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IMPROVEMENT OF THE CONSTANT IN THE NON-UNIFORM VERSION OF THE BERRY-ESSEEN THEOREM

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เราปรับปรุงค่าคงตัวในค่าขอบเขตแบบไม่เอกรูปของทฤษฎีบทเบอร์รี่-เอสซีน โดยใช้วิธีการสองวิธี คือ วิธีการของสไตน์โดยใช้อสมการความเข้มข้น และวิธีการจากทฤษฎีบทของพาดิทซ์-ซิกานอฟ โดยที่ตัวแปรสุ่มไม่จำเป็นต้องมีการแจกแจงแบบเดียวกันและไม่จำเป็นต้องมีโมเมนต์ค่าสัมบูรณ์อันดับที่สาม โดยวิธีการทั้งสอง ได้ผลลัพธ์ว่าวิธีการแรกทำให้ได้ค่าคงตัวที่ดีกว่าเดิมที่ได้มีการบันทึกไว้ก่อนหน้านี้ และวิธีการหลังทำให้ได้ค่าคงตัวที่ดีกว่าที่ได้โดยวิธีการแรก

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We improve the constant in non-uniform version of the Berry-Esseen theorem by using two methods, namely, Stein's method using the concentration inequality and the method obtained from the Paditz-Siganov theorem, on which the random variables are not necessarily identically distributed and the existence of the absolute third moment is not required. By these two methods, the former yields a better constant recorded, while the latter gives an even better constant.

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CHAPTER I
INTRODUCTION

Let $X_1, X_2, ..., X_n$ be independent random variables with zero mean, finite variance and $\sum_{i=1}^{n} EX_i^2 = 1$. Define

$$W_n = X_1 + X_2 + ... + X_n.$$

Let F_n be the distribution function of W_n and Φ the standard normal distribution function. The central limit theorem in probability theory and statistics states that

$$F_n(x) \to \Phi(x) \text{ as } n \to \infty.$$

The Berry-Esseen theorem, also known as the Berry-Esseen inequality, attempts to quantify the rate of this convergence. Statements of the theorem vary, as it was independently discovered by two mathematicians,Andrew C.Berry [2] and Carl-Gustav Esseen [5], who then, along with other authors, refined it repeatly over subsequent decades.

Suppose that $E|X_i|^3 < \infty$ for $i = 1, 2, ..., n$,then we have uniform Berry-Esseen theorem

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \le C_0 \sum_{i=1}^{n} E|X_i|^3 \tag{1.1}$$

and the non-uniform version

$$|F_n(x) - \Phi(x)| \le \frac{C_1}{1 + |x|^3} \sum_{i=1}^{n} E|X_i|^3 \tag{1.2}$$

where both C_0 and C_1 are absolute constants.

In case of uniform bound, Berry [2] and Esseen [5] are the first two persons who obtained (1.1) in case of $X_i's$ are identically distributed. Later,Siganov [11] improved the constant down to 0.7655 in 1986 and 0.7164 by Chen [4] in 2002.

Without assuming the identically distributed of $X_i's$, Beek [14] sharpened the constant to 0.7975 in 1972. The best bound in this case was found in 1986 by Siganov [11].

Theorem 1.1. *(Siganov,1986) Let* $X_1, X_2, ..., X_n$ *be independent random variables such that* $EX_i = 0$ *and* $E|X_i|^3 < \infty$ *for* $i = 1, 2, .., n$. *Assume that* $\sum_{i=1}^{n} EX_i^2 = 1$. *Then*

$$\sup_{x \in \mathbb{R}} |P(W_n \leq x) - \Phi(x)| \leq 0.7915 \sum_{i=1}^{n} E|X_i|^3$$

where $W_n = X_1 + X_2 + ... + X_n$.

For non-uniform bound, Nagaev [7] is the first one who obtained (1.2) in case of $X_i's$ are identically distributed random variables and Bikelis [1] generalized Nagaev's result to the case that $X_i's$ are not necessarily identically distributed. Paditz [9] calculated C_1 which is 114.7 in 1977 and improved his bound to be 31.935 in 1989. His result is in Theorem 1.2.

Theorem 1.2. *(Paditz,1989). Under the assumption of Theorem 1.1, we have*

$$|P(W_n \leq x) - \Phi(x)| \leq \frac{31.935}{1 + |x|^3} \sum_{i=1}^{n} E|X_i|^3.$$

Michel [6] reduced the constant to 30.84 for the independent and identically distributed case.

In 2001, Chen and Shao [3] give the versions of (1.1) and (1.2) without assuming the existence of third moments. Their results stated as follows.

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \leq 4.1 \sum_{i=1}^{n} \{E|X_i|^2 I(|X_i| \geq 1|) + E|X_i|^3 I(|X_i| < 1)\} \qquad (1.3)$$

and

$$|F_n(x) - \Phi(x)| \leq C_2 \sum_{i=1}^{n} \Big\{ \frac{EX_i^2 I(|X_i| \geq 1 + |x|)}{(1 + |x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{(1 + |x|)^3} \Big\} \qquad (1.4)$$

where C_2 is a positive constant and $I(A)$ is an indicator random variable such that

$$I(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

In 2005, Neammanee [8] combined the concentration inequality in [3] with coupling approach to calculate the constant in (1.4). Here is his result.

Theorem 1.3. *Let $X_1, X_2, ..., X_n$ be independent random variables with zero means and $\sum_{i=1}^{n} EX_i^2 = 1$. Let $W_n = X_1 + X_2 + ... + X_n$ and let F_n be the distribution function of W_n. Then*

$$|F_n(x) - \Phi(x)| \leq C_3 \sum_{i=1}^{n} \Big\{ \frac{EX_i^2 I(|X_i| \geq 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^3} \Big\} \tag{1.5}$$

where

$$C_3 = \begin{cases} 21.44 & \text{if } |x| \leq 3, \\ 32 & \text{if } 3 < |x| \leq 3.99, \\ 60 & \text{if } 3.99 < |x| \leq 7.98, \\ 32 & \text{if } 7.98 < |x| < 14, \\ 21.44 & \text{if } |x| \geq 14. \end{cases}$$

We observe that the bounds in (1.5) are given in term of truncated moments and the constant obtained is 21.44 for most values. In Theorem 1.4, we improve the concentration inequality which used in [8] in case of $|x| > 3.99$ and get better constants, i.e., 9.7. for almost x.

