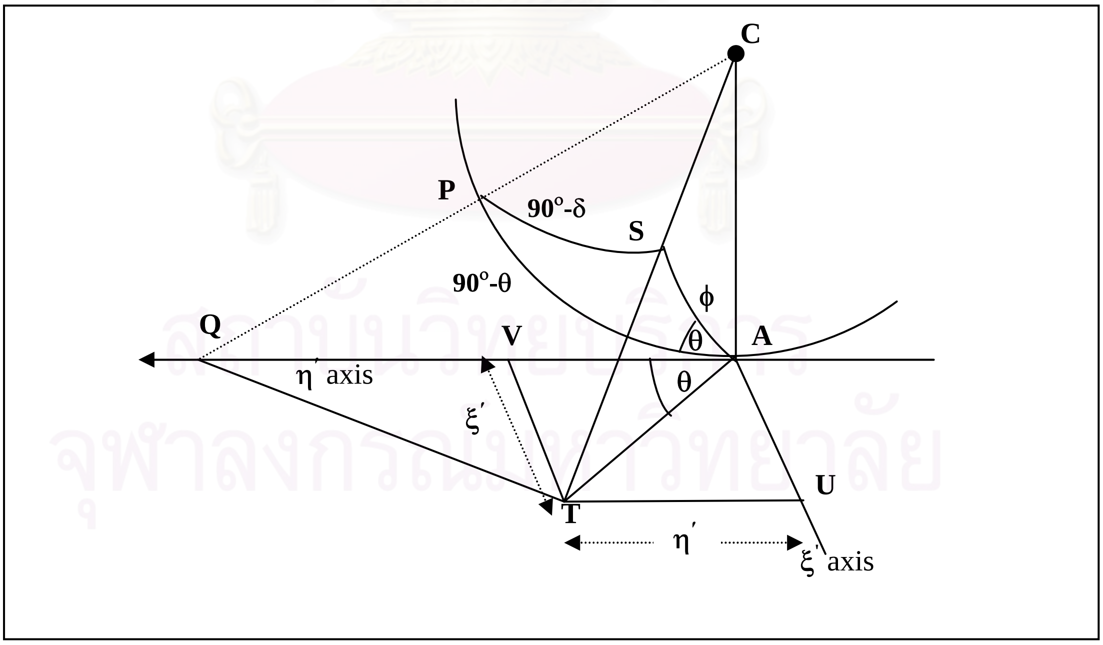


Figure 7.2 The central sphere and the tangent plane [33].

7.2 Standard Coordinates

For the sake of geometrical simplicity we shall suppose the celestial sphere (center **C**) and the tangent plane at **A** to be drawn as in Figure 7.2. It is to be understood that **A** is the point on the celestial sphere towards which the telescope is pointed. If **S** is a star near **A**, its projection **T** on the tangent plane is obtained by joining **C** to S and producing **CS** to meet the tangent plane in **T**. Draw the great circle arc **AS**; then, since the plane of this great circle passes through **C**, it follows that all radii joining **C** to points on **AS** lie in one plane and this plane intersects the tangent plane in a straight line **AT**. More generally, we can say that any great circle projects into a straight line in the tangent plane. Let **P** be the north pole of the celestial sphere. **AS** is the meridian of **A** and it projects into the straight line **AQ**. We shall take **AQ** as the η'- axis of the tangent plane. The ξ'- axis is taken to be **AU**, which is drawn perpendicular to **AQ**, and its positive direction is taken to be eastwards of the meridian **AP** so that increasing values of ξ' correspond to increasing values of the right ascension.

Since **AT** lies in the tangent plane, **AT** is perpendicular to **AC** and is therefore the tangent at **A** to the great circle arc **AS**. Similarly, **AQ** is the tangent to the great circle arc **AP**. Now $\mathbf{Q\hat{A}T}$ defines the angle between any two great circle arcs, intersecting at the tangential point **A**, is exactly reproduced on the tangent plane as the angle between the two straight lines into which the great circles project. This remark holds only for great circles passing through the tangential point; for example, the great circle **SP** projects into the straight line **TQ** and **AP** projects into **AQ**; but $\mathbf{A\hat{Q}T}$ (the angle between **AQ** and **TQ**) is not equal to $\mathbf{S\hat{P}A}$ (the angle between the great circles **AP** and **SP**).

Denote the arc **AS** by ϕ and $\mathbf{S\hat{A}P}$ by θ; then $\mathbf{Q\hat{A}T = \theta}$. Draw perpendiculars **TU**, **TV** to **AU**, **AQ** respectively. Then

$$VT = \xi' = AT\sin\theta \qquad \textbf{(7.4)}$$

$$UT = \eta' = AT\cos\theta\ . \qquad \textbf{(7.5)}$$

Now

$$AT = AC\tan A\hat{C}T = AC\tan\phi\ .$$

Hence

$$\frac{\xi'}{AC} = \tan\phi\sin\theta\ , \qquad \textbf{(7.6)}$$

$$\frac{\eta'}{AC} = \tan\phi\cos\theta\ . \qquad \textbf{(7.7)}$$

Hence, by Equations 7.2 and 7.3,

$$\frac{\xi}{OC} = \tan\phi\sin\theta \qquad \textbf{(7.8)}$$

$$\frac{\eta}{OC} = \tan\phi\cos\theta\ , \qquad \textbf{(7.9)}$$

in which ξ and η are the coordinates of the image of **S** on the photographic plate with reference to rectangular axes through the center **O** of the plate (Figure 7.1), and drawn parallel, but oppositely directed, to the axes **AU**, **AQ** on the tangent plane. **OC** is the focal length of the telescope. Suppose that the focal length is known in millimeters and that the plate coordinates ξ and η are derived also in millimeters by processes which will be described later; then the values of ϕ and θ can be calculated from Equations 7.8 and 7.9. As we shall see immediately, ϕ and θ are functions of the right ascension and declination of **A** are known, the right ascension and declination of **S** can then be deduced from the values of the coordinates ξ and η.

If we take the focal length **OC** to be the unit of length and ξ and η to be expressed in terms of this unit, we have from Equations 7.8 and 7.9,

$$\xi = \tan\phi \sin\theta\,, \qquad \textbf{(7.10)}$$

$$\eta = \tan\phi \cos\theta\,. \qquad \textbf{(7.11)}$$

ξ and η are then called *the standard coordinates* of the star concerned. In this definition of standard coordinates [35], the following points have to be noted: (**i**) the origin of the coordinate axes corresponds to a definite position, whose right ascension and declination are specified with respect to a standard mean equinox; the epoch of this mean equinox is chosen to be 2000.0; (**ii**) the ξ and η axes are correctly oriented for the epoch 2000.0; (**iii**) the definition, being a purely geometrical one, excludes the effects of instrumental imperfections and of refraction and aberration (all of which will be considered later). The standard coordinates of a particular star thus specify the position of the star uniquely, and can therefore be used in place of right ascension and declination.

7.3 Formula for the Standard Coordinates

Let **A**, **D** be the right ascension and declination (referred to 2000.0) of the point **A** on the celestial sphere, and α, δ the corresponding coordinates of the star **S**. We shall now show how the relations between ξ, η and **A**, **D**, α and δ are obtained. In the spherical triangle **ASP** (Figure 7.2) we have: **AP** = 90°-**D**, **SP** = 90°-δ, $A\hat{P}S = \alpha - A$ (in the figure, **S** is eastwards of the meridian **AP**), **AS** = ϕ, $S\hat{A}P = \theta$. Then

$$\cos\phi = \sin\delta\sin D + \cos\delta\cos D\cos(\alpha - A), \qquad \textbf{(7.12)}$$

$$\sin\phi\sin\theta = \cos\delta\sin(\alpha - A), \qquad \textbf{(7.13)}$$

$$\sin\phi\cos\theta = \sin\delta\cos D - \cos\delta\sin D\cos(\alpha - A). \qquad \textbf{(7.14)}$$

Dividing Equation 7.14 by 7.12 we obtain, using Equation 7.11,

$$\eta = \frac{\cos D - \cot\delta\sin D\cos(\alpha - A)}{\sin D + \cot\delta\cos D\cos(\alpha - A)}. \qquad \textbf{(7.15)}$$

Define **q** as follows:

$$\cot q = \cot\delta\cos(\alpha - A). \qquad \textbf{(7.16)}$$

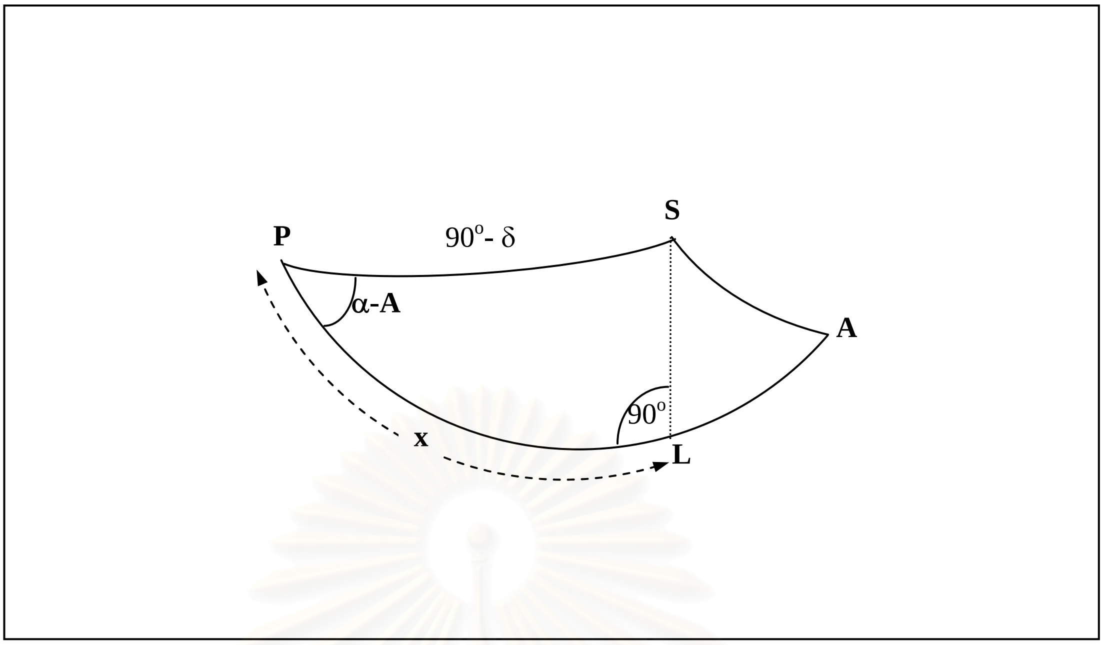


Figure 7.3 The spherical triangle [33].

Then

$$\eta = \frac{\cos D - \sin D \cot q}{\sin D + \cos D \cot q},$$

from which

$$\eta = \tan (q - D). \qquad \textbf{(7.17)}$$

Again, dividing Equation 7.13 by 7.12 and using Equation 7.10, we have

$$\begin{aligned} \xi &= \frac{\cot \delta \sin (\alpha - A)}{\sin D + \cos D \cot \delta \cos (\alpha - A)} \\ &= \frac{\cot q \tan (\alpha - A)}{\sin D + \cos D \cot q} \end{aligned} \qquad \textbf{(7.18)}$$

by Equation 7.16, so that

$$\xi = \frac{\cot q \tan (\alpha - A)}{\cos (q - D)}. \qquad \textbf{(7.19)}$$

The auxiliary quantity **q**, which has been introduced into Equation 7.17 and 7.19 in order to simplify the logarithmic computations of ξ and $\boldsymbol{\eta}$ in the case when **A**, **D**, $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ are all known, is readily seen to have a simple geometrical interpretation. In Figure 7.3, draw a great circle arc **SL** to cut **AP** at right angle in **L**. Denote **PL** by **x**; then

$$\cos x \cos (\alpha - A) = \sin x \tan \delta - \sin (\alpha - A) \cot 90^{\circ},$$

from which

$$\tan x = \cot \delta \cos (\alpha - A).$$

Hence, by Equation 7.16, **x** = 90°- **q**, so that **q** is the declination of **L**. Equations 7.17 and 7.19 enable the calculation of ξ and $\boldsymbol{\eta}$ to be made when **A**, **D**, ξ and $\boldsymbol{\eta}$. We have from Equation 7.15,

$$\eta \sin D + \eta \cot \delta \cos D \cos (\alpha - A)$$
$$= \cos D - \cot \delta \sin D \cos (\alpha - A),$$

from which

$$\cot \delta \cos (\alpha - A)\{\eta \cos D + \sin D\} = \cos D - \eta \sin D\,,$$

and hence,

$$\cot \delta \cos (\alpha - A) = \frac{1 - \eta \tan D}{\eta + \tan D}\,. \qquad \textbf{(7.20)}$$

Again, from Equation 7.18,

$$\cot \delta \sin (\alpha - A) = \xi \sin D + \xi \cos D \cot \delta \cos (\alpha - A)$$
$$= \xi \left\{ \sin D + \frac{\cos D(1 - \eta \tan D)}{\eta + \tan D} \right\},$$

by means of Equation 7.20, whence we derive

$$\cot \delta \sin (\alpha - A) = \frac{\xi \sec D}{\eta + \tan D}\,. \qquad \textbf{(7.21)}$$

Divide Equation 7.21 by 6.20; then

$$\tan (\alpha - A) = \frac{\xi \sec D}{1 - \eta \tan D}\,, \qquad \textbf{(7.22)}$$

from which $\boldsymbol{\alpha - A}$ can be calculated and $\boldsymbol{\alpha}$ obtained. When $\boldsymbol{\alpha - A}$ has been found, $\boldsymbol{\delta}$ can be obtained from Equation 7.20 or 7.21.

In astronomical photography there are two fundamental processes employed directly or indirectly [33]. The first is the calculation of the standard coordinates of one or more stars whose right ascensions and declinations are known; this process involves the use of Equations 7.17 and 7.19. The second is the calculation of right ascensions and declinations of stars from the value of their standard coordinates; this process is carried out by means of Equations 7.22 and 7.20 or 7.21.

7.4 The Measured Coordinates

In defining the standard coordinates of a star we have assumed (**a**) that the optical axis of the telescope passes through the origin of coordinates on the plate, (**b**) that the plate is perpendicular to the optical axis, (**c**) that the η-axis corresponds precisely to the projection of the central meridian, for the epoch 2000.0, on the tangent plane, (**d**) that the ξ-axis is perpendicular to the $\boldsymbol{\eta}$-axis. In practice it is impossible to attain the geometrical perfection just indicated, and consequently the coordinates of a star-image, measured with reference to the axes on the plane, must be expected to differ (generally slightly) from the theoretical standard coordinates. But this is not all. Hitherto, in referring to star-images, we have ignored the effects of refraction and aberration.

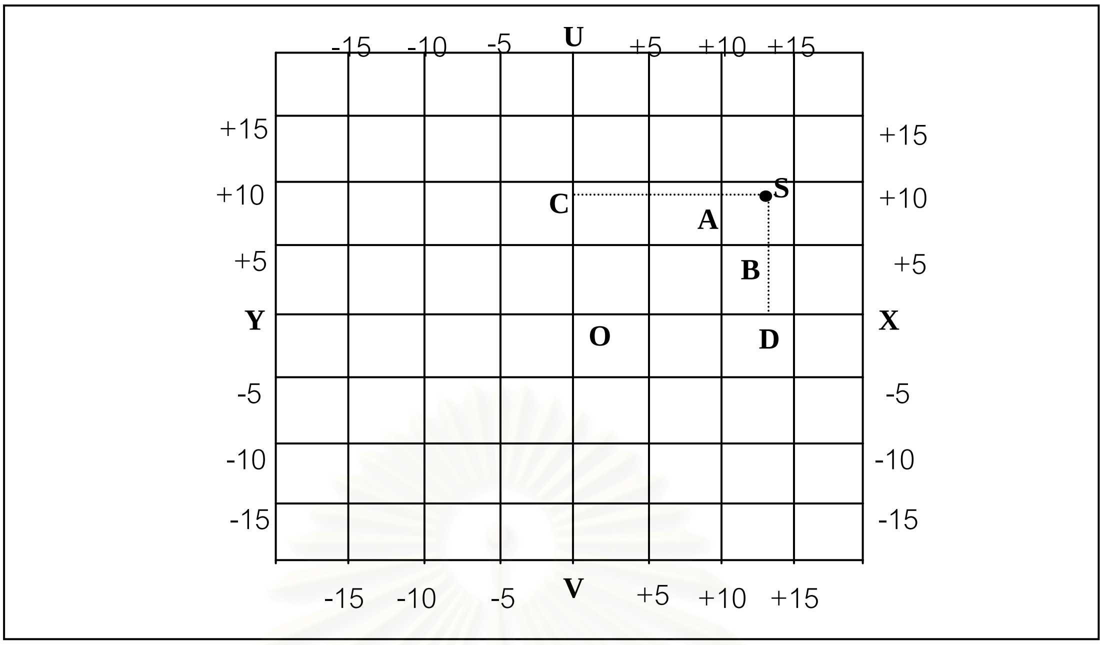


Figure 7.4 The photographic plate measurement [33].

We have seen that owing to refraction and aberration the apparent position of a star on the celestial sphere is displaced by measurable or calculable amounts from its true position; consequently the actual image of a star on the photographic plate will be somewhat displaced from the position it would occupy were these effects inoperative. It will thus be realized that standard coordinates are ideal coordinates, whereas the measured coordinates of a star-image include the effects of geometrical (or mechanical) imperfections and the effects of refraction and aberration. At first sight the problem of deriving the standard coordinates of a star from the measured coordinates of its image on the photographic plate seems one of great difficulty; actually, as we shall see, the solution in practice is extremely simple. We shall now examine in detail the differences between the standard and measured coordinates.

7.5 The Measurement and Scale of Photographic Plates

For convenience we shall consider the plates taken with astrographic telescopes. These instruments, constructed according to a standard design, are in use in about a score of observatories scattered over the globe, the work to which they have been principally devoted being a complete photographic survey of the heavens. In this enterprise the cooperating observatories had definite zones (between certain parallels of declination) assigned to them; several observatories have finished their share of the work, but under existing world-conditions it is uncertain when the survey will be complete.

On each astrographic plate, a network system of parallel lines as shown in Figure 7.4 is photographed, either before or after the plate is exposed to the stars, so that on development the plate shows the stellar images and the *reseau* system of lines. The lines are equally spaced at intervals of five millimeters. We shall suppose that the central lines **XOY** and **UOV** correspond exactly to the ξ-axis and η-axis already defined.Consider a star-image at **S**. The distance **AS**, parallel to **OX**, is measured by a machine and we shall suppose that **AS** = 4.14 mm. The distance of **S** from the axis

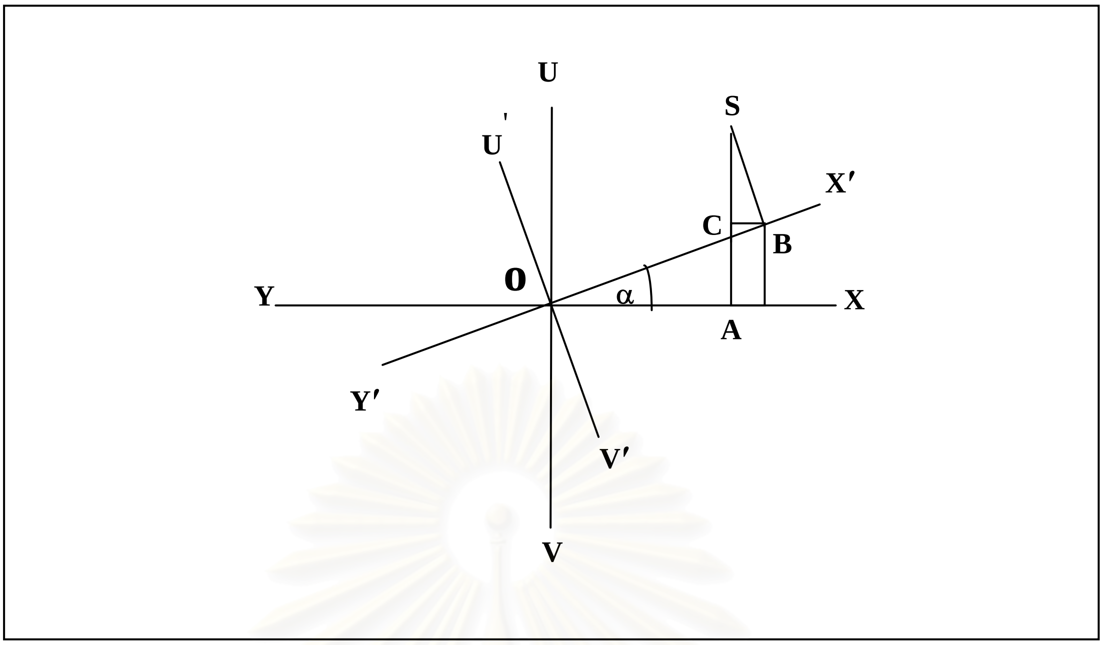


Figure 7.5 Orientation of axes [33].

UOV (that is to say **OC**) is thus 10+4.14 or 14.14 mm. In a similar way, BS is measured and the distance **OD** obtained.

7.6 Discussion of Errors

We shall consider the errors individually. We shall denote by ξ and η the true standard coordinates of a star and by **x** and **y** the coordinates as influenced by the particular error concerned [33].

7.6.1 Error of Orientation

In Figure 7.5 let **XOY** and **UOV** be the axes of coordinates correctly centered and oriented for the epoch 2000.0; let **X′OY′** and **U′OV′** be the axes on the plate correctly centered but erroneously oriented. Let $X'\hat{O}X = \alpha$. Let **S** be a point whose standard coordinates are ξ and η, referred to **OX** and **OY**, and whose coordinates referred to **OX′** and **OY′** are **x** and **y**. Draw perpendiculars **SA**, **SB** to **OX**, **OX′** respectively. Then **OA** = ξ, **AS** = η, **OB** = **x** and **BS** = **y**. We have

$$OA = OB\cos\alpha + BS\cos(90^\circ + \alpha)\,,$$

or

$$\xi = x\cos\alpha - y\sin\alpha\,,$$

whence

$$\xi - x = -2x\sin^2\frac{\alpha}{2} - y\sin\alpha\,.$$

Similarly,

$$\eta - y = x \sin \alpha - 2y \sin^2 \frac{\alpha}{2}.$$

We write these in the form:

$$\left.\begin{aligned} \xi - x &= a_1 x + b_1 y \\ \eta - y &= d_1 x + e_1 y \end{aligned}\right\} \qquad \textbf{(7.23)}$$

where $\mathbf{a_1}$, …., $\mathbf{e_1}$ are simple functions of α. In practice α is always a small angle and in consequence the coefficients $\mathbf{a_1}$,…., $\mathbf{e_1}$ are also small. The formula 7.23 are essentially linear in **x** and **y**.

7.6.2 Centering Error

Suppose firstly that during an exposure the direction of the optical axis corresponds to a given direction (**A**, **D**) referred to the mean equinox of 2000.0, and that ξ and η are the standard coordinates of a stars ($\boldsymbol{\alpha}$, $\boldsymbol{\delta}$) with reference to the position (**A**, **D**) as center. It is hardly to be expected that the straight line joining the center of the object-glass to the origin of the impressed the reseau coordinate axes will coincide exactly with the optical axis, and consequently the origin will correspond to slightly different values of **A** and **D**. Secondly, the optical axis may not be directed quite accurately to the position (**A**, **D**) for 2000.0. As a result, we must therefore assume that the origin of coordinates on the plate corresponds to a position (**A**+Δ **A**, **D**+Δ **D**), where Δ **A** and Δ **D** may be supposed to be small quantities.

All other errors and influences being assumed absent, let **x** and **y** be the coordinates of an image with respect to the reseau axes. Then **x** and **y** may be taken to be the standard coordinates of the star concerned with reference to the position (**A**+Δ **A**, **D**+Δ **D**) as center. We shall denote **x**-ξ and **y**-$\boldsymbol{\eta}$ by $\Delta\xi$ and $\Delta\boldsymbol{\eta}$ respectively.

Now, by Equations 7.10 and 7.11,

$$\xi = \tan\phi \sin\theta, \; \eta = \tan\phi \cos\theta,$$

where ϕ, $\boldsymbol{\theta}$ are functions of **A** and **D**. Corresponding to increments $\mathbf{\Delta A}$ and $\mathbf{\Delta D}$, we shall have increments $\Delta\phi$, $\boldsymbol{\Delta\theta}$. Hence we have

$$\Delta\xi = \Delta\phi(1 + \tan^2\phi) \sin\theta + \Delta\theta \tan\phi \cos\theta,$$

$$\Delta\eta = \Delta\phi(1 + \tan^2\phi) \cos\theta - \Delta\theta \tan\phi \sin\theta.$$

For a star at an angular distance of 1° from (**A**, **D**), $\tan\phi = 1/57$, and in the above formula we can neglect such terms a have factors $\Delta\phi\,\mathbf{tan}^2\,\phi$. We thus have

$$\left.\begin{aligned} \Delta\xi &= \Delta\phi \sin\theta + \eta\Delta\theta \\ \Delta\eta &= \Delta\phi \cos\theta - \xi\Delta\theta. \end{aligned}\right\} \qquad \textbf{(7.24)}$$

We have now to express $\Delta\phi$ and $\boldsymbol{\Delta\theta}$ in terms of $\mathbf{\Delta A}$ and $\mathbf{\Delta D}$.

From Equation 7.12 we have

$$-\sin\phi\Delta\phi = \Delta D\{\sin\delta\cos D - \cos\delta\sin D\cos(\alpha - A)\}$$
$$+\Delta A\cos\delta\cos D\sin(\alpha - a),$$

and, using Equations 7.13 and 7.14, this becomes

$$\Delta\phi = -\Delta D\cos\theta - \Delta A\cos D\sin\theta. \qquad \textbf{(7.25)}$$

From Equation 7.14 we obtain

$$\Delta\theta\sin\phi\cos\theta + \Delta\phi\cos\phi\sin\theta = -\Delta A\cos\delta\cos(\alpha - A),$$

or, using Equation 7.25

$$\Delta\theta\sin\phi\cos\theta = \cos\phi\sin\theta(\Delta D\cos\theta + \Delta A\cos D\sin\theta)$$
$$-\Delta A\cos\delta\cos(\alpha - A).$$

Multiply Equation 7.12 by $\cos D$ and Equation 7.14 by $\sin D$ and subtract. Then

$$\cos\delta\cos(\alpha - A) = \cos\phi\cos D - \sin\phi\cos\theta\sin D.$$

Hence

$$\Delta\theta\sin\phi\cos\theta = \Delta D\cos\phi\sin\theta\cos\theta$$
$$+\Delta A\{\cos\phi\sin^2\theta\cos D - \cos\phi\cos D + \sin\phi\cos\theta\sin D\},$$

from which

$$\Delta\theta\sin\phi = \Delta D\cos\phi\sin\theta + \Delta A(\sin\phi\sin D - \cos\phi\cos\theta\cos D).$$

Multiplying this last equation by $\sec\phi\cos\theta$, we have

$$\eta\Delta\theta = \Delta D\sin\theta\cos\theta + \Delta A(\eta\sin D - \cos^2\theta\cos D).$$

Inserting this expression for $\eta\Delta\theta$ and expression for $\Delta\phi$ given by Equation 7.25 in the first of Equations 7.24, we obtain

$$\Delta\xi = -\Delta A\cos D + \eta\Delta A\sin D,$$

or

$$\xi - x = \Delta A\cos D - \eta\Delta A\sin D.$$

As (ξ -**x**) is of order ΔA, we can write this last equation with sufficient accuracy as

$$\left.\begin{array}{l}\xi - x = \Delta A\cos D - y(\Delta A\sin D)\\ \eta - y = \Delta D + x(\Delta A\sin D)\end{array}\right\}. \qquad \textbf{(7.26)}$$

These formulas have the linear forms:

$$\xi - x = \qquad b_2 y + c_2,$$
$$\eta - y = d_2 x \qquad + f_2.$$

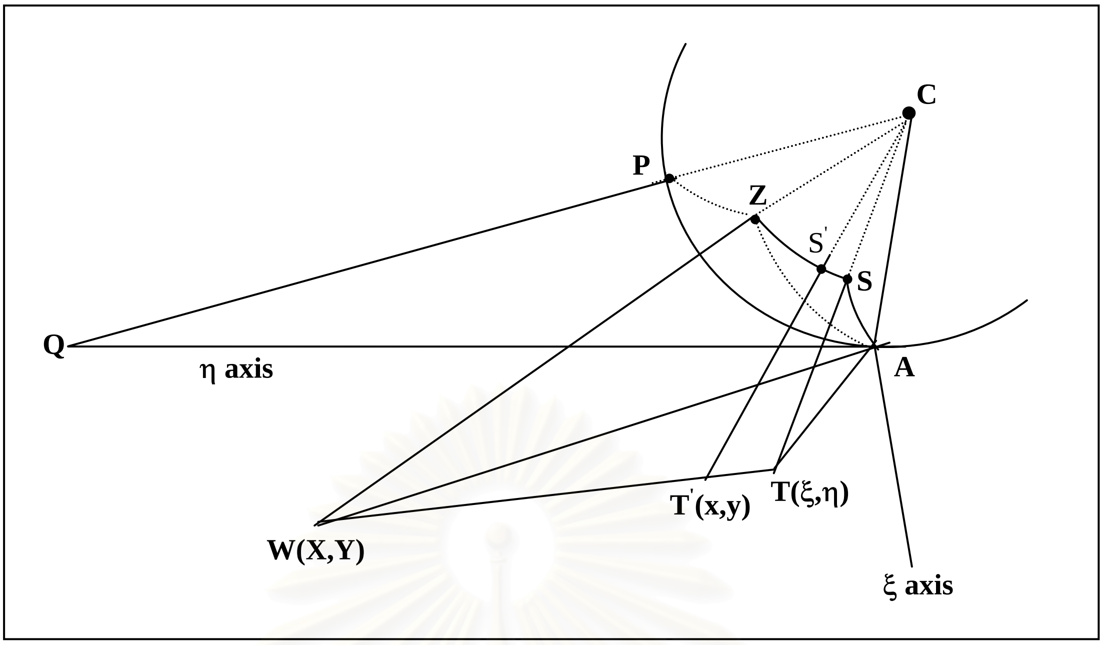


Figure 7.6 The stellar images due to refraction [33].

7.6.3 Error of Tilt

This error is due to the non-perpendicularity of the optical axis to the plane of the plate [36]. If **i** is the angle between the optical axis and normal to the plate, the expressions for **ξ - x**, **η - y** are of the form

$$\xi - x = \tan i(px^2 + qxy)\,,$$
$$\eta - y = \tan i(pxy + qy^2)\,.$$

As the angle **i** is in practice only a few minutes of arc and as the squares and products of x and y only are involve, the correction for tilt can generally be neglected.

The total effect of the various errors considered here is to give the displacements (**ξ - x**) and (**η - y**) in terms of essentially linear expressions; we can thus write the general formulas

$$\left.\begin{aligned} \xi - x &= ax + by + c \\ \eta - y &= dx + ey + f \end{aligned}\right\} \quad \textbf{(7.27)}$$

in which **a**, **b**, …, **f** are small quantities depending on the small errors involved.

7.7 Refraction

We now investigate the displacements of the stellar images due only to refraction. In Figure 7.6, the tangent plane to the celestial sphere at **A** is drawn as in Figure 7.2. **Z** is the zenith and **W** is projection on the tangent plane; let the coordinates of **W** be **X**, **Y**. Owing to refraction, a star **S** is seen at **S′**, the displacement **SS′** being along the great circle arc joining S to **Z**. We have[23]

$$SS' = k \tan ZS\,, \quad \textbf{(7.28)}$$

in which **k** is expressed in circular measure and **ZS** is written, without sensible loss of accuracy, in place of the observed zenith distance **ZS′**.

The great circle **ZS′S** projects into the straight line **WT′T** on the tangent plane, **T** and **T′** being the projections of **S** and **S′** respectively. Let ξ, $\boldsymbol{\eta}$ be the coordinates of **T** and **x**, **y** the coordinates of **T′**.

