การลู่เข้าสู่การแจกแจงปกติของผลบวกสุ่มของตัวแปรสุ่มอิสระที่มีค่าความแปรปรวนจำกัด

นางสาวเพชรรัตน์ รัตนวงษ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2544

ISBN 974-03-0520-2

THE CONVERGENCE TO NORMAL DISTRIBUTION OF RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES WITH FINITE VARIANCES

Miss Petcharat Rattanawong

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2001
ISBN 974-03-0520-2

Thesis title	The Convergence to Normal Distribution of Random Sums of Independent Random Variables with Finite Variances
By	Miss Petcharat Rattanawong
Field of study	Mathematics
Thesis Advisor	Associate Professor Kritsana Neammanee, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

………………………………………………Dean of Faculty of Science
(Associate Professor Wanchai Phothiphichitr, Ph.D.)

Thesis Committee

………………………………………………Chairman
(Assistant Professor Imchit Termwuttipong, Ph.D.)

………………………………………………Thesis Advisor
(Associate Professor Kritsana Neammanee, Ph.D.)

………………………………………………Member
(Nataphan Kitisin, Ph.D.)

เพชรรัตน์ รัตนวงษ์ : การลู่เข้าสู่การแจกแจงปกติของผลบวกสุ่มของตัวแปรสุ่มอิสระที่มีค่าความแปรปรวนจำกัด(THE CONVERGENCE TO NORMAL DISTRIBUTION OF RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES WITH FINITE VARIANCES) อ. ที่ปรึกษา : รองศาสตราจารย์ ดร.กฤษณะ เนียมมณี, 36 หน้า ISBN 974-03-0520-2

กำหนดให้ (X_{nk}) เป็นลำดับสองชั้นของตัวแปรสุ่มที่มีค่าความแปรปรวนจำกัด ให้ (Z_n) เป็นลำดับของตัวแปรสุ่มที่มีค่าเป็นจำนวนเต็มบวก ซึ่งแต่ละจำนวนนับ n, $Z_n, X_{n1}, X_{n2}, \ldots$ เป็นอิสระต่อกัน แต่ละจำนวนนับ n, k, $\mu_{nk} = 0$ เมื่อ μ_{nk} คือ ค่าคาดคะเนของ X_{nk}

ในวิทยานิพนธ์นี้ เราให้เงื่อนไขที่จำเป็นและเพียงพอที่ทำให้ลำดับของฟังก์ชันการแจกแจงของ

$$X_{n1} + X_{n2} + \ldots + X_{nZ_n}$$

ลู่เข้าอย่างอ่อนสู่การแจกแจงปกติมาตรฐาน

ภาควิชา **คณิตศาสตร์**
สาขาวิชา **คณิตศาสตร์**
ปีการศึกษา 2544

ลายมือชื่อนิสิต...
ลายมือชื่ออาจารย์ที่ปรึกษา..............................
ลายมือชื่ออาจารย์ที่ปรึกษาร่วม -

4272358923 MAJOR : MATHEMATICS
KEYWORD : CENTRAL LIMIT THEOREM / RANDOM SUMS / INFINITELY DIVISIBLE

PETCHARAT RATTANAWONG : THE CONVERGENCE TO NORMAL DISTRIBUTION OF RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES WITH FINITE VARIANCES. THESIS ADVISOR:ASSOC.PROF. KRITSANA NEAMMANEE, Ph.D. 36pp. ISBN 974-03-052 0-2

Let (X_{nk}) be a double sequence of random variables with finite variances. Let (Z_n) be a sequence of positive integral-valued random variables such that for each n, $Z_n, X_{n1}, X_{n2},...$ are independent and $\mu_{nk} = 0$,where μ_{nk} is the expectation of X_{nk} .

In this study, we give necessary and sufficient conditions for weak convergence of the distribution functions of random sums

$$X_{n1} + X_{n2} + ... + X_{nZ_n}$$

to the standard normal distribution function.

Department **Mathematics**	Student's signature……………………………..
Field of Study **Mathematics**	Advisor's signature…………………………….
Academic year **2001**	Co-advisor's signature -

ACKNOWLEDGMENTS

I am greatly indebted to Associate Professor Dr.Kritsana Neammanee, my thesis advisor, for his suggestions and helpful advice in preparing and writing this thesis. I would like to thank Assistant Professor Dr.Imchit Termwuttipong and Dr. Nataphan Kitisin, my thesis committee, for their suggestions to this thesis. I also would like to thank all teachers, who have taught me, for my knowledge and skill.

In particular, I would like to express my deep gratitude to my family and my friends for their encouragement throughout my graduate study.

CONTENTS

CHAPTER I

INTRODUCTION

The problem of the convergence of the sequence of distribution functions of sums of a random number of independent random variables has been discussed many times in numerous papers. In this work we investigate the case whose the limit distribution function is the standard normal distribution function Φ.

In the case of one array, let (X_n) be a sequence of independent random variables with zero mean (this is not an essential restriction) and finite variances. Let (Z_n) be a sequence of positive integral-valued random variables which independent of (X_n). Many authors (e.g.[1],[5],[13],[14],[16],[19],[20],[23],[24] and [27]) gave conditions of convergence of the sequence of distribution functions of random sums $X_1 + X_2 + ... + X_{Z_n}$ to the standard normal distribution function Φ.

In this work we consider a double array of random variables. Let (X_{nk}) be a double sequence of random variables with mean 0 and finite variances σ^2_{nk}. For each n, we assume that $Z_n, X_{n1}, X_{n2}, ...$ are independent. In [2],[3],[25] the authors investigated the convergence of the sequence of distribution functions of random sums $X_{n1} + X_{n2} + ... + X_{nZ_n}$ in case of $X_{n1}, X_{n2}, ...$ are identically distributed for every n. The aim of our investigation is to extend the problem to the case of $X_{n1}, X_{n2}, ...$ are not necessary identically distributed. First we state one of the most important versions of central limit theorem of sums.

Theorem 1.1. ([12]) *Let (k_n) be a sequence of positive integers. Assume that* $\lim_{n\to\infty} \sum_{k=1}^{k_n} \sigma^2_{nk} = 1$. *Then* *(i) the sequence of distribution functions of the sums*

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n}$$

converges weakly to Φ *and*

(ii) $(X_{nk}), k = 1, 2, ..., k_n, n = 1, 2, ...$ *is infinitesimal,*

i.e.

$$\max_{1 \le k \le k_n} P(|X_{nk}| \ge \varepsilon) \to 0$$

for every $\varepsilon > 0$,

if and only if $(X_{nk}), k = 1, 2, ..., k_n, n = 1, 2, ...$ *satisfies the Lindeberg condition,*

i.e.

$$\sum_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \to 1$$

for every $\varepsilon > 0$.

In this work, we extend Theorem 1.1 to the case of random sums. In chapter II we summarize known results and notations used in our work. Chapter III contains our main results.

First, we will introduce some conditions:

(α) (X_{nk}) is random infinitesimal with respect to (Z_n),

i.e.

$$\max_{1 \le k \le Z_n} P(|X_{nk}| \ge \varepsilon) \xrightarrow{p} 0$$

for every $\varepsilon > 0$,

(β) $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ and

(γ) for every subsequence (n'), if there exist distribution functions $F^{(q)}$ such that the sequence of distribution functions of the sums

$$X_{n'1} + X_{n'2} + ... + X_{n'l_{n'}(q)}$$

converges weakly to $F^{(q)}$ for a.e. $q \in (0,1)$, then $F^{(q)}(x)$ is measurable in q for every x, where $l_n : (0,1) \to \mathbb{N}$ defined by $l_n(q) = \max\{k \in \mathbb{N} | P(Z_n < k) \leq q\}$. The following is the main theorem.

Theorem 1.2. *Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies conditions (β) and (γ) .Then*

(i) the sequence of distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n}$$

converges weakly to Φ and

(ii) (Z_n, X_{nk}) satisfies (α)

if and only if (Z_n, X_{nk}) satisfies the random Lindeberg condition , i.e.

$$\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1$$

for every $\varepsilon > 0$.

Note that Theorem 1.1 is a special case of Theorem 1.2 when $Z_n = k_n$ for each $n \in \mathbb{N}$.

CHAPTER II

PRELIMINARIES

2.1 Random Variables

A **probability space** is a measure space $(\Omega, \mathcal{E}, P)$ in which P is a measure such that $P(\Omega) = 1$. The set Ω will be refered to as a **sample space**. The elements of $\mathcal{E}$ are called **events**. For any event A, the value $P(A)$ is called the **probability of** A.

A function X from a probability space $(\Omega, \mathcal{E}, P)$ to the set of complex numbers C is said to be a **complex-valued random variable** if for every Borel set B in C, $X^{-1}(B)$ belongs to $\mathcal{E}$. If X is real-valued, we say that it is a **real-valued random variable**, or simply a **random variable**. We note that the composition between a Borel function and a complex-valued random variable is also a complex-valued random variable.