Theorem 1.4. *Under the assumptions of Theorem 1.3, we have*

$$|F_n(x) - \Phi(x)| \leq C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^3} \right\}$$

where

$$C = \begin{cases} 21.44 & \text{if } |x| \le 3, \\ 32 & \text{if } 3 < |x| \le 3.99, \\ 49.18 & \text{if } 3.99 < |x| \le 7.98, \\ 14.69 & \text{if } 7.98 < |x| < 14, \\ 9.7 & \text{if } |x| \ge 14. \end{cases}$$

The method used in Theorem 1.4 is Stein's method which first introduced by Stein [12] in 1972. Besides Stein's method, we give another bounds in Theorem 1.5, Corollary 1.6 and Corollary 1.7 by using Paditz-Siganov theorems.

Theorem 1.5. *Under the assumptions of Theorem 1.3, we have*

$$|F_n(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |x|)}{(1+|x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{(1+|x|)^3} \right\}$$

where

$$C = \begin{cases} 49.89 & \text{if } 0 \le |x| < 1.3, \\ 59.45 & \text{if } 1.3 \le |x| < 2, \\ 73.52 & \text{if } 2 \le |x| < 3, \\ 76.17 & \text{if } 3 \le |x| < 7.98, \\ 45.80 & \text{if } 7.98 \le |x| < 14, \\ 39.39 & \text{if } |x| \ge 14. \end{cases}$$

Corollary 1.6. *Under the assumptions of Theorem 1.3, we have*

$$|F_n(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |\frac{x}{4}|)}{(1+|\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1+|\frac{x}{4}|)^3} \right\}$$

where

$$C = \begin{cases} 9.54 & \text{if } 0 \le |x| < 1.3, \\ 19.74 & \text{if } 1.3 \le |x| < 2, \\ 18.38 & \text{if } 2 \le |x| < 3, \\ 14.63 & \text{if } 3 \le |x| < 7.98, \\ 5.13 & \text{if } 7.98 \le |x| < 14, \\ 3.55 & \text{if } |x| \ge 14. \end{cases}$$

Corollary 1.7. *Under the assumptions of Theorem 1.3, we have*

$$|F_n(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |x|)}{1 + |x|^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{1 + |x|^3} \right\}$$

where

$$C = \begin{cases} 13.11 & \text{if } 0 \le |x| < 1.3, \\ 28.54 & \text{if } 1.3 \le |x| < 2, \\ 46.32 & \text{if } 2 \le |x| < 3, \\ 61.40 & \text{if } 3 \le |x| < 7.98, \\ 40.12 & \text{if } 7.98 \le |x| < 14, \\ 39.39 & \text{if } |x| \ge 14. \end{cases}$$

This thesis is organized as follows. The proofs of Theorem 1.4 is in chapter 3 and Theorem 1.5, Corollary 1.6 and Corollary 1.7 are in chapter 4. Observe that the constant in Corollary 1.6 is sharper than the constant in Theorem 1.3 and Theorem 1.4.

CHAPTER II
PRELIMINARIES

In this chapter, we review some basic knowledges in probability which will be used in our work.

Basic Knowledge in Probability

A **probability space** is a measure space $(\Omega, \mathcal{F}, P)$ for which $P(\Omega) = 1$. The measure P is called a **probability measure**. The set Ω will be referred to as a **sample space** and its elements are called **points** or **elementary events**. The elements of $\mathcal{F}$ are called **events**. For any event A, the value $P(A)$ is called the **probability of** A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if for every Borel set B in $\mathbb{R}$, $X^{-1}(B)$ belongs to $\mathcal{F}$. We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

A random variable X with its distribution function F is said to be a **discrete random variable** if the image of X is countable and said to be a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the

probability function of X.

Now we will give some examples of random variables.

We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

We say that X is a **uniform** random variable with parameter n if there exist $x_1, x_2, \dots, x_n$ such that $P(X = x_i) = \dfrac{1}{n}$ for any $i = 1, 2, \dots, n$ and denoted by $X \sim U(n)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_\alpha$ is a sub σ-algebra of $\mathcal{F}$ for each $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha |\ \alpha \in \Lambda\}$ is **independent** if and only if for any nonempty finite subset $J = \{j_1, j_2, \dots, j_k\}$ of Λ,

$$P(\bigcap_{m=1}^{k} A_m) = \prod_{m=1}^{k} P(A_m)$$

where $A_m \in \mathcal{F}_{j_m}$ for $m = 1, 2, \dots, k$.

Let $\mathcal{E}_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{E}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\mathcal{E}_\alpha) | \alpha \in \Lambda\}$ is independent where $\sigma(\mathcal{E}_\alpha)$ is the smallest σ-algebra with $\mathcal{E}_\alpha \subseteq \sigma(\mathcal{E}_\alpha)$.

We say that the set of random variables $\{X_\alpha |\ \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha) |\ \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B$ is a Borel subset of $\mathbb{R}\}$.

Theorem 2.1. *Random variables* $X_1, X_2, \dots, X_n$ *are* **independent** *if and only if for any Borel sets* $B_1, B_2, \dots, B_n$*, we have*

$$P(\bigcap_{i=1}^{n} \{X_i \in B_i\}) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$. If $\displaystyle\int_\Omega |X| dP < \infty$, then we define its **expected value** or **mean** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.2.

1. *If X is a discrete random variable, then* $E(X) = \sum_{x \in \operatorname{Im} X} xP(X = x)$.

2. *If X is a continuous random variable with probability function f, then*
$$E(X) = \int_{\mathbb{R}} xf(x)dx.$$

Proposition 2.3. *Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$ and $a, b \in \mathbb{R}$. Then we have the followings:*

1. $E(aX + bY) = aE(X) + bE(Y)$.

2. *If $X \leq Y$, then* $E(X) \leq E(Y)$.

3. $|E(X)| \leq E(|X|)$.

Let X be a random variable which $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the **k-th moment** of X about the origin and call $E[(X - E(X))^k]$ the **k-th moment** of X about its mean.

We call the second moment of X about its mean, the **variance** of X, and denote by $\operatorname{Var}(X)$. Then
$$\operatorname{Var}(X) = E[X - E(X)]^2.$$

We note that
$$\operatorname{Var}(X) \ = \ E(X^2) - E^2(X).$$

Proposition 2.4. *If $X_1, X_2, \ldots, X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, \ldots, n$, then*

1. $E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2) \cdots E(X_n)$,

2. $\operatorname{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) \ = \ a_1^2 \operatorname{Var}(X_1) + a_2^2 \operatorname{Var}(X_2) + \cdots + a_n^2 \operatorname{Var}(X_n)$ *for any real number $a_1, a_2, \ldots, a_n$.*

The following inequalities are useful in our work.