If ξ, $\boldsymbol{\eta}$ and **x**, **y** are expressed in terms of **AC** as unit of length, these quantities are the standard coordinates of the star and the measured coordinates of its image on the plate respectively. Since the region to be photographed is generally no more than 2°x2°, the different points on that part of the spherical surface concerned are actually very close to the corresponding projected points on the tangent plane; we accordingly assume that **SS′** = **TT′**. From Equation 7.28 we have

$$TT' = k \tan ZS . \qquad \textbf{(7.29)}$$

Since **T**, **T′** and **W** are collinear, we have

$$\frac{x-\xi}{TT'} = \frac{X-\xi}{TW} \text{ and } \frac{y-\eta}{TT'} = \frac{Y-\eta}{TW} ,$$

and writing $\boldsymbol{\Delta\xi}$ for (ξ - **x**) and $\boldsymbol{\Delta\eta}$ for ($\boldsymbol{\eta}$ - **y**) we obtain

$$\Delta\xi = -\frac{k(X-\xi)}{TW} \tan ZS , \qquad \textbf{(7.30)}$$

$$\Delta\eta = -\frac{k(Y-\eta)}{TW} \tan ZS . \qquad \textbf{(7.31)}$$

We remind the reader that the different coordinates ξ, $\boldsymbol{\eta}$, **x**, **y**, **X**, **Y** are all supposed to be expressed in terms of **AC** as the unit of length. In particular, ξ and $\boldsymbol{\eta}$ will thus be small quantities and in the sequel we shall neglect the much smaller quantities ξ^2, $\xi\boldsymbol{\eta}$, $\boldsymbol{\eta}^2$ and higher power and products of ξ and $\boldsymbol{\eta}$. Now from the plane triangle **TAW**, we have

$$TW^2 = AT^2 + AW^2 - 2AT \cdot AW \cos T\hat{A}W ,$$

so that

$$(X-\xi)^2 + (Y-\eta)^2 = (\xi^2 + \eta^2) + (X^2 + Y^2) - 2AT \cdot AW \cos T\hat{A}W ,$$

which gives us

$$AT \cdot AW \cos T\hat{A}W = X\xi + Y\eta . \qquad \textbf{(7.32)}$$

From the spherical triangle **ZAS**, we have by the cosine-formula

$$\cos ZS = \cos AS \cos AZ + \sin AS \sin AZ \cos ZAS$$
$$= \frac{AC}{CT} \cdot \frac{AC}{CW} + \frac{AT}{CT} \cdot \frac{AW}{CW} \cos ZAS .$$

Since $\mathbf{T\hat{A}W}$ defines the spherical angle **ZAS**, we have from Equation 7.32, putting **AC**=1,

$$\cos ZS = \frac{1 + X\xi + Y\eta}{CT \cdot CW} .$$

But, when we neglect ξ^2 and η^2,

$$\begin{aligned} CT^2 &= 1 + \xi^2 + \eta^2 \\ &= 1 . \end{aligned}$$

Hence

$$\cos ZS = \frac{1 + X\xi + Y\eta}{CW} , \qquad \textbf{(7.33)}$$

from which

$$\begin{aligned} \sin^2 ZS &= \frac{CW^2 - (1 + X\xi + Y\eta)^2}{CW^2} \\ &= \frac{(CW^2 - 1) - 2(X\xi + Y\eta)}{CW^2} \\ &= \frac{AW^2 - 2(X\xi + Y\eta)}{CW^2} . \end{aligned}$$

Hence, using the binomial theorem, and neglecting ξ^2, etc., we obtain

$$\sin ZS = \frac{AW}{CW}\left(1 - \frac{X\xi + Y\eta}{AW^2}\right). \qquad \textbf{(7.34)}$$

From Equations 7.33 and 7.34,

$$\tan ZS = \frac{AW\left(1 - \frac{X\xi + Y\eta}{AW^2}\right)}{1 + X\xi + Y\eta} . \qquad \textbf{(7.35)}$$

Now

$$\begin{aligned} TW^2 &= (X - \xi)^2 + (Y - \eta)^2 \\ &= X^2 + Y^2 - 2(X\xi + Y\eta) \\ &= AW^2 - 2(X\xi + Y\eta) . \end{aligned}$$

So that

$$TW = AW\left(1 - \frac{X\xi + Y\eta}{AW^2}\right). \qquad \textbf{(7.36)}$$

Hence from Equations 7.35 and 7.36,

$$\begin{aligned} \tan ZS &= TW(1 + X\xi + Y\eta)^{-1} \\ &= TW(1 - X\xi - Y\eta) . \end{aligned}$$

We thus obtain from Equation 7.30,

$$\Delta\xi = -k(X - \xi)(1 - X\xi - Y\eta) ,$$

or

$$\Delta\xi = -kX + k\left\{(1 + X^2)\xi + XY\eta\right\}. \qquad \textbf{(7.37)}$$

Since (ξ - **x**) and **k** (expressed in circular measure) are both small, we can write **x** and **y** for ξ and $\boldsymbol{\eta}$ respectively on the right hand side of Equation 7.37 without introducing any appreciable error. Write (ξ - **x**) for $\Delta\xi$; then Equation 7.37 gives us

$$\Delta\xi \equiv \xi - x = -kX + k\{(1 + X^2)x + XYy\}. \qquad \textbf{(7.38)}$$

Similarly, we obtain

$$\Delta\eta \equiv \eta - y = -kY + k\{(1 + Y^2)y + XYx\}. \qquad \textbf{(7.39)}$$

The displacements due to refraction for the center of the plate are **–kX** and **–kY**, and these quantities, since they appear in the values of (ξ - **x**) and ($\boldsymbol{\eta}$ - **y**) for all the images on the plate, may be supposed to be incorporated in the undetermined constants **c** and **f** of Equations 7.27. When **-kX** and **-kY** are omitted from Equations 7.38 and 7.39 the remaining terms express the values of (ξ - **x**) and ($\boldsymbol{\eta}$ - **y**) for the *differential refraction*; these equations are then of the linear form

$$\xi - x = ax + by , \qquad \textbf{(7.40)}$$

$$\eta - y = dx + ey , \qquad \textbf{(7.41)}$$

in which, for example, $a = k(1 + X^2)$.

7.8 Aberration

The investigation of the effect of aberration on the standard coordinates of a star is very similar to that the previous section. Let **F** be the position on the celestial sphere towards which the earth is moving at the time of the observation, the position of a star is displaced from its true position **S** to a position **S′** on the great circle arc **SF**, **S′** being nearer to **F** than **S**. The displacement **SS′** is given by

$$SS' = \kappa \sin FS , \qquad \textbf{(7.42)}$$

where κ is the aberration constant whose value in circular measured is 20.4 **sin** 1″. Confining ourselves to the effects of aberration only, we write, as before, ξ and $\boldsymbol{\eta}$ for the standard coordinates of a star and **x** and **y** for the coordinates of its image on the photographic plate. **F** is, of course, a definite point on the celestial sphere; we shall suppose that its projection on the tangent plane is $\mathbf{W_1}$, with coordinates **U** and **V**. We shall also suppose that **F**, **U** and **V** correspond simply to the time of the middle of the exposure. Following the procedure of the previous section, we have the formula corresponding to Equations 7.30 and 7.31,

$$\Delta\xi = -\frac{\kappa(U - \xi)}{TW_1} \sin FS , \qquad \textbf{(7.43)}$$

$$\Delta\eta = -\frac{\kappa(V - \eta)}{TW_1} \sin FS , \qquad \textbf{(7.44)}$$

and from Equations 7.34 and 7.36,

$$\sin FS = \frac{AW_1}{CW_1}\left(1 - \frac{U\xi + V\eta}{AW_1^2}\right), \qquad \textbf{(7.45)}$$

$$TW_1 = AW_1\left(1 - \frac{U\xi + V\eta}{AW_1^2}\right),$$

so that

$$\frac{\sin FS}{AW_1} = \frac{1}{CW_1}.$$

Now

$$CW_1^2 = 1 + U^2 + V^2.$$

We accordingly obtain

$$\Delta\xi = -\frac{\kappa U}{(1 + U^2 + V^2)^{1/2}} + \frac{\kappa\xi}{(1 + U^2 + V^2)^{1/2}}, \quad \textbf{(7.46)}$$

$$\Delta\eta = -\frac{\kappa V}{(1 + U^2 + V^2)^{1/2}} + \frac{\kappa\eta}{(1 + U^2 + V^2)^{1/2}}. \quad \textbf{(7.47)}$$

As in the previous section, we can write **x** for ξ and y for η without any sensible loss of accuracy on the right hand sides of Equations 7.46 and 7.47; also we can omit the constant terms (independent of ξ and η) on the right of these equations. We then obtain the expressions for *differential aberration* in the form

$$\xi - x = a_1 x, \quad \textbf{(7.48)}$$

$$\eta - y = d_1 y, \quad \textbf{(7.49)}$$

in which $\mathbf{a_1}$ and $\mathbf{d_1}$ are small (they have as a common factor 20.1 **sin** 1″, which is of the order 10^{-4}). Again it is unnecessary, as a rule, to calculate the values of $\mathbf{a_1}$ and $\mathbf{d_1}$ from their theoretical expressions.

7.9 The General Relations between Standard Coordinates and Measured Coordinates

From the previous three sections we have seen that refraction, aberration and the instrumental errors taken separately produce in each coordinate a displacement of image of a star on the plate from the position corresponding to the standard coordinates of the star, and that this displacement is given, generally with sufficient accuracy, as a linear expression in the coordinates. If we combine all the various effects we clearly obtain linear formula for (ξ - **x**) and (η - **y**), which we can write in the general forms

$$\xi - x = ax + by + c, \quad \textbf{(7.50)}$$

$$\eta - y = dx + ey + f, \quad \textbf{(7.51)}$$

where ξ, η are the standard coordinates of the star and **x**, **y** are the measured coordinates of its image on the photographic plate. In these equations **a**, **b**, etc. are small and dependent, in a composite way, on the instrumental errors, on refraction and on aberration.

Thus in general **x** differs from ξ by a small quantity and we can write the equations, without loss of accuracy, in an alternative form, namely,

$$\xi - x = a\xi + b\eta + c\,, \tag{7.52}$$

$$\eta - y = d\xi + e\eta + f\,, \tag{7.53}$$

in which the quantities **a**, **b**, etc. are small. These quantities are called the *plate constants*.

7.10 The Method of Dependeces

The method of dependences was first developed by F. Schlesinger in 1911 [37] in connection with the determination of stellar parallaxes and was subsequently applied to the measurement of the positions of asteroids and comets from photographs [38,39,40].

In the previous section we obtained linear formulas connecting the standard coordinates of a star, or other object, in terms of the measured coordinates and the "plate constants" **a**, **b**, etc. To evaluate the plate constants, we make use of at least three comparison stars. In the astrographic problem, for example, if we have several plates giving the position of a planet, the plate constants have to be determined from each plate and this involves a large amount of numerical work. Consider a series of plates with the same plate-center and in which the positions of the planet differ little from one plate to another. The same comparison stars can be used for each plate and instead of computing the plate constants for each plate we calculate certain quantities which depend on the comparison stars selected and on one position of the planet; these quantities, called *dependences*, are thus independent of the particular plate under investigation. The position of the planet is then expressed in terms of the dependences and certain measured quantities.

7.10.1 The Astrographic Problem: 3 Comparison Stars

In this section we shall suppose that we employ three comparison stars. For a given plate, the measured and standard coordinates of the comparison stars are given by Equations 7.52 and 7.53. We consider the measures in **x** only - the procedure for measures in **y** is similar. For three comparison stars, refer to equatorial coordinate system:

$$\alpha_1 - x_1 = a\alpha_1 + b\delta_1 + c\,, \tag{7.54}$$

$$\alpha_2 - x_2 = a\alpha_2 + b\delta_2 + c\,, \tag{7.55}$$

$$\alpha_3 - x_3 = a\alpha_3 + b\delta_3 + c\,, \tag{7.56}$$

and for the planet

$$\alpha - x = a\alpha + b\delta + c\,. \tag{7.57}$$

If (α_0, δ_0) are the standard coordinates of the planet for one of the plates, we shall refer to it as the "selected plate", we can write Equation 7.57 as

$$\alpha - x = a\alpha_0 + b\delta_0 + c + a(\alpha - \alpha_0) + b(\delta - \delta_0)\,. \tag{7.58}$$

We are assuming for all the plates concerned that (α - α_0) and (δ - δ_0) are small quantities. Also, the constants **a** and **b**, which involve respectively the scale-correction and the orientation of the plate, are to be regarded as small quantities; accordingly, we neglect $a(\alpha - \alpha_0)$ and $b(\delta - \delta_0)$ in Equation 7.58 which then becomes

$$\alpha - x = a\alpha_0 + b\delta_0 + c\,. \qquad \textbf{(7.59)}$$

It is to be remembered that, in Equations 7.54, 7.55, 7.56 and 7.59, $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$ and $\mathbf{x}$ are measured quantities obtained with all necessary accuracy. The standard coordinates ($\boldsymbol{\alpha}_1$, $\boldsymbol{\delta}_1$) etc. of the comparison stars are supposed to be known. We can then, if we please, solve Equations 7.54, 7.55 and 7.56 in order to obtain **a**, **b** and **c** then substitute their values in Equation 7.59. Suppose for the moment that we know the values of $\boldsymbol{\alpha}_0$ and $\boldsymbol{\delta}_0$. We then derive from Equation 7.59 the value of $\boldsymbol{\alpha}$ for the plate concerned. But this procedure is equivalent to the elimination of **a**, **b** and **c** between the four Equations 7.54, 7.55, 7.56 and 7.59, and we can effect this elimination as follows.

Multiply Equations 7.54, 7.55, 7.56 and 7.59 by $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$ and **–1** respectively and add. We obtain

$$\begin{aligned} D_1(\alpha_1 - x_1) + D_2(\alpha_2 - x_2) + D_3(\alpha_3 - x_3) - (\alpha - x) \\ = a[D_1\alpha_1 + D_2\alpha_2 + D_3\alpha_3 - \alpha_0] \\ + b[D_1\delta_1 + D_2\delta_2 + D_3\delta_3 - \delta_0] \\ + c[D_1 + D_2 + D_3 - 1]. \end{aligned} \qquad \textbf{(7.60)}$$

The elimination of **a**, **b** and **c** is effected if

$$D_1\alpha_1 + D_2\alpha_2 + D_3\alpha_3 = \alpha_0\,, \qquad \textbf{(7.61)}$$

$$D_1\delta_1 + D_2\delta_2 + D_3\delta_3 = \delta_0\,, \qquad \textbf{(7.62)}$$

$$D_1 + D_2 + D_3 = 1\,, \qquad \textbf{(7.63)}$$

and these are three equations from which $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ can be obtained. The factors $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ are called the *dependences*. We then have from Equation 7.60

$$\alpha - x = D_1(\alpha_1 - x_1) + D_2(\alpha_2 - x_2) + D_3(\alpha_3 - x_3)\,, \qquad \textbf{(7.64)}$$

from which α can be determined, all the other quantities being now supposed known.

Now ($\boldsymbol{\alpha}_1 - \mathbf{x}_1$), ($\boldsymbol{\alpha}_2 - \mathbf{x}_2$) and ($\boldsymbol{\alpha}_3 - \mathbf{x}_3$) are all small quantities and it will thus be sufficiently accurate to determine $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ if we substitute in Equations 7.61 and 7.62 the *measured* coordinates of the comparison stars and planet for the selected plate. Denoting these by ($\mathbf{X_1}$, $\mathbf{Y_1}$), ($\mathbf{X_2}$, $\mathbf{Y_2}$), ($\mathbf{X_3}$, $\mathbf{Y_3}$) and ($\mathbf{X_0}$, $\mathbf{Y_0}$) respectively, the equations to determine $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ become

$$D_1X_1 + D_2X_2 + D_3X_3 = X_0\,, \qquad \textbf{(7.65)}$$

$$D_1Y_1 + D_2Y_2 + D_3Y_3 = Y_0\,, \qquad \textbf{(7.66)}$$

$$D_1 + D_2 + D_3 = 1\,, \qquad \textbf{(7.67)}$$

from which, solving in determinant form, we have for $\mathbf{D_1}$

$$\frac{D_1}{\begin{vmatrix} X_0 & X_2 & X_3 \\ Y_0 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{vmatrix}}\,. \qquad \textbf{(7.68)}$$

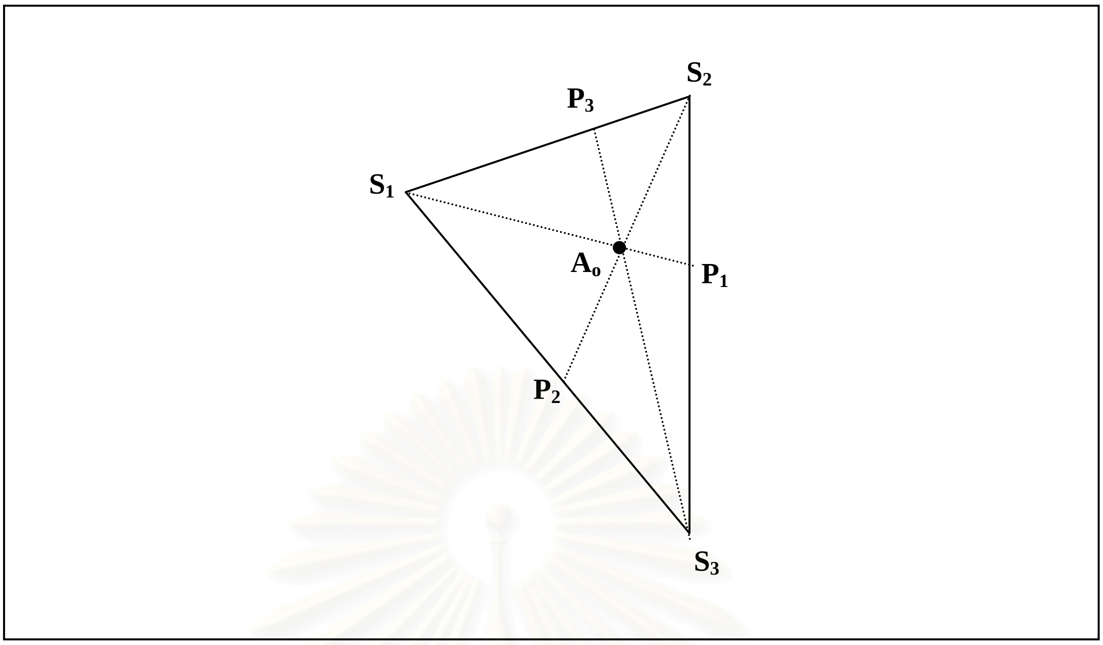


Figure 7.7 Graphical evaluation of dependences for three reference stars [40].

In Figure 7.7, let $\mathbf{S_1}$, $\mathbf{S_2}$ and $\mathbf{S_3}$ be the positions on the selected plate of the images of the comparison stars, and $\mathbf{A_0}$ the position of the image of the planet. Then the determinant under $\mathbf{D_1}$ in Equation 7.68 is simply twice the algebraic measure of the area of the triangle $\mathbf{A_0S_2S_3}$ and second determinant in Equation 7.68 is twice the area of the triangle $\mathbf{S_1S_2S_3}$. We obtain similar results for $\mathbf{D_2}$ and for $\mathbf{D_3}$ and we then have, as the solutions of Equations 7.65, 7.66 and 7.67,

$$\frac{D_1}{A_0S_2S_3} = \frac{D_2}{A_0S_3S_1} = \frac{D_3}{A_0S_1S_2} = \frac{1}{S_1S_2S_3}. \quad \mathbf{(7.69)}$$

Let the straight lines through $\mathbf{A_0}$ and the vertices of the triangle cut the sides in $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$. Then

$$\frac{A_0S_2S_3}{S_1S_2S_3} = \frac{A_0P_1}{S_1P_1}.$$

Hence

$$D_1 = \frac{A_0P_1}{S_1P_1}, \; D_2 = \frac{A_0P_2}{S_2P_2} \quad \text{and } D_3 = 1 - D_1 - D_2. \quad \mathbf{(7.70)}$$

The values of $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ can be readily obtained, with sufficient accuracy, by plotting the positions of the comparison stars and planet on squared paper, from the measured coordinates of the selected plate. It can be shown that the results of applying the method are most accurate when $\mathbf{A_0}$ in Figure 7.7 coincides with the centroid of the triangles $\mathbf{S_1S_2S_3}$.

7.10.2 The Astrographic Problem : n Comparison Stars

We suppose now that the number, **n**, of comparison stars is greater than 3. We now have **n** equations of the form Equations 7.54, 7.55 and 7.56 to give the values of the plate constants **a**, **b** and **c**. They are

$$\left.\begin{array}{l} a\alpha_1 + b\delta_1 + c = \alpha_1 - x_1 \\ a\alpha_2 + b\delta_2 + c = \alpha_2 - x_2 \\ \dots\dots\dots\dots \;\dots\dots\dots\dots\; \dots\dots\dots\dots\;. \\ a\alpha_n + b\delta_n + c = \alpha_n - x_n \end{array}\right\}. \qquad \textbf{(7.71)}$$

The values of **a**, **b** and **c** are to be obtained by the method of least squares. Multiply each by the corresponding coefficient of **a** and, adding, we obtain

$$a\sum_{i=1}^{n}\alpha_i^2 + b\sum_{i=1}^{n}\alpha_i\delta_i + c\sum_{i=1}^{n}\alpha_i = \sum_{i=1}^{n}\alpha_i(\alpha_i - x_i)\;. \qquad \textbf{(7.72)}$$

Similarly, by multiplying each equation of 7.71 by the corresponding coefficient of **b**, and adding, we obtain

$$a\sum_{i=1}^{n}\alpha_i\delta_i + b\sum_{i=1}^{n}\delta_i^2 + c\sum_{i=1}^{n}\delta_i = \sum_{i=1}^{n}\delta_i(\alpha_i - x_i)\;. \qquad \textbf{(7.73)}$$

Also, adding the **n** equations of 7.71, we have

$$a\sum_{i=1}^{n}\alpha_i + b\sum_{i=1}^{n}\delta_i + nc = \sum_{i=1}^{n}(\alpha_i - x_i)\;. \qquad \textbf{(7.74)}$$

As in Equation 7.59, the equation for the planet is

$$a\alpha_0 + b\eta_0 + c = \alpha - x\;, \qquad \textbf{(7.75)}$$

in which, as before, $\boldsymbol{\alpha}_0$ and $\boldsymbol{\eta}_0$ denote the standard coordinates of the planet for a selected plate.

To eliminate **a**, **b** and **c** from the four Equations 7.72 through 7.75 multiply these in order by **P**, **Q**, **R** and –1 and add. Then if **P**, **Q** and **R** are given by

$$P\sum_{i=1}^{n}\alpha_i^2 + Q\sum_{i=1}^{n}\alpha_i\delta_i + R\sum_{i=1}^{n}\alpha_i = \alpha_0\;, \qquad \textbf{(7.76)}$$

$$P\sum_{i=1}^{n}\alpha_i\delta_i + Q\sum_{i=1}^{n}\delta_i^2 + R\sum_{i=1}^{n}\delta_i = \delta_0\;, \qquad \textbf{(7.77)}$$

$$P\sum_{i=1}^{n}\alpha_i + Q\sum_{i=1}^{n}\delta_i + Rn = 1\;, \qquad \textbf{(7.78)}$$

we have

$$\alpha - x = P\sum_{i=1}^{n}\alpha_i(\alpha_i - x_i) + Q\sum_{i=1}^{n}\delta_i(\alpha_i - x_i) + R\sum_{i=1}^{n}(\alpha_i - x_i)\;,$$

which can be written

$$\alpha - x = \sum_{i=1}^{n}(P\alpha_i + Q\delta_i + R)(\alpha_i - x_i)\;.$$

Set

$$D_i \equiv P\alpha_1 + Q\delta_i + R . \quad \textbf{(7.79)}$$

Then

$$\alpha - x = \sum_{i=1}^{n} D_i(\alpha_i - x_i) . \quad \textbf{(7.80)}$$

In this last equation, $\mathbf{D_i}$ is a function of the equatorial coordinates of the **n** comparison stars and of $\boldsymbol{\alpha}_0$ and $\boldsymbol{\eta}_0$. Since $(\boldsymbol{\alpha_i} - \mathbf{x_i})$ is a small quantity for any one of the stars it will be sufficient to regard $\mathbf{D_i}$ as a function of the measured coordinates $\mathbf{x_1},\ldots, \mathbf{y_n}, \mathbf{x_0}, \mathbf{y_0}$ for the selected plate; as before, we denote these coordinates by $\mathbf{X_i}$, $\mathbf{Y_i}$, $\mathbf{X_0}$ and $\mathbf{Y_0}$. Thus **P**, **Q** and **R** are now to be determined from Equations 7.76, 7.77 and 7.78 in which we replace $\boldsymbol{\alpha_i}$ etc. by $\mathbf{X_i}$ etc. Hence

$$P\sum_{i=1}^{n} X_i^2 + Q\sum_{i=1}^{n} X_iY_i + R\sum_{i=1}^{n} X_i = X_0 , \quad \textbf{(7.81)}$$

$$P\sum_{i=1}^{n} X_iY_i + Q\sum_{i=1}^{n} Y_i^2 + R\sum_{i=1}^{n} Y_i = Y_0 , \quad \textbf{(7.82)}$$

$$P\sum_{i=1}^{n} X_i + Q\sum_{i=1}^{n} Y_i + Rn = 1 . \quad \textbf{(7.83)}$$

We can simplify the calculation of **P**, **Q**, and **R** by supposing that for the selected plate the values of $\mathbf{X_i}$, $\mathbf{Y_i}$, $\mathbf{X_0}$ and $\mathbf{Y_0}$ are measured from the centroid of the **n** comparison stars. We then have

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i = 0 ,$$

and **P**, **Q** and **R** are now to be determined from

$$P\sum_{i=1}^{n} X_i^2 + Q\sum_{i=1}^{n} X_iY_i = X_0 , \quad \textbf{(7.84)}$$

$$P\sum_{i=1}^{n} X_iY_i + Q\sum_{i=1}^{n} Y_i^2 = Y_0 , \quad \textbf{(7.85)}$$

$$Rn = 1 . \quad \textbf{(7.86)}$$

Then

$$D_i = PX_i + QY_i + R . \quad \textbf{(7.87)}$$

The quantities $\mathbf{D_i}$ are the *dependences*. The practical procedure is first to calculate **P**, **Q** and **R** by means of Equations 7.84, 7.85 and 7.86 and then to form the dependence $\mathbf{D_i}$ for each star by means of Equation 7.70. As in the previous section,

$$\sum_{i=1}^{n} D_i = 1 , \quad \textbf{(7.88)}$$

as we can see from Equations 7.87 and 7.86, remembering that

$$\sum X_i = \sum Y_i = 0 .$$

When the dependences have been calculated the value of α for the planet can be then found from Equation 7.80 for any number of plates.

CHAPTER 8

THE METHOD OF GAUSS

8.1 Determination of Orbit

In Chapter five we have seen how, once the elements of an orbit are known, the geocentric position on the celestial sphere can be calculated for any time. In this chapter we shall be concerned with the reverse situation, that of finding the elements of an orbit from observations.

The method of Gauss has features which permit the solution to be brought to a higher state of refinement. The excellent utility of this method has encouraged successive researchers to develop increasingly more elegant and powerful variations of the original procedure [2,9]. The approach described in this chapter is one of the simpler versions of the Gaussian method [27].

Orbit determination by the method of Gauss requires exactly three sets of position data [2,9,14,41,42,43,44]. These need not be separated by equal intervals of time. The Guassian method is applicable to orbits of any form, but difficulties may arise when it is used to determine very eccentric orbit because the influence of the radial velocity is neglected in the first approximation.

8.2 Solution by f and g Expressions

Assume that from the available data three sets have been chosen and reduced to the following:

$$\{t_i, \vec{L}_i, \vec{R}_i\} ,$$

where **i** = 1, 2, 3. If we let $\mathbf{t_2}$ be the epoch time, then the observed times can be converted to the modified time intervals

$$\tau_1 = k(t_1 - t_2)$$
$$\tau_3 = k(t_3 - t_2) .$$

Now, the closed or universal **f** and **g** expressions which were developed for the solution of the two-body equation of motion,

$$\ddot{\vec{r}} = -\frac{\mu\vec{r}}{r^3} ,$$

may be employed to describe the dynamic constraints at τ_1 and τ_3 as follows:

$$\vec{r}_1 = f_1\vec{r}_2 + g_1\dot{\vec{r}}_2 \qquad \textbf{(8.1)}$$

$$\vec{r}_3 = f_3\vec{r}_2 + g_3\dot{\vec{r}}_2 . \qquad \textbf{(8.2)}$$

Furthermore, the geometric constraint of fundamental vector triangle must hold at all three times. Thus,

$$\vec{r}_i = p_i\vec{L}_i - \vec{R}_i , \qquad \textbf{(8.3)}$$

for **i** = 1, 2, 3.

Multiplying Equation 8.1 by $\mathbf{g_3}$ and Equation 8.2 by $\mathbf{g_1}$ and subtracting, we eliminate $\dot{\vec{\mathbf{r}}}_2$ to obtain

$$g_3\vec{r}_1 - g_1\vec{r}_3 = (f_1g_3 - f_3g_1)\vec{r}_2 .$$

Dividing each term by the coefficient of $\vec{\mathbf{r}}_2$ and rearranging we can write

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = 0 , \qquad \textbf{(8.4)}$$

where

$$\begin{aligned} c_1 &= +\frac{g_3}{f_1g_3 - f_3g_1} \\ c_2 &= -1 \\ c_3 &= -\frac{g_1}{f_1g_3 - f_3g_1} . \end{aligned} \qquad \textbf{(8.5)}$$

Equation 8.4 reflects the fact that two-body motion requires the three radius vector to lie in the same plane. Substituting Equation 8.3 for each of the $\vec{\mathbf{r}}_\mathbf{i}$

$$c_1p_1\vec{L}_1 + c_2p_2\vec{L}_2 + c_3p_3\vec{L}_3 = c_1\vec{R}_1 + c_2\vec{R}_2 + c_3\vec{R}_3 . \qquad \textbf{(8.6)}$$

The solution of equation for the unknown $\mathbf{p_i}$ is the key to the method of Gauss [2,9].

8.3 The Scalar Equations for the Ranges

We solve Equation 8.6 for the three $\mathbf{p_i}$ by taking appropriate cross and dot products with the vectors $\vec{\mathbf{L}}_\mathbf{i}$. When this is accomplished the result is

$$p_1 = \frac{c_1(\vec{R}_1 \times \vec{L}_2)\cdot\vec{L}_3 + c_2(\vec{R}_2 \times \vec{L}_2)\cdot\vec{L}_3 + c_3(\vec{R}_3 \times \vec{L}_2)\cdot\vec{L}_3}{c_1(\vec{L}_1 \times \vec{L}_2)\cdot\vec{L}_3} \qquad \textbf{(8.7)}$$

$$p_2 = \frac{c_1(\vec{L}_1 \times \vec{R}_1)\cdot\vec{L}_3 + c_2(\vec{L}_1 \times \vec{R}_2)\cdot\vec{L}_3 + c_3(\vec{L}_1 \times \vec{R}_3)\cdot\vec{L}_3}{c_2(\vec{L}_1 \times \vec{L}_2)\cdot\vec{L}_3} \qquad \textbf{(8.8)}$$

$$p_3 = \frac{c_1\vec{L}_1\cdot(\vec{L}_2 \times \vec{R}_1) + c_2\vec{L}_1\cdot(\vec{L}_2 \times \vec{R}_2) + c_3\vec{L}_1\cdot(\vec{L}_2 \times \vec{R}_3)}{c_3\vec{L}_1\cdot(\vec{L}_2 \times \vec{L}_3)} . \qquad \textbf{(8.9)}$$

for convenience, we shall write these equations in the following simpler form:

$$p_1 = \frac{c_1D_{11} + c_2D_{12} + c_3D_{13}}{c_1D_0} \qquad \textbf{(8.10)}$$

$$p_2 = \frac{c_1D_{21} + c_2D_{22} + c_3D_{23}}{c_2D_0} \qquad \textbf{(8.11)}$$

$$p_3 = \frac{c_1D_{31} + c_2D_{32} + c_3D_{33}}{c_3D_0} , \qquad \textbf{(8.12)}$$

where, for $\mathbf{j}$ =1, 2, 3.

$$\begin{aligned} D_{1j} &= (\vec{R}_j \times \vec{L}_2)\cdot\vec{L}_3 \\ D_{2j} &= (\vec{L}_1 \times \vec{R}_j)\cdot\vec{L}_3 \\ D_{3j} &= \vec{L}_1\cdot(\vec{L}_2 \times \vec{R}_j) , \end{aligned} \qquad \textbf{(8.13)}$$

and

$$D_0 = (\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 = \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3)\,. \qquad \textbf{(8.14)}$$

8.4 The First Approximation

Equations 8.10, 8.11, and 8.12 were developed using no assumptions beyond that of two-body motion. Unfortunately, these equations cannot be used to compute the $\mathbf{p_i}$ until the quantities $\mathbf{c_1}$ and $\mathbf{c_3}$ are known. These coefficients were defined in Equation 8.5. Thus, there is a problem because $\mathbf{c_1}$ and $\mathbf{c_3}$ are expressed in terms of the $\mathbf{f_i}$ and $\mathbf{g_i}$, which, in turn, require knowledge of the vector orbital elements. The solution to this dilemma is to assume approximate values for the **f** and **g** expressions which can be improved once initial values for the vector elements have been computed.