We will use the notations $P(X \leq x), P(X \geq x)$ and $P(|X| \geq x)$ to denote $P(\{\omega | X(\omega) \leq x\}), P(\{\omega | X(\omega) \geq x\})$ and $P(\{\omega || X(\omega)| \geq x\})$, respectively.

We define the **expectation** of a complex-valued random variable X to be

$$\int_\Omega X dP$$

provided that the integral $\int_\Omega X dP$ exists. It will be denoted by $E[X]$.

The expectation of a random variable X is known as the **mean** . The expectation of $(X - E[X])^2$ is known as the **variance** of X and it denoted by $\sigma^2(X)$.

Proposition 2.1.1. $([7], p.174)$ *Let $X_1, X_2, ..., X_n$ be random variables. Then*

$$E[X_1 + X_2 + ... + X_n] = \sum_{k=1}^{n} E[X_k],$$

provided that the sums on the right hand side is meaningful.

Let $(\Omega, \mathcal{E}, \mu)$ be a measure space and Y a topological space. Let $X, X_1, X_2, ..., X_n$ be measurable functions from Ω to Y. We will write

$$X_n \to X \text{ a.e. } [\mu]$$

if (X_n) converges to X almost everywhere with respect to μ. In the case that $\Omega = R^k$ and μ is the Lebesgue measure on R^k, we simply write

$$X_n \to X \text{ a.e..}$$

A sequence (X_n) of complex-valued random variables is said to **converges in probability** to a complex-valued random variable X if

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

for every $\varepsilon > 0$. In this case we use the notation

$$X_n \xrightarrow{p} X.$$

Theorem 2.1.2. $([21], p.201)$ *Let $X, X_1, X_2, ...$ and $Y, Y_1, Y_2, ...$ be complex-valued random variables. If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then $X_n + Y_n \xrightarrow{p} X + Y$.*

From now on, we shall assume that all our complex-valued random variables, including real-valued random variables, are defined on a common probability space $(\Omega, \mathcal{E}, P)$.

2.2 Distribution Functions and Characteristic Functions

A function F from $\mathbf{R}$ to $\mathbf{R}$ is said to be a **distribution function** if it is non-decreasing, right-continuous, $F(-\infty) = 0$ and $F(+\infty) = 1$.

For any random variable X, the function $F : \mathbf{R} \to \mathbf{R}$ defined by

$$F(x) = P(X \leq x)$$

is a distribution function. It is called the **distribution function of the random variable** X.

Now we will give some examples of random variables.

Example 2.2.1. we say that X is a **standard normal random variable** if the distribution function of X is defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

Example 2.2.2. we say that X is a **degenerate random variable** with parameter a, if the distribution function of X is defined by

$$F(x) = \begin{cases} 0 & if\ x < a \\ 1 & if\ x \geq a. \end{cases}$$

Proposition 2.2.3. ([15], p.28) *Let X be a random variable with the distribution function F. If $E[X]$ exists, then*

$$E[X] = \int_{-\infty}^{\infty} x dF(x).$$

Let F be a distribution function. The function $\varphi : R \to C$ defined by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

is called the **characteristic function of the distribution function** F. If F is the distribution function of a random variable X, then φ is also called the **characteristic function of** X.

Proposition 2.2.4. $([18], p.45)$

(i) The product of two characteristic functions is a characteristic function.

(ii) If φ is a characteristic function, then $|\varphi|^2$ is also a characteristic function.

Proposition 2.2.5. $([8], p.477)$ *Let (F_n) be a sequence of distribution functions and (φ_n) a sequence of corresponding characteristic functions. Let (p_n) be a sequence of non-negative numbers such that $\sum\limits_{k=1}^{\infty} p_k = 1$. Then the function*

$$F(x) = \sum_{k=1}^{\infty} p_k F_k(x)$$

is a distribution function and the function

$$\varphi(t) = \sum_{k=1}^{\infty} p_k \varphi_k(t)$$

is the characteristic function of F.

Any random variables $X_1, X_2, ..., X_n$ are called **independent** if

$$P(\bigcap_{k=1}^{n} \{\omega | X_k(\omega) \leq x_k\}) = \prod_{k=1}^{n} P(X_k \leq x_k)$$

holds for every real numbers $x_1, x_2, ..., x_n$.

A sequence of random variables (X_n) is said to be a **sequence of independent random variables** if $X_{i_1}, X_{i_2}, ..., X_{i_n}$ are independent for all distinct $i_1, i_2, ..., i_n$.

Theorem 2.2.6. $([7], p.188, 191)$ *Let $X_1, X_2, ..., X_n$ be random variables with the characteristic functions $\varphi_1, \varphi_2, ..., \varphi_n$, respectively. Assume that $X_1, X_2, ..., X_n$ are independent. Then the followings hold.*

(i) The characteristic function φ of $X_1 + X_2 + ... + X_n$ is given by

$$\varphi(t) = \varphi_1(t)\varphi_2(t)...\varphi_n(t) \quad \textit{for all} \quad t \in R.$$

(ii) $\sigma^2(X_1+X_2+...+X_n)=\sigma^2(X_1)+\sigma^2(X_2)+...+\sigma^2(X_n)$ *if* $\sigma^2(X_i)<\infty$ *for* $i=1,2,...,n$.

Let $F, F_1, F_2, ...$ be bounded non-decreasing functions. The sequence (F_n) is said to **converges weakly** to F if

(i) for every continuity point x of F, $F_n(x) \to F(x)$ and

(ii) $F_n(+\infty) \to F(+\infty)$ and $F_n(-\infty) \to F(-\infty)$.

We will write

$$F_n \xrightarrow{w} F$$

if (F_n) converges weakly to F. Note that the weak limit of the sequence (F_n), if it exists, is unique. In the following theorems we state some facts of weak convergence which will be used in our work.

Theorem 2.2.7. ([17]) *Let* (Y_n) *be a sequence of random variables and put* $H_n(x) = P(Y_n \leq x)$. *Suppose* $\sup\limits_{n\in\mathbb{N}} E[Y_n^2] < \infty$. *If* $H_n \xrightarrow{w} H$ *for some distribution function* H *then we have* $\lim\limits_{n\to\infty} E[Y_n] = \int_{-\infty}^{\infty} x\,dH(x) < \infty$.

Theorem 2.2.8. *(Helly's Theorem,* [15], *p.133) Let* (F_n) *be a sequence of uniformly bounded, non-decreasing, right-continuous functions. Then* (F_n) *contains a subsequence which converges weakly to a bounded, non-decreasing, right-continuous function.*

Let $\mathcal{M}$ be the set of bounded, non-decreasing, right-continuous functions M from $\mathbf{R}$ into $[0,\infty)$ which vanish at $-\infty$. The function L defined for any $M_1, M_2 \in \mathcal{M}$ by

$$L(M_1, M_2) = \inf_{h\geq 0}\{h \mid M_1(x-h)-h \leq M_2(x) \leq M_1(x+h)+h \;\text{ for every } x \text{ in } \mathbf{R}\}$$

is a complete metric on $\mathcal{M}$. ([11], p.39)

The following corollary follows from Theorem 2.2.8 and the fact that the elements in $\mathcal{M}$ vanish at $-\infty$.

Corollary 2.2.9. *Let (M_n) be a uniformly bounded sequence of elements in $\mathcal{M}$. Then it contains a subsequence which converges weakly to an element in $\mathcal{M}$.*

Theorem 2.2.10. $([10], p.39)$ *Let $M, M_1, M_2, ...$ be elements in $\mathcal{M}$. Then the following statements are equivalent.*

(i) $M_n \xrightarrow{w} M$.

(ii) *For every bounded continuous function g on $\mathcal{R}$,*

$$\int_{-\infty}^{\infty} g(x)dM_n(x) \to \int_{-\infty}^{\infty} g(x)dM(x).$$

(iii) $L(M_n, M) \to 0$.

Theorem 2.2.11. $([26], p.15)$ *Let (F_n) and (φ_n) be sequences of distribution functions and their characteristic functions, respectively. Let F be a distribution function with the characteristic φ. If $F_n \xrightarrow{w} F$, then (φ_n) converges to φ uniformly on arbitrary finite interval.*

Theorem 2.2.12. $([26], p.15)$ *Let (F_n) and (φ_n) be sequences of distribution functions and their characteristic functions, respectively. Let φ be a complex-valued function which is continuous at 0. If (φ_n) converges to φ for every t, then there exists a distribution function F such that $F_n \xrightarrow{w} F$ and the characteristic function of F is φ.*

Let F_1 and F_2 be distribution functions. The **convolution** of F_1 and F_2 is defined by

$$(F_1 * F_2)(x) = \int_{-\infty}^{\infty} F_2(x-y)dF_1(y) = \int_{-\infty}^{\infty} F_1(x-y)dF_2(y) \quad \text{for all } x \in \mathbf{R}.$$

Theorem 2.2.13. $([9], p.252)$ *Let $F, G, F_n, G_n, n = 1, 2, ...$ be distribution functions. If $F_n \xrightarrow{w} F$ and $G_n \xrightarrow{w} G$, then $F_n * G_n \xrightarrow{w} F * G$.*

2.3 Infinitely Divisible Distribution Functions

A characteristic function φ is said to be **infinitely divisible** if for every natural number n, there exists a characteristic functions φ_n such that for every t,

$$\varphi(t) = \{\varphi_n(t)\}^n.$$

The distribution function of any infinitely divisible characteristic function is also said to be **infinitely divisible**. A random variable is said to be **infinitely divisible** if its characteristic function is infinitely divisible.