1. **Hölder's inequality** :

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p)E^{\frac{1}{q}}(|Y|^q)$$

where $1 \leq p, q \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $E(|X|^p) < \infty$, $E(|Y|^q) < \infty$.

2. **Chebyshev's inequality** :

$$P(|X| \geq \varepsilon) \leq \frac{E|X|^p}{\varepsilon^p} \quad \text{for all} \quad \varepsilon, p > 0$$

where $E|X|^p < \infty$.

CHAPTER III

MAIN RESULT VIA STEIN'S METHOD

In this chapter, let $X_1, X_2, ..., X_n$ be independent random variables with zero mean, finite variance and $\sum_{i=1}^{n} EX_i^2 = 1$. Define

$$W_n = X_1 + X_2 + \cdots + X_n.$$

Let F_n be the distribution function of W_n, and Φ the standard normal distribution function. Then $VarW_n = 1$ and $EW_n^2 = 1$.

In 2001, Chen and Shao [3] investigated the constant in the non-uniform version of the Berry-Esseen theorem in term of truncated moments without assuming the existence of third moments. Their result states as follows.

Theorem 3.1.

$$|F_n(x) - \Phi(x)| \leq C \sum_{i=1}^{n} \Big\{ \frac{EX_i^2 I(|X_i| \geq 1 + |x|)}{(1+|x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{(1+|x|)^3} \Big\}$$

where C is a positive constant.

In 2005, Neammanee [8] combined the concentration inequality approach which used in [3] and the coupling approach to calculate the constant C. Here is his result.

Theorem 3.2.

$$|F_n(x) - \Phi(x)| \leq C \sum_{i=1}^{n} \Big\{ \frac{EX_i^2 I(|X_i| \geq 1 + |\frac{x}{4}|)}{(1+|\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1+|\frac{x}{4}|)^3} \Big\}$$

where

$$C = \begin{cases} 21.44 & \textit{if} \ \ |x| \leq 3, \\ 32 & \textit{if} \ \ 3 < |x| \leq 3.99, \\ 60 & \textit{if} \ \ 3.99 < |x| \leq 7.98, \\ 32 & \textit{if} \ \ 7.98 < |x| < 14, \\ 21.44 & \textit{if} \ \ |x| \geq 14. \end{cases}$$

The aim of this chapter is to improve the constant in Theorem 3.2. Here is our result.

Theorem 3.3. *(Main theorem)*

$$|F_n(x) - \Phi(x)| \leq C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^3} \right\}$$

where

$$C = \begin{cases} 21.44 & \textit{if} \ \ |x| \leq 3, \\ 32 & \textit{if} \ \ 3 < |x| \leq 3.99, \\ 49.18 & \textit{if} \ \ 3.99 < |x| \leq 7.98, \\ 14.69 & \textit{if} \ \ 7.98 < |x| < 14, \\ 9.7 & \textit{if} \ \ |x| \geq 14. \end{cases}$$

The tool which was used to prove these results is the Stein's technique. Stein(1972) introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation. This technique is free from Fourier methods but relied instead on the differential equation. Stein's method has been widely applied in the area of normal approximation. The method is as follows.

Let Z be a standard normal distributed random variable and C_{bd} the set of continuous and piecewise continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$. For $f \in C_{bd}$ and a real valued measurable function h with $E|N(h)| < \infty$, the equation

$$f'(w) - wf(w) = h(w) - N(h) \quad \text{for } w \in \mathbb{R}$$

is called Stein's equation for normal distribution, where

$$N(h) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(x) e^{-\frac{x^2}{2}} dx.$$

If $h = h_x$ where $h_x : \mathbb{R} \to \mathbb{R}$ is defined by

$$h_x(w) = \begin{cases} 1 & \text{if} \quad w \leq x, \\ 0 & \text{otherwise} \end{cases}$$

for $w \in \mathbb{R}$, then the Stein's equation becomes

$$f'(w) - wf(w) = h_x(w) - \Phi(x). \tag{3.1}$$

Hence

$$E(f'(W) - Wf(W)) = P(W \leq x) - \Phi(x)$$

for any random variable W. It is well-known that the solution f_x of Stein's equation in (3.1) is given by

$$f_x(w) = \begin{cases} \sqrt{2\pi} e^{\frac{w^2}{2}} \Phi(w)[1 - \Phi(x)] & \text{if} \quad w \leq x, \\ \sqrt{2\pi} e^{\frac{w^2}{2}} \Phi(x)[1 - \Phi(w)] & \text{if} \quad w > x, \end{cases}$$

and for all real number w, u, we have

$$|wf_x(w)| < 1, \tag{3.2}$$

$$|f_x'(w)| \leq 1, \tag{3.3}$$

$$|f_x'(w) - f_x'(u)| \leq 1, \tag{3.4}$$

$$\text{and} \quad 0 < f_x(w) \leq \min(\frac{\sqrt{2\pi}}{4}, \frac{1}{x}) \quad \text{for} \quad x > 0 \tag{3.5}$$

(see Chen and Shao [3], pp.246). To prove our main result, we divide the proof into two parts, auxiliary results and the main theorem.

3.1 Auxiliary Results

For each $x > 0$, let $Y_{i,x} = X_i I(|X_i| < 1 + x)$, $S_x = \sum_{i=1}^{n} Y_{i,x}$,

$$\alpha_x = \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1 + x),\ \beta_x = \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1 + x),$$

$$\gamma_x = \frac{\beta_x}{2}, \quad \text{and} \quad \delta_x = \frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3}.$$

Proposition 3.4. *Let* $0 \leq x \leq y$. *Then*

1. $\delta_x \geq \delta_y$,
2. $\delta_y \leq \dfrac{(1+x)^2}{(1+y)^2}\delta_x$ *and*
3. $\delta_x \leq \dfrac{(1+y)^3}{(1+x)^3}\delta_y$.