Consider the first few terms of the **f** and **g** series developed in Section 4.4

$$\begin{aligned} f_i &= 1 - \frac{1}{2}u_2\tau_i^2 + \frac{1}{2}u_2 z_2 \tau_i^3 + \ldots.. \\ g_i &= \tau_i - \frac{1}{6}u_2\tau_i^3 + \frac{1}{4}u_2 z_2 \tau_i^4 + \ldots.\,, \end{aligned} \qquad \textbf{(8.15)}$$

where **i** =1, 2, 3, and

$$u_2 = \frac{\mu}{r_2^3}$$

$$z_2 = \frac{\vec{r}_2 \cdot \dot{\vec{r}}_2}{r_2^2}\,.$$

If the $\mathbf{f_i}$ and $\mathbf{g_i}$ series are truncated to eliminate all terms beyond the second, the resulting approximations do not require the vector elements. Thus, we make the following simplifying assumptions:

$$\begin{aligned} f_i &= 1 - \frac{1}{2}u_2\tau_i^2 \\ g_i &= \tau_1 - \frac{1}{6}u_2\tau_i^3\,, \end{aligned} \qquad \textbf{(8.16)}$$

where the contributions of the vectors have been ignored so the only unknown is the scalar $\mathbf{u_2}$. Using these approximations, it is possible to form

$$f_1 g_3 - f_3 g_1 \approx \tau - \frac{u_2}{6}\tau^3\,, \qquad \textbf{(8.17)}$$

where all orders above τ_i^3 have been neglected and $\tau_3 - \tau_1$ has been replaced by τ.

Therefore, substituting Equations 8.16 and 8.17 into the expressions for $\mathbf{c_1}$ and $\mathbf{c_3}$, we obtain

$$c_1 \approx \frac{\tau_3 - u_2\tau_3^3/6}{\tau - u_2\tau^3/6}$$

$$c_3 \approx \frac{-\tau_1 + u_2\tau_1^3/6}{\tau - u_2\tau^3/6}\,.$$

Carrying out the division and neglecting all terms with order greater than three, the result is

$$c_1 \approx +\frac{\tau_3}{\tau} + \frac{u_2\tau_3}{6\tau}(\tau^2 - \tau_3^2)$$
$$c_3 \approx -\frac{\tau_1}{\tau} + \frac{u_2\tau_1}{6\tau}(\tau^2 - \tau_1^2)\,. \qquad \textbf{(8.18)}$$

For convenience, we rewrite Equations 8.18 as follows:

$$c_1 \approx A_1 + \frac{\mu B_1}{r_2^3}$$
$$c_3 \approx A_3 + \frac{\mu B_3}{r_2^3}\,, \qquad \textbf{(8.19)}$$

where $\mathbf{u_2}$ has been replaced by $\boldsymbol{\mu}/\mathbf{r}_2^3$ and

$$A_1 = +\frac{\tau_3}{\tau}$$
$$B_1 = +\frac{1}{6}A_1(\tau^2 - \tau_3^2)$$
$$A_3 = -\frac{\tau_1}{\tau}$$
$$B_3 = +\frac{1}{6}A_3(\tau^2 - \tau_1^2)\,. \qquad \textbf{(8.20)}$$

8.5 The Scalar Equations Relating p and r at Epoch

Equation 8.11 can now be solved for the range $\mathbf{p_2}$ by letting $\mathbf{c_2}$ = -1 and replacing $\mathbf{c_1}$ and $\mathbf{c_3}$ by Equations 8.19. Making these substitutions and collecting terms, we can obtain the following expression:

$$p_2 = A + \frac{\mu B}{r_2^3}\,, \qquad \textbf{(8.21)}$$

where

$$A = -\frac{A_1 D_{21} - D_{22} + A_3 D_{23}}{D_0}$$
$$B = -\frac{B_1 D_{21} + B_3 D_{23}}{D_0}\,. \qquad \textbf{(8.22)}$$

A second equation containing the unknowns $\mathbf{p_2}$ and $\mathbf{r_2}$ can be derived by taking the dot product of Equation 8.3 with itself when **i** = 2. Thus,

$$\vec{r}_2 \cdot \vec{r}_2 = (p_2\vec{L}_2 - \vec{R}_2)\cdot(p_2\vec{L}_2 - \vec{R}_2)\,.$$

Making use of the fact that $\mathbf{L}_2^2$ =1, since $\vec{\mathbf{L}}_2$ is a unit vector, the above equation can be reduced to

$$r_2^3 = p_2^3 + p_2 E + F\,, \qquad \textbf{(8.23)}$$

where

$$E = -2(\vec{L}_2 \cdot \vec{R}_2)$$
$$F = R_2^2 \ . \qquad \textbf{(8.24)}$$

We now have two independent scalar equations relating $\mathbf{p_2}$ and $\mathbf{r_2}$

8.6 The Scalar Equation of Lagrange

Substituting Equation 8.21 for the range $\mathbf{p_2}$ in Equation 8.23, we obtain the equation of Lagrange, that is

$$r_2^8 + ar_2^6 + br_2^3 + c = 0 \ , \qquad \textbf{(8.25)}$$

where

$$a = -(A^2 + AE + F)$$
$$b = -\mu(2AB + BE) \qquad \textbf{(8.26)}$$
$$c = -\mu^2 B^2 \ .$$

If we let the symbol $\mathbf{x}$ represent the unknown value of $\mathbf{r_2}$ which will satisfy Equation 8.25, we can write

$$f(x) = x^8 + ax^6 + bx^3 + c \ .$$

Differentiating **f(x)** with respect to **x** produces

$$f'(x) = 8x^7 + 6ax^5 + 3bx^2 \ .$$

When the above equations are solved for **x** by the Newton-Raphson method, the result will be the value of $\mathbf{r_2}$ needed to complete Gauss' method.

8.7 The Vector Orbital Elements

8.7.1 Initial Position Vector

The $\mathbf{f_i}$ and $\mathbf{g_i}$ can now be computed by Equations 8.16 using the value of $\mathbf{r_2}$ found from Equation 8.25. Thus,

$$f_i \approx 1 - \frac{1}{2} u_2 \tau_i^2$$
$$g_i \approx \tau_i - \frac{1}{6} u_2 \tau_i^3 \ , \qquad \textbf{(8.27)}$$

where **i** =1, 3,

and

$$u_2 = \frac{\mu}{r_2^3} \ . \qquad \textbf{(8.28)}$$

Employing these values of $\mathbf{f_i}$ and $\mathbf{g_i}$, the quantities $\mathbf{c_1}$, $\mathbf{c_2}$, and $\mathbf{c_3}$ are obtained from Equations 8.5:

$$\begin{aligned} c_1 &= +\frac{g_3}{f_1 g_3 - f_3 g_1} \\ c_2 &= -1 \\ c_3 &= -\frac{g_1}{f_1 g_3 - f_3 g_1} \ . \end{aligned} \qquad \textbf{(8.29)}$$

Finally, substituting $\mathbf{c_1}$, $\mathbf{c_2}$, and $\mathbf{c_3}$ into

$$\begin{aligned} p_1 &= \frac{c_1 D_{11} + c_2 D_{12} + c_3 D_{13}}{c_1 D_0} \\ p_2 &= \frac{c_1 D_{21} + c_2 D_{22} + c_3 D_{23}}{c_2 D_0} \\ p_3 &= \frac{c_1 D_{31} + c_2 D_{32} + c_3 D_{33}}{c_3 D_0} \end{aligned} \qquad \textbf{(8.30)}$$

produces the three $\mathbf{p_i}$ which can be used in the general geometric constraint,

$$\vec{r}_i = p_i \vec{L}_i - \vec{R}_i , \qquad \textbf{(8.31)}$$

to determine the vector element $\vec{\mathbf{r}}_2$ along with the other two radius vectors.

8.7.2 Initial Velocity Vector

We now have more than enough information to determine the vector element $\dot{\vec{\mathbf{r}}}_2$. Consider again the dynamic constraints at τ_1 and τ_3:

$$\begin{aligned} \vec{r}_1 &= f_1 \vec{r}_2 + g_1 \dot{\vec{r}}_2 \\ \vec{r}_3 &= f_3 \vec{r}_2 + g_3 \dot{\vec{r}}_2 \ . \end{aligned}$$

Multiplying the first equation by $\mathbf{f_3}$ and the second equation by $\mathbf{f_1}$, we subtract and eliminate $\vec{r}_2$ to obtain

$$f_3 \vec{r}_1 - f_1 \vec{r}_3 = -(f_1 g_3 - f_3 g_1) \dot{\vec{r}}_2 \ .$$

Dividing each term by the coefficient of $\dot{\vec{\mathbf{r}}}_2$ and rearranging, we can write

$$\dot{\vec{r}}_2 = d_1 \vec{r}_1 + d_3 \vec{r}_3 , \qquad \textbf{(8.32)}$$

where

$$\begin{aligned} d_1 &= -\frac{f_3}{f_1 g_3 - f_3 g_1} \\ d_3 &= +\frac{f_1}{f_1 g_3 - f_3 g_1} \ . \end{aligned} \qquad \textbf{(8.33)}$$

Therefore, since approximate values for $\mathbf{f_i}$ and $\mathbf{g_i}$, are known, $\mathbf{d_1}$ and $\mathbf{d_3}$ can be calculated, and Equation 8.32 yields the velocity $\dot{\vec{\mathbf{r}}}_2$.

In the case of heliocentric orbits, the correction for light-time should be introduced before going on to refine the initial approximations of $\vec{\mathbf{r}}_2$ and $\dot{\vec{\mathbf{r}}}_2$. Thus, for all three observation times $\mathbf{t_i}$, the corrected values are

$$t_{ci} = t_i - \frac{p_i}{c}, \tag{8.34}$$

where

$$c = 173.1446 \quad \text{AU/day} .$$

Consequently, the modified time intervals should also be computed anew from

$$\begin{aligned} \tau_1 &= k(t_{c1} - t_{c2}) \\ \tau_2 &= k(t_{c3} - t_{c2}) \\ \tau &= \tau_3 - \tau_1 . \end{aligned} \tag{8.35}$$

8.7.3 Refinement of the Elements

The vector orbital elements may be refined by using the initial values of $\vec{\mathbf{r}}_2$ and $\dot{\vec{\mathbf{r}}}_2$ to recompute the $\mathbf{f_i}$ and $\mathbf{g_i}$ from their universal formulations. When this has been accomplished, improved values of $\mathbf{c_i}$ and $\mathbf{d_i}$ can be determined. The results permit better values of the $\mathbf{p_i}$ to be found from

$$\begin{aligned} p_1 &= \frac{c_1 D_{11} + c_2 D_{12} + c_3 D_{13}}{c_1 D_0} \\ p_2 &= \frac{c_1 D_{21} + c_2 D_{22} + c_3 D_{23}}{c_2 D_0}, \\ p_3 &= \frac{c_1 D_{31} + c_2 D_{32} + c_3 D_{33}}{c_3 D_0} \end{aligned}$$

which may then be used with the geometric constraint,

$$\vec{r}_i = p_i \vec{L}_i - \vec{R}_i,$$

to improve the element $\vec{\mathbf{r}}_2$ along with the other two position vectors. Finally, the new $\mathbf{d_i}$ and $\mathbf{r_i}$ are used in

$$\dot{\vec{r}}_2 = d_1 \vec{r}_1 + d_3 \vec{r}_3 ,$$

to find an improved $\dot{\vec{\mathbf{r}}}_2$. The whole process is repeated until the magnitudes of the $\mathbf{p_i}$ stop changing and converge to stable values. When this occurs, we have the final vector element set, and the preliminary orbit is determined.

The orbital elements derived from the method of Gauss will represent all three observations satisfactorily because the corresponding $\vec{\mathbf{L}}_i$ were used to determine the $\vec{\mathbf{r}}_i$ when the geometric constraint was applied. Therefore, the accuracy of the elements must be tested by comparing computed positions with observed positions not used in the solution.

CHAPTER 9

WORKING, RESULTS AND DISCUSSION

This chapter presents the results of applying the Gauss's method to calculate the position and velocity vector elements at a given epoch time of Jupiter and Mars. To apply the technique of astronomical photography, we obtain the observational data needed to calculate the orbit of our planets. Consequently, in the last portion, the results are discussed and compared with the results from the astronomical almanac [45,46].

9.1 Working

From the fact that the orbit of celestial body around the Sun which has a conic section (without regarding the perturbation) we can get the classical elements which characterize the solution of the two-body problem [27]. It is sufficiently to determine the orbit of celestial body. The observation from the Earth can be done in many ways such as the analysis by radio wave or photography which used in this thesis.

Normally, the method of observation (after usually process by the reduction method in Chapter 7) provides the *angle only data* [9] which is sufficient for the determination of orbit by Gauss method. The observed data containing the time of observations, right ascension and declination in equatorial coordinate system.

The celestial bodies such as Jupiter and Mars are observed in this thesis and the instruments have use consist of

1. Nikon FM single lens reflex (SLR) camera
2. Standard lens Nikor 50 mm F 1.4
3. Tripod (Slik 6000)
4. Daylight films (ISO 400, 24 x 36 mm)
5. Star chart 1999 [47-52]

The observable location is Salaya, where is the subdivision of a Nakorn pathom. Salaya district performing the observation has a clear sky, no cloud or dust making it preferable to Bangkok in which the artificial light obscure the heavenly objects. Salaya can be easily travel from Bangkok. We facilitated to take photographs and sometimes we need to take the photographs in late at night. So we have to wait until the celestial body appear in the appropriate position (the zenith point or the region near the declination at zero degree). At the position, the objects appear at the east and move along the ecliptic line (depend on the inclination of orbital plane of observable object) to the west, following the Earth's rotation that rotate from west to east. Example, to take photographs of Mars in January 17, 1999 has, approximately 2:35 a.m.

The starting day of the observation was January 16, 1999. At 7:45 p.m. on the west near the horizon we took the photographs of Jupiter as shown in Figure 9.1 with place the planets (Jupiter and Mars) at the center of films. In the method, we need to include the stars that we know their positions in equatorial coordinate system (right ascension and declination) as much as possible. Then it was selected to be the reference stars.

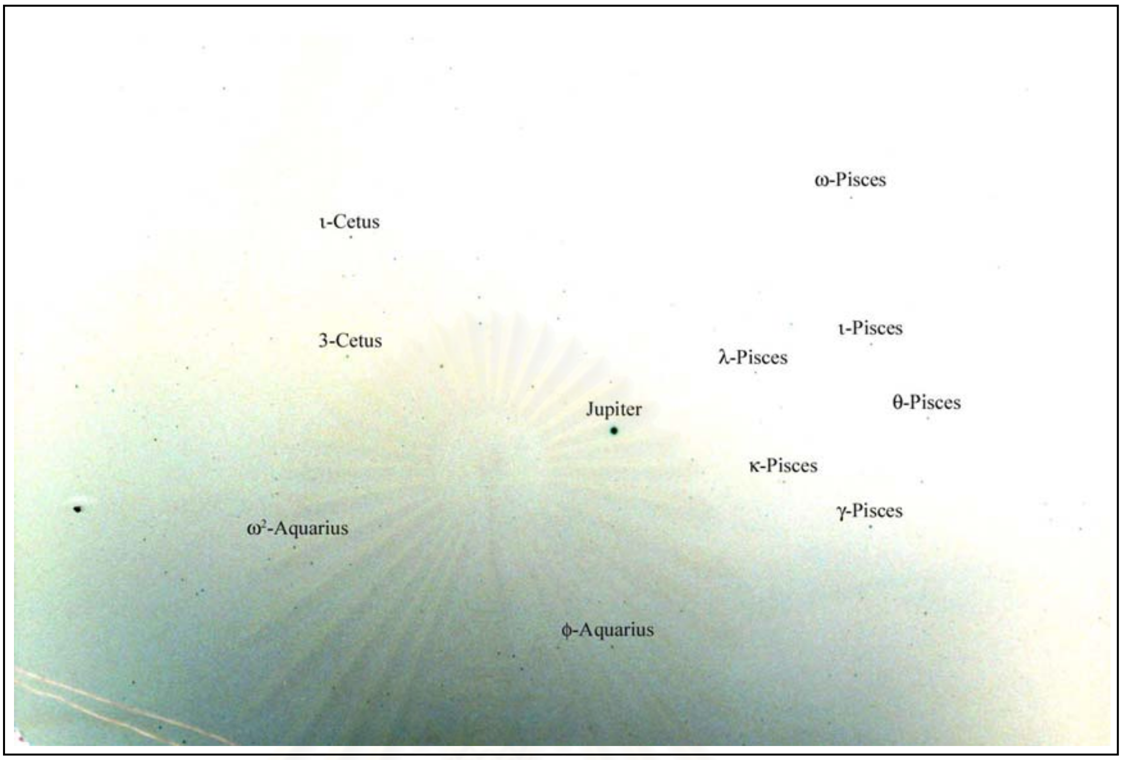


Figure 9.1 The photograph of Jupiter, January 16, 1999.

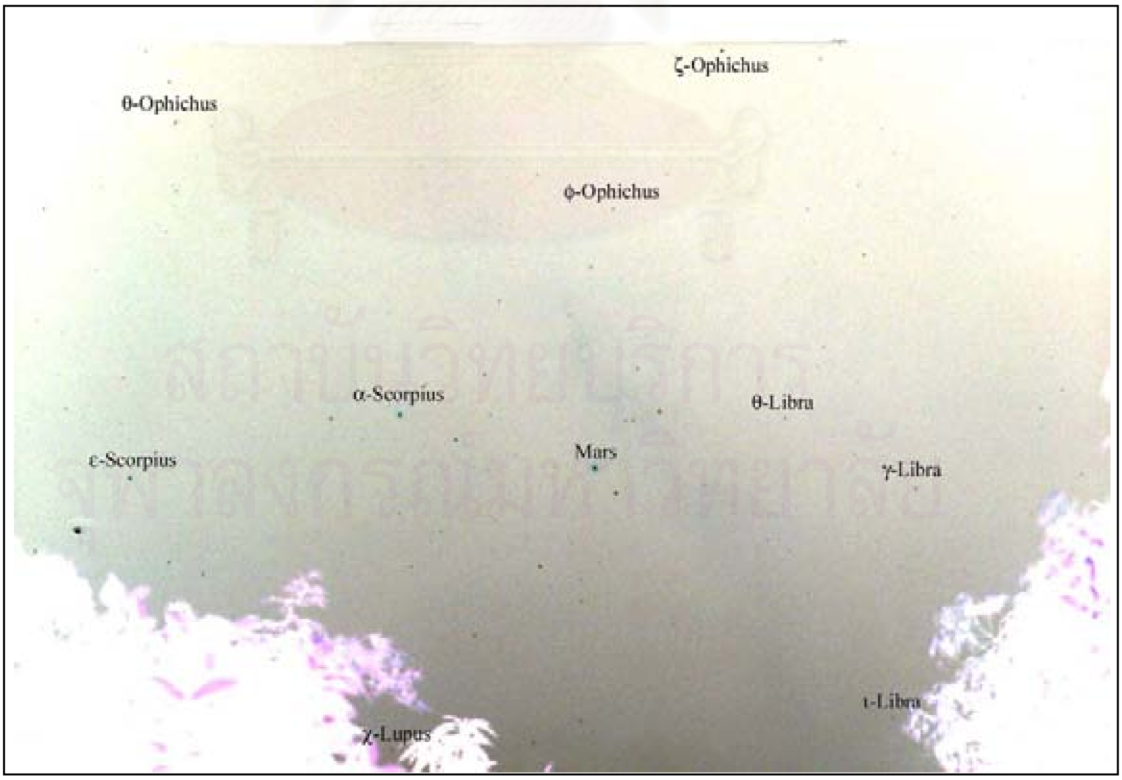


Figure 9.2 The photograph of Mars, September 8, 1999.

For convenience, the positions of both of the stars and the planets are obtained by opening the aperture in the lens as widest at f-stop 1.4 [53,54] and setting the shutter speed about 5 seconds (if we set the shutter speed longer than 5 seconds for the standard lens 50 mm., the object would appear as a line because the Earth rotates around itself).

In Figure 9.1 the picture has been color inverted. The black spot at the center is Jupiter and the reference stars have less brightness. The next day we took photographs of Mars at 2:35 a.m. Unfortunately, there is an artificial light from houses. This light reduced the brightness of the stars. So we choose the photograph that is taken on September 8, 1999 in Figure 9.2. The camera setting detail for the Mars is the same as for Jupiter. Later times, to obtain the best photographs we have to wait until the object reached the zenith point (less artificial light.)

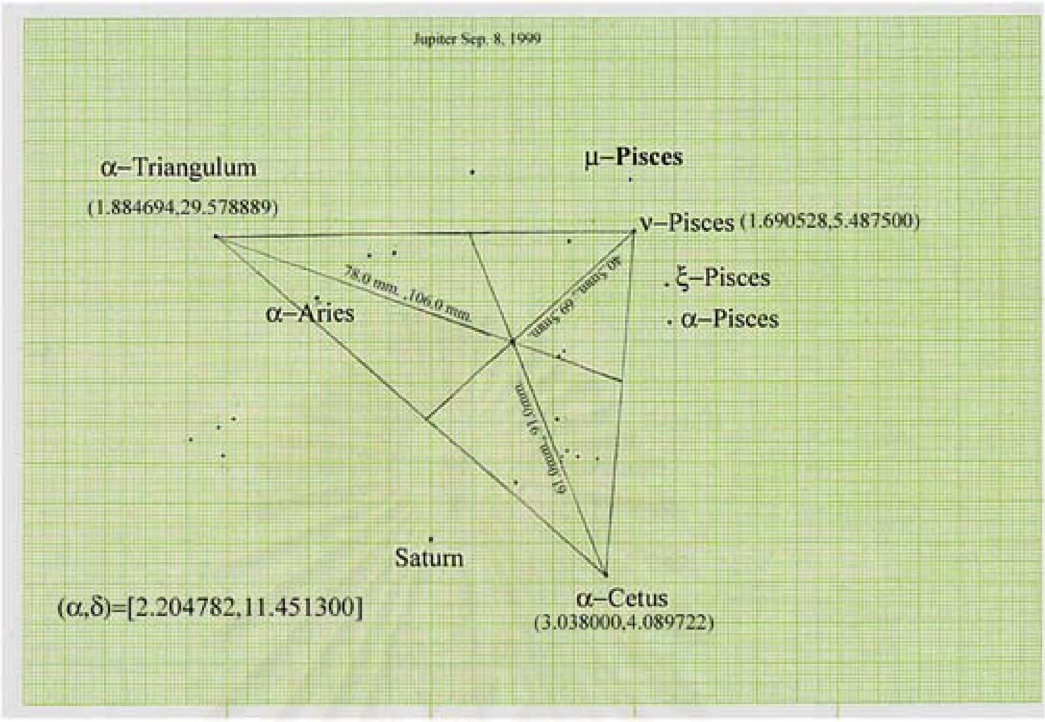
We took the Mars and Jupiter in the next week as well. We choose the period of observation to be about one week because we want to have the pictures of the planets which relate to the reference stars. Yet sometimes, the observation of the celestial body can't be taken the photograph because of many reasons such as the objects appears near the moon so the moon's light is brighter than the reference stars or the objects appears at during the day. These problems occur in the observation of Jupiter that we can't take the photograph from February to August in 1999. The data of Mars and Jupiter are shown in the tables 9.1 through 9.4

From the photographs we can see the name of the reference stars, right ascension and declination by comparing from the star chart [45-52]. It is named after the constellation in Greek era [55] as shown in Figures 9.1 and 9.2.

After we know the right ascension and declination of the stars, the next step is to obtain the right ascension and declination of Jupiter and Mars. This can be done by the method of dependences. It can be derived into 2 types: 3 reference stars and 5 reference stars. Because of the limitation in our observation (too little stars), the latter type is done in a less frequency. However we try to use both method by using the same data.

9.1.1 Method of Dependency: 3 Reference Stars.

From the photographs we sketched them on the graphing paper with resolution 1 mm. as shown in Figure 9.3. From the right ascension, declination of the stars and other quantities in Figure 7.7 we can find the right ascension and declination of the planets by using Equations 7.61, 7.62 and 7.63. The results are shown in tables 9.1 and 9.2 where X, Y, Z are the topocentric positions of the Sun (from table in the astronomical almanac [45,46]) and using Julian date to convenience for the specification of time in Gauss's method.



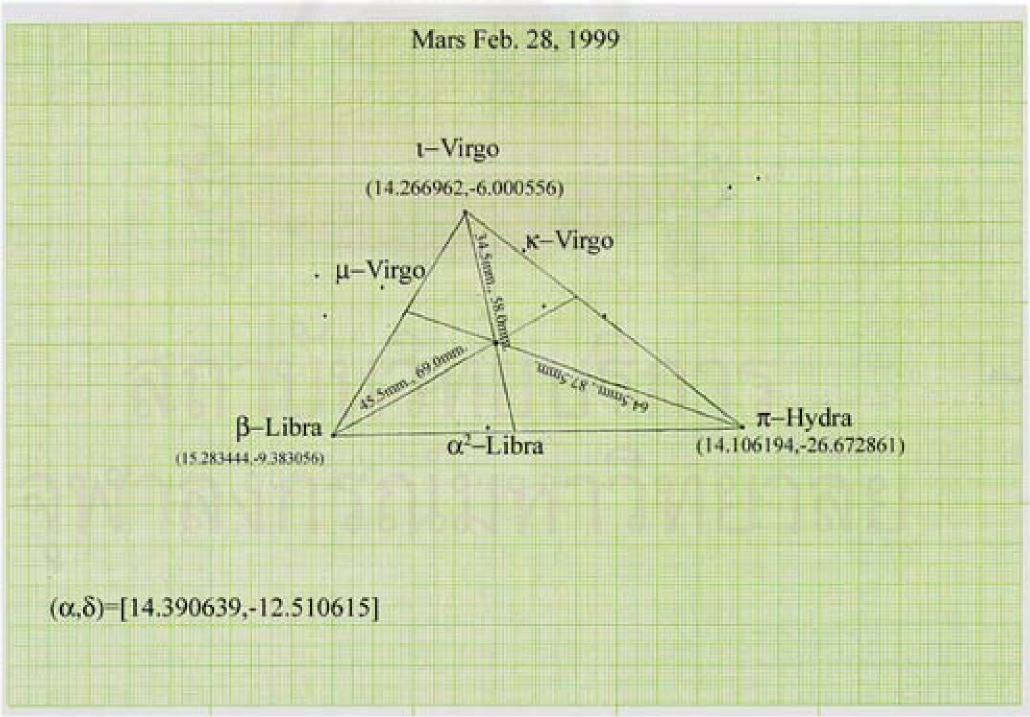


Figure 9.3 Sketching pictures (Jupiter and Mars) on the graphing paper.

JUPITER							
D/M/Y	Time	Julian Date	Right Ascension	Declination	X	Y	Z
16/1/1999	19:45	2451195.322917	18.254915	-3.121560	0.4350922	-0.8094893	-0.3509578
23/1/1999	19:00	2451202.291667	17.616314	-2.535857	0.5410324	-0.7544064	-0.3270807
30/1/1999	19:40	2451209.319444	12.847877	-2.126235	0.6395634	-0.6873894	-0.2980237
8/9/1999	23:30	2451430.479167	2.204782	11.451300	-0.9769745	0.2255460	0.0977893
19/9/1999	0:30	2451440.520833	2.165645	11.319543	-1.0018289	0.0698396	0.0302830
5/10/1999	0:05	2451456.503472	2.083568	10.830186	-0.9807426	-0.1801977	-0.0781227
19/10/1999	22:52	2451471.452778	1.917320	10.311054	-0.8943174	-0.4022344	-0.1743884
27/10/1999	21:31	2451479.396528	1.863612	9.962421	-0.8235861	-0.5101480	-0.2211762
17/11/1999	1:05	2451499.545139	1.732379	9.058393	-0.5778776	-0.7361569	-0.3191625
1/12/1999	18:50	2451514.284722	1.620075	8.641591	-0.3511939	-0.8453251	-0.3664889
7/12/1999	19:00	2451520.291667	1.636643	8.632747	-0.2511452	-0.8739995	-0.3789209
16/12/1999	21:57	2451529.414583	1.606796	8.590354	-0.0942162	-0.8987501	-0.3896568
23/12/1999	18:53	2451536.286806	1.595872	8.667984	0.0257542	-0.9020897	-0.3911031
4/1/2000	19:10	2451548.298611	1.637291	8.703891	0.2335655	-0.8763429	-0.3799378
12/1/2000	19:15	2451556.302083	1.657360	8.413147	0.3668161	-0.8372123	-0.3629776
26/1/2000	19:30	2451570.312500	1.738010	9.432903	0.5808886	-0.7293575	-0.3162115
1/2/2000	19:20	2451576.305556	1.817611	9.934944	0.6624552	-0.6692618	-0.2901584
8/2/2000	20:15	2451583.343750	1.861046	10.117069	0.7488010	-0.5892035	-0.2554534
13/3/2000	19:15	2451617.302083	2.233649	12.357944	0.9878263	-0.1028619	-0.0445986

Table 9.1 The observational data of Jupiter from the method of dependences: 3 - reference stars.

MARS							
D/M/Y	Time	Julian Date	Right Ascension	Declination	X	Y	Z
17/1/1999	2:35	2451195.607639	13.918710	-8.3835228	0.4395625	-0.8074811	-0.3500872
24/1/1999	3:15	2451202.635417	14.266305	-9.275035	0.5460620	-0.7513933	-0.3257744
31/1/1999	2:30	2451209.604166	14.028503	-10.056338	0.6433622	-0.6844542	-0.2967509
7/2/1999	2:50	2451216.618056	14.278954	-10.919104	0.7315683	-0.6067593	-0.2630618
21/2/1999	2:30	2451230.604167	14.210536	-11.779351	0.8733278	-0.4255904	-0.1845210
28/2/1999	2:00	2451237.583333	14.390639	-12.510615	0.9249294	-0.3249595	-0.1408895
8/3/1999	0:30	2451245.520833	15.024819	-13.526650	0.9669905	-0.2047959	-0.0887880
14/3/1999	2:10	2451251.590278	14.993534	-13.535273	0.9867765	-0.1101129	-0.0477401
25/3/1999	1:30	2451262.562500	15.085812	-13.593778	0.9946700	0.0634662	0.0275135
12/4/1999	0:10	2451280.506944	14.625239	-12.601398	0.9313965	0.3395688	0.1472210
21/4/1999	23:00	2451290.458333	14.578763	-11.906317	0.8577205	0.4805546	0.2083452
25/4/1999	23:00	2451294.458333	14.445790	-11.595872	0.8209863	0.5335016	0.2313033
2/5/1999	22:15	2451301.427083	13.875640	-10.401241	0.7480403	0.6196814	0.2686688
16/5/1999	0:00	2451314.500000	14.040348	-10.182779	0.5838654	0.7572277	0.3282973
24/5/1999	19:45	2451323.322917	13.516521	-9.422362	0.4563260	0.8294336	0.3596077
19/6/1999	19:50	2451349.326389	13.547916	-10.587300	0.0335536	0.9317591	0.4039711

Table 9.2 The observational data of Mars from the method of dependences: 3 - reference stars.