Theorem 2.3.1. $([18], p.81)$

(i) If φ is an infinitely divisible characteristic function, then for every t , $\varphi(t) \neq 0$.

(ii) If φ is an infinitely divisible characteristic function, then $|\varphi|^2$ is also infinitely divisible characteristic function.

(iii) The product of a finite number of infinitely divisible characteristic functions is infinitely divisible.

(iv) A characteristic function which is the limit of a sequence of infinitely divisible characteristic functions is infinitely divisible.

Theorem 2.3.2. $([26], p.32)$ *A function $\varphi(t)$ is the characteristic function of an infinitely divisible with finite variance if and only if it admits the representation*

$$\ln \varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x) \tag{2.1}$$

where

$$f(t,x) = \begin{cases} (e^{itx} - 1 - itx)\frac{1}{x^2} & if\ x \neq 0 \\ -\frac{t^2}{2} & if\ x = 0, \end{cases}$$

*μ is a real constant, K is non-decreasing bounded function. The formula (2.1) is known as **Kolmogorov's formula**.*

There is another representation of the logarithm of an infinitely divisible characteristic function φ, known as **Levy's formula**:

$$\begin{aligned} \ln \varphi(t) = & iat - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0^-} (e^{itx} - 1 - \frac{itx}{1+x^2}) dM(x) \\ & + \int_{0^+}^{+\infty} (e^{itx} - 1 - \frac{itx}{1+x^2}) dN(x) \end{aligned} \tag{2.2}$$

where $\sigma^2 \geq 0$ and a are real constants, M and N are non-decreasing functions defined on $(-\infty, 0)$ and $(0, +\infty)$ respectively with $M(-\infty) = N(\infty) = 0$ and

$$\int_{-\varepsilon}^{0^-} x^2 dM(x) + \int_{0^+}^{\varepsilon} x^2 dN(x) < +\infty$$

for every positive real number ε.

We will write $F = L(a, \sigma^2, M, N)$ if an infinitely divisible distribution function F is represented by **Levy's formula**(2.2).

Remark 2.3.3. For the standard normal distribution function Φ, we know that $\Phi = L(a_0, \sigma_0^2, M_0, N_0)$ where $a_0 = 0, \sigma_0^2 = 1, M_0(u) = 0 (u < 0)$ and $N_0(u) = 0 (u > 0)$.

Theorem 2.3.4. ([15], p.246) *For each infinitely divisible distribution function, the function K in Theorem 2.3.2 can be chosen to be right-continuous and $K(-\infty) = 0$. The function K in this theorem is unique.*

Theorem 2.3.5. ([11], p.85) *Let X be an infinitely divisible random variable with finite variance. Let the constant μ and the function K be given in the Kolmogorov's formula of the characteristic function of X. Then*

(i) $E[X] = \mu$

(ii) $\sigma^2(X) = K(+\infty)$.

2.4 Kolmogorov Theorems

In this section, we let $(X_{nk}), k = 1, 2, ..., k_n, n = 1, 2, ...$ be a double sequence of random variables with finite variances. For each n and k, we let μ_{nk}, σ^2_{nk} and F_{nk} be the expectation, variance and distribution function of X_{nk}, respectively.

In [11], Kolmogorov gave necessary and sufficient conditions for weak convergence of the sequence of distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n} - A_n$$

where (A_n) is a sequence of real numbers. There is an important convergence theorem (Theorem 2.4.1). In this theorem (X_{nk}) must satisfy the following conditions.

$(\tilde{\alpha})$ $(X_{nk} - \mu_{nk})$ is infinitesimal, i.e., for every $\varepsilon > 0$

$$\max_{1 \le k \le k_n} P(|X_{nk} - \mu_{nk}| \ge \varepsilon) \to 0.$$

$(\tilde{\beta})$ There exists a real number C such that

$$\sum_{k=1}^{k_n} \sigma^2_{nk} < C.$$

In order to prove the theorem, Kolmogorov defined **the accompanying distribution function** of the sums

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n} - A_n$$

to be the distribution function whose logarithm of its characteristic function is given by

$$\ln \psi_n(t) = -iA_n t + it \sum_{k=1}^{k_n} \mu_{nk} + \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} (e^{itx} - 1) dF_{nk}(x + \mu_{nk}).$$

Theorem 2.4.1. $([11], p.98)$ *Assume that (X_{nk}) satisfies the conditions $(\tilde{\alpha})$, $(\tilde{\beta})$ and for each $n, X_{n1}, X_{n2}, ..., X_{nk_n}$ are independent. Then there exists a sequence (A_n) of real numbers such that the sequence of distribution functions of the sums*

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n} - A_n$$

converges weakly to a limit distribution function if and only if the sequence of accompanying distribution functions of S_n converges weakly to the same limit distribution function.

Theorem 2.4.2. $([11], p.116)$ *In order that for some suitably chosen constants A_n the sequence of distributions of the sums*

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n} - A_n$$

of independent infinitesimal random variables converges to a limit, it is necessary and sufficient that there exist non-decreasing functions M and N defined in the intervals $(-\infty, 0)$ and $(0, +\infty)$, respectively, such that $M(-\infty) = 0$ and $N(+\infty) = 0$ and a constant $\sigma \geq 0$ such that

(1) *at every continuity point u of M and N*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(u) = M(u),$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F_{nk}(u) - 1) = N(u),$$

(2)

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \sum_{k=1}^{k_n} \{\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - (\int_{|x|<\varepsilon} x dF_{nk}(x))^2\}$$

$$= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{k=1}^{k_n} \{\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - (\int_{|x|<\varepsilon} x dF_{nk}(x))^2\}$$

$$= \sigma^2.$$

The constants A_n may be chosen according to the formula

$$A_n = \sum_{k=1}^{k_n} \int_{|x|<\tau} x dF_{nk}(x) - \gamma(\tau)$$

where $\gamma(\tau)$ *is any constant and* $-\tau$ *and* τ *are continuity points of* M *and* N*, respectively.*

CHAPTER III

CONVERGENCE TO NORMAL DISTRIBUTION OF RANDOM SUMS

The purpose of this chapter is to find necessary and sufficient conditions for the weak convergence of the sequence of distribution functions of random sums to the standard normal distribution function. One of the important tools is what is known as the "q-quantiles of Z_n".

3.1 Definition and properties of q-quantiles

Let Z be a positive integral-valued random variable. Let $l : (0,1) \to \mathbb{N}$ be defined by

$$l(q) = \max\{k \in \mathbb{N} \big| P(Z < k) \leqslant q\}.$$

The function l is called the **q-quantiles** of Z.

Remark 3.1.1. For a positive integral-valued random variable Z, the function q-quantiles of Z is non-decreasing.

Theorem 3.1.2. ([4]) *For every n, let* (a_{nk}) , *k=1,2,... be a nondecreasing sequence of non-negative real numbers and* Z_n *an integral-valued random variable.Further,let* $a \geq 0$ *be fixed.Then we have* $a_{nZ_n} \xrightarrow{p} a$ *if and only if* $a_{nl_n(q)} \to a$ *for all* $q \in (0,1)$ *where* l_n *is the q-quantiles of* Z_n.

3.2 Convergence to normal distribution of random sums

Let (X_{nk}) be a double sequence of random variables with zero means and finite variances σ_{nk}^2 and (Z_n) a sequence of positive integral-valued random variables. Assume that for each n, Z_n, X_{n1}, X_{n2},...are independent. For $q \in (0,1)$, let

$$S_n^{(q)} = X_{n1} + X_{n2} + ... + X_{nl_n(q)}$$

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n}$$

and let $F_n^{(q)}$ and F_n be the distribution functions of $S_n^{(q)}$ and S_{Z_n}, respectively.

To prove the main theorem(Theorem 3.2.12), we need the concept of random infinitesimal. We say that (X_{nk}) is **random infinitesimal** with respect to (Z_n) if for every $\varepsilon > 0$,

$$\max_{1 \le k \le Z_n} P(|X_{nk}| \ge \varepsilon) \xrightarrow{p} 0.$$

The following results are useful in our work.