Proof. 1. It follows from the fact that

$$\begin{aligned}
&\frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3} \\
&= \frac{EX_i^2 I(|X_i| \geq 1+y)}{(1+x)^2} + \frac{EX_i^2 I(1+x \leq |X_i| < 1+y)}{(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3} \\
&\geq \frac{EX_i^2 I(|X_i| \geq 1+y)}{(1+x)^2} + \frac{E|X_i|^3 I(1+x \leq |X_i| < 1+y)}{(1+y)(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3} \\
&\geq \frac{EX_i^2 I(|X_i| \geq 1+y)}{(1+y)^2} + \frac{E|X_i|^3 I(1+x \leq |X_i| < 1+y)}{(1+y)^3} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+y)^3} \\
&= \frac{EX_i^2 I(|X_i| \geq 1+y)}{(1+y)^2} + \frac{E|X_i|^3 I(|X_i| < 1+y)}{(1+y)^3}.
\end{aligned}$$

2. Note that

$$\begin{aligned}
&(1+y)^2\Big[\frac{EX_i^2 I(|X_i| \geq 1+y)}{(1+y)^2} + \frac{E|X_i|^3 I(|X_i| < 1+y)}{(1+y)^3}\Big] \\
&= EX_i^2 I(|X_i| \geq 1+y) + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+y)} + \frac{E|X_i|^3 I(1+x \leq |X_i| < 1+y)}{(1+y)} \\
&\leq EX_i^2 I(|X_i| \geq 1+y) + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)} + \frac{(1+y)EX_i^2 I(1+x \leq |X_i| < 1+y)}{(1+y)}
\end{aligned}$$

$$= \; EX_i^2 I(|X_i| \geq 1+x) + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)}$$
$$= \; (1+x)^2 [\frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3}].$$

Hence 2. holds.

3. By the fact that

$$(1+x)^3 [\frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3}]$$
$$= \; (1+x) EX_i^2 I(|X_i| \geq 1+y) + (1+x) EX_i^2 I(1+x \leq |X_i| < 1+y)$$
$$+ E|X_i|^3 I(|X_i| < 1+x)$$
$$\leq (1+y) EX_i^2 I(|X_i| \geq 1+y) + \frac{(1+x) E|X_i|^3 I(1+x \leq |X_i| < 1+y)}{(1+x)}$$
$$+ E|X_i|^3 I(|X_i| < 1+x)$$
$$= \; (1+y) EX_i^2 I(|X_i| \geq 1+y) + E|X_i|^3 I(|X_i| < 1+y)$$
$$= (1+y)^3 [\frac{EX_i^2 I(|X_i| \geq 1+y)}{(1+y)^2} + \frac{E|X_i|^3 I(|X_i| < 1+y)}{(1+y)^3}],$$

we have $(1+x)^3 \delta_x \leq (1+y)^3 \delta_y$. □

Remark. 1. and 3. in Proposition 3.4 are stated in Chen and Shao [3], but the proof was not given.

Proposition 3.5. *For a nonempty subset* Λ *of* $\{1, 2, ..., n\}$, *let*

$$S_{\Lambda,x} = \sum_{i \in \Lambda} Y_{i,x} \;\; and \;\; U^i_{\Lambda,x} = \sum_{\substack{j \in \Lambda \\ j \neq i}} |Y_{j,x}| \min(\gamma_x, |Y_{j,x}|).$$

Then

1. $|ES_{\Lambda,x}| \leq \dfrac{\alpha_x}{1+x}$,
2. $ES^4_{\Lambda,x} \leq \; (1+x)\beta_x + 1 + \dfrac{\alpha_x \beta_x}{1+x} + (\dfrac{\alpha_x}{1+x})^2 + (\dfrac{\alpha_x}{1+x})^4$, *and*
3. $E|U^i_{\Lambda,x} - EU^i_{\Lambda,x}|^4 \leq (16(1+x)\beta_x + 1)\gamma_x^4$.

Proof. 1. It follows from the fact that

$$\begin{aligned}
|ES_{\Lambda,x}| &= |\sum_{i\in\Lambda} EY_{i,x}| \\
&= |\sum_{i\in\Lambda} EX_i I(|X_i| < 1+x)| \\
&= |\sum_{i\in\Lambda} EX_i - \sum_{i\in\Lambda} EX_i I(|X_i| \geq 1+x)| \\
&= |\sum_{i\in\Lambda} EX_i I(|X_i| \geq 1+x)| \\
&\leq \sum_{i=1}^{n} E|X_i| I(|X_i| \geq 1+x) \\
&\leq \sum_{i=1}^{n} \frac{E|X_i|^2 I(|X_i| \geq 1+x)}{1+x} \\
&= \frac{\alpha_x}{1+x}.
\end{aligned}$$

2. From 1. and the fact that $|Y_{i,x}| = |X_i| I(|X_i| < 1+x) < 1+x$ and $\sum_{i\in\Lambda} EY_{i,x}^2 \leq \sum_{i=1}^{n} EX_i^2 = 1$, we have

$$\begin{aligned}
ES_{\Lambda,x}^4 &= E(\sum_{i\in\Lambda} Y_{i,x})^4 \\
&= E[\sum_{i\in\Lambda} Y_{i,x}^4 + \sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda \\ j\neq i}} Y_{i,x}^2 Y_{j,x}^2 + \sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda \\ j\neq i}} Y_{i,x}^3 Y_{j,x} \\
&\quad + \sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda \\ j\neq i}}\sum_{\substack{k\in\Lambda \\ k\neq i,j}} Y_{i,x}^2 Y_{j,x} Y_{k,x} + \sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda \\ j\neq i}}\sum_{\substack{k\in\Lambda \\ k\neq i,j}}\sum_{\substack{l\in\Lambda \\ l\neq i,j,k}} Y_{i,x} Y_{j,x} Y_{k,x} Y_{l,x}] \\
&\leq \sum_{i\in\Lambda} E|Y_{i,x}|^3 |Y_{i,x}| + |\sum_{i\in\Lambda} EY_{i,x}^2||\sum_{j\in\Lambda} EY_{j,x}^2| \\
&\quad + \sum_{i\in\Lambda} E|Y_{i,x}|^3 |\sum_{\substack{j\in\Lambda \\ j\neq i}} EY_{j,x}| + |\sum_{i\in\Lambda} EY_{i,x}^2||\sum_{\substack{j\in\Lambda \\ j\neq i}} EY_{j,x}||\sum_{\substack{k\in\Lambda \\ k\neq i,j}} EY_{k,x}| \\
&\quad + |\sum_{i\in\Lambda} EY_{i,x}||\sum_{\substack{j\in\Lambda \\ i\neq j}} EY_{j,x}||\sum_{\substack{k\in\Lambda \\ k\neq i,j}} EY_{k,x}||\sum_{\substack{l\in\Lambda \\ l\neq i,j,k}} EY_{l,x}| \\
&\leq (1+x)\beta_x + 1 + \frac{\alpha_x\beta_x}{1+x} + (\frac{\alpha_x}{1+x})^2 + (\frac{\alpha_x}{1+x})^4.
\end{aligned}$$