MARS							
D/M/Y	Time	Julian date	Right Ascension	Declination	X	Y	Z
21/6/1999	20:45	2451351.364583	13.666162	-10.855177	-0.0260454	0.9323888	0.4042451
2/7/1999	20:15	2451362.343750	13.875643	-11.599048	-0.1856244	0.9170884	0.3976083
8/7/1999	19:45	2451368.322917	14.138485	-13.144876	-0.2839913	0.8956636	0.3883172
21/8/1999	20:10	2451412.340278	15.447101	-20.264874	-0.8610417	0.4871323	0.2112020
3/9/1999	20:45	2451425.364583	16.251115	-22.240242	-0.9532276	0.3025965	0.1311914
8/9/1999	20:00	2451430.333333	15.762411	-21.407613	-0.9763829	0.2277659	0.0987516
10/10/1999	19:00	2451462.291667	17.281906	-24.547597	-0.9543424	-0.2684323	-0.1163744
1/12/1999	18:55	2451514.288194	20.395553	-20.527726	-0.3511327	-0.8453463	-0.3664981

Table 2 (Continued) The observational data of Mars for three reference stars.

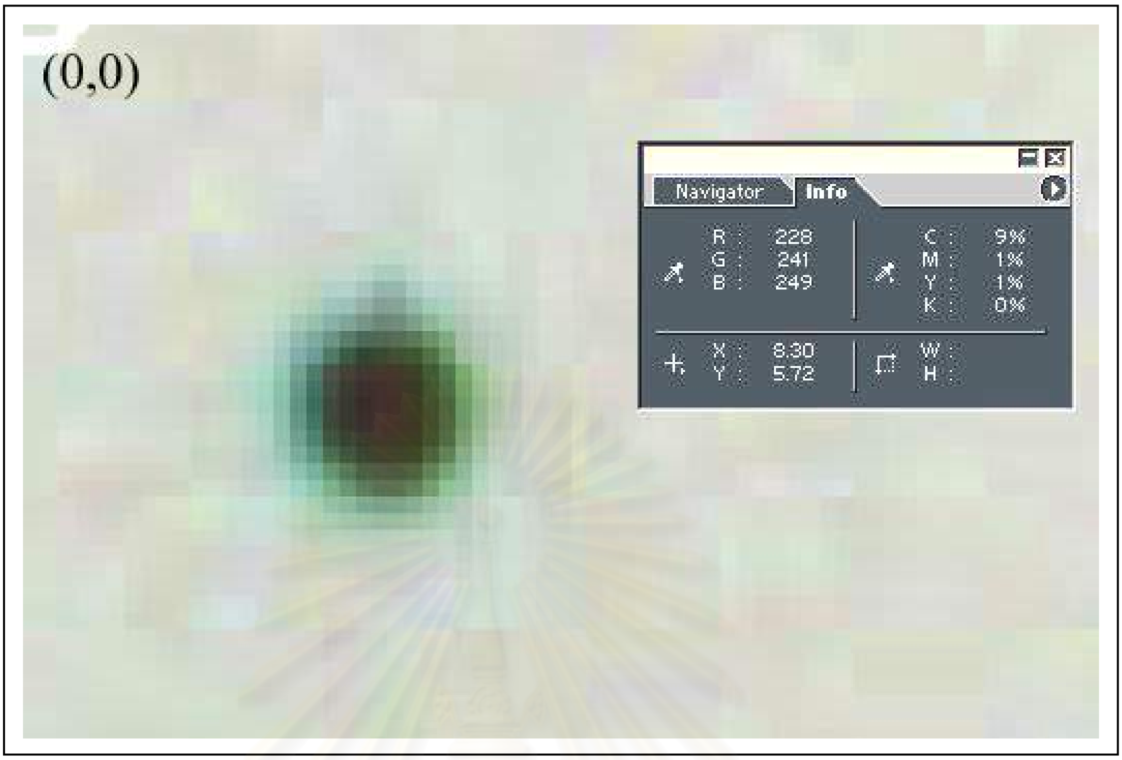


Figure 9.4 The picture of Jupiter is extended by Adobe PhotoShop.

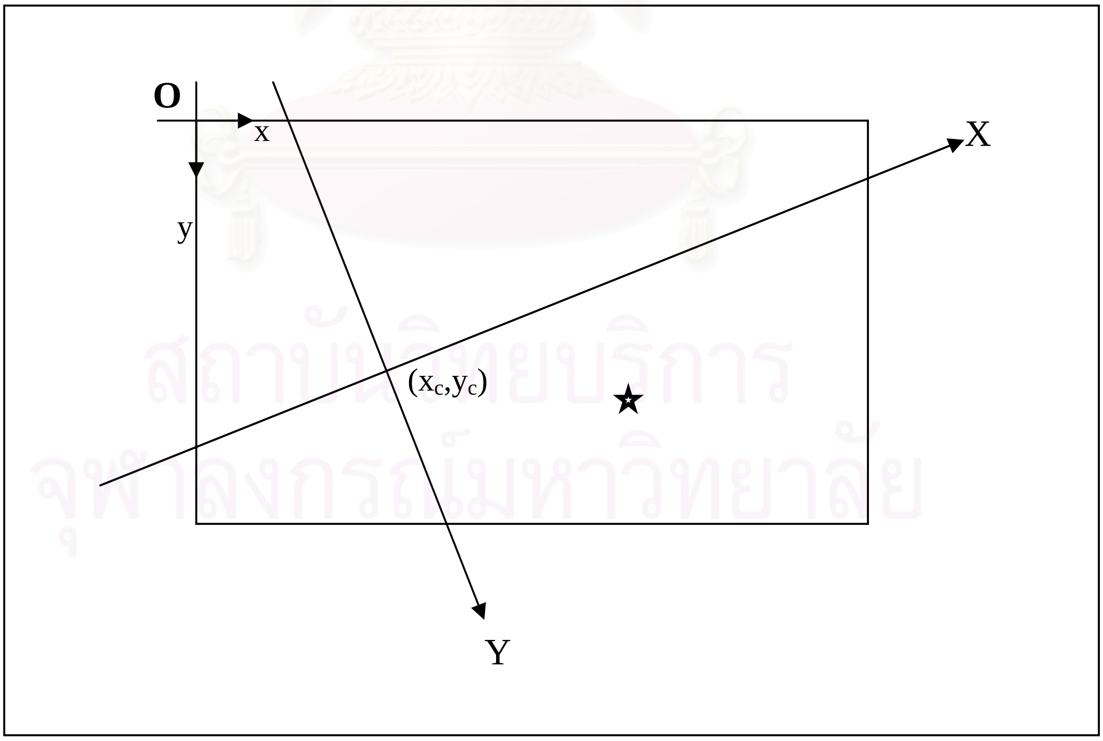


Figure 9.5 The coordinates of Adobe PhotoShop (x, y) and centriod (X, Y).

9.1.2 Method of Dependency: 5 Reference Stars.

After the photographs are scanned, to enlarge this picture and get the accurate position, we use Adobe PhotoShop 6.0 to find the coordinates of the celestial bodies relative to the origin of this program (x, y) as shown in Figure 9.4. We choose the upper left corner as the origin of the coordinates, where +x axis is on the right and +y axis is under the origin. According to the program, the coordinates of the planets and the reference stars are shown as follows:

Jupiter 16/1/1999	
Celestial Body	Photographic Coordinates
Jupiter	x = 46.88 , y = 28.23
ω^2 – Aquarius	x = 22.56 , y = 37.47
ι – Cetus	x = 26.86 , y = 13.20
φ – Aquarius	x = 46.76 , y = 45.14
ω – Pisces	x = 64.90 , y = 9.99
θ – Pisces	x = 70.63 , y = 27.18

Jupiter 8/9/1999	
Celestial Body	Photographic Coordinates
Jupiter	x = 46.76 , y = 22.37
α – Triangulum	x = 6.39 , y = 7.33
α – Aries	x = 20.18 , y = 16.47
μ – Cetus	x = 47.47 , y = 41.76
α – Cetus	x = 59.72 , y = 53.76
ν – Pisces	x = 63.29 , y = 7.15

Jupiter 19/9/1999	
Celestial Body	Photographic Coordinates
Jupiter	x = 43.22 , y = 28.57
α – Triangulum	x = 1.33 , y = 16.89
α – Aries	x = 15.61 , y = 25.31
μ – Cetus	x = 44.23 , y = 49.04
ν – Pisces	x = 58.05 , y = 13.85
γ – Cetus	x = 60.30 , y = 49.90

Jupiter 5/10/1999	
Celestial Body	Photographic Coordinates
Jupiter	x = 46.76 , y = 22.37
α – Triangulum	x = 6.39 , y = 7.33
α – Aries	x = 20.18 , y = 16.47
ζ – Pisces	x = 47.47 , y = 41.76
ν – Pisces	x = 59.72 , y = 53.76
γ – Cetus	x = 63.29 , y = 7.15

Jupiter 17/11/1999	
Celestial Body	Photographic Coordinates
Jupiter	x = 7.01 , y = 4.54
α – Aries	x = 1.50 , y = 8.26
γ – Aries	x = 3.01 , y = 6.57
ο – Pisces	x = 4.17 , y = 4.08
ε – Pisces	x = 6.53 , y = 0.51
ο – Cetus	x = 12.76 , y = 7.27

Jupiter 7/12/1999	
Celestial Body	Photographic Coordinates
Jupiter	x = 7.84 , y = 5.23
α – Aries	x = 1.05 , y = 7.31
η – Pisces	x = 5.08 , y = 8.27
δ – Pisces	x = 8.87 , y = 5.23
α – Pisces	x = 9.80 , y = 8.27
θ – Cetus	x = 15.10 , y = 5.04

Jupiter 4/1/2000	
Celestial Body	Photographic Coordinates
Jupiter	x = 40.77 , y = 27.34
μ – Cetus	x = 0.63 , y = 25.76
α – Pisces	x = 26.60 , y = 41.84
γ – Aries	x = 29.37 , y = 3.37
η – Pisces	x = 42.21 , y = 11.85
δ – Pisces	x = 68.01 , y = 27.55

Jupiter 1/2/2000	
Celestial Body	Photographic Coordinates
Jupiter	x = 7.33 , y = 5.18
θ – Cetus	x = 0.23 , y = 8.68
ο – Cetus	x = 1.57 , y = 2.53
ε – Pisces	x = 7.31 , y = 9.69
χ – Pisces	x = 12.54 , y = 7.83
α – Aries	x = 12.67 , y = 2.28

Jupiter 8/2/2000	
Celestial Body	Photographic Coordinates
Jupiter	x = 7.95 , y = 5.19
γ – Cetus	x = 4.08 , y = 0.21
α – Pisces	x = 4.70 , y = 4.51
η – Pisces	x = 10.48 , y = 6.53
β – Aries	x = 12.29 , y = 3.83
α – Aries	x = 13.19 , y = 2.41

Jupiter 13/3/2000	
Celestial Body	Photographic Coordinates
Jupiter	x = 7.58 , y = 5.43
δ – Cetus	x = 2.26 , y = 3.14
α – Pisces	x = 3.87 , y = 7.14
λ – Cetus	x = 5.55 , y = 0.87
ε – Aries	x = 10.71 , y = 0.37
β – Aries	x = 11.33 , y = 6.66

Mars 17/1/1999	
Celestial Body	Photographic Coordinates
Mars	x = 34.44 , y = 38.55
τ – Virgo	x = 8.45 , y = 43.89
ζ – Virgo	x = 18.71 , y = 30.56
ψ – Virgo	x = 46.07 , y = 15.97
89 – Virgo	x = 54.40 , y = 51.82
γ – Hydra	x = 70.98 , y = 39.67

Mars 14/3/1999	
Celestial Body	Photographic Coordinates
Mars	x = 42.40 , y = 26.95
β – Libra	x = 20.97 , y = 32.86
μ – Virgo	x = 30.47 , y = 13.80
α^2 – Libra	x = 42.13 , y = 34.99
κ – Virgo	x = 50.49 , y = 11.46
π – Hydra	x = 76.38 , y = 39.74

Mars 25/3/1999	
Celestial Body	Photographic Coordinates
Mars	x = 8.55 , y = 5.29
β – Libra	x = 5.04 , y = 6.62
μ – Virgo	x = 6.27 , y = 3.37
ι – Virgo	x = 8.27 , y = 1.62
ι – Libra	x = 8.28 , y = 9.04
λ – Virgo	x = 10.02 , y = 3.89

Mars 25/4/1999	
Celestial Body	Photographic Coordinates
Mars	x = 6.64 , y = 5.89
μ – Virgo	x = 2.80 , y = 8.04
ζ – Virgo	x = 4.18 , y = 0.73
α – Virgo	x = 8.57 , y = 2.01
π – Hydra	x = 12.45 , y = 8.66
γ – Hydra	x = 13.36 , y = 3.96

Mars 24/5/1999	
Celestial Body	Photographic Coordinates
Mars	x = 7.61 , y = 5.16
ζ – Virgo	x = 4.28 , y = 3.33
κ – Virgo	x = 5.80 , y = 8.75
θ – Virgo	x = 7.38 , y = 2.20
α – Virgo	x = 8.57 , y = 4.74
γ – Hydra	x = 13.27 , y = 6.82

Mars 19/6/1999	
Celestial Body	Photographic Coordinates
Mars	x = 7.79 , y = 6.13
μ – Virgo	x = 0.36 , y = 5.19
α^2 – Libra	x = 0.39 , y = 9.76
ζ – Virgo	x = 7.20 , y = 1.97
γ – Virgo	x = 12.85 , y = 1.56
δ – Corvus	x = 14.80 , y = 1.97

Mars 21/6/1999	
Celestial Body	Photographic Coordinates
Mars	x = 8.28 , y = 5.89
π – Hydra	x = 2.31 , y = 10.12
κ – Virgo	x = 5.19 , y = 3.78
α – Virgo	x = 9.21 , y = 6.55
ζ – Virgo	x = 10.68 , y = 2.31
θ – Virgo	x = 11.82 , y = 5.40

Mars 2/7/1999	
Celestial Body	Photographic Coordinates
Mars	x = 42.82 , y = 30.95
α^2 – Libra	x = 5.65 , y = 28.70
κ – Virgo	x = 30.33 , y = 22.06
ι – Virgo	x = 31.69 , y = 12.16
ζ – Virgo	x = 58.34 , y = 7.89
θ – Virgo	x = 67.94 , y = 23.57

Mars 8/7/1999	
Celestial Body	Photographic Coordinates
Mars	x = 7.36 , y = 5.77
β – Libra	x = 0.25 , y = 0.53
α^2 – Libra	x = 1.64 , y = 4.32
π – Hydra	x = 9.00 , y = 20.28
α – Virgo	x = 10.36 , y = 6.53
ζ – Virgo	x = 11.69, y = 2.21

Mars 8/9/1999	
Celestial Body	Photographic Coordinates
Mars	x = 44.98 , y = 30.70
ε – Scorpius	x = 9.50 , y = 31.45
χ – Lupus	x = 29.53 , y = 52.02
ϕ – Ophiuchus	x = 46.47 , y = 11.02
θ – Libra	x = 59.45 , y = 26.92
ι – Libra	x = 67.44 , y = 49.61

The coordinates obtained by the program lead to the *dependences* $\mathbf{D_i}$. D_i is related to the centroid coordinates as in Equations 7.84 through 7.87. So we have to transformed our data to the centroid (X, Y) as shown in Figure 9.5.

To define (x_c, y_c) as the centroid coordinates relative to our origin (**O**). We use the following equations:

$$\sum_i x_i - nx_c = 0$$

$$\sum_i y_i - ny_c = 0 .$$

Where x_i and y_i are the coordinates of the stars (i) compare with the origin and **n** is the number of the reference stars, in this case **n** = 5.

We can also transform the coordinates by the relation [33].

$$X = x - x_c$$

$$Y = y - y_c .$$

Using Equations 7.84 through 7.87 we obtain the dependences of the reference stars. These quantities lead to the right ascension and declination of Jupiter and Mars as in Equation 7.80. We create the program using Turbo **C++** version 3.0 to calculate the right ascension and declination in both 3 and 5 reference stars (see Appendix C.2 and C.3). The numerical results are an observational data and shown in tables 9.3 and 9.4.

The data are obtained from method of dependency in both 3 and 5 reference stars, we select only three data at each observed time to determine the orbit of Jupiter and Mars by Gauss's method [2,9].

JUPITER							
D/M/Y	Time	Julian Date	Right Ascension	Declination	X	Y	Z
16/1/1999	19:45	2451195.322917	23.695737	-3.285087	0.4350922	-0.8094893	-0.3509578
8/9/1999	23:30	2451430.479167	2.188110	11.594247	-0.9769745	0.2255460	0.0977893
19/9/1999	0:30	2451440.520833	2.149041	11.326239	-1.0018289	0.0698396	0.0302830
5/10/1999	0:05	2451456.503472	2.047980	10.757947	-0.9807426	-0.1801977	-0.0781227
17/11/1999	1:05	2451499.545139	1.746747	7.687088	-0.5778776	-0.7361569	-0.3191625
7/12/1999	19:00	2451520.291667	1.603731	8.471869	-0.2511452	-0.8739995	-0.3789209
4/1/2000	19:10	2451548.298611	1.597450	8.536461	0.2335655	-0.8763429	-0.3799378
1/2/2000	19:20	2451576.305556	1.768776	9.722489	0.6624552	-0.6692618	-0.2901584
8/2/2000	20:15	2451583.343750	1.821094	10.047594	0.7488010	-0.5892035	-0.2554534
13/3/2000	19:15	2451617.302083	2.207587	12.230821	0.9878263	-0.1028619	-0.0445986

Table 9.3 The observational data of Jupiter from the method of dependences: 5 - reference stars.

MARS							
D/M/Y	Time	Julian Date	Right Ascension	Declination	X	Y	Z
17/1/1999	2:35	2451195.607639	13.642020	-8.1586716	0.4395625	-0.8074811	-0.3500872
31/1/1999	2:30	2451209.604166	13.828924	-10.174488	0.6433622	-0.6844542	-0.2967509
14/3/1999	2:10	2451251.590278	14.680361	-13.275144	0.9867765	-0.1101129	-0.0477401
25/4/1999	23:00	2451294.458333	14.126893	-11.376284	0.8209863	0.5335016	0.2313033
24/5/1999	19:45	2451323.322917	13.517633	-9.273174	0.4563260	0.8294336	0.3596077
19/6/1999	19:50	2451349.326389	13.580618	-10.272986	0.0335536	0.9317591	0.4039711
21/6/1999	20:45	2451351.364583	13.614814	-10.848755	-0.0260454	0.9323888	0.4042451
2/7/1999	20:15	2451362.343750	13.794871	-11.936112	-0.1856244	0.9170884	0.3976083
8/7/1999	19:45	2451368.322917	14.117446	-10.766748	-0.2839913	0.8956636	0.3883172
8/9/1999	20:00	2451430.333333	16.111443	-22.066737	-0.9763829	0.2277659	0.0987516

Table 9.4 The observational data of Mars from the method of dependences: 5 - reference stars.

9.2 Results

Method of Gauss

A method of Gauss is employed to obtain position and velocity $(\vec{r}, \dot{\vec{r}})$ at the epoch time of celestial body leading to six classical elements. We have written the program for this purpose (see Appendix C.4 and C.5). There are 2 cases, Jupiter and Mars, as below

We work in the units of solar days, in distance units of **AU** (the radius of Earth's orbit), and masses of a solar mass (**M**). Then astronomers like to use the Gaussian gravitational constant **k**. In addition, We use mass units of 1 solar mass **M**.

$$\mathbf{k} = 0.0172029895 \qquad\qquad \mathbf{M} = 1.$$

The coordinates of three different planet measurements are the heliocentric vectors 1, 2 and 3. We start with the right ascension and the declination measured here on Earth. We use α for right ascension angle, and δ for the declination. The data is entered in decimal degrees, and converted into radians by the parameter Q1

$$Q1 = \frac{\pi}{180}.$$

We use time units of Julian dates, and then multiply them by the Gaussian gravitational constant **k**, to make the time come out in units of mean solar day.

Jupiter	Mars
t_1 = 2451195.322917 Jan. 16, 1999 7:45 p.m.	t_1 = 2451195.607639 Jan. 17, 1999 2:35 a.m.
t_2 = 2451440.520833 Sep. 19, 1999 0:30 a.m.	t_2 = 2451251.590278 Mar. 14, 1999 2:10 a.m.
t_3 = 2451548.298611 Jan. 4, 2000 7:10 p.m.	t_3 = 2451362.343750 July 2, 1999 8:15 p.m.

$$\tau_1 = k(t_1 - t_2)$$
$$\tau_3 = k(t_3 - t_2)$$
$$\tau_2 = \tau_3 - \tau_1.$$

Jupiter		Mars	
Right Ascension	Declination	Right Ascension	Declination
$\alpha_1 = 18.254915$	$\delta_1 = -3.121560$	$\alpha_1 = 13.642020$	$\delta_1 = -8.158672$
$\alpha_2 = 2.165645$	$\delta_2 = 11.319543$	$\alpha_2 = 14.680361$	$\delta_2 = -13.275144$
$\alpha_3 = 1.637291$	$\delta_3 = 8.703891$	$\alpha_3 = 13.794871$	$\delta_3 = -11.936112$

These are three normalized differences associated with three separate observations. Now we need the geocentric Sun positions for the three times. These are from the 1999 astronomical almanac book [45], and we interpolate between two adjacent days in these tables to get the right Julian time.

Solar Position in Case of Jupiter		
X_1 = 0.4350922	Y_1 = -0.8094893	Z_1 = -0.3509578
X_2 = -1.0018289	Y_2 = 0.0698396	Z_2 = 0.0302830
X_3 = 0.2335655	Y_3 = -0.8763429	Z_3 = -0.3799378

Solar Position in Case of Mars		
X_1 = 0.4395625	Y_1 = -0.8074811	Z_1 = -0.3500872
X_2 = 0.9867765	Y_2 = -0.1101129	Z_2 = -0.0477401
X_3 = -0.1856244	Y_3 = 0.9170884	Z_3 = 0.3976083

$\vec{R}_1 = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$, $\vec{R}_2 = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$, $\vec{R}_3 = \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}$ $\vec{R}$ is the solar position for each of the three times.

Next we calculate the geocentric unit vectors for each observation from the right ascension and the declinations measured. These are basically the angles to the planet from the Earth but we don't know the distance, let

$$(L_{xi} \;\; L_{yi} \;\; L_{zi}) := (\cos\alpha_i \cos\delta_i \quad \sin\alpha_i \cos\delta_i \quad \sin\delta_i),$$

$$\vec{L}_1 = \begin{bmatrix} L_{x1} \\ L_{y1} \\ L_{z1} \end{bmatrix} \qquad \vec{L}_2 = \begin{bmatrix} L_{x2} \\ L_{y2} \\ L_{z2} \end{bmatrix} \qquad \vec{L}_3 = \begin{bmatrix} L_{x3} \\ L_{y3} \\ L_{z3} \end{bmatrix},$$

$\vec{L}_1$, $\vec{L}_2$, and $\vec{L}_3$ are the unit vectors (directions) from the Earth to the planet for each of the three time

$|\vec{L}_1| = 1.000000$, $|\vec{L}_2| = 1.000000$, $|\vec{L}_3| = 1.000000$. The vectors do have length 1.

Now Gauss determines the positions at a given time t_2, by using information from times t_1 and t_3 [27, 41], and the Newtonian force law.

Since a simple two-body orbit will be in one plane, then it is possible to describe the third position vector for the object as some linear combination of the other two positions, as long as the other two are not parallel to each other. So, for example $\vec{r}_2 = c_1\vec{r}_1 + c_3\vec{r}_3$, what we do is to calculate the geometrical coefficients $\mathbf{c_1}$ and $\mathbf{c_3}$. We calculate the geometrical coefficients from the relation of areas swept out per unit time and using Kepler's second law. c_1 and c_3 is described in the form of **f** and **g** series in Equations 8.29. We have obtained **f** and **g** by finding the distance between the Sun and the planet (the heliocentric distance at epoch time r_2) which is carried out by solving Lagrange equation in Section 8.6. The results are

Jupiter	Mars
r_2 = 4.925931 f_1 = 0.925507 f_3 = 0.985607 g_1 = -4.113184 g_3 = 1.845109 c_1 = 0.320240 c_3 = 0.713890 c_2 = -1	r_2 = 1.607091 f_1 = 0.888283 f_3 = 0.562753 g_1 = -0.927157 g_3 = 1.627512 c_1 = 0.827218 c_3 = 0.471248 c_2 = -1.

From these results, We can evaluate the distance from the Earth to the planet using Equations 8.30

Jupiter	Mars
p_1 = 5.405957 p_2 = 4.097718 p_3 = 4.624786	p_1 = 1.247038 p_2 = 0.755195 p_3 = 0.836244

Finally we have solved for the position vector of Jupiter and Mars at the three times. Now we multiply the distance by their unit vector from the Earth and get the Sun-planet heliocentric rectangular equatorial vector positions

$$\vec{r}_i = p_i \vec{L}_i - \vec{R}_i .$$

At epoch time t_2 we have

Jupiter	Mars
$\lvert \vec{r}_2 \rvert = 4.925676$	$\lvert \vec{r}_2 \rvert = 1.595389$

This is the first loop of the entire process, next the heliocentric position vector is improved to higher order of accuracy. We are going to correct the observation times for the speed of light and then iterate once on these coefficients, for an improved calculation which shown in Equation 8.34 and getting the modified time interval described in Equations 8.35.

$$t_{c1} = t_1 - \frac{p_1}{173.1446} \quad , \quad t_{c2} = t_2 - \frac{p_2}{173.1446} \quad , \quad t_{c3} = t_3 - \frac{p_3}{173.1446}$$

$$\tau_1 = k(t_{c1} - t_{c2}) \quad , \qquad \tau_3 = k(t_{c3} - t_{c2}) \quad , \qquad \tau_2 = \tau_3 - \tau_1 .$$

Now, we can recompute the geometrical coefficients by **f** and **g** series. We have used the “universal formulation” in Section 6.3 which more appropriate than the approximation of **f** and **g** by neglecting the term order higher than 3 as done in the first loop.

Getting

Jupiter	Mars
$f_1 = 0.926888$ $f_3 = 0.985613$ $g_1 = -4.115132$ $g_3 = 1.845044$	$f_1 = 0.889690$ $f_3 = 0.564451$ $g_1 = 0.927629$ $g_3 = 1.610266$

and the geometrical coefficients

Jupiter	Mars
$c_1 = 0.320111$ $c_3 = 0.713785$ $c_2 = -1$	$c_1 = 0.825184$ $c_3 = 0.472713$ $c_2 = -1$

as done in the first loop. We have

Jupiter	Mars
$p_1 = 5.406717$ $p_2 = 4.097742$ $p_3 = 4.626073$	$p_1 = 1.265940$ $p_2 = 0.767151$ $p_3 = 0.829529$

To terminate the program we have set the limits of convergence by compare the recently obtained for **p** and the previous one. The difference should less than 10^{-6} in magnitude for acceptance. The recent **p** leads to the heliocentric vector of the planets which can be deduced from the definition of the geocentric constraint

$$\vec{r}_i = p_i\vec{L}_i - \vec{R}_i.$$

At epoch time t_2 we have

Jupiter	Mars
$\lvert\vec{r}_2\rvert = 4.925726$	$\lvert\vec{r}_2\rvert = 1.621668$

The position and velocity components at epoch time are

Jupiter	Mars
$x = 4.400741$ $y = 2.073352$ $z = 0.774499$ $\dot{x} = -0.210888$ $\dot{y} = 0.374655$ $\dot{z} = 0.164454$	$x = -1.570208$ $y = -0.383017$ $z = -0.132492$ $\dot{x} = 0.226879$ $\dot{y} = -0.657073$ $\dot{z} = -0.304540$

($\dot{x}$, $\dot{y}$ and $\dot{z}$ can be readily obtained as described in Section 8.7.2).

Now we have to go to the ecliptic plane. Since the Earth axis tilt by 23 degree, 27 minutes to the plane of the ecliptic, this means that we have to rotate the coordinates (using Equations 3.16, 3.17 and 3.18)

$$D(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{bmatrix}$$ $D(\varepsilon)$ is the 2D rotation matrix though an angle ε radians.

Since we are running out of $\mathbf{r_2}$'s to use. We will choose $\mathbf{r}$ to denote the new heliocentric rectangular ecliptic coordinates, at epoch time t_2

$$\vec{r} = D(\varepsilon)\cdot\vec{r}_2$$

Jupiter	Mars
$\lvert\vec{r}\rvert = 4.925726$	$\lvert\vec{r}\rvert = 1.621668$

Just checking that the coordinate-rotation did not change the distances.

SO, FROM THE KNOWN POSITION AT A GIVEN EPOCH TIME, THE ORBIT PARAMETERS CAN BE CALCULATED NEXT.

Finally, we can calculate the six standard orbital elements, in the heliocentric ecliptic coordinate system as shown in Section 5.3. Define "the ascending node vector $\vec{N}$" that

$$\vec{N} = \vec{k}\times\vec{h}$$

$\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{h} = \vec{r}\times\vec{v}$ is the angular momentum vector, and

$a{:} = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)$, **a** is the semimajor axis.

Jupiter	Mars
a = 5.145403	**a** = 1.521296

$$e_x = \left(\frac{v^2}{\mu} - \frac{1}{r}\right)x - \left(\frac{r\dot{r}}{\mu}\right)\dot{x}$$

$$e_y = \left(\frac{v^2}{\mu} - \frac{1}{r}\right)y - \left(\frac{r\dot{r}}{\mu}\right)\dot{y}$$

$$e_z = \left(\frac{v^2}{\mu} - \frac{1}{r}\right)z - \left(\frac{r\dot{r}}{\mu}\right)\dot{z},$$

$e := \sqrt{e_x^2 + e_y^2 + e_z^2}$, **e** is the eccentricity.

Jupiter	Mars
e = 0.043968	**e** = 0.084052

$i := a\cos\left(\frac{h_z}{h}\right)/Q1$

Jupiter	Mars
i = 1.345439	**i** = 1.700862

iota is the inclination of the orbit with respect to the ecliptic

$\Omega := a\cos\left(\frac{N_x}{N}\right)/Q1$

Jupiter	Mars
Ω = 107.390368	Ω = 54.199783

Omega is the longitude of the ascending node of Jupiter with respect to the line of the vernal equinox (if $N_y < 0.0$ then $\Omega = 360 - \Omega$).

$\omega := a\cos\left(\frac{\vec{N}\cdot\vec{e}}{Ne}\right)/Q1$

Jupiter	Mars
ω = 293.718743	ω = 284.830516

ω is the argument of the perifocus (if $\mathbf{e_z} < 0.0$ then $\omega = 360 - \omega$).

$E := a\cos\left(\frac{h/\mu - r}{ae} - e\right)/Q1$

Jupiter	Mars
E = 13.824567	E = 141.717454

E is eccentric anomaly angle at epoch time t_2.

$M := (E - e \sin E)/Q1$

Jupiter	Mars
M = 346.777385	M = 221.266138

M is the mean anomaly angle at epoch time t_2.

$T := t_2 - \frac{M}{\mu}$, $n = k\sqrt{\frac{\mu}{a^3}}$

Jupiter	Mars
T = 2447335.958245	T = 2450830.348361

T is the time of perihelion passage, in Julian date unit.