Proposition 3.2.1. ([28])*Let (X_{nk}) be random infinitesimal with respect to (Z_n). If $F_n \xrightarrow{w} F$ for some distribution function F, then there exists a subsequence (n') such that for a.e. $q \in (0,1)$, there exist distribution functions $\overline{F}^{(q)}$ and a bounded sequence of real numbers $(a_{n'}^{(q)})$ such that*

$$F_{n'}^{(q)} * E_{a_{n'}^{(q)}} \xrightarrow{w} \overline{F}^{(q)}$$

where E_a stands for the degenerated distribution function with parameter $a \in \mathbf{R}$.

Proposition 3.2.2. ([5]) *If for a.e. $q \in (0,1)$, there exists distribution function $F^{(q)}$ such that $F_n^{(q)} \xrightarrow{w} F^{(q)}$ and for each $x \in \mathbf{R}$, $F^{(q)}(x)$ is a measurable function in q , then $F_n \xrightarrow{w} F$ where F is a distribution function defined by $F(x) = \int_0^1 F^{(q)}(x)dq$.*

Proposition 3.2.3. ([3])*For every $q \in (0,1)$, let $F^{(q)} = L(a_q, \sigma_q^2, M_q, N_q)$ be an infinitely divisible distribution function with zero mean. Suppose that σ_q^2 and the* *function* $M_q, |N_q|$ *are non-decreasing in* q *and that the integral* $F(x) = \int_0^1 F^{(q)}(x)dq$ *exists for all* $x \in R$*. Then we have* $F = \Phi$ *if and only if* $F^{(q)} = \Phi$ *for all* $q \in (0,1)$*.*

Corollary 3.2.4. *For a.e.* $q \in (0,1)$ *let* $F^{(q)} = L(a_q, \sigma_q^2, M_q, N_q)$ *be an infinitely divisible distribution function with zero mean. Suppose that* σ_q^2 *and the function* $M_q, |N_q|$ *are non-decreasing in* q *and that the integral* $F(x) = \int_0^1 F^{(q)}(x)dq$ *exists for all* $x \in R$*. Then we have* $F = \Phi$ *if and only if* $F^{(q)} = \Phi$ *a.e.* $q \in (0,1)$*.*

Theorem 3.2.5. *Let* (X_{nk}) *be random infinitesimal with respect to* (Z_n) *and* $\sum\limits_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$*. Assume that* $F_n \xrightarrow{w} \Phi$*. Then the followings hold.*
(i) there exists a subsequence (n') *such that for a.e.* $q \in (0,1)$*, there exists a distribution function* $F^{(q)}$ *which* $F_{n'}^{(q)} \xrightarrow{w} F^{(q)}$ *,*
(ii) if for each $x \in \mathbf{R}, F^{(q)}(x)$ *is a measurable function in* q*, then* $F_n^{(q)} \xrightarrow{w} \Phi$ *for every* $q \in (0,1)$*.*

Proof. (i) By Proposition 3.2.1 , there exist a subsequence (n') of (n) such that for a.e. $q \in (0,1)$, there exist distribution function $\overline{F}^{(q)}$ and bounded sequence $(a_{n'}^{(q)})$ such that

$$F_{n'}^{(q)} * E_{a_{n'}^{(q)}} \xrightarrow{w} \overline{F}^{(q)}. \tag{3.1}$$

It follows from $\sum\limits_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ and Theorem 3.1.2 that $\sum\limits_{k=1}^{l_n(q)} \sigma_{nk}^2 \to 1$ for all $q \in (0,1)$. Then for each $q \in (0,1)$,$\sup\limits_{n\in\mathbb{N}} \sum\limits_{k=1}^{l_n(q)} \sigma_{nk}^2 < \infty$. So for all $q \in (0,1)$

$$\sup_{n\in\mathbb{N}} E[(S_n^{(q)})^2] = \sup_{n\in\mathbb{N}} \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 < \infty. \tag{3.2}$$

Thus from (3.2) and the boundedness of $(a_{n'}^{(q)})$, we have for a.e. $q \in (0,1)$,

$$\begin{aligned}\sup_{n'\in\mathbb{N}} E[(S_{n'}^{(q)} + a_{n'}^{(q)})^2] &= \sup_{n'\in\mathbb{N}} E[(S_{n'}^{(q)})^2 + 2a_{n'}^{(q)}(S_{n'}^{(q)}) + (a_{n'}^{(q)})^2] \\ &= \sup_{n'\in\mathbb{N}}[E[(S_{n'}^{(q)})^2] + 2a_{n'}^{(q)}E(S_{n'}^{(q)}) + (a_{n'}^{(q)})^2] \\ &= \sup_{n'\in\mathbb{N}}[E[(S_{n'}^{(q)})^2] + (a_{n'}^{(q)})^2] \\ &\le \sup_{n'\in\mathbb{N}} E[(S_{n'}^{(q)})^2] + \sup_{n'\in\mathbb{N}}(a_{n'}^{(q)})^2 \\ &< \infty.\end{aligned}$$

From this fact and (3.1) we can apply Theorem 2.2.7 with $Y_{n'} = S_{n'}^{(q)} + a_{n'}^{(q)}$ for a.e. $q \in (0,1)$. Then

$$\lim_{n'\to\infty} a_{n'}^{(q)} = \lim_{n'\to\infty}(E[S_{n'}^{(q)}] + a_{n'}^{(q)}) = \lim_{n'\to\infty} E[S_{n'}^{(q)} + a_{n'}^{(q)}] = \int_{-\infty}^{\infty} x d\overline{F}^{(q)}$$

for a.e. $q \in (0,1)$. Let $a^{(q)} = \int_{-\infty}^{\infty} x d\overline{F}^{(q)}$. Thus $\lim\limits_{n'\to\infty} a_{n'}^{(q)} = a^{(q)} < \infty$ for a.e. $q \in (0,1)$. It is easy to check that $E_{-a_{n'}^{(q)}} \xrightarrow{w} E_{-a^{(q)}}$ for a.e. $q \in (0,1)$. From this fact, (3.1) and Theorem 2.2.13 we see that

$$F_{n'}^{(q)} \xrightarrow{w} \overline{F}^{(q)} * E_{-a^{(q)}}$$

for a.e. $q \in (0,1)$. Let $F^{(q)} = \overline{F}^{(q)} * E_{-a^{(q)}}$. Thus $F_{n'}^{(q)} \xrightarrow{w} F^{(q)}$ for a.e. $q \in (0,1)$. Hence we have (i)

(ii)Assume that for each $x \in \mathbf{R}$, $F^{(q)}(x)$ is a measurable function in q. Let (n') be an arbitrary subsequence of (n). By (i) , there exists another subsequence $(n'') \subset (n')$ such that , for a.e. $q \in (0,1)$,

$$F_{n''}^{(q)} \xrightarrow{w} F^{(q)} \tag{3.3}$$

for some distribution function $F^{(q)}$ and by Proposition 3.2.2 , $\Phi(x) = \int_0^1 F^{(q)}(x)dq$. First we show that the $F^{(q)}$ satisfies all conditions of Corollary 3.2.4 . By (3.2), (3.3) , and applying Theorem 2.2.7 with $Y_{n''} = S_{n''}^{(q)}$, we have

$$\int_{-\infty}^{\infty} x dF^{(q)}(x) = \lim_{n''\to\infty} E[S_{n''}^{(q)}] = 0 \tag{3.4}$$

for a.e. $q \in (0,1)$. Thus $F^{(q)}$ has zero mean. Applying Theorem 3.1.2 with $a_{nk} = \sup\limits_{1\le l\le k} P(|X_{nl}| \ge \varepsilon)$ and (X_{nl}) is random infinitesimal with respect to (Z_n), we obtain

$$\lim_{n\to\infty} \sup_{1\le l\le l_n(q)} P(|X_{nl}| \ge \varepsilon) = 0$$

for all $q \in (0,1)$. By (3.3) and Theorem 2.4.1, we have that the accompanying distribution function of $S_{n''}^{(q)}$ converge weakly to $F^{(q)}$ for a.e. $q \in (0,1)$. By Theorem 2.3.1 (iv), $F^{(q)} = L(a_q, \sigma_q^2, M_q, N_q)$ are infinite divisible. From Theorem 2.4.2 and the non-decreasing monotonicity of the $l_n(q)$ it can be easily seen that $\sigma_q^2, M_q, |N_q|$ are non-decreasing in q. Therefore Corollary 3.2.4 can be applied and it follows that $F^{(q)} = \Phi$ for a.e. $q \in (0,1)$. So $F_{n''}^{(q)} \xrightarrow{w} \Phi$ for a.e. $q \in (0,1)$. Next we will show that $F_{n''}^{(q)} \xrightarrow{w} \Phi$ for all $q \in (0,1)$.Let $q \in (0,1)$ and $A = \{q \in (0,1) | F_{n''}^{(q)} \xrightarrow{w} \Phi\}$. Then there exist q_1, q_2 in A such that $q_1 < q < q_2$. From Theorem 2.4.2, Remark 2.3.3 and the non-decreasing monotonicity of the $l_{n''}(q)$, we have for $u < 0$,