We note that $\sum_{i\in\phi} a_i = 0$ for real numbers $a_i's$

3. For each $j \in \Lambda$, let

$$\bar{Y}_{j,x} = |Y_{j,x}|\min(\gamma_x, |Y_{j,x}|) - E|Y_{j,x}|\min(\gamma_x, |Y_{j,x}|).$$

Then $E\bar{Y}_{j,x} = 0$ and $|\bar{Y}_{j,x}| \leq \gamma_x(|Y_{j,x}| + E|Y_{j,x}|) \leq 2(1+x)\gamma_x$. Similar to (2), we can show that

$$\begin{aligned}
& E|U^i_{\Lambda,x} - EU^i_{\Lambda,x}|^4 \\
&\quad = E[\sum_{\substack{j\in\Lambda\\ j\neq i}} \bar{Y}_{j,x}]^4 \\
&\quad = E\Big[\sum_{\substack{j\in\Lambda\\ j\neq i}} \bar{Y}^4_{j,x} + \sum_{\substack{j\in\Lambda\\ j\neq i}}\sum_{\substack{k\in\Lambda\\ k\neq i,j}} \bar{Y}^2_{j,x}\bar{Y}^2_{k,x} + \sum_{\substack{j\in\Lambda\\ j\neq i}}\sum_{\substack{k\in\Lambda\\ k\neq i,j}} \bar{Y}^3_{j,x}\bar{Y}_{k,x} \\
&\qquad + \sum_{\substack{j\in\Lambda\\ j\neq i}}\sum_{\substack{k\in\Lambda\\ k\neq i,j}}\sum_{\substack{l\in\Lambda\\ l\neq i,j,k}} \bar{Y}^2_{j,x}\bar{Y}_{k,x}\bar{Y}_{l,x} + \sum_{\substack{j\in\Lambda\\ j\neq i}}\sum_{\substack{k\in\Lambda\\ k\neq i,j}}\sum_{\substack{l\in\Lambda\\ l\neq i,j,k}}\sum_{\substack{m\in\Lambda\\ m\neq i,j,k,l}} \bar{Y}_{j,x}\bar{Y}_{k,x}\bar{Y}_{l,x}\bar{Y}_{m,x}\Big] \\
&\quad \leq \sum_{\substack{j\in\Lambda\\ j\neq i}} E|\bar{Y}_{j,x}|^3|\bar{Y}_{j,x}| + |\sum_{\substack{j\in\Lambda\\ j\neq i}} E\bar{Y}^2_{j,x}||\sum_{\substack{k\in\Lambda\\ k\neq i,j}} E\bar{Y}^2_{k,x}| \\
&\quad \leq 2(1+x)\gamma_x \sum_{j\in\Lambda} E|\bar{Y}_{j,x}|^3 + |\sum_{j\in\Lambda} E\bar{Y}^2_{j,x}|^2 \\
&\quad \leq 2(1+x)\gamma_x \sum_{j\in\Lambda} E\Big[|Y_{j,x}|\min(\gamma_x,|Y_{j,x}|) + E|Y_{j,x}|\min(\gamma_x,|Y_{j,x}|)\Big]^3 \\
&\qquad + \Big[\sum_{j\in\Lambda} Var(|Y_{j,x}|\min(\gamma_x,|Y_{j,x}|))\Big]^2 \\
&\quad \leq 2(1+x)\gamma^4_x \sum_{j\in\Lambda} E\Big[|Y_{j,x}| + E|Y_{j,x}|\Big]^3 + \Big[\sum_{j\in\Lambda} E(|Y_{j,x}|\min(\gamma_x,|Y_{j,x}|))^2\Big]^2 \\
&\quad \leq 8(1+x)\gamma^4_x \sum_{j\in\Lambda} E(|Y_{j,x}|^3 + E|Y_{j,x}|^3) + \gamma^4_x\Big[\sum_{j\in\Lambda} EY^2_{j,x}\Big]^2 \\
&\quad \leq 16(1+x)\gamma^4_x \sum_{j\in\Lambda} E|Y_{j,x}|^3 + \gamma^4_x \\
&\quad \leq (16(1+x)\beta_x + 1)\gamma^4_x
\end{aligned}$$

where we use the fact that $(a+b)^3 \leq 2^2(a^3+b^3)$ for $a,b>0$ and $\sum_{j\in\Lambda} EY_{j,x}^2 \leq \sum_{j=1}^{n} EX_j^2 \leq 1$ in the fifth and the sixth inequality, respectively. □

Proposition 3.6. *Let x be a positive real number and $g:\mathbb{R}\to\mathbb{R}$ defined by*

$$g(w) = (wf_x(w))'.$$

Then

1. *g is an increasing non-negative function on $[0,x)$ and*
2. *$|g(x)| \leq 1+x$ for $x \geq 1$.*

Proof. 1. See Chen and Shao [3], pp. 249.

2. By Chen and Shao [3], pp. 248, we have

$$g(w) = \begin{cases} (\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}(1-\Phi(w)) - w)\Phi(x) & \text{if} \quad w \geq x, \\ (\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\Phi(w)) + w)(1-\Phi(x)) & \text{if} \quad w < x. \end{cases}$$

Then for $x \geq 1$,

$$\begin{aligned} |g(x)| &\leq \Big(\sqrt{2\pi}(1+x^2)e^{\frac{x^2}{2}}(1-\Phi(x)) - x\Big)\Phi(x) \\ &\leq \Big(\sqrt{2\pi}(1+x^2)e^{\frac{x^2}{2}}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} - x\Big)\Phi(x) \\ &= (\frac{1+x^2}{x} - x)\Phi(x) \\ &\leq 1+x \end{aligned}$$

where we used the fact that

$$1-\Phi(x) \leq \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} \tag{3.6}$$

for $x>0$ (see [13], pp. 23) in the second inequality. □

In order to prove the main theorem, we use the idea of Neammanee [8]. Then, we need the following concentration inequality and proposition 3.8.