The calculated orbital elements of these data are:

Jupiter

----

a = 5.145403	distance of perihelion closet approach the Sun (in AU)
e = 0.043968	eccentricity
i = 1.345439	inclination angle from ecliptic
Ω = 107.390368	longitude of ascending node
ω = 293.718743	Argument of perihelion
T = 2447335.958245	Time of perihelion passage
= June. 23, 1988 11:00 a.m.	

Mars

----

a = 1.521296	distance of perihelion closet approach the Sun (in AU)
e = 0.084052	eccentricity
i = 1.700862	inclination angle from ecliptic
Ω = 54.199783	longitude of ascending node
ω = 284.830516	Argument of perihelion
T = 2450830.348361	Time of perihelion passage
= Jan. 16, 1998 8:22 p.m.	

Moreover, we select the other sets of preliminary orbit data to calculate the classical elements as shown in tables 9.5 and 9.6.

Now we check the accuracy of the orbital elements using these parameters to calculate the position and velocity vector at epoch time ($\vec{r}_2$ and $\vec{v}_2$), and then $\vec{r}_2$ and $\vec{v}_2$ are calculated to find the right ascension and the declination of arbitrary time which are the preliminary orbit data as we have observed and determined the Jupiter and Mars orbits.

We calculated the position and velocity from the classical elements by using the formulae state in Section 5.4. Now,

$$\begin{aligned} i &= i \cdot Q1 \\ \Omega &= \Omega \cdot Q1 \\ \omega &= \omega \cdot Q1 \\ M &= M \cdot Q1 \\ E &= E \cdot Q1, \end{aligned}$$

We will improve the parameter **E** by using Newton-Raphson method. The new **E** lead to the scalar components of position and velocity appears in Equations 5.59. Consequently, the position and velocity vector can be found from Equation 5.52

$$\begin{aligned} \vec{r} &= \bar{x}\vec{P} + \bar{y}\vec{Q} \\ \vec{v} &= \dot{\bar{x}}\vec{P} + \dot{\bar{y}}\vec{Q}, \end{aligned}$$

where $\vec{P}$ and $\vec{Q}$ are defined from Equations 5.64 and 5.65 respectively. For celestial equatorial systems, rotating $\vec{r}$ and $\vec{v}$ by matrix **D**(-ε)**,** we have

$$\vec{r}_2 = D(-\varepsilon) \cdot \vec{r}\,, \qquad \vec{v}_2 = D(-\varepsilon) \cdot \vec{v}\,,$$

Jupiter	Mars
$\vec{r}_2 = \begin{pmatrix} 4.400741 \\ 2.073351 \\ 0.774499 \end{pmatrix}$	$\vec{r}_2 = \begin{pmatrix} -1.570208 \\ -0.383017 \\ -0.132492 \end{pmatrix}$
$\vec{v}_2 = \begin{pmatrix} -0.210888 \\ 0.374655 \\ 0.164454 \end{pmatrix}$	$\vec{v}_2 = \begin{pmatrix} 0.226879 \\ -0.657073 \\ -0.304546 \end{pmatrix}$

Comparing between $\vec{r}_2$ and $\vec{v}_2$ obtained by this method and Gauss's method, the results have shown to be identical results.

Using position and velocity vector to calculate the right ascension and declination at epoch time t_0 employed by the formulae in Chapter 6. Let $\vec{r}_0$ and $\vec{v}_0$ to be position and velocity at epoch time.

.

Jupiter								
t_1	**t_2**	**t_3**	**a**	**e**	**i**	**Ω**	**ω**	**T**
*16/1/1999	17/11/1999	26/1/2000	5.9469322	0.2466846	1.3540095	108.7109930	205.1449707	2450736.1099344
*16/1/1999	*8/9/1999	7/12/1999	6.1081910	0.1711398	1.3376510	106.0952679	270.4925113	2451330.0982264
*16/1/1999	*8/9/1999	*1/2/2000	5.1512353	0.1800390	1.3282304	104.0279993	356.0793152	2447821.1759242
*16/1/1999	*8/9/1999	13/3/2000	5.4783254	0.0905595	1.3352446	105.4683411	279.8518335	2451425.0657365
*16/1/1999	*8/9/1999	*13/3/2000	4.9775410	0.0148916	1.3335166	104.9033378	305.0942882	2447642.1046518
*16/1/1999	*19/9/1999	*4/1/2000	5.1454074	0.0439738	1.3454377	107.3907045	293.7324469	2447336.1037951
*16/1/1999	*19/9/1999	*1/2/2000	5.2399013	0.3159066	1.3308744	104.8367959	5.0586085	2447661.1873539
*16/1/1999	19/9/1999	8/2/2000	5.3050847	0.1600296	1.3949344	114.3359630	332.8262524	2447543.2377190
*16/1/1999	5/10/1999	8/2/2000	5.0306579	0.1180199	1.3950657	114.3328740	342.1130985	2447976.9257016
19/9/1999	*4/1/2000	13/3/2000	5.3101483	0.3186870	1.4498441	97.8678706	207.9274757	2450865.1968487

Table 9.5 The classical elements of Jupiter at epoch time t_2.
(Since * are the data from 5-reference stars).

Mars								
t_1	**t_2**	**t_3**	**a**	**e**	**i**	**Ω**	**ω**	**T**
*17/1/1999	14/3/1999	25/3/1999	1.5226900	0.0978765	1.6009529	98.0348986	203.8995500	2450743.6510137
*17/1/1999	*14/3/1999	19/6/1999	1.5133042	0.1544338	1.7852079	52.5114181	269.8353769	2450784.5564166
*17/1/1999	*14/3/1999	*2/7/1999	1.5212958	0.0840533	1.7008563	54.1999100	284.8293777	2450830.3462507
*17/1/1999	25/4/1999	8/7/1999	1.4638548	0.0832216	1.5068264	73.9704726	257.0677906	2450837.8759703
14/3/1999	25/4/1999	8/7/1999	1.5241081	0.0940631	1.9996421	67.1341040	25.5649153	2451076.2289969
*14/3/1999	24/5/1999	21/6/1999	1.5172155	0.1932683	1.2977409	49.6898106	64.4774712	2451147.8985234
*14/3/1999	*2/7/1999	*8/9/1999	1.5197647	0.0868060	1.6928876	54.3883849	283.2582335	2450827.6155798

Table 9.6 The classical elements of Mars at epoch time t_2.
(Since * are the data from 5-reference stars).

Now, $\vec{r}$ can be computed from the orbital elements by using the numerical integration of a universal formulation, then we correct the time by improving the speed of light. And we get the topocentric position vector at time t_0, $\vec{p}_0$. From the geometric constraint

$$\vec{p}_0 = \vec{r}_0 - \vec{R}_0 ,$$

this lead to the geometric unit vector at time t_0

$$\vec{L}_0 = \frac{\vec{p}_0}{|\vec{p}_0|} .$$

Using Equations 6.93 through 6.96, We have the right ascension and declination of Jupiter and Mars at time epoch t_0 as shown below:

Jupiter			
(α, δ)	Test	Observation	Astronomical Almanac
α	2.148738	2.165145	2.150000
δ	11.324716	11.319543	11.433333

Mars			
(α, δ)	Test	Observation	Astronomical Almanac
α	14.680106	14.680361	14.685333
δ	-13.273794	-13.275144	-13.375000

From these six elements, we choose the best data of Jupiter and Mars to sketch their orbits in space (heliocentric orbit) as shown in Figure 9.6 and 9.7

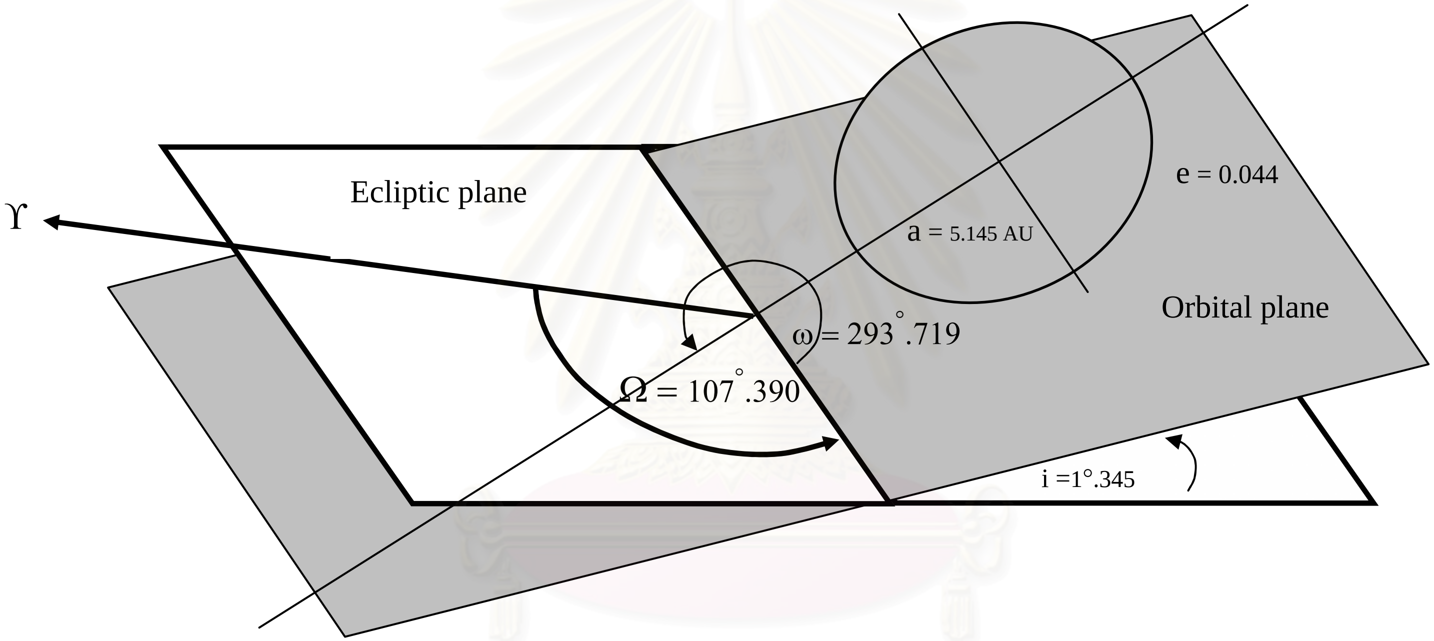


Figure 9.6 The orbit of Jupiter in space.

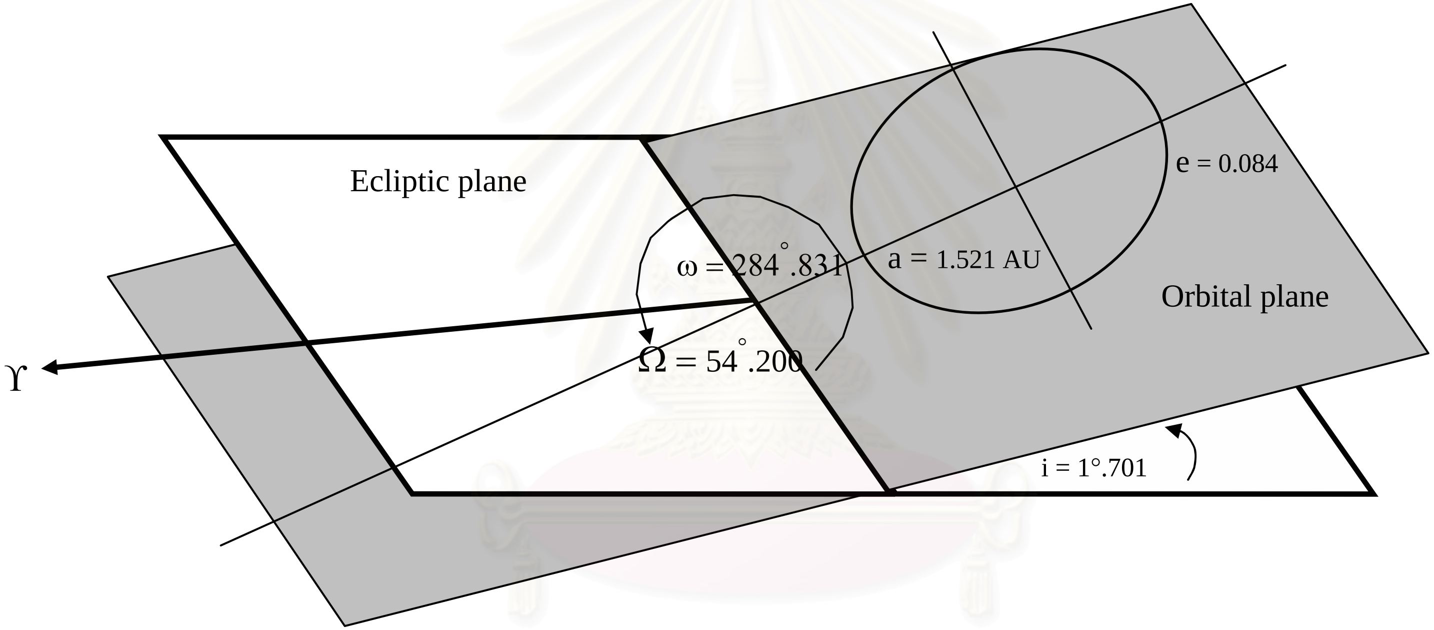


Figure 9.7 The orbit of Mars in space.

Jupiter		
Classical Elements	**Gauss's Method**	**Astronomical Almanac**
a	5.145	5.202603
e	0.044	0.048493
i	1.345	1.3033
Ω	107.390	100.4628
ω	293.719	273.8665
T	2447335.943	2447267.308399

Table 9.7 Comparison the classical elements of Jupiter from Gauss's method with the astronomical almanac.

Mars		
Classical Elements	**Gauss's Method**	**Astronomical Almanac**
a	1.521	1.523679
e	0.084	0.093400
i	1.701	1.8498
Ω	54.200	49.5609
ω	284.831	286.4951
T	2450830.349	2450829.358385

Table 9.8 Comparison the classical elements of Mars from Gauss's method with the astronomical almanac.

9.3 Discussion

A comparison of the classical elements from the Gauss's method (Figures 9.6 and 9.7) with the astronomical almanac (1999) [45] in tables 9.7 and 9.8. There are errors in the orbital determination. These errors are accepted in some cases of consideration. The Gauss's method has limitation. If the arc of the celestial body's apparent path has too little curvature, the coefficient D_0 in Equations 8.10, 8.11 and 8.12 will approach zero. This method may be failed [2,9,14,27,42,43,44].

Consider the time duration of observation of Jupiter and Mars, tables 9.1 through 9.4, compares to the time duration of planet's orbit around the Sun. The observation time interval is quite small. So the errors may arise from this reason.

The other way, there are errors in the observational data. Our photographs were taken from the standard lens (short focus 50 mm). This gives the accuracy of the pictures less than 1 minute. If one wants to improves this accuracy, the picture should be $10'' \times 12''$ or more. The picture of that size is more expensive than the smaller size. The pictures in this thesis are $4'' \times 6''$. The reference stars are located in the margin of the photograph. There are distortions in our pictures when we take measurement for the distance of the celestial bodies. Distortions are decreased when the reference stars are located in the middle of the photograph. If one uses the long focus, this problem will be solved.

We can neglect the time error during observational data. The Earth spends one day for rotating 1 degree. If we spend 20 minuets for time error, the Earth rotates less than 1 minute. That why we can neglect the observational time error.

The other problem is a divergence of our computer's results. This is the limitation of the Gauss's method to determine the orbit of the celestial body with a three very closely spaced observations. There are a few observational data give a computer's results converge (tables 9.5 and 9.6). A good observational data are shown in tables 9.7 and 9.8.

CHAPTER 10

SUMMARY

Gauss's method can determine the position and velocity vectors of the planets in solar system while a direct observation is not accomplished. Gauss's method needs only three observational data, time and angular data (right ascension and declination) to determine the orbit of celestial body.

From the observation, we know only the fixed star's coordinates in the equatorial coordinate system. The right ascension and declination of the planets can be determined by using the method of dependences. This method needs to know the angular data (right ascension and declination) of three reference stars or five reference stars to compute the angular data (right ascension and declination) of the planets. For three-reference star case, we plot it in the graphing paper. The distance between these celestial bodies in this thesis is found. We get the dependence D_i values. Finally, the right ascension and declination of the planets at a given epoch time can be computed.

In the case of five reference stars, the coordinates of these celestial bodies are found by using the microcomputer (not graphing paper such as in previous case). This case gives more accurate position of the reference stars and planets than previous case. However, during the observation, some problems such as the moonlight, sunlight, artificial light from houses, clouds make pictures not clear. The observation data is not completely smooth, for example, the Jupiter's data in the thesis. However, if data is correct for only three positions of the celestial body, Gauss's method can determines the correct orbit.

Preliminary approximation of the Gaussian method is a very fundamental and useful transformation from a set of angular data to a set of position and velocity in inertial space. By considering two constraints, one states the position vector of the planet, the Sun and the Earth are the triangular forms. This constraint is called a geometric constraint. Another one represents the heliocentric vector of the planet expressed in the term of **f** and **g** series. This constraint is called a dynamics constraint.

In this thesis, the position and velocity vectors at epoch time of the planet in heliocentric coordinates system are calculated by using the numerical iteration. The first part of approximate of the geometrical coefficients c_1 and c_3 involve the second order **f** and **g** series. After the first loop of iteration, these coefficients are adjusted by the universal formulation. The iteration terminates when the topocentric vector, $\vec{\rho}$ gives a suitable value. The topocentric vector is determined by two cases of our constraints.

Their components of position and velocity from Gauss's method do not clearly reveal the orbit's size, shape and orientation in space. However, this problem can be solved in the celestial mechanics. The classical elements are introduced to explain the orbit in the heliocentric system clearer.

In this thesis we use Gauss's method to transform a set of angular data from observation to a set of position and velocity vectors in the solar system at epoch time. And the orbit geometry in celestial mechanics gives six classical elements of these planets (Jupiter and Mars). These values are shown in table 9.5 and table 9.6 in Chapter 9. The next step is to compare the classical elements from our calculation with the astronomical almanac [45,46] (tables 9.7 and 9.8 in Chapter 9). There are many errors. These errors probably come from the observation instruments mentioned in Chapter 9.

We write another program for checking our program (Method of Gauss). Position and velocity components of Jupiter and Mars from Gauss method are the initial conditions. In this procedure, the right ascension and declination of Jupiter and Mars at epoch time are calculated again and compared with the astronomical almanac [45] and the right ascension and declination from observations. We are agreed with each other that means can improve the input data in Gauss method and our programs for determination of orbit of Jupiter and Mars. Moreover, if we have the input data of other planet or comet then our program can find its orbit accurately.

However, Marsden [9] reported that the Gauss's method works for calculating the position of the celestial body in the past and future approximately using only three sets of the observational data. If the instrument can give an accurate data, the orbit determination can probably give an accurate result.

REFERENCES

1. Szebehely, V. G. Adventures in Celestial Mechanics. Texas: University of Texas, 1989.

2. Danby, J. M. A. Fundamentals of Celestial Mechanics. Richmond: Willmann-Bell, 1992.

3. Peterson, I. Newton's Clock: Chaos in the Solar System. New York: W. H. Freeman, 1993.

4. Peterson, I. Gauss's Orbit. MAA Online. Available from: file:///A|/Gauss's orbits.htm (1999).

5. Teets, D., and Whitehead K. The Discovery of Ceres: How Gauss Became Famous. Mathematics Magazine 72 (1999): 83.

6. Cunningham, C. S. Giuseppe Piazzi and the Missing Planet. Sky & Telescope (1992): 274.

7. Stewart, I. Gauss. Scientific American (1977): 123.

8. Taff, L. G. On Initial Orbit Determination. Astron., J. 89 (1984): 1426-1428.

9. Marsden, B. G. Initial Orbit Determination: The Pragmatist's Point of View. Astron., J. 90 (1985): 1541-1547.

10. Stumpff, K. Astron. Nachr. 275 (1974): 108.

11. Herrick, S. Astron., J. 70 (1965): 309.

12. Encke, J. F. Berliner Astron., Jahrbuch fur. (1851): 377.

13. Merton, G. Mon. Not. R. Astron. Soc. 85 (1925): 693.

14. Escobal, P. R. Method of Orbit Determination. New York: John Wiley & Sons, 1976.

15. The Astronomical Almanac 1984. Washington D. C.: U. S. Government Printing Office, 1983.

16. Woolard, E. W., and Clemence, G. M. Spherical Astronomy. New York: Academic Press, 1966.

17. The Astronomical Almanac 1985. Washington D. C.: U. S. Government Printing Offices, 1984.

18. Hestences, D. New Foundations for Classical Mechanics. Dordrecht: Kluwer Academic, 1993.

19. Chow, T. L. Classical Mechanics. New York: John Wiley & Suns, 1995.

20. Sirisomboonlarp, P. An Introduction to Celestial Mechanics in Thai Version. Bangkok: Chulalongkorn University, 1991.

21. Smart, W. M. Celestial Mechanics. London: Longmans-Green, 1953.

22. Brouwer, D., and Clemence, G. M. Method of Celestial Mechanics. New York: Academic Press, 1961.

23. Roy, A. E. Orbital Motion. Bristol: Adam Hilger, 1978.

24. Taff, L. G. Celestial Mechanics. New York: John Wiley & Sons, 1985.

25. Spiegel, M. R. Mathematical Handbook of Formulas and Tables. New York: McGraw-Hill, 1968.

26. Fowles, G. R., and Cassiday, G. L. Analytical Mechanics. Philadelphia: Harcourt Brance & Company, 1993.

27. Boulet, D. L. Methods of Orbit Determination for the Microcomputer. Richmond: Willmann-Bell, 1991.

28. Baker, R. M. L., and Makemson, M. W. An Introduction to Astrodynamics. New York: Academic Press, 1967.

29. Bate, R. A.; Mueller, D. D.; and White, J. E. Fundamentals of Astrodynamics. New York: Dover, 1971.

30. Boccaleti, D., and Pucacco, G. Theory of Orbits Volume 1. Berlin: Springer-Verlag, 1996.

31. Sirisomboonlarp, P. The General Theory of Celestial Mechanics in Thai Version. Bangkok: Chulalongkorn University, 1990.

32. Barker, R. M. L. Astrodynamics: Applications and Advanced Topics. New York: Academic Press, 1967.

33. Smart, W. M. Text-Book on Spherical Astronomy. London: Cambridge University, 1971.

34. Konig, A. Astronomical Techniques Volume 2. California: University of Chicago, 1962.

35. Turner, H. H. Monthly Notices. 115 (1893): 13.

36. Schlesinger, F. The Trigonometrical Parallaxes of 851 Stars. Astronomical Observatory of Yale University. 8 (1936): 3-14.

37. Schlesinger, F. Astrophysical Journal. 33 (1911): 161.

38. Schlesinger F. A. Short Method for Deriving Positions of Asteroids, Comets, etc., from Photographs. Astron., J. 37 (1926): 77-84.

39. Comrie, L. J. Note on the Reduction of Photographic Plates. Journal of the British Astronomical Association. 39 (1929): 203-209.

40. Van de Kamp, P. Principles of Astrometry. San Francisco: W. H. Freeman and Company, 1967.

41. Gauss, C. F. Theoria Motus Corporum Coelestium. (Hamburg: Perthes and Besser, 1809), English translation by C. H. Davis (1857). New York: Dover, 1963.

42. Herget, P. The Computation of Orbits. Cincinnati: Ann Arbor, MI: Edwards Bros, 1948.

43. Dubyago, A. D. The Determination of Orbits. New York: Macmillan, 1961.

44. Mounton, F. R. An Introduction to Celestial Mechanics. New York: Dover, 1970.

45. Temmonkansokunempyoo 1999. Tokyo: Chijinshokan, 1999.

46. Temmonkansokunempyoo 2000. Tokyo: Chijinshokan, 2000.

47. Temmongaido, January 1999. Tokyo: Seibundooshinkosha, 1999.

48. Temmongaido, Febuary 1999. Tokyo: Seibundooshinkosha, 1999.

49. Temmongaido, March 1999. Tokyo: Seibundooshinkosha, 1999.

50. Temmongaido, August 1999. Tokyo: Seibundooshinkosha, 1999.

51. Temmongaido, November 1999. Tokyo: Seibundooshinkosha, 1999.

52. Rika Nempyoo Heisei Kyu Nen (Chronological Scientific Table 1999). Tokyo: Maruzen National Astronomical Observatory, 2000.

53. Chetrakarn, S. Fundamentals of Photography in Thai Version. Bangkok: Technique Printing, 1998.

54. Cheurwiwat, T. Amateur Photography in Thai Version. Bangkok: Dwangkamol, 1996.

55. Roy, A. Oxford Illustrated Encyclopedia of the Universe. New York: Oxford University, 1992.

56. Arfken, G. B., and Weber, H. J. Mathematical Methods for Physicists. San Diego: Academic Press, 1995.

APPENDICES

APPENDIX A

ORBIT IMPROVEMENT

When preliminary orbital elements are used to predict the motion of a celestial body, it is normally found that the observed and computed motion is not in satisfactory agreement. Thus, observed-minus-computed (**O-C**) residuals can be determined for each time of observation by subtracting the computed coordinates from the measured coordinates. When there exist at least three reliable observations which cover a significant part of the orbital motion, yet are not too distant from the epoch of the preliminary elements, it is often possible to improve the elements by a straightforward differential correction process which ignores perturbations. If the situation does not permit this simple approach, then perturbations should be taken into account each time the residuals are determined.

In contrast to the dynamical problem of determining the orbital parameters initially, differential correction is primarily a numerical procedure which uses a multiple linear least squares regression to make small changes to the elements in order to minimize the **O-C** residuals. There is no guarantee that the resulting element set will ultimately prove to be better than the preliminary one. Only time and additional observations can finally decide. However, given an initial element set which is not too far off the mark and observations sufficiently accurate and spaced to get a representative sample of the conic section of the orbit, the least squares differential correction process can be very effective [14,22,32].

A.1 The Differential Equations of Condition

A celestial body's right ascension $\boldsymbol{\alpha}$ and declination $\boldsymbol{\delta}$ are complex functions of the orbital elements and the components of $\vec{\mathbf{R}}$, the position of the dynamical center. However, since the vector $\vec{\mathbf{R}}$ may be regarded as accurately known, in need of no improvement, we can simplify our problem by considering only the functional dependence of $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ on the position and velocity elements

$$\vec{r}_0 = \{x_0, y_0, z_0\}$$
$$\vec{v}_0 = \{\dot{x}_0, \dot{y}_0, \dot{z}_0\}'$$

at the arbitrary epoch $\mathbf{t_0}$. Thus, we let

$$\alpha = \alpha(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0) \qquad \textbf{(A.1)}$$

$$\delta = \delta(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)\,, \qquad \textbf{(A.2)}$$

and apply the definition of the total derivative of a function to obtain [14,56]

$$d\alpha = \frac{\partial \alpha}{\partial x_0} dx_0 + \frac{\partial \alpha}{\partial y_0} dy_0 + \frac{\partial \alpha}{\partial z_0} dz_0 + \dots + \frac{\partial \alpha}{\partial \dot{z}_0} d\dot{z}_0 \qquad \textbf{(A.3)}$$

$$d\delta = \frac{\partial \delta}{\partial x_0} dx_0 + \frac{\partial \delta}{\partial y_0} dy_0 + \frac{\partial \delta}{\partial z_0} dz_0 + \dots + \frac{\partial \delta}{\partial \dot{z}_0} d\dot{z}_0\,. \qquad \textbf{(A.4)}$$

Equations A.3 and A.4 express the amount of change produced in $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ in response to independent changes in one or more of the scalar components of the position and velocity vectors. The partial derivatives represent the individual *rates* at which $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ change with respect to each of the orbital elements. In practice, the differentials can be replaced by finite differences, so that we can write

$$\Delta\alpha = \frac{\partial\alpha}{\partial x_0}\Delta x_0 + \frac{\partial\alpha}{\partial y_0}\Delta y_0 + \frac{\partial\alpha}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\alpha}{\partial \dot{z}}\Delta \dot{z}_0 \qquad \textbf{(A.5)}$$

$$\Delta\delta = \frac{\partial\delta}{\partial x_0}\Delta x_0 + \frac{\partial\delta}{\partial y_0}\Delta y_0 + \frac{\partial\delta}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\delta}{\partial \dot{z}}\Delta \dot{z}_0, \qquad \textbf{(A.6)}$$

where $\boldsymbol{\Delta\alpha}$ and $\boldsymbol{\Delta\delta}$ are the measured **O-C** residuals in right ascension and declination, respectively, and $\mathbf{\Delta x}_0$, $\mathbf{\Delta y}_0$, …, $\mathbf{\Delta \dot{z}}_0$ are the small changes needed to improve the orbital elements so that the residuals in α and δ are eliminated.

The residuals $\boldsymbol{\Delta\alpha}$ and $\boldsymbol{\Delta\delta}$ in Equations A.5 and A.6 are known quantities obtained from measurements of the orbiting body's position on the celestial sphere. Three such observations would enable us to write six independent linear equations of condition. However, since there may be more than three sets of residuals available, we have in general

$$\begin{aligned}
\Delta\alpha_1 &= \frac{\partial\alpha_1}{\partial x_0}\Delta x_0 + \frac{\partial\alpha_1}{\partial y_0}\Delta y_0 + \frac{\partial\alpha_1}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\alpha_1}{\partial \dot{z}}\Delta \dot{z}_0 \\
\Delta\alpha_2 &= \frac{\partial\alpha_2}{\partial x_0}\Delta x_0 + \frac{\partial\alpha_2}{\partial y_0}\Delta y_0 + \frac{\partial\alpha_2}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\alpha_2}{\partial \dot{z}}\Delta \dot{z}_0 \\
\Delta\alpha_3 &= \frac{\partial\alpha_3}{\partial x_0}\Delta x_0 + \frac{\partial\alpha_3}{\partial y_0}\Delta y_0 + \frac{\partial\alpha_3}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\alpha_3}{\partial \dot{z}}\Delta \dot{z}_0 \\
&\ \ \vdots \\
\Delta\alpha_n &= \frac{\partial\alpha_n}{\partial x_0}\Delta x_0 + \frac{\partial\alpha_n}{\partial y_0}\Delta y_0 + \frac{\partial\alpha_n}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\alpha_n}{\partial \dot{z}}\Delta \dot{z}_0 \\
\Delta\delta_1 &= \frac{\partial\delta_1}{\partial x_0}\Delta x_0 + \frac{\partial\delta_1}{\partial y_0}\Delta y_0 + \frac{\partial\delta_1}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\delta_1}{\partial \dot{z}}\Delta \dot{z}_0 \\
\Delta\delta_2 &= \frac{\partial\delta_2}{\partial x_0}\Delta x_0 + \frac{\partial\delta_2}{\partial y_0}\Delta y_0 + \frac{\partial\delta_2}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\delta_2}{\partial \dot{z}}\Delta \dot{z}_0 \\
\Delta\delta_3 &= \frac{\partial\delta_3}{\partial x_0}\Delta x_0 + \frac{\partial\delta_3}{\partial y_0}\Delta y_0 + \frac{\partial\delta_3}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\delta_3}{\partial \dot{z}}\Delta \dot{z}_0 \qquad \textbf{(A.7)} \\
&\ \ \vdots \\
\Delta\delta_n &= \frac{\partial\delta_n}{\partial x_0}\Delta x_0 + \frac{\partial\delta_n}{\partial y_0}\Delta y_0 + \frac{\partial\delta_n}{\partial z_0}\Delta z_0 + \ldots + \frac{\partial\delta_n}{\partial \dot{z}}\Delta \dot{z}_0,
\end{aligned}$$

where $\mathbf{n} \geq 3$. Now, if we are somehow able to obtain reasonable values for the partial derivatives, then these equations can be solved by multiple linear least squares regression to yield values for the corrections $\mathbf{\Delta x}_0$, $\mathbf{\Delta y}_0$, …, $\mathbf{\Delta \dot{z}}_0$ which best fit all the date

A.2 Numerical Evaluation of the Partial Derivatives

Values for the partial derivatives in Equations A.7 can be determined to sufficient accuracy by a simple numerical process which is accomplished by the computer. If $\boldsymbol{\varepsilon}$ represents any one of the six orbital elements, and $\boldsymbol{\Delta\varepsilon}$ is some small change introduced in that typical element, then the partial derivatives of $\boldsymbol{\alpha_i}$ and $\boldsymbol{\delta_i}$ with respect to $\boldsymbol{\varepsilon}$ can be approximated as follows [14]:

$$\frac{\partial \alpha_i}{\partial \varepsilon} \approx \frac{\alpha_i(x_0,\ldots,\varepsilon_0+\Delta\varepsilon,..,\dot{z}_0)-\alpha_i(x_0,\ldots,\varepsilon_0,\ldots,\dot{z}_0)}{\Delta\varepsilon} \qquad \textbf{(A.8)}$$

$$\frac{\partial \delta_i}{\partial \varepsilon} \approx \frac{\delta_i(x_0,\ldots,\varepsilon_0+\Delta\varepsilon,\ldots,\dot{z}_0)-\delta_i(x_0,\ldots,\varepsilon_0,\ldots,\dot{z}_0)}{\Delta\varepsilon}, \qquad \textbf{(A.9)}$$

where **i** = 1 to **n**. Therefore, by incrementing each element in turn while the others maintain their original values, Equations A.8 and A.9 will produce the six partial derivatives of $\boldsymbol{\alpha_i}$ and $\boldsymbol{\delta_i}$ at each observation time $\mathbf{t_i}$. In most cases, selecting an incrementation $\Delta\varepsilon$ which is equal to a few percent of $\boldsymbol{\varepsilon}$ will produce a small change in $\boldsymbol{\alpha_i}$ and $\boldsymbol{\delta_i}$ which is sufficient to yield satisfactory approximations of he partial derivatives [14,32,56].