$$0 = \lim_{n''\to\infty} \sum_{k=1}^{l_{n''}(q_1)} F_{n''k}(u) \le \lim_{n''\to\infty} \sum_{k=1}^{l_{n''}(q)} F_{n''k}(u) \le \lim_{n''\to\infty} \sum_{k=1}^{l_{n''}(q_2)} F_{n''k}(u) = 0$$

for $u > 0$,

$$0 = \lim_{n''\to\infty} \sum_{k=1}^{l_{n''}(q_1)} (F_{n''k}(u) - 1) \le \lim_{n''\to\infty} \sum_{k=1}^{l_{n''}(q)} (F_{n''k}(u) - 1) \le \lim_{n''\to\infty} \sum_{k=1}^{l_{n''}(q_2)} (F_{n''k}(u) - 1) = 0$$

and

$$\begin{aligned} 1 &= \lim_{\varepsilon\to 0} \liminf_{n''\to\infty} \sum_{k=1}^{l_{n''}(q_1)} \{\int_{|x|<\varepsilon} x^2 dF_{n''k}(x) - (\int_{|x|<\varepsilon} x dF_{n''k}(x))^2\} \\ &\le \lim_{\varepsilon\to 0} \liminf_{n''\to\infty} \sum_{k=1}^{l_{n''}(q)} \{\int_{|x|<\varepsilon} x^2 dF_{n''k}(x) - (\int_{|x|<\varepsilon} x dF_{n''k}(x))^2\} \\ &\le \lim_{\varepsilon\to 0} \liminf_{n''\to\infty} \sum_{k=1}^{l_{n''}(q_2)} \{\int_{|x|<\varepsilon} x^2 dF_{n''k}(x) - (\int_{|x|<\varepsilon} x dF_{n''k}(x))^2\} = 1, \end{aligned}$$

$$
\begin{aligned}
1 &= \lim_{\varepsilon\to 0}\limsup_{n''\to\infty}\sum_{k=1}^{l_{n''}(q_1)}\{\int_{|x|<\varepsilon}x^2dF_{n''k}(x)-(\int_{|x|<\varepsilon}xdF_{n''k}(x))^2\}\\
&\le \lim_{\varepsilon\to 0}\limsup_{n''\to\infty}\sum_{k=1}^{l_{n''}(q)}\{\int_{|x|<\varepsilon}x^2dF_{n''k}(x)-(\int_{|x|<\varepsilon}xdF_{n''k}(x))^2\}\\
&\le \lim_{\varepsilon\to 0}\limsup_{n''\to\infty}\sum_{k=1}^{l_{n''}(q_2)}\{\int_{|x|<\varepsilon}x^2dF_{n''k}(x)-(\int_{|x|<\varepsilon}xdF_{n''k}(x))^2\}=1.
\end{aligned}
$$

From Theorem 2.4.2 and remark 2.3.3, we also obtain that $F_{n''}^{(q)}\xrightarrow{w}\Phi$. So $F_{n''}^{(q)}\xrightarrow{w}\Phi$ for all $q\in(0,1)$. That is every convergent subsequence of $(F_n^{(q)})$ converges weakly to Φ for all q. Thus $(F_n^{(q)})$ converges weakly to Φ for all q. Hence we have (ii). □

Proposition 3.2.6. *Let (X_{nk}) be random infinitesimal with respect to (Z_n). If $F_n^{(q)}\xrightarrow{w}\Phi$ for all $q\in(0,1)$, then $F_n\xrightarrow{w}\Phi$.*

Proof. Suppose that $F_n^{(q)}\xrightarrow{w}\Phi$ for all $q\in(0,1)$. Let $x\in\mathbf{R}$, we will show that $F_n(x)\to\Phi(x)$. First we show that , for each $n\in\mathbb{N}$, $F_n^{(q)}(x)$ is a measurable function in q. Let $n\in\mathbb{N}$. Let $ImZ_n=\{k_{ni}|k_{ni}<k_{n(i+1)}\}$, $q_{ni}=\sum_{k=1}^{k_{ni}}P(Z_n=k)$ and $q_{n0}=0$. Then for each $q\in[q_{n(i-1)},q_{ni})$, we have $l_n(q)=k_{ni}$.

<u>Case 1</u> ImZ_n is finite.

$$
\begin{aligned}
F_n^{(q)}(x) &= P(S_n^{(q)}\le x)\\
&= P(\sum_{j=1}^{l_n(q)}X_{nj}\le x)\\
&= \sum_{i=1}^{|ImZ_n|}P(\sum_{j=1}^{k_{ni}}X_{nj}\le x)\chi_{[q_{n(i-1)},q_{ni})}(q),
\end{aligned}
$$

where $\chi_A(x)=\begin{cases}1 & ;\ x\in A\\ 0 & ;\ x\notin A.\end{cases}$

So, $F_n^{(q)}(x)$ is a simple function. Hence $F_n^{(q)}(x)$ is a measurable function in q.

Case 2 ImZ_n is infinite.

Let

$$S_m^x(q) = \sum_{i=1}^{m} P(\sum_{j=1}^{k_{ni}} X_{nj} \leq x)\chi_{[q_{n(i-1)},q_{ni})}(q).$$

It is easy to check that for each $m \in \mathbb{N}$, $S_m^x(q) \leq S_{m+1}^x(q)$ for all $q \in (0,1)$.Then (S_m^x) is an increasing sequence of non-negative simple functions. By Monotone Convergence Theorem , we have

$$\lim_{m\to\infty} S_m^x(q) = \sum_{i=1}^{\infty} P(\sum_{j=1}^{k_{ni}} X_{nj} \leq x)\chi_{[q_{n(i-1)},q_{ni})}(q) = F_n^{(q)}(x)$$

is measurable in q. Hence for each $n \in \mathbb{N}$, $F_n^{(q)}(x)$ is a measurable function in q. It follows from this fact and Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_0^1 F_n^{(q)}(x)dq = \int_0^1 \Phi(x)dq = \Phi(x) \tag{3.5}$$

for all $q \in (0,1)$.

Since

$$\begin{aligned}
F_n(x) &= P(S_{Z_n} \leq x) \\
&= \sum_{k_{nj}\in ImZ_n} P(Z_n = k_{nj})P(X_{n1} + X_{n2} + ... + X_{nk_{nj}} \leq x) \\
&= \sum_{k_{nj}\in ImZ_n} (q_{nj} - q_{n(j-1)})P(X_{n1} + X_{n2} + ... + X_{nk_{nj}} \leq x) \\
&= \sum_{k_{nj}\in ImZ_n} \int_{q_{n(j-1)}}^{q_{nj}} P(S_n^{(q)} \leq x)dq \\
&= \int_0^1 P(S_n^{(q)} \leq x)dq \\
&= \int_0^1 F_n^{(q)}(x)dq,
\end{aligned}$$

we have $\lim_{n\to\infty} F_n(x) = \Phi(x)$. Hence $F_n \xrightarrow{w} \Phi$. □

Lemma 3.2.7. *Let $K, K_1, K_2, ...$ be elements in $\mathcal{M}$. Assume that the following conditions are satisfies:*

(a)

$$\int_{-\infty}^{\infty} f(t,x)dK_n(x) \to \int_{-\infty}^{\infty} f(t,x)dK(x)$$

for every real number t and

(b) $(K_n(+\infty))$ *is bounded.*

Then $K_n \xrightarrow{w} K$.

Proof. Since K_n is non-decreasing and $(K_n(+\infty))$ is bounded, we have (K_n) is uniformly bounded. By Corollary 2.2.9, there exist a subsequence (K_{n_k}) of (K_n) and a function $\overline{K}$ in $\mathcal{M}$ such that $K_{n_k} \xrightarrow{w} \overline{K}$. Since for each $t \in \mathbf{R}$, and $x \neq 0$.

$$\begin{aligned}
|f(t,x)| &= |(e^{itx} - 1 - itx)\frac{1}{x^2}| \\
&= |(\sum_{k=0}^{1} \frac{(itx)^k}{k!} + \theta\frac{t^2x^2}{2!} - 1 - itx)\frac{1}{x^2}|, \text{ where } |\theta| < 1 \\
&= |(1 + itx + \theta\frac{t^2x^2}{2} - 1 - itx)\frac{1}{x^2}| \\
&< \frac{t^2}{2}.
\end{aligned}$$

So $|f(t,x)|$ is bounded for all real number t and x, it follows from Theorem 2.2.10 that

$$\int_{-\infty}^{\infty} f(t,x)dK_{n_k}(x) \to \int_{-\infty}^{\infty} f(t,x)d\overline{K}(x)$$

for every real number t. From this fact and (a) we have

$$\int_{-\infty}^{\infty} f(t,x)d\overline{K}(x) = \int_{-\infty}^{\infty} f(t,x)dK(x).$$

By Theorem 2.3.4, we have $\overline{K} = K$. So $K_{n_k} \xrightarrow{w} K$. That is every subsequence of (K_n), it contains a subsequence which converges weakly to K. By using Theorem 2.2.10, we have that every subsequence of (K_n), it contains a subsequence which converges to K with the metric L. This implies that (K_n) converges to K with respect to the metric L. So by Theorem 2.2.10, we have $K_n \xrightarrow{w} K$. $\square$

In the following theorems, we assume that (Z_n, X_{nk}) satisfies the following conditions:

(α) (X_{nk}) is random infinitesimal with respect to (Z_n).