Proposition 3.7. *Concentration Inequality*
Let $i \in \{1, 2, \ldots, n\}$, $W_n^{(i)} = W_n - X_i$, *and* $S_{i,a} = S_{\Lambda,a}$ *where* $\Lambda = \{1, 2, ..., n\} - \{i\}$. *Then for* $1 \le a < b < \infty$ *and* $(1+a)^2\alpha_a + (1+a)\beta_a < \frac{1}{80}$, *we have*

$$P(a \le W_n^{(i)} \le b) \le \frac{(b-a+2\gamma_a)}{C(1+a)^3}\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3} + \frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2} + \frac{3(1+\gamma_a)}{(a-\gamma_a)} + 1\Big)ES_{i,a}^4 + \frac{1.465 \times 10^{-7}\beta_a}{[0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C]^4(1+a)^3} + \frac{\alpha_a}{(1+a)^2}$$

for any positive constant C *such that* $C < 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a$. *Furthermore,*

1. $P(a \le W_n^{(i)} \le b) \le \frac{7.417}{(1+a)^3}(b-a) + 8.125\delta_a$ *for* $a \ge 2$,

2. $P(a \le W_n^{(i)} \le b) \le \frac{5.264}{(1+a)^3}(b-a) + 7.018\delta_a$ *for* $a \ge 3$,

3. $P(a \le W_n^{(i)} \le b) \le \frac{3.522}{(1+a)^3}(b-a) + 3.916\delta_a$ *for* $a \ge 6$.

Proof. By the fact that $(1+a)^2\alpha_a + (1+a)\beta_a < \frac{1}{80}$, we have $\alpha_a < \frac{1}{80}$ and $\beta_a < \frac{1}{80}$. Hence $a - \gamma_a > 0$ and $0.5 - (\beta_a)^{\frac{2}{3}} - 2\gamma_a > 0$.
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{for} \quad t < a - \gamma_a, \\ (1+t+\gamma_a)^3(t-a+\gamma_a) & \text{for} \quad a-\gamma_a \le t \le b+\gamma_a, \\ (1+t+\gamma_a)^3(b-a+2\gamma_a) & \text{for} \quad t > b+\gamma_a. \end{cases}$$

By equations (2.19) and (2.23), pp. 1958-1959 of Neammanee [8], we know that

$$P(a \le W_n^{(i)} \le b) \le \frac{1}{C(1+a)^3}ES_{i,a}f(S_{i,a}) + P(U_{\Lambda,a}^i \le C) + \frac{\alpha_a}{(1+a)^2}, \quad (3.7)$$

for every positive constant C.
To bound the right hand side of (3.7), we divide the proof into two steps.
Step 1. We will show that

$$ES_{i,a}f(S_{i,a}) \le (b-a+2\gamma_a)\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3} + \frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2} + \frac{3(1+\gamma_a)}{(a-\gamma_a)} + 1\Big)ES_{i,a}^4. \quad (3.8)$$

First, we will show that

$$ES_{i,a}f(S_{i,a}) \le ES_{i,a}I(S_{i,a} \ge a-\gamma_a)(1+S_{i,a}+\gamma_a)^3(b-a+2\gamma_a). \tag{3.9}$$

It is obvious that (3.9) holds in case of $S_{i,a} < a-\gamma_a$ and $S_{i,a} > b+\gamma_a$.
Assume that $a-\gamma_a \le S_{i,a} \le b+\gamma_a$. Then

$$\begin{aligned} ES_{i,a}f(S_{i,a}) &= ES_{i,a}(1+S_{i,a}+\gamma_a)^3(S_{i,a}-a+\gamma_a) \\ &= ES_{i,a}I(S_{i,a} \ge a-\gamma_a)(1+S_{i,a}+\gamma_a)^3(S_{i,a}-a+\gamma_a) \\ &\le ES_{i,a}I(S_{i,a} \ge a-\gamma_a)(1+S_{i,a}+\gamma_a)^3((b+\gamma_a)-a+\gamma_a) \\ &= ES_{i,a}I(S_{i,a} \ge a-\gamma_a)(1+S_{i,a}+\gamma_a)^3(b-a+2\gamma_a). \end{aligned}$$

Hence, (3.9) holds. By (3.9),

$$\begin{aligned} &ES_{i,a}f(S_{i,a}) \\ &\quad \le (b-a+2\gamma_a)|ES_{i,a}I(S_{i,a} \ge a-\gamma_a)(1+S_{i,a}+\gamma_a)^3| \\ &\quad = (b-a+2\gamma_a)|ES_{i,a}I(S_{i,a} \ge a-\gamma_a)\{(1+\gamma_a)^3+3(1+\gamma_a)^2S_{i,a} \\ &\qquad +3(1+\gamma_a)S_{i,a}^2+S_{i,a}^3\}| \\ &\quad \le (b-a+2\gamma_a)\Big\{(1+\gamma_a)^3|ES_{i,a}I(S_{i,a} \ge a-\gamma_a)| \\ &\qquad +3(1+\gamma_a)^2ES_{i,a}^2I(S_{i,a} \ge a-\gamma_a)+3(1+\gamma_a)|ES_{i,a}^3I(S_{i,a} \ge a-\gamma_a)| \\ &\qquad +ES_{i,a}^4\Big\}. \end{aligned}$$

From this fact and the fact that

$$\begin{aligned} |ES_{i,a}I(S_{i,a} \ge a-\gamma_a)| &\le \frac{ES_{i,a}^4I(S_{i,a} \ge a-\gamma_a)}{(a-\gamma_a)^3} \le \frac{ES_{i,a}^4}{(a-\gamma_a)^3}, \\ |ES_{i,a}^2I(S_{i,a} \ge a-\gamma_a)| &\le \frac{ES_{i,a}^4I(S_{i,a} \ge a-\gamma_a)}{(a-\gamma_a)^2} \le \frac{ES_{i,a}^4}{(a-\gamma_a)^2}, \\ |ES_{i,a}^3I(S_{i,a} \ge a-\gamma_a)| &\le \frac{ES_{i,a}^4I(S_{i,a} \ge a-\gamma_a)}{(a-\gamma_a)} \le \frac{ES_{i,a}^4}{(a-\gamma_a)}, \end{aligned}$$

we have

$$ES_{i,a}f(S_{i,a}) \le (b-a+2\gamma_a)\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3}+\frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2}+\frac{3(1+\gamma_a)}{(a-\gamma_a)}+1\Big)ES_{i,a}^4.$$