Once the partial derivatives have been determined, Equations A.7 are solved for the corrections $\boldsymbol{\Delta x_0}$, $\boldsymbol{\Delta y_0},\ldots,$ $\boldsymbol{\Delta \dot{z}_0}$. When these are added to the original elements, a new set

$$\begin{aligned}\vec{r}_0 &= \{x_0+\Delta x_0, y_0+\Delta y_0, z_0+\Delta z_0\}\\ \vec{v}_0 &= \{\dot{x}_0+\Delta\dot{x}_0, \dot{y}_0+\Delta\dot{y}_0, \dot{z}_0+\Delta\dot{z}_0\}\end{aligned} \qquad \textbf{(A.10)}$$

is obtained for the arbitrary epoch $\mathbf{t_0}$. Finally, new angular positions are generated and compared to the observations. If significant residuals remain, these may be used to compute further improvements to the elements and the entire process repeated until the observed and computed $\boldsymbol{\alpha_i}$ and $\boldsymbol{\delta_i}$ agree within the limits of the accuracy of each angular measurement.

From the results in tables 9.5 and 9.6 we found that the classical elements are differ for different set samples. So there is a possibility that the orbit determination by three-observation is insufficient. We then improve the position and velocity elements from observational data by a program C (see Appendices C.7 and C.8). The improved data were shown tables A.1 and A.2

Which we have chosen position and velocity elements of the planets from Figures 9.7 and 9.8. Form tables A.1 and A.2 we found that in Mars's case, there are deviations in classical elements when compare to the astronomical almanac [45,46] but these are acceptable. In Jupiter case, it can not be improve since the calculation is diverge. This may be caused by right ascension and declination of Jupiter with reduction technique. The information has shown to be divided into two periods. The former is in January and latter is in September – March. This is shown in tables 9.1 through 9.4.

Jupiter			
Classical Elements from Gauss's method		**Classical Elements from Improved Data**	
a	5.145	**a**	-
e	0.044	**e**	-
i	1.345	**i**	-
Ω	107.390	**Ω**	-
ω	293.719	**ω**	-
T	2447335.943	**T**	-

Table A.1

Mars			
Classical Elements from Gauss's Method		**Classical Elements from Improved Data**	
a	1.521	**a**	1.494
e	0.084	**e**	0.066
i	1.701	**i**	1.863
Ω	54.200	**Ω**	61.294
ω	284.831	**ω**	295.298
T	2450830.349	**T**	2540879.421

Table A.2

Classical Elements (C.E.)	Jupiter		
	Astronomical Almanac	Method of Gauss (C.E.)	Orbit Improvement (C.E.)
a	5.202603	5.145	-
e	0.048493	0.044	-
i	1.3033	1.345	-
Ω	100.4628	107.390	-
ω	273.8665	293.719	-
T	2447267.308399	2447335.943	-

Table A.3

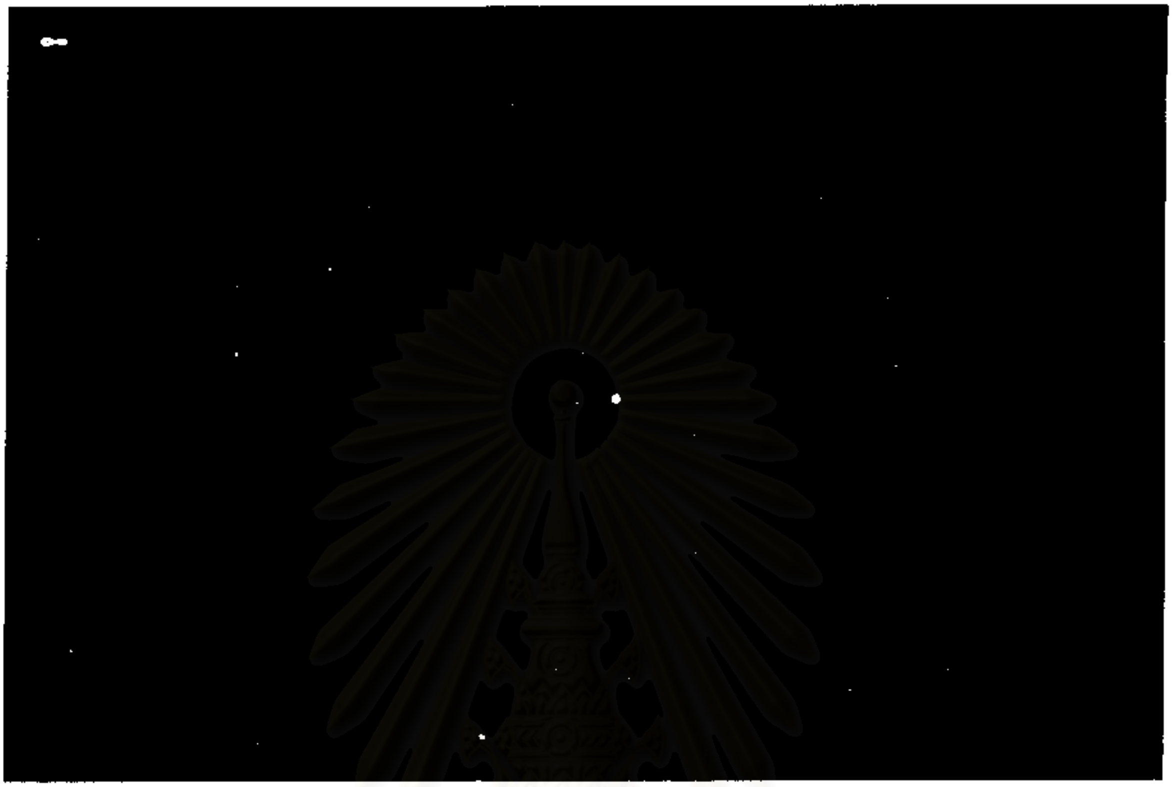
Classical Elements (C.E.)	Mars		
	Astronomical Almanac	Method of Gauss (C.E.)	Orbit Improvement (C.E.)
a	1.523679	1.521	1.494
e	0.093400	0.084	0.066
i	1.8498	1.701	1.863
Ω	49.5609	54.200	61.294
ω	286.4951	284.831	295.298
T	2450829.358385	2450830.349	2540879.421

Table A.4

APPENDIX B

SOME PHOTOGRAPHS ARE USED IN THIS THESIS

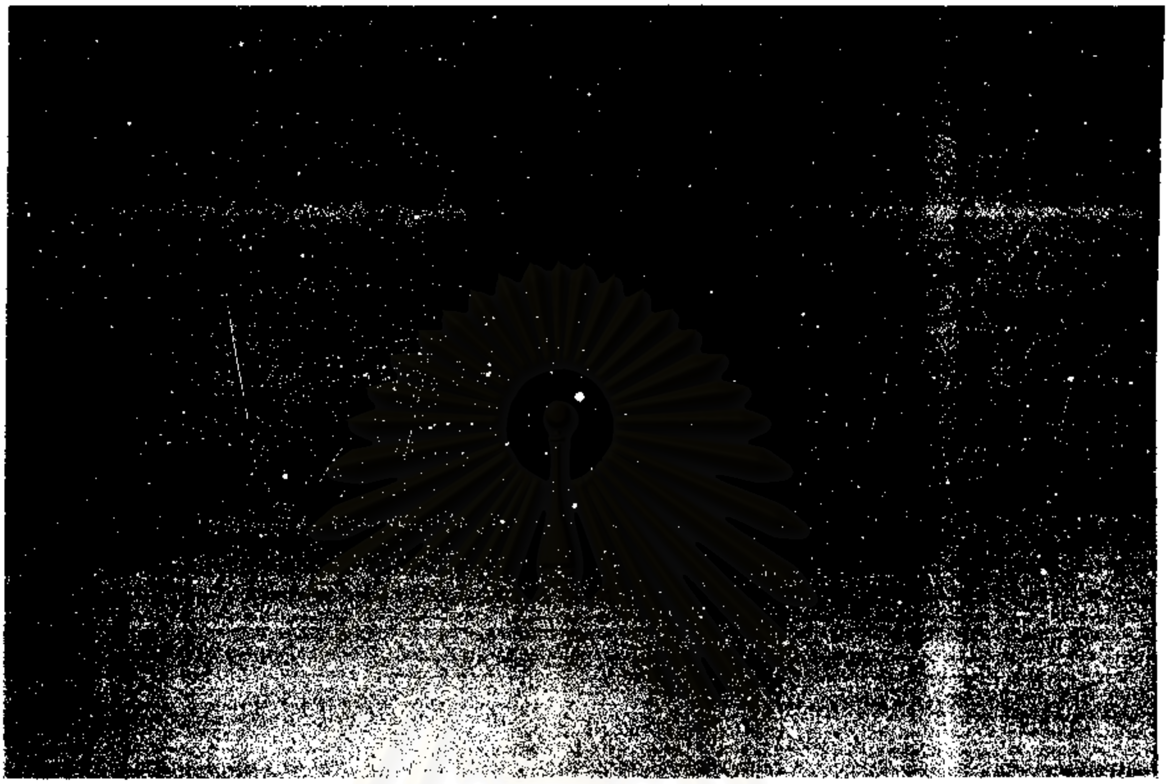
B. 1 Jupiter



The picture was taken at Salaya, Nakornpathom, September 19, 1999.

The picture was taken at Bangkok, January 4, 2000.

B. 2 Mars



The picture was taken at Salaya, Nakornpathom, March 14, 1999.

The picture was taken at Salaya, Nakornpathom, July 2, 1999.

APPENDIX C

COMPUTER PROGRAMS

Appendix C.1

Julian Date

```
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <conio.h>
main()
{
 int D,M,Y;
 double J0,UT,JD,m,h;
 clrscr();
 printf(" Date = ");scanf("%d",&D);
 printf(" month = ");scanf("%d",&M);
 printf(" year  = ");scanf("%d",&Y);
 J0 = 367.0*Y-floor(floor((7.0/4.0)*(Y+floor((M+9.0)/12.0))))
      +floor(275.0*M/9.0)+D+1721013.5;
  printf("\n J0 = %0.8lf\n\n",J0);
  printf(" hour = ");scanf("%lf",&h);
  printf(" minus : m = ");scanf("%lf",&m);
UT = h+m/60;
 printf("\n UT = %0.8lf\n ",UT);
JD=J0+UT/24;
printf(" \n JD = %0.8lf\n ",JD);
getch();
 return(0);
}
```

Appendix C.2

Method of Dependence (3-reference stars)

```
#include<stdio.h>
#include<math.h>
#include<conio.h>

main()
{
double AP,SP,BQ,SQ,CR,SR,AS,CS,BS,D,Da,Db,Dc,a1,b1,a2,b2,a3,b3,a,b;
clrscr();
printf("Input AS and AP\n");
scanf("%lf %lf",&AS,&AP);
printf("Input right ascension and declination of star 1\n");
scanf("%lf %lf",&a1,&b1);
printf("Input BS and BQ\n");
scanf("%lf %lf",&BS,&BQ);
printf("Input right ascension and declination of star 2\n");
scanf("%lf %lf",&a2,&b2);
printf("Input CS and CR\n");
scanf("%lf %lf",&CS,&CR);
printf("Input right ascension and declination of star 3\n");
scanf("%lf %lf",&a3,&b3);
SP=AP-AS;
SQ=BQ-BS;
SR=CR-CS;
Da=SP/AP;
Db=SQ/BQ;
Dc=SR/CR;
D=Da+Db+Dc;
a=a1*Da+a2*Db+a3*Dc;
b=b1*Da+b2*Db+b3*Dc;
printf("-------------------------------------------------\n");
printf("Dependences = %.7lf\n",D);
printf("Right ascension = %.7lf\n",a);
printf("Declination = %.7lf\n",b);
getch();
return(0);
}
```

Appendix C.3

Method of Dependence (5-reference stars)

```
#include<stdio.h>
#include<math.h>
#include<conio.h>

main()
{
double x1,x2,x3,x4,x5,y1,y2,y3,y4,y5,xs,ys,sumx,sumy,xr,yr,X1,X2,X3,
       X4,X5,Y1,Y2,Y3,Y4,Y5,XS,YS,SUMX,SUMY,SUMXY,SUMX2,SUMY2,D1,D2,
       D3,D4,D5,P,Q,R,SUMD,SUMDr,SUMDx,SUMDd,SUMDy,d1,d2,d3,d4,d5,r1,
       r2,r3,r4,r5,ds,rs;
clrscr();
printf("Please enter xs,ys\n");
scanf("%lf %lf",&xs,&ys);
printf ("Please enter x1,y1,ritht ascension1 and declination1\n");
scanf ("%lf %lf %lf %lf",&x1,&y1,&r1,&d1);
printf ("Please enter x2,y2,right ascension2 and declination2\n");
scanf ("%lf %lf %lf %lf",&x2,&y2,&r2,&d2);
printf ("please enter x3,y3,right ascension3 and declination3\n");
scanf ("%lf %lf %lf %lf",&x3,&y3,&r3,&d3);
printf ("please enter x4,y4,right ascension4 and declination4\n");
scanf ("%lf %lf %lf %lf",&x4,&y4,&r4,&d4);
printf ("please enter x5,y5,right ascension5 and declination5\n");
scanf ("%lf %lf %lf %lf",&x5,&y5,&r5,&d5);


sumx=x1+x2+x3+x4+x5;
sumy=y1+y2+y3+y4+y5;
xr=sumx/5.0;   yr=sumy/5.0;
X1=x1-xr;   X2=x2-xr;   X3=x3-xr;   X4=x4-xr;   X5=x5-xr;   XS=xs-xr;
Y1=y1-yr;   Y2=y2-yr;   Y3=y3-yr;   Y4=y4-yr;   Y5=y5-yr;   YS=ys-yr;

SUMX=X1+X2+X3+X4+X5;
printf ("SUMX=%.7e\n",SUMX);
SUMY=Y1+Y2+Y3+Y4+Y5;
printf ("SUMY=%.7e\n",SUMY);
SUMX2=X1*X1+X2*X2+X3*X3+X4*X4+X5*X5;
SUMY2=Y1*Y1+Y2*Y2+Y3*Y3+Y4*Y4+Y5*Y5;
SUMXY=X1*Y1+X2*Y2+X3*Y3+X4*Y4+X5*Y5;
P=(XS*SUMY2-YS*SUMXY)/(SUMX2*SUMY2-SUMXY*SUMXY);
Q=(XS*SUMXY-YS*SUMX2)/(SUMXY*SUMXY-SUMY2*SUMX2);
R=0.2;
D1=P*X1+Q*Y1+R;
D2=P*X2+Q*Y2+R;
D3=P*X3+Q*Y3+R;
D4=P*X4+Q*Y4+R;
D5=P*X5+Q*Y5+R;
SUMD=D1+D2+D3+D4+D5;
printf ("SUMD=%.7e\n",SUMD);
SUMDr=D1*r1+D2*r2+D3*r3+D4*r4+D5*r5;
SUMDx=D1*x1+D2*x2+D3*x3+D4*x4+D5*x5;
rs=SUMDr+xs-SUMDx;
SUMDd=D1*d1+D2*d2+D3*d3+D4*d4+D5*d5;
SUMDy=D1*y1+D2*y2+D3*y3+D4*y4+D5*y5;
ds=SUMDd+ys-SUMDy;

printf("Right Ascension of star number 1,2,3,4,and 5 are\n");
printf("%lf\t",r1);printf("%lf\t",r2);printf("%lf\t",r3);printf("%lf\t",r4);
printf("%lf\t",r5);printf("Declination of star number 1,2,3,4,and 5 are\n");
printf("%lf\t",d1);printf("%lf\t",d2);printf("%lf\t",d3);printf("%lf\t",d4);
printf("%lf\t",d5);printf("x-coordinate of star number 1,2,3,4,and 5 are\n");
printf("%lf\t",x1);printf("%lf\t",x2);printf("%lf\t",x3);printf("%lf\t",x4);
printf("%lf\t",x5);printf("y-coordinate of star number 1,2,3,4,and 5 are\n");
```

```
printf("%lf\t",y1);printf("%lf\t",y2);printf("%lf\t",y3);printf("%lf\t",y4);
printf("%lf\t",y5);printf("!!!!Result!!!!\n");
printf("Right Ascention is ---->%.7lf\n",rs);
printf("Declination is -------->%.7lf\n",ds);
getch();
return(0);
}
```

Appendix C.4

Method of Gauss (Jupiter)

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

double dij(double a[4][4],double b[4][4],double c[4][4],int x,int y,int z)
{
   double r[4],sum;
  int i;
  sum=0.0;
    r[1]=a[x][2]*b[y][3]-(a[x][3]*b[y][2]);
    r[2]=a[x][3]*b[y][1]-(a[x][1]*b[y][3]);
    r[3]=a[x][1]*b[y][2]-(a[x][2]*b[y][1]);
    for (i=1;i<=3;i=i+1)
    {
      sum=sum+(r[i]*c[z][i]);
    }
return sum;
}

double dot(double a[4][4],double b[4][4],int x,int y)
{
 double sum=0.0;
 int i;
  for (i=1;i<=3;i++)
   {
    sum=sum+a[x][i]*b[y][i];
   }
return sum;
}

double newton(double tiu[3],double l[4][4],double rr[4][4],double mass,
       double d0,double gv)
{
    double a,b,cc,A,B,E,F,r,x1,x2;
  int k=0;
    x1=tiu[3]/tiu[2];
    x2=-tiu[1]/tiu[2];
     E=-2.0*dot(l,rr,2,2);
     F=dot(rr,rr,2,2);
   A=-(x1*dij(l,rr,l,1,1,3)-dij(l,rr,l,1,2,3)+(x2)*dij(l,rr,l,1,3,3))/d0;
    B=-((x1*(tiu[2]*tiu[2]-tiu[3]*tiu[3])/6.0)*dij(l,rr,l,1,1,3)+(x2
      *(tiu[2]*tiu[2]-tiu[1]*tiu[1])/6.0)*dij(l,rr,l,1,3,3))/d0;
    a=-(A*A+A*E+F);
    b=-mass*(2.0*A*B+B*E);
    cc=-(mass*mass)*(B*B);
    do
    {
   r=gv; //gv calculated values,r old
   gv=gv-((pow(gv,8.0)+a*pow(gv,6.0)+b*(gv*gv*gv)+cc)/(8.0*pow(gv,7.0)
      +(6.0*a)*pow(gv,5.0)+(3.0*gv*gv*gv)));
   k=k+1;
    }while(fabs(gv-r)>=10e-6);
   return r;
}

double findl(double all[4],double bll[4],int x,int y)
{
    double l[4][4],L,a,b,Q1=M_PI/180.0;
    int i;
   for (i=1;i<=3;i++)
    {
      a=all[i]*15.0*Q1;
```

```
        b=bll[i]*Q1;
        l[i][1]=cos(b)*cos(a);
        l[i][2]=cos(b)*sin(a);
        l[i][3]=sin(b);
       }
     L=l[x][y];
return L;
}

double findp(double f[4],double g[4],double r[4][4],double l[4][4],int x)
{
 double p[4];
 double d0,pp,c1,c2,c3,dfg;
  dfg=f[1]*g[3]-(f[3]*g[1]);
   c1=g[3]/dfg;
   c2=-1.0;
   c3=-g[1]/dfg;
   d0=dij(l,l,l,1,2,3);
  p[1]=(c1*dij(r,l,l,1,2,3)+c2*dij(r,l,l,2,2,3)+c3*dij(r,l,l,3,2,3))/(c1*d0);
  p[2]=(c1*dij(l,r,l,1,1,3)+c2*dij(l,r,l,1,2,3)+c3*dij(l,r,l,1,3,3))/(c2*d0);
  p[3]=(c1*dij(l,r,l,2,1,1)+c2*dij(l,r,l,2,2,1)+c3*dij(l,r,l,2,3,1))/(c3*d0);
 pp=p[x];
 return pp;
}

double findv(double r[4][4],double f[4],double g[4],int x)
{
  double v[4],dgf,d1,d3,v2;
   int i;
   dgf=f[1]*g[3]-(f[3]*g[1]);
   d1=-f[3]/dgf;
   d3=f[1]/dgf;
 for (i=1;i<=3;i++)
     {
       v[i]=d1*r[1][i]+d3*r[3][i];
      }
   v2=v[x];
   return v2;
}

double findfg(double r[4][4],double v[4][4],double mu,double dt,double k,int
{
 double r0,d0,ai,c0,ww,xx,x1,xa,x3,B[20];
  double fx,df,x2,cc,uu,ss,f,g,fp,gp,rr,num;
  int i;
  B[1]=1.0;
     for(i=2;i<=19;i=i+1)
    {
   B[i]=B[i-1]/i;
    }
   r0=sqrt(dot(r,r,2,2));
   d0=dot(r,v,2,2)/sqrt(mu);
   ai=2.0/r0-dot(v,v,2,2)/(mu);
   c0=1.0-r0*ai;
   ww=k*dt*sqrt(mu);
   xx=ww/r0;
   fx=0.0;df=1.0;
  do
   {
      xx=xx-fx/df;
      x2=xx*xx;
      xa=x2*ai;
      x3=x2*xx;
      cc=x2*(B[2]-xa*(B[4]-xa*(B[6]-xa*(B[8]-xa*(B[10]-xa*(B[12]
          -xa*(B[14]-xa*(B[16]-xa*(B[18])))))))));
      uu=x3*(B[3]-xa*(B[5]-xa*(B[7]-xa*(B[9]-xa*(B[11]-xa*(B[13]
          -xa*(B[15]-xa*(B[17]-xa*(B[19])))))))));
      ss=xx-uu*ai;
      df=r0+c0*cc+d0*ss;
      fx=r0*xx+c0*uu+d0*cc-ww;
    }while(fabs(fx)>1.0e-6);
```

```
     f=1.0-cc/r0;
     g=(r0*ss+d0-cc)/sqrt(mu);
    rr=r0+c0*cc+d0*ss;
    fp=-sqrt(mu)*ss/(rr*r0);
    gp=1.0-cc/rr;
       if (j==1)    num = f;
       else if (j==2) num = g;
       else if (j==3) num = fp;
       else if (j==4) num = gp;
    return num;
 }

 main()
 {
    double t[4],L[4][4],R[4][4],p[4],c[4],f[4],g[4],tua[4],r[4][4],pt[4];
    double m,k,d0,u2,H,con,R0,D0,AI,C0,CC,WW,XX,X2,XA,UU,SS,X3,v[4][4];
    double sr,sr2,FX,DF,p1,p2,p3,r1,r2,F,G,r3,n=0.0,al[4],bl[4],B[21];
    double d[4],ff[4],gg[4],tt[4],b,X,mu,rv,rr[4][4],hh,HH,T,ee,V,a,e;
    double h[4][4],P,O,I,W,E,M,vv[4][4],xb,yb,sx,cx,srr,er[4][4],N[4][4];
    double NN,Q1=M_PI/180.0,wr,Rr,EE,XP,YP,XB,et,YB,Ff,DFf,rR[4],Vv[4];
    double Bb,EP,PP[4],QQ[4],lrl,qq,nn,rRO[4],VvO[4];
    int i,j,q;
    clrscr();
    con=173.1446;
    k=0.01720209895;
    m=0.000954791;
    mu=1.0+m;

 tt[1]=2451195.322917;
 al[1]=23.695737;
 bl[1]=-3.285087;
 R[1][1]=0.4350922;
 R[1][2]=-0.8094893;
 R[1][3]=-0.3509578;
 tt[2]=2451440.520833;
 al[2]=2.149041;
 bl[2]=11.326239;
 R[2][1]=-1.0018289;
 R[2][2]=0.0698396;
 R[2][3]=0.0302830;
 tt[3]=2451548.298611;
 al[3]=1.597450;
 bl[3]=8.536461;
 R[3][1]=0.2335655;
 R[3][2]=-0.8763429;
 R[3][3]=-0.3799178;

    T=(tt[2]-2451545.0)/36525.0;
    ee=(23.439291-0.0130042*T-0.00000016*T*T)*Q1;
    for (j=1;j<=3;j++)
    {
       for (i=1;i<=3;i++)
       {
    L[j][i]=findl(al,bl,j,i);
       }
    }
    tua[1]=k*(tt[1]-tt[2]);
    tua[3]=k*(tt[3]-tt[2]);
    tua[2]=tua[3]-tua[1];
    sr2=newton(tua,L,R,1.0+m,dij(L,L,L,1,2,3),10.0);
    for (i=1;i<=3;i++)
    {
    p[i]=0.0;
    }
    printf("***Method of Gauss*** \n");
    printf("Determination of Orbit of Jupiter\n");

 do
 {
 sr=sr2;
 u2=(1.0+m)/(sr*sr*sr);
```

```
if (n>0.0)
{
B[1]=1.0;
for(q=2;q<=19;q++)
{
B[q]=B[q-1]/q;
}
for (i=1;i<=3;i++)
{
   H=tua[i];
   R0=sr;
   D0=(dot(r,v,2,2))/sqrt(1.0+m);
   AI=2.0/R0-dot(v,v,2,2)/(1.0+m);
   C0=1.0-R0*AI;
   WW=H*sqrt(1.0+m);
   XX=WW/R0;
   FX=0.0;DF=1.0;
do
{
   XX=XX-FX/DF;
   X2=XX*XX;
   XA=X2*AI;
   X3=X2*XX;
   CC=X2*(B[2]-XA*(B[4]-XA*(B[6]-XA*(B[8]-XA*(B[10]-XA*(B[12]
      -XA*(B[14]-XA*(B[16]-XA*(B[18])))))))));
   UU=X3*(B[3]-XA*(B[5]-XA*(B[7]-XA*(B[9]-XA*(B[11]-XA*(B[13]
     -XA*(B[15]-XA*(B[17]-XA*(B[19])))))))));
   SS=XX-UU*AI;
   FX=R0*XX+C0*UU+D0*CC-WW;
   DF=R0+C0*CC+D0*SS;
}while(fabs(FX)>1.0e-6);
f[i]=1.0-CC/R0;
g[i]=(R0*SS+D0*CC)/sqrt(1.0+m);
f[i]=(ff[i]+f[i])/2.0;
g[i]=(gg[i]+g[i])/2.0;
}
}
else
{
for (i=1;i<=3;i++)
   {
   f[i]=1.0-(u2*tua[i]*tua[i])/2.0;
   g[i]=tua[i]-(u2*tua[i]*tua[i]*tua[i])/6.0;
   }
}
for (j=1;j<=3;j++)
{
   pt[j]=p[j];
   p[j]=findp(f,g,R,L,j);
   v[2][j]=findv(r,f,g,j);
   t[j]=tt[j]-(p[j]/con);
   ff[j]=f[j];
   gg[j]=g[j];
for (i=1;i<=3;i++)
   {
r[j][i]=(p[j]*L[j][i])-R[j][i];
   }
}
sr2=sqrt(dot(r,r,2,2));
tua[1]=k*(t[1]-t[2]);
tua[3]=k*(t[3]-t[2]);
tua[2]=tua[3]-tua[1];
p1=fabs(p[1]-pt[1]);
p2=fabs(p[2]-pt[2]);
p3=fabs(p[3]-pt[3]);
n=n+1;
}while(p1>1.0e-6 || p2>1.0e-6 || p3>1.0e-6);
printf("At epoch Julian date: %lf\n",tt[2]);
printf("The heliocentric distance of Jupiter at epoch time = %.6lf AU\n",sr);
printf("------------------------------------------------------------------------\n");
printf("The position components of Jupiter at epoch time:\n");
```

```
printf("x = %.6lf\t y = %.6lf\t z = %.6lf\n",r[2][1],r[2][2],r[2][3]);
printf("The velocity components of Jupiter at epoch time:\n");
printf("Vx= %.6lf\t Vy= %.6lf\t Vz= %.6lf\n",v[2][1],v[2][2],v[2][3]);

rr[2][1]=r[2][1];
rr[2][2]=r[2][3]*sin(ee)+r[2][2]*cos(ee);
rr[2][3]=r[2][3]*cos(ee)-r[2][2]*sin(ee);
vv[2][1]=v[2][1];
vv[2][2]=v[2][2]*cos(ee)+v[2][3]*sin(ee);
vv[2][3]=v[2][3]*cos(ee)-v[2][2]*sin(ee);

h[2][1]=rr[2][2]*vv[2][3]-rr[2][3]*vv[2][2];
h[2][2]=rr[2][3]*vv[2][1]-rr[2][1]*vv[2][3];
h[2][3]=rr[2][1]*vv[2][2]-rr[2][2]*vv[2][1];
hh=dot(h,h,2,2);
HH=sqrt(hh);

N[2][1]=-h[2][2];
N[2][2]=h[2][1];
N[2][3]=0.0;

NN=sqrt(dot(N,N,2,2));
srr=sqrt(dot(rr,rr,2,2));
rv=dot(rr,vv,2,2);
V=dot(vv,vv,2,2);
a=1.0/((2.0/srr)-(V/mu));

for(i=1;i<=3;i++)
    {
    er[2][i]=((V/mu-1.0/srr)*rr[2][i])-((rv/mu)*vv[2][i]);
    }

P=hh/mu;
e=sqrt(dot(er,er,2,2));
O=acos(N[2][1]/NN)/Q1;
   if(N[2][2]<0.0)
   {
   O=360.0-O;
   }

I=acos(h[2][3]/HH)/Q1;
W=acos(dot(N,er,2,2)/(NN*e))/Q1;
   if (er[2][3]<0.0)
   {
      W=360.0-W;
   }

xb=(P-srr)/e;
yb=rv*sqrt(P/mu)/e;
b=a*sqrt(1.0-e*e);
cx=xb/a+e;
sx=yb/b;
   if(fabs(sx)<=0.707107)
   {
      X=asin(fabs(sx));
   }
   if(fabs(cx)<=0.707107)
   {
      X=acos(fabs(cx));
   }
   if(cx>=0.0 && sx>=0.0)
   {
      X=X;
   }
   if(cx<0.0 && sx>=0.0)
   {
      X=180.0*Q1-X;
   }
   if(cx<0.0 && sx<0.0)
   {
      X=180.0*Q1+X;
```