(β) $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$.

(γ) for every subsequence (n'), if there exist distribution functions $F^{(q)}$ such that the sequence of distribution functions of the sums

$$X_{n'1} + X_{n'2} + ... + X_{n'l_{n'}(q)}$$

converges weakly to $F^{(q)}$ for a.e. $q \in (0,1)$, then $F^{(q)}(x)$ is measurable in q for every x.

Theorem 3.2.8. *Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the conditions $(\alpha), (\beta)$ and (γ). Then the sequence of distribution functions of random sums*

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n}$$

converges weakly to Φ if and only if $\hat{\varphi}_{l_n(q)}(t) \to e^{-\frac{t^2}{2}}$ for every $q \in (0,1)$ and every real number t, where $\hat{\varphi}_{l_n(q)}(t)$ be the characteristic function of the accompanying distribution function of

$$S_n^{(q)} = X_{n1} + X_{n2} + ... + X_{nl_n(q)}.$$

Proof. ($\rightarrow$) By Theorem 3.2.5 (ii), we have $F_n^{(q)} \xrightarrow{w} \Phi$ for all $q \in (0,1)$. By Theorem 2.4.1, the sequence of accompanying distribution functions of $S_n^{(q)}$ converges weakly to Φ for all $q \in (0,1)$. Hence $\hat{\varphi}_{l_n(q)}(t) \to e^{-\frac{t^2}{2}}$ for every $q \in (0,1)$ and $t \in \mathrm{R}$.

($\leftarrow$) Suppose that $\hat{\varphi}_{l_n(q)}(t) \to e^{-\frac{t^2}{2}}$ for every $q \in (0,1)$ and every real number t. Then the sequence of accompanying distribution functions of $S_n^{(q)}$ converges weakly to Φ for all $q \in (0,1)$. By Theorem 2.4.1, $F_n^{(q)} \xrightarrow{w} \Phi$ for all $q \in (0,1)$. By proposition 3.2.6, $F_n \xrightarrow{w} \Phi$. □

Theorem 3.2.9. *Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the conditions $(\alpha), (\beta)$ and (γ). Then the sequence of distribution functions of the random sums*

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n}$$

converges weakly to Φ if and only if

(i') $K_{Z_n}(u) \xrightarrow{p} K(u)$ for every continuity point u of K and

(ii') $K_{Z_n}(+\infty) \xrightarrow{p} K(+\infty)$

where $K_{Z_n}(u) = \sum\limits_{k=1}^{Z_n} \int_{-\infty}^{u} x^2 dF_{nk}(x)$ and

$$K(u) = \begin{cases} 0 & for\ u < 0 \\ 1 & for\ u \geq 0. \end{cases}$$

Proof. $(\rightarrow)$ From the fact $K_{Z_n}(+\infty) = \sum\limits_{k=1}^{Z_n} \sigma_{nk}^2$, $K(+\infty) = 1$ and (β), we have (ii').

To prove (i') , let u be any continuity point of K.

For each n and j , let

$$a_{nj}(u) = \sum_{k=1}^{j} \int_{-\infty}^{u} x^2 dF_{nk}(x).$$

Hence $a_{nZ_n}(u) = K_{Z_n}(u)$.

To prove $K_{Z_n}(u) \xrightarrow{p} K(u)$, by Theorem 3.1.2 , it suffices to show that

$$a_{nl_n(q)}(u) \rightarrow K(u)$$

for every $q \in (0,1)$,i.e.

$$K_{l_n(q)}(u) \rightarrow K(u)$$

for every $q \in (0,1)$.

To do this, we will apply Lemma 3.2.7 to a sequence $(K_{l_n(q)})$ for all $q \in (0,1)$. Let $q \in (0,1)$. By Theorem 3.2.8, $\hat{\varphi}_{l_n(q)}(t) \rightarrow e^{-\frac{t^2}{2}}$ for every real number t. This implies

$$\ln \hat{\varphi}_{l_n(q)}(t) \rightarrow -\frac{t^2}{2}$$

for every real number t.

Note that $\int_{-\infty}^{\infty} f(t,x)dK(x) = -\frac{t^2}{2}$ and

$$\ln \hat{\varphi}_{l_n(q)}(t) = \sum_{k=1}^{l_n(q)} \int_{-\infty}^{\infty} f(t,x)dF_{nk}(x) = \int_{-\infty}^{\infty} f(t,x)dK_{l_n(q)}(x). \tag{3.6}$$

Then

$$\int_{-\infty}^{\infty} f(t,x)dK_{l_n(q)}(x) \to \int_{-\infty}^{\infty} f(t,x)dK(x).$$

So (a) of Lemma 3.2.7 is satisfied. By (β), it follows from Theorem 3.1.2 that

$$\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 \to 1.$$

This implies that $(K_{l_n(q)}(+\infty))$ is bounded. Therefore the condition (b) of Lemma 3.2.7 is satisfied. Thus $K_{l_n(q)}(u) \to K(u)$.

$(\leftarrow)$ To prove the sufficient condition,by Theorem 3.2.8 it suffices to show that $\hat{\varphi}_{l_n(q)}(t) \to e^{-\frac{t^2}{2}}$ for every $q \in (0,1)$ and every $t \in \mathbf{R}$.

Let $q \in (0,1)$ and t be any real number. It follows from (i') and Theorem 3.1.2 that

$$K_{l_n(q)} \xrightarrow{w} K.$$

By Theorem 2.2.10 ,

$$\int_{-\infty}^{\infty} f(t,x)dK_{l_n(q)}(x) \to \int_{-\infty}^{\infty} f(t,x)dK(x).$$

By (3.6),we have $\ln \hat{\varphi}_{l_n(q)}(t) \to \frac{-t^2}{2}$. Hence $\hat{\varphi}_{l_n(q)}(t) \to e^{-\frac{t^2}{2}}$. $\square$

Theorem 3.2.10. *Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the conditions $(\alpha), (\beta)$ and (γ). Then the sequence of distribution functions of random sums*

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n}$$

converges weakly to Φ if and only if

(i') $\quad \sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1$ *for every* $\varepsilon > 0$ *and*

(ii') $\quad \sum_{k=1}^{Z_n} \int_{-\infty}^{\infty} x^2 dF_{nk}(x) \xrightarrow{p} 1.$

Proof. By Theorem 3.2.9, it suffices to show the conditions (i') and (ii') are equivalent to the following conditions

(1) $K_{Z_n}(u) \xrightarrow{p} K(u)$ for every continuity point u of K , and

(2) $K_{Z_n}(+\infty) \xrightarrow{p} K(+\infty)$.

($\rightarrow$) Assume that (i') and (ii') hold. Since (2) is equivalent to (ii'), we have (2). To prove (1), let u be the continuity point of K.

<u>Case 1</u> $u < 0$.

From (i'), we have $\sum_{k=1}^{Z_n} \int_{|x|<-u} x^2 dF_{nk}(x) \xrightarrow{p} 1$. From this fact and (ii') we have $\sum_{k=1}^{Z_n} \int_{|x|>-u} x^2 dF_{nk}(x) = \sum_{k=1}^{Z_n} \int_{-\infty}^{\infty} x^2 dF_{nk}(x) - \sum_{k=1}^{Z_n} \int_{|x|<-u} x^2 dF_{nk}(x) \xrightarrow{p} 1 - 1 = 0.$ Thus $\sum_{k=1}^{Z_n} \int_{-\infty}^{u} x^2 dF_{nk}(x) \xrightarrow{p} 0$. That is $K_{Z_n}(u) \xrightarrow{p} 0$.

<u>Case 2</u> $u > 0$.

From (i'), we have $\sum_{k=1}^{Z_n} \int_{|x|<u} x^2 dF_{nk}(x) \xrightarrow{p} 1$. From this fact and (ii') we have $\sum_{k=1}^{Z_n} \int_{|x|>u} x^2 dF_{nk}(x) = \sum_{k=1}^{Z_n} \int_{-\infty}^{\infty} x^2 dF_{nk}(x) - \sum_{k=1}^{Z_n} \int_{|x|<u} x^2 dF_{nk}(x) \xrightarrow{p} 1 - 1 = 0.$ Then $\sum_{k=1}^{Z_n} \int_{-\infty}^{-u} x^2 dF_{nk}(x) \xrightarrow{p} 0$. Thus $\sum_{k=1}^{Z_n} \int_{-\infty}^{u} x^2 dF_{nk}(x) = \sum_{k=1}^{Z_n} \int_{-\infty}^{-u} x^2 dF_{nk}(x) + \sum_{k=1}^{Z_n} \int_{|x|<u} x^2 dF_{nk}(x) \xrightarrow{p} 0 + 1 = 1$. That is $K_{Z_n}(u) \xrightarrow{p} 1$. From case 1 and case 2, we have (1).