Step 2. We will show that

$$P(U^i_{\Lambda,a} \le C) \le \frac{1.464 \times 10^{-7}\beta_a}{[0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C]^4(1+a)^3}. \tag{3.10}$$

To bound $P(U^i_{\Lambda,a} \le C)$, we note that

$$EU^i_{\Lambda,a} \ge 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a$$

(see Neammanee [8], pp.1959). By proposition (3.5)(3) and the fact that $(1+a)\beta_a < \frac{1}{80}$, we have

$$\begin{aligned} &E|U^i_{\Lambda,a} - EU^i_{\Lambda,a}|^4 \\ &\le (16(\frac{1}{80}) + 1)\gamma_a^4 = 1.2\gamma_a^4 = 0.075\beta_a^4 = 0.075\beta_a^3\beta_a \le 1.465 \times 10^{-7}\frac{\beta_a}{(1+a)^3}. \end{aligned}$$

By Chebyshev's inequality, we have, for $C < 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a$,

$$\begin{aligned} P(U^i_{\Lambda,a} \le C) &\le P(EU^i_{\Lambda,a} - U^i_{\Lambda,a} \ge 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C) \\ &\le \frac{E|U^i_{\Lambda,a} - EU^i_{\Lambda,a}|^4}{[0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C]^4} \\ &\le \frac{1.465 \times 10^{-7}\beta_a}{[0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C]^4(1+a)^3}. \end{aligned}$$

By (3.7),(3.8), and (3.10), we have

$$\begin{aligned} P(a \le W_n^{(i)} \le b) \le{}& \frac{(b-a+2\gamma_a)}{C(1+a)^3}\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3} + \frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2} + \frac{3(1+\gamma_a)}{(a-\gamma_a)} + 1\Big)ES^4_{i,a} \\ &+ \frac{1.465 \times 10^{-7}\beta_a}{[0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C]^4(1+a)^3} + \frac{\alpha_a}{(1+a)^2} \end{aligned} \tag{3.11}$$

for $0 < C < 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a$.

Next, we will prove the proposition in case of $a \ge 6$. Let

$$C = 0.46. \tag{3.12}$$

Since $(1+a)^2\alpha_a + (1+a)\beta_a < \frac{1}{80}$ and $a \ge 6$,

$$\beta_a < \frac{1}{80(1+a)} \le 0.00179, \quad \alpha_a < \frac{1}{80(1+a)^2} \le 0.000255 \tag{3.13}$$

$$\gamma_a < 0.000893, \quad (1+a)\beta_a < \frac{1}{80}, \quad \text{and} \quad \frac{\alpha_a}{1+a} < 0.000037. \tag{3.14}$$

By (3.13), we have

$$
\begin{aligned}
0.5-(\beta_a)^{\frac{2}{3}}-2\alpha_a &\geq 0.5-(0.00179)^{\frac{2}{3}}-2(0.000255)\\
&\geq 0.4847. \qquad (3.15)
\end{aligned}
$$

Then, by (3.11) with $C=0.46$, we have

$$
\begin{aligned}
P(a\leq W_n^{(i)}\leq b) \leq{}& \frac{(b-a+2\gamma_a)}{0.46(1+a)^3}\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3}+\frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2}+\frac{3(1+\gamma_a)}{(a-\gamma_a)}+1\Big)ES_{i,a}^4\\
&+\frac{1.465\times 10^{-7}\beta_a}{[0.5-(\beta_a)^{\frac{2}{3}}-2\alpha_a-0.46]^4(1+a)^3}+\frac{\alpha_a}{(1+a)^2}. \qquad (3.16)
\end{aligned}
$$

From (3.14), proposition (3.5)(2), and $a\geq 6$, we have

$$
\begin{aligned}
\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3}+\frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2}+\frac{3(1+\gamma_a)}{(a-\gamma_a)}+1\Big) &\leq \Big((\frac{1.001}{5.999})^3+3(\frac{1.001}{5.999})^2+3(\frac{1.001}{5.999})+1\Big)\\
&\leq 1.589
\end{aligned}
$$

and

$$
ES_{i,a}^4\leq \frac{1}{80}+1+\frac{1}{(80)(7)}+(0.000037)^2+(0.000037)^4\leq 1.014,
$$

which implies that

$$
\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3}+\frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2}+\frac{3(1+\gamma_a)}{(a-\gamma_a)}+1\Big)ES_{i,a}^4\leq 1.62. \qquad (3.17)
$$

Hence, by (3.15)-(3.17),

$$
\begin{aligned}
P(a\leq W_n^{(i)}\leq b) &\leq \frac{1.62(b-a+2\gamma_a)}{0.46(1+a)^3}+\frac{1.465\times 10^{-7}\beta_a}{(0.0247)^4(1+a)^3}+\frac{\alpha_a}{(1+a)^2}\\
&\leq \frac{3.522(b-a)}{(1+a)^3}+\frac{3.522\beta_a}{(1+a)^3}+\frac{0.394\beta_a}{(1+a)^3}+\frac{\alpha_a}{(1+a)^2}\\
&\leq \frac{3.522}{(1+a)^3}(b-a)+3.916\delta_a.
\end{aligned}
$$

Similar to case $a\geq 6$, for case $a\geq 1$, $a\geq 2$ and $a\geq 3$, we choose C in (3.12) to be $0.44, 0.43$, and 0.46, respectively. □

Proposition 3.8. *Let x be a positive real number and g defined as in proposition 3.6. If $(1+x)^2\alpha_x + (1+x)\beta_x < \frac{1}{80}$, then for $|u| \le 1 + \frac{x}{4}$, we have*

1. $Eg(W_n^{(i)} + u) \le \dfrac{0.458}{(1+\frac{x}{4})^3} + 0.903\delta_{\frac{x}{4}}(1+x)$ *for* $x \ge 14$,

2. $Eg(W_n^{(i)} + u) \le \dfrac{1.344}{(1+\frac{x}{4})^3} + 2.534\delta_{\frac{x}{4}}(1+x)$ *for* $7.98 \le x < 14$,

3. $Eg(W_n^{(i)} + u) \le \dfrac{20.319}{(1+\frac{x}{4})^3} + 19.828\delta_{\frac{x}{4}}(1+x)$ *for* $3.99 \le x < 7.98$.

Proof. We will prove the proposition in case of $x \ge 14$ and for the other cases we can use the same argument.