```
    }
    if(cx>=0.0 && sx<0.0)
    {
       X=360.0*Q1-X;
    }
 M=(X-e*sx)/Q1;
 E=acos((xb/a)+e)/Q1;
 nn=k*sqrt(mu/(a*a*a));
 T=tt[2]-(M*Q1/nn);
 printf("------------------------------------------------------------------\n");
 printf("The classical elements of Jupiter at epoch time : \n");
 printf("Semimajor axis                         a= %.6lf\n",a);
 printf("Eccentricity                           e= %.6lf\n",e);
 printf("Inclination angle from ecliptic        i= %.6lf\n",I);
 printf("Longitude of ascending node            O= %.6lf\n",O);
 printf("Argument of perihelion                 W= %.6lf\n",W);
 printf("Time of perifocal passage              T= %.6lf\n",T);
 I=I*Q1;
 O=O*Q1;
 W=W*Q1;
 M=M*Q1;
 E=E*Q1;
 E=M;
    do
    {
    Ff=E-e*sin(E)-M;
    DFf=1.0-e*cos(E);
    E=E-(Ff/DFf);
    }while(fabs(Ff)>=1.0e-6);
 Rr=a*(1.0-e*cos(E));
 EP=sqrt(mu/a)/Rr;
 Bb=a*sqrt(1.0-e*e);
 XB=a*(cos(E)-e);
 YB=Bb*sin(E);
 XP=-a*EP*sin(E);
 YP=+Bb*EP*cos(E);
 PP[1]=+cos(W)*cos(O)-sin(W)*sin(O)*cos(I);
 PP[2]=+cos(W)*sin(O)+sin(W)*cos(O)*cos(I);
 PP[3]=+sin(W)*sin(I);
 QQ[1]=-sin(W)*cos(O)-cos(W)*sin(O)*cos(I);
 QQ[2]=-sin(W)*sin(O)+cos(W)*cos(O)*cos(I);
 QQ[3]=+cos(W)*sin(I);
 for(i=1;i<=3;i++)
    {
    rR[i]=XB*PP[i]+YB*QQ[i];
    Vv[i]=XP*PP[i]+YP*QQ[i];
    rRO[1]=rR[1];
    rRO[2]=rR[2]*cos(ee)-rR[3]*sin(ee);
    rRO[3]=rR[3]*cos(ee)+rR[2]*sin(ee);
    VvO[1]=Vv[1];
    VvO[2]=Vv[2]*cos(ee)-Vv[3]*sin(ee);
    VvO[3]=Vv[3]*cos(ee)+Vv[2]*sin(ee);
    }
    printf("------------------------------------------------------------------\n");
    printf("Recompute the position and velocity components from the classical
           elements\n");
    printf("x = %.6lf\t y = %.6lf\t z = %.6lf\n",rRO[1],rRO[2],rRO[3]);
    printf("Vx= %.6lf\t Vy= %.6lf\t Vz= %.6lf\n",VvO[1],VvO[2],VvO[3]);
getch();
return(0);
}
```

Appendix C.5

Method of Gauss (Mars)

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

double dij(double a[4][4],double b[4][4],double c[4][4],int x,int y,int z)
{
   double r[4],sum;
   int i;
   sum=0.0;
   r[1]=a[x][2]*b[y][3]-(a[x][3]*b[y][2]);
   r[2]=a[x][3]*b[y][1]-(a[x][1]*b[y][3]);
   r[3]=a[x][1]*b[y][2]-(a[x][2]*b[y][1]);
   for (i=1;i<=3;i++)
   {
      sum=sum+(r[i]*c[z][i]);
   }
return sum;
}

double dot(double a[4][4],double b[4][4],int x,int y)
{
double sum=0.0;
int i;
for (i=1;i<=3;i++)
   {
   sum=sum+a[x][i]*b[y][i];
   }
return sum;
}

double newton(double tiu[3],double l[4][4],double rr[4][4],double mass,
       double d0,double gv)
{
   double a,b,cc,A,B,E,F,r,x1,x2;
   int k=0;
   x1=tiu[3]/tiu[2];
   x2=-tiu[1]/tiu[2];
   E=-2.0*dot(l,rr,2,2);
   F=dot(rr,rr,2,2);
   A=-(x1*dij(l,rr,l,1,1,3)-dij(l,rr,l,1,2,3)+(x2)*dij(l,rr,l,1,3,3))/d0;
   B=-((x1*(tiu[2]*tiu[2]-tiu[3]*tiu[3])/6.0)*dij(l,rr,l,1,1,3)+(x2*(tiu[2]
     *tiu[2]-tiu[1]*tiu[1])/6.0)*dij(l,rr,l,1,3,3))/d0;
   a=-(A*A+A*E+F);
   b=-mass*(2.0*A*B+B*E);
   cc=-(mass*mass)*(B*B);
   do
   {
   r=gv; //gv calculated values,r old
   gv=gv-((pow(gv,8.0)+a*pow(gv,6.0)+b*(gv*gv*gv)+cc)/(8.0*pow(gv,7.0)+
      (6.0*a)*pow(gv,5.0)+(3.0*gv*gv*gv)));
   k=k+1;
   }while(fabs(gv-r)>=10e-6);
   return r;
}

double findl(double all[4],double bll[4],int x,int y)
{
   double l[4][4],L,a,b,Q1=M_PI/180.0;
   int i;
   for (i=1;i<=3;i++)
   {
      a=all[i]*15.0*Q1;
```

```
        b=bll[i]*Q1;
        l[i][1]=cos(b)*cos(a);
        l[i][2]=cos(b)*sin(a);
        l[i][3]=sin(b);
     }
     L=l[x][y];
  return L;
  }

  double findp(double f[4],double g[4],double r[4][4],double l[4][4],int x)
  {
  double p[4];
  double d0,pp,c1,c2,c3,dfg;
     dfg=f[1]*g[3]-(f[3]*g[1]);
     c1=g[3]/dfg;
     c2=-1.0;
     c3=-g[1]/dfg;
     d0=dij(l,l,l,1,2,3);
     p[1]=(c1*dij(r,l,l,1,2,3)+c2*dij(r,l,l,2,2,3)+c3*dij(r,l,l,3,2,3))/(c1*d0
     p[2]=(c1*dij(l,r,l,1,1,3)+c2*dij(l,r,l,1,2,3)+c3*dij(l,r,l,1,3,3))/(c2*d0
     p[3]=(c1*dij(l,r,l,2,1,1)+c2*dij(l,r,l,2,2,1)+c3*dij(l,r,l,2,3,1))/(c3*d0
  pp=p[x];
  return pp;
  }

  double findv(double r[4][4],double f[4],double g[4],int x)
  {
     double v[4],dgf,d1,d3,v2;
     int i;
     dgf=f[1]*g[3]-(f[3]*g[1]);
     d1=-f[3]/dgf;
     d3=f[1]/dgf;
     for (i=1;i<=3;i++)
     {
        v[i]=d1*r[1][i]+d3*r[3][i];
     }
     v2=v[x];
     return v2;
  }

  double findfg(double r[4][4],double v[4][4],double mu,double dt,double k,int
  {
     double r0,d0,ai,c0,ww,xx,x1,xa,x3,B[20];
     double fx,df,x2,cc,uu,ss,f,g,fp,gp,rr,num;
     int i;
     B[1]=1.0;
        for(i=2;i<=19;i=i++)
        {
     B[i]=B[i-1]/i;
        }
     r0=sqrt(dot(r,r,2,2));
     d0=dot(r,v,2,2)/sqrt(mu);
     ai=2.0/r0-dot(v,v,2,2)/(mu);
     c0=1.0-r0*ai;
     ww=k*dt*sqrt(mu);
     xx=ww/r0;
     fx=0.0;df=1.0;
     do
     {
        xx=xx-fx/df;
        x2=xx*xx;
        xa=x2*ai;
        x3=x2*xx;
        cc=x2*(B[2]-xa*(B[4]-xa*(B[6]-xa*(B[8]-xa*(B[10]-xa*(B[12]
          -xa*(B[14]-xa*(B[16]-xa*(B[18])))))))));
        uu=x3*(B[3]-xa*(B[5]-xa*(B[7]-xa*(B[9]-xa*(B[11]-xa*(B[13]
          -xa*(B[15]-xa*(B[17]-xa*(B[19])))))))));
        ss=xx-uu*ai;
        df=r0+c0*cc+d0*ss;
        fx=r0*xx+c0*uu+d0*cc-ww;
     }while(fabs(fx)>1.0e-6);
```

```
    f=1.0-cc/r0;
    g=(r0*ss+d0-cc)/sqrt(mu);
    rr=r0+c0*cc+d0*ss;
    fp=-sqrt(mu)*ss/(rr*r0);
    gp=1.0-cc/rr;
       if (j==1)   num = f;
       else if (j==2) num = g;
       else if (j==3) num = fp;
       else if (j==4) num = gp;
    return num;
 }

 main()
 {
    double t[4],L[4][4],R[4][4],p[4],c[4],f[4],g[4],tua[4],r[4][4],pt[4];
    double m,k,d0,u2,H,con,R0,D0,AI,C0,CC,WW,XX,X2,XA,UU,SS,X3,v[4][4];
    double sr,sr2,FX,DF,p1,p2,p3,r1,r2,F,G,r3,n=0.0,al[4],bl[4],B[21];
    double d[4],ff[4],gg[4],tt[4],b,X,mu,rv,rr[4][4],hh,HH,T,ee,V,a,e;
    double h[4][4],P,O,I,W,E,M,vv[4][4],xb,yb,sx,cx,srr,er[4][4],N[4][4];
    double NN,Q1=M_PI/180.0,wr,Rr,EE,XP,YP,XB,et,YB,Ff,DFf,rR[4],Vv[4];
    double Bb,EP,PP[4],QQ[4],lrl,qq,nn,rRO[4],VvO[4];
    int i,j,q;
    clrscr();
    con=173.1446;
    k=0.01720209895;
    m=0.000000323;
    mu=1.0+m;

 tt[1]=2451195.607639;
 al[1]=13.6420208;
 bl[1]=-8.158672;
 R[1][1]=0.4395625;
 R[1][2]=-0.8074811;
 R[1][3]=-0.3500872;
 tt[2]=2451251.590278;
 al[2]=14.680361;
 bl[2]=-13.275144;
 R[2][1]=0.9867762;
 R[2][2]=-0.1101151;
 R[2][3]=-0.0477410;
 tt[3]=2451362.343750;
 al[3]=13.794871;
 bl[3]=-11.936112;
 R[3][1]=-0.1856244;
 R[3][2]=0.9170884;
 R[3][3]=0.3976083;

    T=(tt[2]-2451545.0)/36525.0;
    ee=(23.439291-0.0130042*T-0.00000016*T*T)*Q1;
    for (j=1;j<=3;j++)
    {
       for (i=1;i<=3;i++)
       {
    L[j][i]=findl(al,bl,j,i);
       }
    }
    tua[1]=k*(tt[1]-tt[2]);
    tua[3]=k*(tt[3]-tt[2]);
    tua[2]=tua[3]-tua[1];
    sr2=newton(tua,L,R,1.0+m,dij(L,L,L,1,2,3),10.0);
    for (i=1;i<=3;i++)
    {
    p[i]=0.0;
    }

    printf("***Method of Gauss*** \n");
    printf("Determination of Orbit of Mars\n");
 do
 {
 sr=sr2;
 u2=(1.0+m)/(sr*sr*sr);
```

```
    if (n>0.0)
    {
       B[1]=1.0;
    for(q=2;q<=19;q++)
    {
    B[q]=B[q-1]/q;
    }
    for (i=1;i<=3;i++)
    {
       H=tua[i];
       R0=sr;
       D0=(dot(r,v,2,2))/sqrt(1.0+m);
       AI=2.0/R0-dot(v,v,2,2)/(1.0+m);
       C0=1.0-R0*AI;
       WW=H*sqrt(1.0+m);
       XX=WW/R0;
       FX=0.0;DF=1.0;
    do
    {
       XX=XX-FX/DF;
       X2=XX*XX;
       XA=X2*AI;
       X3=X2*XX;
       CC=X2*(B[2]-XA*(B[4]-XA*(B[6]-XA*(B[8]-XA*(B[10]-XA*(B[12]
         -XA*(B[14]-XA*(B[16]-XA*(B[18])))))))));
       UU=X3*(B[3]-XA*(B[5]-XA*(B[7]-XA*(B[9]-XA*(B[11]-XA*(B[13]
         -XA*(B[15]-XA*(B[17]-XA*(B[19])))))))));
       SS=XX-UU*AI;
       FX=R0*XX+C0*UU+D0*CC-WW;
       DF=R0+C0*CC+D0*SS;
    }while(fabs(FX)>1.0e-6);
    f[i]=1.0-CC/R0;
    g[i]=(R0*SS+D0*CC)/sqrt(1.0+m);
    f[i]=(ff[i]+f[i])/2.0;
    g[i]=(gg[i]+g[i])/2.0;
    }
    }
    else
    {
    for (i=1;i<=3;i++)
       {
       f[i]=1.0-(u2*tua[i]*tua[i])/2.0;
       g[i]=tua[i]-(u2*tua[i]*tua[i]*tua[i])/6.0;
       }
    }
    for (j=1;j<=3;j++)
    {
       pt[j]=p[j];
       p[j]=findp(f,g,R,L,j);
       v[2][j]=findv(r,f,g,j);
       t[j]=tt[j]-(p[j]/con);
       ff[j]=f[j];
       gg[j]=g[j];
    for (i=1;i<=3;i++)
       {
    r[j][i]=(p[j]*L[j][i])-R[j][i];
       }
    }
    sr2=sqrt(dot(r,r,2,2));
    tua[1]=k*(t[1]-t[2]);
    tua[3]=k*(t[3]-t[2]);
    tua[2]=tua[3]-tua[1];
    p1=fabs(p[1]-pt[1]);
    p2=fabs(p[2]-pt[2]);
    p3=fabs(p[3]-pt[3]);
    n=n+1;
}while(p1>1.0e-6 || p2>1.0e-6 || p3>1.0e-6);
printf("At epoch Julian date: %lf\n",tt[2]);
printf("The heliocentric distance of Mars at epoch time = %.6lf AU\n",sr);
printf("------------------------------------------------------------------------\n");
printf("The position components of Mars at epoch time:\n");
```

```
printf("x = %.6lf\t y = %.6lf\t z = %.6lf\n",r[2][1],r[2][2],r[2][3]);
printf("The velocity components of Mars at epoch time:\n");
printf("Vx= %.6lf\t Vy= %.6lf\t Vz= %.6lf\n",v[2][1],v[2][2],v[2][3]);

rr[2][1]=r[2][1];
rr[2][2]=r[2][3]*sin(ee)+r[2][2]*cos(ee);
rr[2][3]=r[2][3]*cos(ee)-r[2][2]*sin(ee);
vv[2][1]=v[2][1];
vv[2][2]=v[2][2]*cos(ee)+v[2][3]*sin(ee);
vv[2][3]=v[2][3]*cos(ee)-v[2][2]*sin(ee);
h[2][1]=rr[2][2]*vv[2][3]-rr[2][3]*vv[2][2];
 h[2][2]=rr[2][3]*vv[2][1]-rr[2][1]*vv[2][3];
 h[2][3]=rr[2][1]*vv[2][2]-rr[2][2]*vv[2][1];
hh=dot(h,h,2,2);
 HH=sqrt(hh);
 N[2][1]=-h[2][2];
 N[2][2]=h[2][1];
 N[2][3]=0.0;

 NN=sqrt(dot(N,N,2,2));
 srr=sqrt(dot(rr,rr,2,2));
rv=dot(rr,vv,2,2);
 V=dot(vv,vv,2,2);
 a=1.0/((2.0/srr)-(V/mu));

for(i=1;i<=3;i++)
   {
   er[2][i]=((V/mu-1.0/srr)*rr[2][i])-((rv/mu)*vv[2][i]);
   }

P=hh/mu;
e=sqrt(dot(er,er,2,2));
O=acos(N[2][1]/NN)/Q1;
   if(N[2][2]<0.0)
   {
   O=360.0-O;
   }

I=acos(h[2][3]/HH)/Q1;
W=acos(dot(N,er,2,2)/(NN*e))/Q1;
if (er[2][3]<0.0)
   {
      W=360.0-W;
   }

xb=(P-srr)/e;
yb=rv*sqrt(P/mu)/e;
b=a*sqrt(1.0-e*e);
cx=xb/a+e;
sx=yb/b;
   if(fabs(sx)<=0.707107)
   {
      X=asin(fabs(sx));
   }
   if(fabs(cx)<=0.707107)
   {
      X=acos(fabs(cx));
   }
   if(cx>=0.0 && sx>=0.0)
   {
      X=X;
   }
   if(cx<0.0 && sx>=0.0)
   {
      X=180.0*Q1-X;
   }
   if(cx<0.0 && sx<0.0)
   {
      X=180.0*Q1+X;
   }
   if(cx>=0.0 && sx<0.0)
```

```
    {
    X=360.0*Q1-X;
    }
M=(X-e*sx)/Q1;
E=acos((xb/a)+e)/Q1;
nn=k*sqrt(mu/(a*a*a));
T=tt[2]-(M*Q1/nn);
printf("-----------------------------------------------------------------\n");
printf("The classical elements of Mars at epoch time : \n");
printf("Semimajor axis                      a= %.6lf\n",a);
printf("Eccentricity                        e= %.6lf\n",e);
printf("Inclination angle from ecliptic     i= %.6lf\n",I);
printf("Longitude of ascending node         O= %.6lf\n",O);
printf("Argument of perihelion              W= %.6lf\n",W);
printf("Time of perifocal passage           T= %.6lf\n",T);

I=I*Q1;
O=O*Q1;
W=W*Q1;
M=M*Q1;
E=E*Q1;
E=M;
   do
   {
   Ff=E-e*sin(E)-M;
   DFf=1.0-e*cos(E);
   E=E-(Ff/DFf);
   }while(fabs(Ff)>=1.0e-6);
Rr=a*(1.0-e*cos(E));
EP=sqrt(mu/a)/Rr;
Bb=a*sqrt(1.0-e*e);
XB=a*(cos(E)-e);
YB=Bb*sin(E);
XP=-a*EP*sin(E);
YP=+Bb*EP*cos(E);
PP[1]=+cos(W)*cos(O)-sin(W)*sin(O)*cos(I);
PP[2]=+cos(W)*sin(O)+sin(W)*cos(O)*cos(I);
PP[3]=+sin(W)*sin(I);
QQ[1]=-sin(W)*cos(O)-cos(W)*sin(O)*cos(I);
QQ[2]=-sin(W)*sin(O)+cos(W)*cos(O)*cos(I);
QQ[3]=+cos(W)*sin(I);
for(i=1;i<=3;i++)
   {
   rR[i]=XB*PP[i]+YB*QQ[i];
   Vv[i]=XP*PP[i]+YP*QQ[i];
   rRO[1]=rR[1];
   rRO[2]=rR[2]*cos(ee)-rR[3]*sin(ee);
   rRO[3]=rR[3]*cos(ee)+rR[2]*sin(ee);
   VvO[1]=Vv[1];
   VvO[2]=Vv[2]*cos(ee)-Vv[3]*sin(ee);
   VvO[3]=Vv[3]*cos(ee)+Vv[2]*sin(ee);
   }
   printf("-------------------------------------------------------------n");
   printf("Recompute the position and velocity components from the classical
          elements:\n");
   printf("x = %.6lf\t ẏ = %.6lf\t z = %.6lf\n",rRO[1],rRO[2],rRO[3]);
   printf("Vx= %.6lf\t Vy= %.6lf\t Vz= %.6lf\n",VvO[1],VvO[2],VvO[3]);
getch();
return(0);
}
```

Appendix C.6

Computation of the Astrometric Positions (Jupiter)

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

double findfg(double *a,double *b,double dt,double m,double k,int choose)
{
   double r0,d0,ai,c0,ww,xx,x2,x3,xa,cc,uu,ss,B[20],fx,df;
   double f,g,r,fp,gp,result;
   int i;
   B[1]=1.0;
      for (i=2;i<=19;i=i+1)
      {
   B[i]=B[i-1]/i;
      }
   r0=sqrt((a[1]*a[1])+(a[2]*a[2])+(a[3]*a[3]));
   d0=((a[1]*b[1])+(a[2]*b[2])+(a[3]*b[3]))/sqrt(m);
   ai=(2.0/r0)-((b[1]*b[1])+(b[2]*b[2])+(b[3]*b[3]))/m;
   c0=1.0-(r0*ai);
   ww=k*sqrt(m)*dt;
   xx=ww/r0;
   do
   {
      x2=xx*xx;
      xa=x2*ai;
      x3=x2*xx;
      cc=x2*(B[2]-xa*(B[4]-xa*(B[6]-xa*(B[8]-xa*(B[10]-xa*(B[12]
         -xa*(B[14]-xa*(B[16]-xa*(B[18])))))))));
      uu=x3*(B[3]-xa*(B[5]-xa*(B[7]-xa*(B[9]-xa*(B[11]-xa*(B[13]
         -xa*(B[15]-xa*(B[17]-xa*(B[19])))))))));
      ss=xx-uu*ai;
      fx=r0*xx+c0*uu+d0*cc-ww;
      df=r0+c0*cc+d0*ss;
      xx=xx-(fx/df);
   }while(fabs(fx)>1e-8);
      f=1.0-(cc/r0);
      g=(r0*ss+d0*cc)/sqrt(m);
      r=r0+c0*cc+d0*ss;
      fp=-sqrt(m)*ss/(r*r0);
      gp=1.0-(cc/r);
   if (choose ==1)
      result=f;
   else if (choose ==2)
      result=g;
   else if (choose ==3)
      result=fp;
   else if (choose ==4)
      result=gp;
   return result;
}
double mag(double x[4][4],double y[4][4],int a,int b)
{
   double s=0.0;
   int i;
   for (i=1;i<=3;i=i+1)
      {
   s=s+(x[a][i]*y[b][i]);
      }
   return sqrt(s);
}

main()
{
```

```
    double ab,n,e,k,m,t[101],r[101][4],v[101][4],tf,ns;
    double rr[4],dt,f,g,fg,gp,R1[4],V1[4];
    double pp[4],ap,p,l1[4],Q1;
    double x,cd,cx,sx,a,d,am,ah,ac,dd,dm,dc;
    int i,j,l;
    clrscr();
        ab=1.0/173.1446;
        rr[1]=-1.0018289;
        rr[2]=0.0698396;
        rr[3]=0.0302830;
        Q1=M_PI/180.0;
        k=0.01720209895;
        m=0.000954791+1.0;
        tf=2451440.520833;
        ns=1.0;
        t[0]=2451440.520833;
        dt=(tf-t[0])/ns;
        r[0][1]=4.400741;
        r[0][2]=2.073351;
        r[0][3]=0.774499;
        v[0][1]=-0.210888;
        v[0][2]=0.374655;
        v[0][3]=0.164454;
        for (i=0;i<=ns;i=i+1)
        {
    if (i==0)
    {
printf("t(%d)=%.6lf,r[1]=%.6lf,r[2]=%.6lf,r[3]=%.6lf,mag(r)=%.6lf\n",i,t[i],
        r[0][1],r[0][2],r[0][3],mag(r,r,0,0));
    }
    else
    {
        t[i]=t[i-1]+dt;
        R1[1]=r[i-1][1];R1[2]=r[i-1][2];R1[3]=r[i-1][3];
        V1[1]=v[i-1][1];V1[2]=v[i-1][2];V1[3]=v[i-1][3];
            f=findfg(R1,V1,dt,m,k,1);
            g=findfg(R1,V1,dt,m,k,2);
            fg=findfg(R1,V1,dt,m,k,3);
            gp=findfg(R1,V1,dt,m,k,4);
            for (j=1;j<=3;j=j+1)
            {
        r[i][j]=f*r[i-1][j]+g*v[i-1][j];
        v[i][j]=fg*r[i-1][j]+gp*v[i-1][j];
            }
printf("t(%d)=%.6lf,r[1]=%.6lf,r[2]=%.6lf,r[3]=%.6lf,mag(r)=%.6lf\n",i,
        t[i],r[i][1],r[i][2],r[i][3],mag(r,r,i,i));
printf("v[1]=%.6lf,v[2]=%.6lf,v[3]=%.6lf,mag(v)=%.6lf\n",v[i][1],v[i][2],
        v[i][3],mag(v,v,i,i));
    }
        }

    for (l=i-1;l>=1;l=l-1)
    {
        for (j=1;j<=3;j=j+1)
        {
            pp[j]=r[l][j]+rr[j];
        }
            p=sqrt(pp[1]*pp[1]+pp[2]*pp[2]+pp[3]*pp[3]);
            ap=ab*p;
        while(t[l]-(tf-ap)>1e-6)
        {
            t[l]=tf-ap;
            dt=t[l]-t[l-1];
            R1[1]=r[l-1][1];R1[2]=r[l-1][2];R1[3]=r[l-1][3];
            V1[1]=v[l-1][1];V1[2]=v[l-1][2];V1[3]=v[l-1][3];
        f=findfg(R1,V1,dt,m,k,1);
        g=findfg(R1,V1,dt,m,k,2);
        for (j=1;j<=3;j=j+1)
        {
            r[l][j]=f*(r[l-1][j])+g*(v[l-1][j]);
        }
```

```
        for (j=1;j<=3;j=j+1)
        {
           pp[j]=r[1][j]+rr[j];
        }
        p=sqrt(pp[1]*pp[1]+pp[2]*pp[2]+pp[3]*pp[3]);
        ap=ab*p;
           }
        for (j=1;j<=3;j=j+1)
        {
        ll[j]=pp[j]/p;
        }
        cd=sqrt(1.0-(ll[3]*ll[3]));
        cx=ll[1]/cd;
        sx=ll[2]/cd;
           if (sx<=0.707107)
           {
              x=asin(fabs(sx));
           }
           if (cx<=0.707107)
           {
              x=acos(fabs(cx));
           }
           if (cx>=0.0 && sx>=0.0) x=x;
           else if (cx<0.0 && sx>=0.0) x=180.0*Q1-x;
           else if (cx<0.0 && sx<0.0) x=180.0*Q1+x;
           else if (cx>=0.0 && sx<0.0) x=360.0*Q1-x;
        a=x/(15.0*Q1);
        d=(asin(ll[3]))/Q1;
  printf("\n");
  printf("after light time...\n");
  printf("r[1]=%.6lf,r[2]=%.6lf,r[3]=%.6lf,mag(r)=%.6lf\n",r[1][1],r[1][2],
        r[1][3],mag(r,r,1,1));
  printf("v[1]=%.6lf,v[2]=%.6lf,v[3]=%.6lf,mag(v)=%.6lf\n",v[1][1],v[1][2],
        v[1][3],mag(v,v,1,1));
  printf("\n");
  printf("Right ascension =%.6lf   Declination =%.6lf \n",a,d);
     }
     getch();
     return(0);
  }
```

Appendix C.7

Computation of the Astrometric Positions (Mars)

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

double findfg(double *a,double *b,double dt,double m,double k,int choos
{
   double r0,d0,ai,c0,ww,xx,x2,x3,xa,cc,uu,ss,B[20],fx,df;
   double f,g,r,fp,gp,result;
   int i;
   B[1]=1.0;
      for (i=2;i<=19;i=i+1)
      {
   B[i]=B[i-1]/i;
      }
   r0=sqrt((a[1]*a[1])+(a[2]*a[2])+(a[3]*a[3]));
   d0=((a[1]*b[1])+(a[2]*b[2])+(a[3]*b[3]))/sqrt(m);
   ai=(2.0/r0)-((b[1]*b[1])+(b[2]*b[2])+(b[3]*b[3]))/m;
   c0=1.0-(r0*ai);
   ww=k*sqrt(m)*dt;
   xx=ww/r0;
   do
   {
      x2=xx*xx;
      xa=x2*ai;
      x3=x2*xx;
      cc=x2*(B[2]-xa*(B[4]-xa*(B[6]-xa*(B[8]-xa*(B[10]-xa*(B[12]-xa*(B[14
         -xa*(B[16]-xa*(B[18]))))))));
      uu=x3*(B[3]-xa*(B[5]-xa*(B[7]-xa*(B[9]-xa*(B[11]-xa*(B[13]-xa*(B[15
         -xa*(B[17]-xa*(B[19]))))))));
      ss=xx-uu*ai;
      fx=r0*xx+c0*uu+d0*cc-ww;
      df=r0+c0*cc+d0*ss;
      xx=xx-(fx/df);
   }while(fabs(fx)>1e-8);
      f=1.0-(cc/r0);
      g=(r0*ss+d0*cc)/sqrt(m);
      r=r0+c0*cc+d0*ss;
      fp=-sqrt(m)*ss/(r*r0);
      gp=1.0-(cc/r);
   if (choose ==1)
      result=f;
   else if (choose ==2)
      result=g;
   else if (choose ==3)
      result=fp;
   else if (choose ==4)
      result=gp;
   return result;
}
double mag(double x[4][4],double y[4][4],int a,int b)
{
   double s=0.0;
   int i;
   for (i=1;i<=3;i=i+1)
      {
      s=s+(x[a][i]*y[b][i]);
      }
   return sqrt(s);
}

main()
{
```

```
    double ab,n,e,k,m,t[101],r[101][4],v[101][4],tf,ns;
    double rr[4],dt,f,g,fg,gp,R1[4],V1[4];
    double pp[4],ap,p,l1[4],Q1;
    double x,cd,cx,sx,a,d,am,ah,ac,dd,dm,dc;
    int i,j,l;
    clrscr();
    /*  Initial values  */
       ab=1.0/173.1446;
       rr[1]=0.9867762;
       rr[2]=-0.1101151;
       rr[3]=-0.0477410;
       Q1=M_PI/180.0;
       k=0.01720209895;
       m=0.000000323+1.0;
       tf=2451251.590278;
       ns=1.0;
       t[0]=2451251.590278;
       dt=(tf-t[0])/ns;
       r[0][1]=-1.570208;
       r[0][2]=-0.383017;
       r[0][3]=-0.132492;
       v[0][1]=0.226879;
       v[0][2]=-0.657073;
       v[0][3]=-0.304540;

       for (i=0;i<=ns;i=i+1)
       {
    if (i==0)
    {
 printf("t(%d)=%.6lf,r[1]=%.6lf,r[2]=%.6lf,r[3]=%.6lf,mag(r)=%.6lf\n",i,t[i],
       r[0][1],r[0][2],r[0][3],mag(r,r,0,0));
    }
    else
    {
       t[i]=t[i-1]+dt;
       R1[1]=r[i-1][1];R1[2]=r[i-1][2];R1[3]=r[i-1][3];
       V1[1]=v[i-1][1];V1[2]=v[i-1][2];V1[3]=v[i-1][3];
       f=findfg(R1,V1,dt,m,k,1);
       g=findfg(R1,V1,dt,m,k,2);
       fg=findfg(R1,V1,dt,m,k,3);
       gp=findfg(R1,V1,dt,m,k,4);
       for (j=1;j<=3;j=j+1)
          {
       r[i][j]=f*r[i-1][j]+g*v[i-1][j];
       v[i][j]=fg*r[i-1][j]+gp*v[i-1][j];
          }
 printf("t(%d)=%.6lf,r[1]=%.6lf,r[2]=%.6lf,r[3]=%.6lf,mag(r)=%.6lf\n",i,
       t[i],r[i][1],r[i][2],r[i][3],mag(r,r,i,i));
 printf(" v[1]=%.6lf,v[2]=%.6lf,v[3]=%.6lf,mag(v)=%.6lf\n",v[i][1],v[i][2],
       v[i][3],mag(v,v,i,i));
    }
       }

    for (l=i-1;l>=1;l=l-1)
    {
       for (j=1;j<=3;j=j+1)
       {
          pp[j]=r[l][j]+rr[j];
       }
          p=sqrt(pp[1]*pp[1]+pp[2]*pp[2]+pp[3]*pp[3]);
          ap=ab*p;
       while(t[l]-(tf-ap)>1e-5)
       {
          t[l]=tf-ap;
          dt=t[l]-t[l-1];
          R1[1]=r[l-1][1];R1[2]=r[l-1][2];R1[3]=r[l-1][3];
          V1[1]=v[l-1][1];V1[2]=v[l-1][2];V1[3]=v[l-1][3];
       f=findfg(R1,V1,dt,m,k,1);
       g=findfg(R1,V1,dt,m,k,2);
       for (j=1;j<=3;j=j+1)
       {
```

```
            r[l][j]=f*(r[l-1][j])+g*(v[l-1][j]);
         }
         for (j=1;j<=3;j=j+1)
         {
            pp[j]=r[l][j]+rr[j];
         }
         p=sqrt(pp[1]*pp[1]+pp[2]*pp[2]+pp[3]*pp[3]);
         ap=ab*p;
            }
         for (j=1;j<=3;j=j+1)
         {
         ll[j]=pp[j]/p;
         }
         cd=sqrt(1.0-(ll[3]*ll[3]));
         cx=ll[1]/cd;
         sx=ll[2]/cd;
            if (sx<=0.707107)
            {
               x=asin(fabs(sx));
            }
            if (cx<=0.707107)
            {
               x=acos(fabs(cx));
            }
            if (cx>=0.0 && sx>=0.0) x=x;
            else if (cx<0.0 && sx>=0.0) x=180.0*Q1-x;
            else if (cx<0.0 && sx<0.0) x=180.0*Q1+x;
            else if (cx>=0.0 && sx<0.0) x=360.0*Q1-x;
         a=x/(15.0*Q1);
         d=(asin(ll[3]))/Q1;
printf("\n");
printf("after light time...\n");
printf("r[1]=%.6lf,r[2]=%.6lf,r[3]=%.6lf,mag(r)=%.6lf\n",r[1][1],r[1][2],
         r[1][3],mag(r,r,1,1));
printf("          v[1]=%.6lf,v[2]=%.6lf,v[3]=%.6lf,mag(v)=%.6lf\n",v[1][1],
         v[1][2],v[1][3],mag(v,v,1,1));
printf("\n");
printf("Right ascension =%.6lf   Declination =%.6lf  \n",a,d);
   }
   getch();
   return(0);
}
```