($\leftarrow$) Assume (1) and (2) holds. Since (2) is equivalent to (ii'), we have (ii'). From (1), for $\varepsilon > 0$,

$\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) = \sum_{k=1}^{Z_n} \int_{-\infty}^{\varepsilon} x^2 dF_{nk}(x) - \sum_{k=1}^{Z_n} \int_{-\infty}^{-\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1 - 0 = 1$. Thus we have (i'). Hence (i') and (ii') are equivalent to (1) and (2), respectively. $\square$

Proposition 3.2.11. ([21], p.63) *Let* (Z_n, X_{nk}) *be a random double sequence of random variables which satisfies the following conditions.*

(i′) $\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1$ *for every* $\varepsilon > 0$ *and*

(ii′) (Z_n, X_{nk}) *satisfies the condition* (β).

Then (Z_n, X_{nk}) *satisfies the condition* (α).

The following theorem is the main theorem of this chapter.

Theorem 3.2.12. *Let* (Z_n, X_{nk}) *be a random double sequence of random variables which satisfies the conditions* (β) *and* (γ). *Then*

(i) the sequence of distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n}$$

converges weakly to Φ *and*

(ii) (Z_n, X_{nk}) *satisfies* (α)

if and only if (Z_n, X_{nk}) *satisfies the random Lindeberg condition , i.e.*

$$\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1$$

for every $\varepsilon > 0$.

Proof. It follows from Proposition 3.2.11 and Theorem 3.2.10. □

Corollary 3.2.13. *Let* (k_n) *be a sequence of positive integers. Assume that* $\lim_{n\to\infty} \sum_{k=1}^{k_n} \sigma_{nk}^2 = 1$. *Then*

(i) the sequence of distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n}$$

converges weakly to Φ *and*

(ii) (X_{nk}) , $k = 1, 2, ..., k_n$, $n = 1, 2, ...$ *is infinitesimal, i.e.*

$$\max_{1\le k\le k_n} P(|X_{nk}| \ge \varepsilon) \to 0$$

for every $\varepsilon > 0$

if and only if $(X_{nk}),\ k = 1, 2, ..., k_n,\ n = 1, 2, ...$ *satisfies the Lindeberg condition, i.e.*

$$\sum_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \to 1$$

for every $\varepsilon > 0$.

Proof. In order that S_n can be viewed as a random sums, we define Z_n and $\tilde{X}_{nk}$ as follows. For any positive integer n, we define

$$Z_n(\omega) = k_n$$

for all $\omega \in \Omega$.

For $k = 1, 2, ..., k_n$, define $\tilde{X}_{nk}(\omega) = X_{nk}(\omega)$ for all $\omega \in \Omega$ and for $k > k_n$, define $\tilde{X}_{nk}(\omega) = 0$ for all $\omega \in \Omega$. It follows that $(Z_n, \tilde{X}_{nk})$ is a random double sequence of random variables which are independent in each row. We denote the distribution function, characteristic function, mean and variance of $\tilde{X}_{nk}$ by $\tilde{F}_{nk},\ \tilde{\varphi}_{nk},\ \tilde{\mu}_{nk}$ and $\tilde{\sigma}^2_{nk}$, respectively. Then for each n and k, $\tilde{\mu}_{nk} = 0$. Since $Z_n(\omega) = k_n$ for all $\omega \in \Omega,\ l_n(q) = k_n$ for all $q \in (0, 1)$. First, we will show that $(Z_n, \tilde{X}_{nk})$ satisfies the condition (β). Let $a_{nl_n(q)} = \sum\limits_{k=1}^{l_n(q)} \tilde{\sigma}^2_{nk} = \sum\limits_{k=1}^{k_n} \tilde{\sigma}^2_{nk} = \sum\limits_{k=1}^{k_n} \sigma^2_{nk}$ for all $q \in (0, 1)$. So $a_{nl_n(q)} \to 1$ for all $q \in (0, 1)$. By Theorem 3.1.2, $a_{nZ_n} \xrightarrow{p} 1$. Therefore $\sum\limits_{k=1}^{Z_n} \tilde{\sigma}^2_{nk} \xrightarrow{p} 1$. That is $(Z_n, \tilde{X}_{nk})$ satisfies the condition (β). Let (n') be a subsequence of (n). Since $l_n(q) = k_n$ for all $q \in (0, 1)$, $P(\tilde{X}_{n'1} + \tilde{X}_{n'2} + ... + \tilde{X}_{n'l_{n'}(q)} \leq x) = P(X_{n'1} + X_{n'2} + ... + X_{n'k_{n'}} \leq x)$ for all $q \in (0, 1)$. Then the weak limit distribution function $F^{(q)}(x)$, if it exists, of a sequence of distribution function of the sums

$$\tilde{X}_{n'1} + \tilde{X}_{n'2} + ... + \tilde{X}_{n'l_{n'}(q)}$$

is measurable in q for every x. That is $(Z_n, \tilde{X}_{nk})$ satisfies the condition (γ).

Next , we will prove that the sequence of the distribution functions of the sums

$$\tilde{S}_{Z_n} = \tilde{X}_{n1} + \tilde{X}_{n2} + ... + \tilde{X}_{nZ_n}$$

converges weakly to Φ if and only if the sequence of the distribution functions of sums

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n}$$

converges weakly to Φ.

According to the fact that $P(Z_n = k_n) = 1$, we have the characteristic function $\tilde{\varphi}_n$ of $\tilde{S}_{Z_n}$ is given by

$$\begin{aligned}\tilde{\varphi}_n(t) = E[\prod_{k=1}^{Z_n} \tilde{\varphi}_{nk}(t)] \\ &= \sum_{j=1}^{\infty} P(Z_n = j) \prod_{k=1}^{j} \tilde{\varphi}_{nk}(t) \\ &= P(Z_n = k_n) \prod_{k=1}^{k_n} \tilde{\varphi}_{nk}(t) \\ &= \prod_{k=1}^{k_n} \varphi_{nk}(t)\end{aligned}$$

which is the characteristic function φ_n of S_n. Then $\varphi_n(t) \to e^{\frac{-t^2}{2}}$ for all $t \in \mathbf{R}$ if and only if $\tilde{\varphi}_n(t) \to e^{\frac{-t^2}{2}}$ for all $t \in \mathbf{R}$. Hence the sequence of the distribution functions of $\tilde{S}_{Z_n}$ converges weakly to Φ if and only if the sequence of the distribution functions of S_n converges weakly to Φ.

($\rightarrow$) Let $\varepsilon > 0$ be given and $a_{nl_n(q)} = \max\limits_{1\leq k\leq l_n(q)} P(|X_{nk}| \geq \varepsilon)$. So $a_{nl_n(q)} = a_{nk_n} \to 0$ for all $q \in (0,1)$. By Theorem 3.1.2, $a_{nZ_n} \xrightarrow{p} 0$. That is $\max\limits_{1\leq k\leq Zn} P(|X_{nk}| \geq \varepsilon)$ $\xrightarrow{p}$ 0. Therefore $(Z_n, \tilde{X}_{nk})$ is random infinitesimal. By Theorem 3.2.12, $\sum\limits_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1$ for every $\varepsilon > 0$. It follows from Theorem 3.1.2 , $\sum\limits_{k=1}^{l_n(q)} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \to 1$ for every $q \in (0,1)$ and for every $\varepsilon > 0$. Since $l_n(q) = k_n$ for all $q \in (0,1)$, $\sum\limits_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \to 1$ for every $\varepsilon > 0$. Therefore (X_{nk}) satisfies the Lindeberg condition.

($\leftarrow$) Let $\varepsilon > 0$ be given. Since (X_{nk}) satisfies the Lindeberg condition, we have $\sum\limits_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \to 1$. Since $l_n(q) = k_n$ for all $q \in (0,1)$, $\sum\limits_{k=1}^{l_n(q)} \int_{|x|<\varepsilon} x^2 dF_{nk}(x)$

$\to 1$ for all $q \in (0,1)$. By Theorem 3.1.2, $\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1$. Hence (Z_n, X_{nk}) satisfies the random Lindeberg condition. Therefore the sufficiency follows from Theorem 3.2.12. $\square$

Example 3.2.14. For each n, let Z_n be such that

$$P(Z_n = n) = 1 - \frac{1}{n^2} \text{ and } P(Z_n = n+1) = \frac{1}{n^2}.$$

For each n and k, defined X_{nk} as follows:

If $k \neq n+1$, let X_{nk} be defined by

$$P(X_{nk} = \frac{1}{\sqrt{n}}) = P(X_{nk} = -\frac{1}{\sqrt{n}}) = \frac{1}{2}.$$

In case $k = n+1$, let X_{nk} be defined by

$$P(X_{nk} = 2^n) = P(X_{nk} = -2^n) = \frac{1}{2}.$$

It can be seen that $\mu_{nk} = 0$ for every n and k, and

$$\sigma_{nk}^2 = \begin{cases} \frac{1}{n} & \text{if } \ k \neq n+1 \\ 2^{2n} & \text{if } \ k = n+1. \end{cases}$$

Assume that for each $n, Z_n, X_{n1}, X_{n2}, \ldots$ are independent.

Then

1. For $q \in (0,1)$ and $n \geq 2$,

$$l_n(q) = \begin{cases} n & \text{if } \ 0 < q < 1 - \frac{1}{n^2} \\ n+1 & \text{if } \ 1 - \frac{1}{n^2} \leq q < 1. \end{cases}$$

2. (Z_n, X_{nk}) satisfies the condition (β).

3. (Z_n, X_{nk}) satisfies the condition (γ).

4. (Z_n, X_{nk}) satisfies the random Lindeberg condition.

5.The sequence of the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n}$$

converges weakly to Φ.

Next, we will show that 1-5 hold.

1.For $q \in (0,1)$ and $n \geq 2$.

Case 1 $0 < q < 1 - \frac{1}{n^2}$.

$$P(Z_n < n) = 0 < q \text{ and } P(Z_n < n+1) = P(Z_n = n) = 1 - \frac{1}{n^2} > q.$$

Then $l_n(q) = n$.

Case 2 $1 - \frac{1}{n^2} \leq q < 1$.

$P(Z_n < n+1) = P(Z_n = n) = 1 - \frac{1}{n^2} \leq q$ and

$$P(Z_n < n+2) = P(Z_n = n) + P(Z_n = n+1) = 1 - \frac{1}{n^2} + \frac{1}{n^2} = 1 > q.$$

Then $l_n(q) = n + 1$.

2. For every $\varepsilon > 0$, we have

$$P(|\sum_{k=1}^{Z_n} \sigma_{nk}^2 - 1| \geq \varepsilon) \leq P(Z_n = n+1) = \frac{1}{n^2}$$

which converge to 0. So $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$. Hence (Z_n, X_{nk}) satisfies the condition (β).

3. Let (n') be a subsequence of (n). Let $x \in \mathbf{R}$ and $q \in (0,1)$. Let $N_1 \in \mathbb{N}$ such that $\frac{1}{N_1^2} < 1 - q$. For each $n' \geq N_1$,

$$F_{n'}^{(q)}(x) = P(X_{n'1} + X_{n'2} + ... + X_{n'n'} \leq x).$$

Then the weak limit distribution function $F^{(q)}(x)$ of $F_{n'}^{(q)}(x)$, if it exists, is measurable in q for every x. Therefore (Z_n, X_{nk}) satisfies the condition (γ).

4. Let $\varepsilon > 0$ and $q \in (0,1)$. Let $N_1 \in \mathbb{N}$ such that $\frac{1}{N_1^2} < 1 - q$.

Let $N_2 \in \mathbb{N}$ be such that $\varepsilon > \frac{1}{N_2}$. Let $N = max\{N_1, N_2\}$.

For $n \geq N$,

$$\begin{aligned}
\sum_{k=1}^{l_n(q)} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) &= \sum_{k=1}^{n} \int_{-\varepsilon}^{\varepsilon} x^2 dF_{nk}(x) \\
&= \sum_{k=1}^{n} \Big(\int_{\{-\frac{1}{\sqrt{n}}\}} x^2 dF_{nk}(x) + \int_{\{\frac{1}{\sqrt{n}}\}} x^2 dF_{nk}(x) \Big) \\
&= \sum_{k=1}^{n} \Big(\big(\frac{1}{2} - 0\big)\big(-\frac{1}{\sqrt{n}}\big)^2 + \big(1 - \frac{1}{2}\big)\big(\frac{1}{\sqrt{n}}\big)^2 \Big) \\
&= \sum_{k=1}^{n} \frac{1}{n} = \frac{n}{n} = 1.
\end{aligned}$$

Then $\sum\limits_{k=1}^{l_n(q)} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \to 1$ for all $q \in (0,1)$. Thus $\sum\limits_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x+\mu_{nk})$ $\xrightarrow{p} 1$. Therefore (Z_n, X_{nk}) satisfies the random Lindeberg condition.

4. Follows from Theorem 3.2.12. $\square$

REFERENCES

1. Batirov, H. ;Manevic, D.V. and Nagaev, S.V. On Esseen′s inequality for the sum of a random number of non-identically distributed random variables. Mat.Zametke 22, (1977): 143-146.
2. Belov, E.G. and Pecinkin, A.V. A limit theorem for sums of a random number of random term. Math Operations forsch.Statist.Ser.Stalist.10, (1979): 461-467.
3. Bethmann, J. Limit behaviour of sums of random number of independent and identically distributed random variables.Theory Probab.Applications. 32,(1987): 562-564.
4. Bethmann, J. The Lindeberg-Feller theorem for sum of random number of independent random variables in a triangle array.Theory of Prop. Applications.(1988): 334-339
5. Blum, J.R. ;Hanson, D.L. and Rosenblatt, J.I.On the central limit theorem for the sum of a random number of independent random variables. Z.Wahrscheinlichkeitstheoric und Verw.Gebiete1 (1962): 389-393.
6. Chaidee, N. The convergence of distribution functions of random sums of independent random variables with finite variances .M.D.Thesis, Department of Mathematics, Graduate School, Chulalongkorn University, 2000.
7. Cramer, H. Mathematical method of statistics. Uppsala:Almqvist Wilesells, 1945.
8. Feller,W. An induction to probability theory and its applications. New York: John Wiley*Sons, 1966.
9. Fristedt, B. and Gray, L. A Modern Approach to Probability Theory.Boston: Birkhauser, 1997.
10. Gnedenko, B.V. Theory of probability. New York: Chelsca Publishing 1962.

11. Gnedenko, B.V.and Kolmogorov, A.N. Limit distributions for sums of independent random variables. Cambridge: Addison-Wesley, 1954.

12. Jesiak, B. ;Rossberg H.J. and Siegel,G., Analytic Methods of Probability Theory, Akademic-Verlag, Berlin, 1985.

13. Krajka,A. and Rychlik,Z. Necessary and sufficient conditions for weak convergence of random sums of independent random variables. Comment.Math.Univ Carolin.34, (1993): 465-482.

14. Kruglov, V.M.The convergence of the distributions of sums of a random number of independent random variables to the normal distribution. Moscow Univ. Math.Bull.31, (1976): 5-12.

15. Laha, R.G. and Rohatgi, V.K. Probability Theory. New York: John Wiley*Sons, 1979.

16. Landers, D. and Rogge, L.The exact approximation order in the central-limit -theorem for random summation, Z.Wahrscheinlichkeitstheoric und Verw. Gebiete 36 (1976): 269-283.

17. Loeve, M. Probability Theory. Third Edition,Van Nostrand, Princeton, NJ, 1963.

18. Lukacs, E. Characteristic functions. New York: Hafner Publishing 1960.

19. Mogyoro di, J. A central limit theorem for the sum of a random number of independent random variables. Magyar Tud.Akad.Mat. Kurato Int. Kozl.7, (1962): 409- 424.

20. Mogyoro di,J. A central limit theorem for sum of a random number of random variables. Ann.Univ.Sci.Budapest.Eotvos Sect.Math.10 (1967): 171-182.

21. Munroe, M.E. Measure and integration.California: Addison-Wesley, 1971.

22. Neammanee, K. A limit theorem for random sums of independent random variables with finite variances. Doctoral dissertation, Department of Mathematics, Graduate School, Chulalongkorn University, 1993.

23. Orazov, G. Refinement of theorems on the asumptotic distribution of sums of a random number of random summands under various normalization.

Taskent.Gos.Univ. Naucn. Trudy Vyp.402 (1972) : 87-93.

24. Ouyang ,Guang Zhong, On the central limit theorem for the sum of a random number of independent random variables, Fudan Xue bau 20,(1981): 89-93.

25. Pecinkin, A,V. The convergence to normal law of sums of a random number of random summands. Teor.Verojatnost.iPrimenen.18, (1973) : 380-382.

26. Petrov, V.V. Sums of independent random variables. Berlin:Springer Verlag, 1975.

27. Shanthikumar, J.G. A central limit theorem for random sums of random variables. Oper.Res.Lett.3, (1984): 153-155.

28. Shapiro, J.M. Error estimates for certain probability limit theorems. Ann.Math Statist. (1955): 617-630.

29. Szasz, D. Limit theorems for the distributions of the sums of random number of random variables.Ann.Math.Statist. (1972): 1902-1913.

VITA

Miss Petcharat Rattanawong was born on September 20, 1977 in Prachinburi, Thailand. She graduated with a Bachelor Degree of Science in Mathematics from Chulalongkorn University in 1998. In 1999, she was admitted into a Master Degree program in Mathematics at Chulalongkorn University. During her study for her Master's degree, she received a financial support from University Development Commission (UDC) scholarship for two years.