Appendix C.8

Orbit Improvement (Jupiter)

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define NP 20
#define PC 3.0
#define NM 6

double findfg(double *a,double *b,double dt,double m,double k,int choose)
{
   double r0,d0,ai,c0,ww,xx,x2,x3,xa,cc,uu,ss,B[20],fx,df;
   double f,g,r,fp,gp,result;
   int i;
   B[1]=1.0;
   for (i=2;i<=19;i=i+1)
      {
   B[i]=B[i-1]/i;
      }
   r0=sqrt((a[1]*a[1])+(a[2]*a[2])+(a[3]*a[3]));
   d0=((a[1]*b[1])+(a[2]*b[2])+(a[3]*b[3]))/sqrt(m);
   ai=(2.0/r0)-((b[1]*b[1])+(b[2]*b[2])+(b[3]*b[3]))/m;
   c0=1.0-(r0*ai);
   ww=k*sqrt(m)*dt;
   xx=ww/r0;
   do
   {
      x2=xx*xx;
      xa=x2*ai;
      x3=x2*xx;
      cc=x2*(B[2]-xa*(B[4]-xa*(B[6]-xa*(B[8]-xa*(B[10]-xa*(B[12]-xa*(B[14]
         -xa*(B[16]-xa*(B[18])))))))));
      uu=x3*(B[3]-xa*(B[5]-xa*(B[7]-xa*(B[9]-xa*(B[11]-xa*(B[13]-xa*(B[15]
         -xa*(B[17]-xa*(B[19])))))))));
      ss=xx-uu*ai;
      fx=r0*xx+c0*uu+d0*cc-ww;
      df=r0+c0*cc+d0*ss;
      xx=xx-(fx/df);
   }while(fabs(fx)>1e-8);
      f=1.0-(cc/r0);
      g=(r0*ss+d0*cc)/sqrt(m);
      r=r0+c0*cc+d0*ss;
      fp=-sqrt(m)*ss/(r*r0);
      gp=1.0-(cc/r);
   if (choose ==1)
      result=f;
   else if (choose ==2)
      result=g;
   else if (choose ==3)
      result=fp;
   else if (choose ==4)
      result=gp;
   return result;
}

double findad(double *p,double q,int sel)
{
      double sx,cx,l[4],pp,x,cd,result;
      int k;
   pp=sqrt((p[1]*p[1])+(p[2]*p[2])+(p[3]*p[3]));
      for (k=1;k<=3;k=k+1)
      {
         l[k]=p[k]/pp;
```

```
        }
        cd=sqrt(1.0-l[3]*l[3]);
        cx=l[1]/cd;
        sx=l[2]/cd;
        if (sx<=0.707107)
        {
    x=asin(fabs(sx));
        }
        if (cx<=0.707107)
        {
    x=acos(fabs(cx));
        }
        if (cx>=0.0 && sx>=0.0) x=x;
        else if (cx<0.0 && sx>=0.0) x=180.0*q-x;
        else if (cx<0.0 && sx<0.0) x=180.0*q+x;
        else if (cx>=0.0 && sx<0.0) x=360.0*q-x;
        if (sel==1) result=x/(15.0*q);
        else if (sel==2) result=(asin(l[3]))/q;
    return result;
}
double finddc(double x[2*NP+1][NM+2],int sel,int nm,int np)
{
    double a[2*NP+1][NM+2],pe,cd,he,ss,result,xu[NM+1],ce;
    int j,i,k,l,ll,nn,ne,jp;
    ne=2*np;
        for (j=1;j<=nm;j=j+1)
        {
        for (k=1;k<=nm+1;k=k+1)
        {
            a[j][k]=0.0;
        for (i=1;i<=ne;i=i+1)
        {
            a[j][k]=a[j][k]+(x[i][j]*x[i][k]);
        }
        }
        }
        nn=nm;
            for (i=1;i<=nn-1;i=i+1)
            {
        jp=i;
        pe=fabs(a[i][i]);
            for (j=i+1;j<=nn;j=j+1)
            {
                ce=fabs(a[j][i]);
            if (ce-pe < 0.0)
            {
                if (jp==i)
                {
                for (l=i+1;l<=nn;l=l+1)
                {
                    for (ll=i+1;ll<=nn+1;ll=ll+1)
                    {
                        a[l][ll]=a[l][ll]-(a[l][i]*a[i][ll])/a[i][i];
                    }
                    a[l][i]=0.0;
                }
                }
                else
                {
                for (k=i;k<=nn+1;k=k+1)
                {
                    he=a[i][k];
                    a[i][k]=a[jp][k];
                    a[jp][k]=he;
                }
                for (l=i+1;l<=nn;l=l+1)
                {
                    for (ll=i+1;ll<=nn+1;ll=ll+1)
                    {
                        a[l][ll]=a[l][ll]-(a[l][i]*a[i][ll])/a[i][i];
                    }
```

```
                a[l][i]=0.0;
             }
             }
          }
          else
          {
             pe=ce;
             jp=j;
          }
          }
          }
   xu[nn]=a[nn][nn+1]/a[nn][nn];
      for (i=nn-1;i>=1;i=i-1)
      {
      ss=0.0;
      for (k=i+1;k<=nn;k=k+1)
      {
             ss=ss+a[i][k]*xu[k];
      }
         xu[i]=(a[i][nn+1]-ss)/a[i][i];
      }
   if (sel==1) result = xu[1];
   else if (sel==2) result = xu[2];
   else if (sel==3) result = xu[3];
   else if (sel==4) result = xu[4];
   else if (sel==5) result = xu[5];
   else if (sel==6) result = xu[6];
   return result;
}
main()
{
   double ta[NP+1],a0[NP+1],d0[NP+1],t[NP+1],R[NP+1][4],V[NP+1][4];
   double RR[NP+1][4],eo[NM+1],x[2*NP+1][NM+2],sm[2*NP+1],s[2*NP+1];
   double r[4],v[4];
   double ab,E,K,M,ne,PE,CE,jp,Q1,pp,ap;
   double dd[NP+1],da[NP+1],pe[NM+1],ac[NP+1][NM+1],pa[NP+1][NM+1];
   double ll[4],p[4],xu[NM+1],A[NM][NM+2],pl[4],pd[NP+1][NM+1];
   double dc[NP+1][NM+1],ce[NM+1];
   int i,j,k,l,m,z,zz;
   double al,ba,dt,f,g,rr,fp,gp,Rr[NP+1];
   char c;
   clrscr();
   ab=1.0/173.1446;
   K=0.01720209895;
   Q1=M_PI/180.0;
   M=0.000954791+1.0;
   t[0]=2451440.5208333;
ta[1]= 2451195.3229167;     a0[1]= 23.6957366;     d0[1]= -3.2850870;
ta[2]= 2451202.2916667;     a0[2]= 17.6163136;     d0[2]= -2.5358570;
ta[3]= 2451209.3194444;     a0[3]= 12.8478768;     d0[3]= -2.1262348;
ta[4]= 2451430.4791667;     a0[4]= 2.2047824;      d0[4]= 11.4512997;
ta[5]= 2451440.5208333;     a0[5]= 2.1490413;      d0[5]= 11.3262390;
ta[6]= 2451456.5034722;     a0[6]= 2.0835679;      d0[6]= 10.8301864;
ta[7]= 2451471.4527778;     a0[7]= 1.9173204;      d0[7]= 10.3110544;
ta[8]= 2451479.3965278;     a0[8]= 1.8636123;      d0[8]= 9.9624210;
ta[9]= 2451499.5451389;     a0[9]= 1.7323786;      d0[9]= 9.0583931;
ta[10]=2451490.3750000;     a0[10]=1.7655548;      d0[10]=9.3103931;
ta[11]= 2451514.2847222;    a0[11]= 1.6200747;     d0[11]= 8.6415906;
ta[12]= 2451520.2916667;    a0[12]= 1.6366431;     d0[12]= 8.6327469;
ta[13]= 2451529.4145833;    a0[13]= 1.6067960;     d0[13]= 8.5903537;
ta[14]= 2451536.2868056;    a0[14]= 1.5958717;     d0[14]= 8.6679843;
ta[15]= 2451548.2986111;    a0[15]= 1.5974504;     d0[15]= 8.5364612;
ta[16]= 2451556.3020833;    a0[16]= 1.6573597;     d0[16]= 8.4131468;
ta[17]= 2451570.3125000;    a0[17]= 1.7380096;     d0[17]= 9.4329030;
ta[18]= 2451576.3055556;    a0[18]= 1.7687759;     d0[18]= 9.7224891;
ta[19]= 2451583.3437500;    a0[19]= 1.8610464;     d0[19]= 10.1170691;
ta[20]= 2451617.3020833;    a0[20]= 2.2075871;     d0[20]= 12.2308209;

      eo[1]=4.4004595;
      eo[2]=2.0733442;
      eo[3]=0.7744964;
```

```
        eo[4]=-0.2108931;
        eo[5]=0.3746540;
        eo[6]=0.1644540;

    RR[1][1]=0.4350922;      RR[1][2]=-0.8094893;     RR[1][3]=-0.3509578;
    RR[2][1]=0.5410324;      RR[2][2]=-0.7544064;     RR[2][3]=-0.3270807;
    RR[3][1]=0.6395634;      RR[3][2]=-0.6873894;     RR[3][3]=-0.2980237;
    RR[4][1]=-0.9769745;     RR[4][2]=0.2255460;      RR[4][3]=0.0977893;
    RR[5][1]=-1.0018289;     RR[5][2]=0.0698396;      RR[5][3]=0.0302830;
    RR[6][1]=-0.9807426;     RR[6][2]=-0.1801977;     RR[6][3]=-0.0781227;
    RR[7][1]=-0.8943174;     RR[7][2]=-0.4022344;     RR[7][3]=-0.1743884;
    RR[8][1]=-0.8235861;     RR[8][2]=-0.5101480;     RR[8][3]=-0.2211762;
    RR[9][1]=-0.5778776;     RR[9][2]=-0.7361569;     RR[9][3]=-0.3191625;
    RR[10][1]=-0.7005456;    RR[10][2]=-0.6430284;    RR[10][3]=-0.2787817;
    RR[11][1]=-0.3511939;    RR[11][2]=-0.8453251;    RR[11][3]=-0.3664889;
    RR[12][1]=-0.2511452;    RR[12][2]=-0.8739995;    RR[12][3]=-0.3789209;
    RR[13][1]=-0.0942162;    RR[13][2]=-0.8987501;    RR[13][3]=-0.3896568;
    RR[14][1]=0.0257542;     RR[14][2]=-0.9020897;    RR[14][3]=-0.3911031;
    RR[15][1]=0.2335655;     RR[15][2]=-0.8763429;    RR[15][3]=-0.3799378;
    RR[16][1]=0.3668161;     RR[16][2]=-0.8372123;    RR[16][3]=-0.3629776;
    RR[17][1]=0.5808886;     RR[17][2]=-0.7293575;    RR[17][3]=-0.3162115;
    RR[18][1]=0.6624552;     RR[18][2]=-0.6692618;    RR[18][3]=-0.2901584;
    RR[19][1]=0.7488010;     RR[19][2]=-0.5892035;    RR[19][3]=-0.2554534;
    RR[20][1]=0.9878263;     RR[20][2]=-0.1028619;    RR[20][3]=-0.0445986;
do
{
    for (z=0;z<=NM;z=z+1)
    {
    for (j=1;j<=NM;j=j+1)
    {
        ce[j]=eo[j];
    }
            for (j=1;j<=NP;j=j+1)
            {
        t[j]=ta[j];
            }
            if (z!=0)
            {
        pe[z]=fabs(eo[z]*PC/100.0);
        ce[z]=eo[z]+pe[z];
            }
        for (k=1;k<=3;k=k+1)
        {
        R[0][k]=ce[k];
        V[0][k]=ce[k+3];
        }
        for (i=1;i<=NP;i=i+1)
        {
        al=t[i];
        r[1]=R[i-1][1];r[2]=R[i-1][2];r[3]=R[i-1][3];
        v[1]=V[i-1][1];v[2]=V[i-1][2];v[3]=V[i-1][3];
            for (k=1;k<=3;k=k+1) p[k]=0.0;
        do
        {
            for (k=1;k<=3;k=k+1) pl[k]=p[k];
            t[i]=al;
            dt=t[i]-t[i-1];
            f=findfg(r,v,dt,M,K,1);
            g=findfg(r,v,dt,M,K,2);
            fp=findfg(r,v,dt,M,K,3);
            gp=findfg(r,v,dt,M,K,4);
            for (j=1;j<=3;j=j+1)
            {
            R[i][j]=(f*R[i-1][j])+(g*V[i-1][j]);
            V[i][j]=(fp*R[i-1][j])+(gp*V[i-1][j]);
                p[j]=R[i][j]+RR[i][j];
            }
            pp=sqrt((p[1]*p[1])+(p[2]*p[2])+(p[3]*p[3]));
            ap=ab*pp;
            al=ta[i]-ap;
        }while(t[i]-al>1e-5);
```

```
                    ac[i][z]=findad(pl,Q1,1);
                    dc[i][z]=findad(pl,Q1,2);
                }
                    if (z==0)
                    {
                  for (k=1;k<=NP;k=k+1)
                  {
                    da[k]=a0[k]-ac[k][z];
                    dd[k]=d0[k]-dc[k][z];
//          printf("ta(%d)=%lf  da(%d)=%.5lf  dd(%d)=%.5lf\n",k,ta[k],k,da[k],k,dd[k])
                  }
                  printf("Continue? ");
                  scanf("%s",&c);
                  printf("\n");
                    if (c=='y')
                        zz=1;
                    else
                    {
                        zz=NM;
                        z=NM;
                    }
                    }

          }
          for (i=1;i<=NP;i=i+1)
          {
          for (j=1;j<=NM;j=j+1)
          {
              pa[i][j]=Q1*15.0*(ac[i][j]-ac[i][0])/pe[j];
              pd[i][j]=Q1*(dc[i][j]-dc[i][0])/pe[j];
//            printf("pa(%d,%d)=%lf,pd(%d,%d)=%lf\n",i,j,pa[i][j],i,j,pd[i][j]);
          }
      }
              for (i=1;i<=NP;i=i+1)
              {
                  for (m=1;m<=NM;m=m+1)
                  {
                  x[i][m]=pa[i][m];
                  x[i+NP][m]=pd[i][m];
                  }
                  x[i][7]=Q1*15.0*da[i];
                  x[i+NP][7]=Q1*dd[i];
              }
                  for (i=1;i<=NM;i=i+1)
                  {
              xu[i]=finddc(x,i,NM,NP);
              eo[i]=eo[i]+xu[i];
              if (c=='y')
              {
                  printf("de[%d]=%.10lf\n",i,xu[i]);
              }
                  }
                  printf("\n");
              if (c=='y')
              {
              for (i=1;i<=NM;i=i+1) printf("eo(%d)=%.7lf\n",i,eo[i]);
              }
      }while(zz<=NM-1);
              return(0);
      }
```

Appendix C.9

Orbit Improvement (Mars)

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define NP 24
#define PC 3.0
#define NM 6

double findfg(double *a,double *b,double dt,double m,double k,int choose)
{
   double r0,d0,ai,c0,ww,xx,x2,x3,xa,cc,uu,ss,B[20],fx,df;
   double f,g,r,fp,gp,result;
   int i;
   B[1]=1.0;
      for (i=2;i<=19;i=i+1)
      {
   B[i]=B[i-1]/i;
      }
   r0=sqrt((a[1]*a[1])+(a[2]*a[2])+(a[3]*a[3]));
   d0=((a[1]*b[1])+(a[2]*b[2])+(a[3]*b[3]))/sqrt(m);
   ai=(2.0/r0)-((b[1]*b[1])+(b[2]*b[2])+(b[3]*b[3]))/m;
   c0=1.0-(r0*ai);
   ww=k*sqrt(m)*dt;
   xx=ww/r0;
   do
   {
      x2=xx*xx;
      xa=x2*ai;
      x3=x2*xx;
      cc=x2*(B[2]-xa*(B[4]-xa*(B[6]-xa*(B[8]-xa*(B[10]-xa*(B[12]-xa*(B[14]
         -xa*(B[16]-xa*(B[18])))))))));
      uu=x3*(B[3]-xa*(B[5]-xa*(B[7]-xa*(B[9]-xa*(B[11]-xa*(B[13]-xa*(B[15]
         -xa*(B[17]-xa*(B[19])))))))));
      ss=xx-uu*ai;
      fx=r0*xx+c0*uu+d0*cc-ww;
      df=r0+c0*cc+d0*ss;
      xx=xx-(fx/df);
   }while(fabs(fx)>1e-8);
      f=1.0-(cc/r0);
      g=(r0*ss+d0*cc)/sqrt(m);
      r=r0+c0*cc+d0*ss;
      fp=-sqrt(m)*ss/(r*r0);
      gp=1.0-(cc/r);
   if (choose ==1)
      result=f;
   else if (choose ==2)
      result=g;
   else if (choose ==3)
      result=fp;
   else if (choose ==4)
      result=gp;
   return result;
}

double findad(double *p,double q,int sel)
{
      double sx,cx,l[4],pp,x,cd,result;
      int k;
   pp=sqrt((p[1]*p[1])+(p[2]*p[2])+(p[3]*p[3]));
      for (k=1;k<=3;k=k+1)
      {
         l[k]=p[k]/pp;
```

```
        }
        cd=sqrt(1.0-l[3]*l[3]);
        cx=l[1]/cd;
        sx=l[2]/cd;
        if (sx<=0.707107)
        {
    x=asin(fabs(sx));
        }
        if (cx<=0.707107)
        {
    x=acos(fabs(cx));
        }
        if (cx>=0.0 && sx>=0.0) x=x;
        else if (cx<0.0 && sx>=0.0) x=180.0*q-x;
        else if (cx<0.0 && sx<0.0) x=180.0*q+x;
        else if (cx>=0.0 && sx<0.0) x=360.0*q-x;
        if (sel==1) result=x/(15.0*q);
        else if (sel==2) result=(asin(l[3]))/q;
    return result;
}
double finddc(double x[2*NP+1][NM+2],int sel,int nm,int np)
{
    double a[2*NP+1][NM+2],pe,cd,he,ss,result,xu[NM+1],ce,ne;
    int j,i,k,l,ll,nn,jp;
    ne=2*np;
        for (j=1;j<=nm;j=j+1)
        {
        for (k=1;k<=nm+1;k=k+1)
        {
            a[j][k]=0.0;
        for (i=1;i<=ne;i=i+1)
        {
            a[j][k]=a[j][k]+(x[i][j]*x[i][k]);
        }
        }
        }
        nn=nm;
            for (i=1;i<=nn-1;i=i+1)
            {
        jp=i;
        pe=fabs(a[i][i]);
            for (j=i+1;j<=nn;j=j+1)
            {
                ce=fabs(a[j][i]);
            if (ce-pe < 0.0)
            {
                if (jp==i)
                {
                for (l=i+1;l<=nn;l=l+1)
                {
                    for (ll=i+1;ll<=nn+1;ll=ll+1)
                    {
                        a[l][ll]=a[l][ll]-(a[l][i]*a[i][ll])/a[i][i];
                    }
                    a[l][i]=0.0;
                }
                }
                else
                {
                for (k=i;k<=nn+1;k=k+1)
                {
                    he=a[i][k];
                    a[i][k]=a[jp][k];
                    a[jp][k]=he;
                }
                for (l=i+1;l<=nn;l=l+1)
                {
                    for (ll=i+1;ll<=nn+1;ll=ll+1)
                    {
                        a[l][ll]=a[l][ll]-(a[l][i]*a[i][ll])/a[i][i];
                    }
```

```
            ]
            }
         }
         else
         {
            pe=ce;
            jp=j;
         }
         }
         }
   xu[nn]=a[nn][nn+1]/a[nn][nn];
      for (i=nn-1;i>=1;i=i-1)
      {
      ss=0.0;
      for (k=i+1;k<=nn;k=k+1)
      {
            ss=ss+a[i][k]*xu[k];
      }
         xu[i]=(a[i][nn+1]-ss)/a[i][i];
      }
   if (sel==1) result = xu[1];
   else if (sel==2) result = xu[2];
   else if (sel==3) result = xu[3];
   else if (sel==4) result = xu[4];
   else if (sel==5) result = xu[5];
   else if (sel==6) result = xu[6];
   return result;
}
main()
{
   double ta[NP+1],a0[NP+1],d0[NP+1],t[NP+1],R[NP+1][4],V[NP+1][4];
   double RR[NP+1][4],eo[NM+1],x[2*NP+1][NM+2],sm[2*NP+1],s[2*NP+1];
   double r[4],v[4];
   double ab,E,K,M,ne,PE,CE,jp,Q1,pp,ap;
   double dd[NP+1],da[NP+1],pe[NM+1],ac[NP+1][NM+1],pa[NP+1][NM+1];
   double ll[4],p[4],xu[NM+1],A[NM][NM+2],pl[4],pd[NP+1][NM+1];
   double dc[NP+1][NM+1],ce[NM+1];
   int i,j,k,l,m,z,zz;
   double al,ba,dt,f,g,rr,fp,gp,Rr[NP+1];
   char c;
   clrscr();
   ab=1.0/173.1446;
   K=0.01720209895;
   Q1=M_PI/180.0;
   M=0.000000323+1.0;
   t[0]=2451251.5902778;
ta[1]= 2451195.6076389;     a0[1]= 13.6420198;       d0[1]= -8.1586716;
ta[2]= 2451202.6354167;     a0[2]= 14.2663048;       d0[2]= -9.2750346;
ta[3]= 2451209.6041660;     a0[3]= 14.0285034;       d0[3]= -10.0563382;
ta[4]= 2451216.6180556;     a0[4]= 14.2789537;       d0[4]= -10.9191036;
ta[5]= 2451230.6041667;     a0[5]= 14.2105357;       d0[5]= -11.7793506;
ta[6]= 2451237.5833333;     a0[6]= 14.3906392;       d0[6]= -12.5106152;
ta[7]= 2451245.5208333;     a0[7]= 15.0248192;       d0[7]= -13.5266501;
ta[8]= 2451251.5902778;     a0[8]= 14.9935338;       d0[8]= -13.5352734;
ta[9]= 2451262.5625000;     a0[9]= 15.0858119;       d0[9]= -13.5937782;
ta[10]= 2451280.5069444;    a0[10]= 14.6252393;      d0[10]= -12.6013976;
ta[11]= 2451290.4583333;    a0[11]= 14.5787627;      d0[11]= -11.9063137;
ta[12]= 2451294.4583333;    a0[12]= 14.4457902;      d0[12]= -11.5958720;
ta[13]= 2451301.4270833;    a0[13]= 13.8756396;      d0[13]= -10.4012411;
ta[14]= 2451314.5000000;    a0[14]= 14.0403476;      d0[14]= -10.1827789;
ta[15]= 2451323.3229167;    a0[15]= 13.5165213;      d0[15]= -9.4223617;
ta[16]= 2451349.3263889;    a0[16]= 13.5479155;      d0[16]= -10.5873001;
ta[17]= 2451351.3645833;    a0[17]= 13.6661617;      d0[17]= -10.8551768;
ta[18]= 2451362.3437500;    a0[18]= 13.8756430;      d0[18]= -11.5990476;
ta[19]= 2451368.3229167;    a0[19]= 14.1384853;      d0[19]= -13.1448762;
ta[20]= 2451412.3402778;    a0[20]= 15.4471007;      d0[20]= -20.2648736;
ta[21]= 2451425.3645833;    a0[21]= 16.2511150;      d0[21]= -22.2402421;
ta[22]= 2451430.3333333;    a0[22]= 15.7624112;      d0[22]= -21.4076129;
ta[23]= 2451462.2916667;    a0[23]= 17.2819063;      d0[23]= -24.5475968;
ta[24]= 2451514.2881944;    a0[24]= 20.3955527;      d0[24]= -20.5277262;
```

```
      eo[1]=-1.4906253;
      eo[2]=-0.3920329;
      eo[3]=-0.1235015;
      eo[4]=0.2843121;
      eo[5]=-0.6817588;
      eo[6]=-0.3018053;

    RR[1][1]=0.4395625;     RR[1][2]=-0.8074811;    RR[1][3]=-0.3500872;
    RR[2][1]=0.5460620;     RR[2][2]=-0.7513933;    RR[2][3]=-0.3257744;
    RR[3][1]=0.6433622;     RR[3][2]=-0.6844542;    RR[3][3]=-0.2967509;
    RR[4][1]=0.7315683;     RR[4][2]=-0.6067593;    RR[4][3]=-0.2630618;
    RR[5][1]=0.8733278;     RR[5][2]=-0.4255904;    RR[5][3]=-0.1845210;
    RR[6][1]=0.9249294;     RR[6][2]=-0.3249595;    RR[6][3]=-0.1408895;
    RR[7][1]=0.9669905;     RR[7][2]=-0.2047959;    RR[7][3]=-0.0887880;
    RR[8][1]=0.9867765;     RR[8][2]=-0.1101129;    RR[8][3]=-0.0477410;
    RR[9][1]=0.9946700;     RR[9][2]=0.0634662;     RR[9][3]=0.0275135;
    RR[10][1]=0.9313965;    RR[10][2]=0.3395688;    RR[10][3]=0.1472210;
    RR[11][1]=0.8577205;    RR[11][2]=0.4805546;    RR[11][3]=0.2083452;
    RR[12][1]=0.8209863;    RR[12][2]=0.5335016;    RR[12][3]=0.2313033;
    RR[13][1]=0.7480403;    RR[13][2]=0.6196814;    RR[13][3]=0.2686688;
    RR[14][1]=0.5838654;    RR[14][2]=0.7572277;    RR[14][3]=0.3282973;
    RR[15][1]=0.4563260;    RR[15][2]=0.8294336;    RR[15][3]=0.3596077;
    RR[16][1]=0.0335536;    RR[16][2]=0.9317591;    RR[16][3]=0.4039711;
    RR[17][1]=-0.0260454;   RR[17][2]=0.9323888;    RR[17][3]=0.4042451;
    RR[18][1]=-0.1856244;   RR[18][2]=0.9170884;    RR[18][3]=0.3976083;
    RR[19][1]=-0.2839913;   RR[19][2]=0.8956636;    RR[19][3]=0.3883172;
    RR[20][1]=-0.8610417;   RR[20][2]=0.4871323;    RR[20][3]=0.2112020;
    RR[21][1]=-0.9532276;   RR[21][2]=0.3025965;    RR[21][3]=0.1311914;
    RR[22][1]=-0.9763829;   RR[22][2]=0.2277659;    RR[22][3]=0.0987516;
    RR[23][1]=-0.9543424;   RR[23][2]=-0.2684323;   RR[23][3]=-0.1163744;
    RR[24][1]=-0.3511327;   RR[24][2]=-0.8453463;   RR[24][3]=-0.3664981;
do
{
    for (z=0;z<=NM;z=z+1)
    {
    for (j=1;j<=NM;j=j+1)
    {
            ce[j]=eo[j];
    }
            for (j=1;j<=NP;j=j+1)
            {
        t[j]=ta[j];
        }
            if (z!=0)
            {
        pe[z]=fabs(eo[z]*PC/100.0);
        ce[z]=eo[z]+pe[z];
            }
        for (k=1;k<=3;k=k+1)
        {
        R[0][k]=ce[k];
        V[0][k]=ce[k+3];
        }
        for (i=1;i<=NP;i=i+1)
        {
        al=t[i];
        r[1]=R[i-1][1];r[2]=R[i-1][2];r[3]=R[i-1][3];
        v[1]=V[i-1][1];v[2]=V[i-1][2];v[3]=V[i-1][3];
            for (k=1;k<=3;k=k+1) p[k]=0.0;
        do
        {
            for (k=1;k<=3;k=k+1) p1[k]=p[k];
            t[i]=al;
            dt=t[i]-t[i-1];
            f=findfg(r,v,dt,M,K,1);
            g=findfg(r,v,dt,M,K,2);
            fp=findfg(r,v,dt,M,K,3);
            gp=findfg(r,v,dt,M,K,4);
            for (j=1;j<=3;j=j+1)
            {
```

```
                  R[i][j]=(f*R[i-1][j])+(g*V[i-1][j]);
                  V[i][j]=(fp*R[i-1][j])+(gp*V[i-1][j]);
                  p[j]=R[i][j]+RR[i][j];
                  }
                  pp=sqrt((p[1]*p[1])+(p[2]*p[2])+(p[3]*p[3]));
                  ap=ab*pp;
                  al=ta[i]-ap;
              }while(t[i]-al>1e-5);
                      ac[i][z]=findad(pl,Q1,1);
                      dc[i][z]=findad(pl,Q1,2);
              }
                      if (z==0)
                      {
                  for (k=1;k<=NP;k=k+1)
                  {
                      da[k]=a0[k]-ac[k][z];
                      dd[k]=d0[k]-dc[k][z];
//                    printf("ta(%d)=%lf  da(%d)=%.5lf  dd(%d)=%.5lf\n",k,ta[k],k,
//                              da[k],k,dd[k]);
                  }
                  printf("Continue? ");
                  scanf("%s",&c);
                  printf("\n");
                      if (c=='y')
                          zz=1;
                      else
                      {
                          zz=NM;
                          z=NM;
                      }
                      }

              }
          for (i=1;i<=NP;i=i+1)
          {
          for (j=1;j<=NM;j=j+1)
          {
              pa[i][j]=Q1*15.0*(ac[i][j]-ac[i][0])/pe[j];
              pd[i][j]=Q1*(dc[i][j]-dc[i][0])/pe[j];
          }
          }
              for (i=1;i<=NP;i=i+1)
              {
                  for (m=1;m<=NM;m=m+1)
                  {
                  x[i][m]=pa[i][m];
                  x[i+NP][m]=pd[i][m];
                  }
                  x[i][7]=Q1*15.0*da[i];
                  x[i+NP][7]=Q1*dd[i];
              }
              for (i=1;i<=NM;i=i+1)
                  {
              xu[i]=finddc(x,i,NM,NP);
              eo[i]=eo[i]+xu[i];
              if (c=='y')
              {
                  printf("de[%d]=%.10lf\n",i,xu[i]);
              }
                  }
                  printf("\n");
              if (c=='y')
              {
              for (i=1;i<=NM;i=i+1) printf("eo(%d)=%.7lf\n",i,eo[i]);
              }
        }while(zz<=NM-1);
              return(0);
        }
```

BIOGRAPHY

Mr. Yuttakan Ratthanachai was born at January 17, 1976 at the amphur Chiangkham, Phayao Thailand. In 1997, he graduated in Bachelor’s degree of science (B.Sc. Physics) from Srinakharinwirot university then he is studying Master’s degree of science (Physics) in Chulalongkorn university.