From equations (2.44) and (2.45) of proposition 2.4 in Neammanee [8], we have, for $x \ge 14$

$$\begin{aligned} Eg(W_n^{(i)} + u) \;&\le \frac{2.517}{(1+x)^3} + g(x-1)P(x-1 < W_n^{(i)} + u < x) \\ &\quad + \int_{x-1}^{x} g'(w)P(w < W_n^{(i)} + u < x)dw \end{aligned} \tag{3.18}$$

and

$$g(x-1) \le \frac{0.056}{(1+x)^3}. \tag{3.19}$$

Since $(x-1) - u \ge (x-1) - (1+\frac{x}{4}) \ge 8.4$ for $x \ge 14$,

$$\begin{aligned} P(x-1 < W_n^{(i)} + u < x) \;&\le\; P(W_n^{(i)} > x-1-u) \\ &\le\; P(W_n^{(i)} > 8.4) \\ &\le\; \frac{E(W_n^{(i)})^2}{(8.4)^2} \\ &\le\; \frac{EW_n^2}{70.56} \\ &=\; \frac{1}{70.56} \\ &=\; 0.0142 \end{aligned} \tag{3.20}$$

where we have used Chebyshev's inequality in the third inequality.
So, by (3.18)-(3.20), and proposition 3.7(3)

$$
\begin{aligned}
Eg(W_n^{(i)}+u) &\le \frac{2.517}{(1+x)^3} + \frac{(0.056)(0.0142)}{(1+x)^3} + \int_{x-1}^{x} g'(w)P(w < W_n^{(i)}+u < x)dw \\
&= \frac{2.518}{(1+x)^3} + \int_{x-1}^{x} g'(w)P(w-u < W_n^{(i)} < x-u)dw \\
&\le \frac{2.518}{(1+x)^3} + \int_{x-1}^{x} g'(w)[\frac{3.522}{(1+w-u)^3}(x-w) + 3.916\delta_{w-u}]dw.
\end{aligned} \tag{3.21}
$$

Note that for $x \ge 14$,

$$
(x-1)-u \ge (x-1)-(1+\frac{x}{4}) = \frac{3x}{4} - 2 \ge \frac{3x}{5}. \tag{3.22}
$$

By (3.5), proposition 3.6(1-2), (3.22), proposition 3.4(1), we have, for $x \ge 14$

$$
\begin{aligned}
&\int_{x-1}^{x} g'(w)[\frac{3.522}{(1+w-u)^3}(x-w) + 3.916\delta_{w-u}]dw \\
&\le \frac{3.522}{(1+(x-1)-u)^3}\int_{x-1}^{x}(x-w)g'(w)dw + 3.916\int_{x-1}^{x} g'(w)\delta_{w-u}dw \\
&\le \frac{3.522}{(1+\frac{3x}{5})^3}\int_{x-1}^{x}(x-w)dg(w) + 3.916\delta_{(x-1)-u}\int_{x-1}^{x} g'(w)dw \\
&\le \frac{3.522}{(1+\frac{3x}{5})^3}\Big[\int_{x-1}^{x} g(w)dw - g(x-1)\Big] + 3.916[g(x)-g(x-1)]\delta_{\frac{3x}{5}} \\
&\le \frac{3.522}{(1+\frac{3x}{5})^3}\int_{x-1}^{x} g(w)dw + 3.916g(x)\delta_{\frac{3x}{5}} \\
&\le \frac{3.522}{(1+\frac{3x}{5})^3}(xf_x(x)-(x-1)f_x(x-1)) + 3.916(1+x)\delta_{\frac{3x}{5}} \\
&\le \frac{3.522}{(1+\frac{3x}{5})^3}(xf_x(x)) + 3.916(1+x)\delta_{\frac{3x}{5}} \\
&\le \frac{3.522}{(1+\frac{3x}{5})^3} + 3.916(1+x)\delta_{\frac{3x}{5}}
\end{aligned} \tag{3.23}
$$

where we have used the fact that

$$
0 \le |xf_x(x)| < |x\min(\frac{\sqrt{2\pi}}{4}, \frac{1}{x})| \le 1
$$

in the last inequality.
By proposition 3.4(2) and the fact that for $x \geq 14$,

$$\frac{x}{4} \leq \frac{3x}{5}, \quad \frac{1+\frac{x}{4}}{1+x} \leq 0.3 \ \text{ and } \ \frac{1+\frac{x}{4}}{1+\frac{3x}{5}} \leq 0.48,$$

we have

$$\delta_{\frac{3x}{5}} \leq \frac{(1+\frac{x}{4})^2}{(1+\frac{3x}{5})^2}\delta_{\frac{x}{4}} \leq 0.2304\delta_{\frac{x}{4}}.$$

From this fact, (3.21) and (3.23), we have

$$\begin{aligned} Eg(W_n^{(i)}+u) &\leq \frac{2.518}{(1+x)^3} + \frac{3.522}{(1+\frac{3x}{5})^3} + 3.916(1+x)\delta_{\frac{3x}{5}} \\ &\leq \frac{0.458}{(1+\frac{x}{4})^3} + 0.903\delta_{\frac{x}{4}}(1+x). \end{aligned}$$

□

We note that proposition 3.7 and proposition 3.8 improve the following results from Neammanee [8].

Proposition 3.9. *Let* $i \in \{1, 2, \ldots, n\}$ *and* $W_n^{(i)} = W_n - X_i$. *Then for* $3 \leq a < b < \infty$ *and* $(1+a)^2\alpha_a + (1+a)\beta_a < \frac{1}{80}$, *we have*

$$P(a \leq W_n^{(i)} \leq b) \leq \frac{40.98}{(1+a)^3}(b-a) + 46.38\delta_a.$$

Proposition 3.10. *Let* $x \geq 14$. *If* $(1+x)^2\alpha_x + (1+x)\beta_x < \frac{1}{80}$, *then for* $|u| \leq 1 + \frac{x}{4}$, *we have*

$$Eg(W_n^{(i)}+u) \leq \frac{4.60}{(1+\frac{x}{4})^3} + 5.13\delta_{\frac{x}{4}}(1+x).$$

We are now ready to prove the main theorem.

3.2 Proof of the main theorem.

Proof We will show the proof for $x \geq 0$ as we can simply apply the result to $-W_n$ when $x < 0$.
By the fact that

$$\begin{aligned} &P(W_n \leq x, W_n = S_x) \leq P(S_x \leq x) \\ \text{and} \quad &P(S_x \leq x, W_n = S_x) \leq P(W_n \leq x), \end{aligned}$$

we have

$$\begin{aligned} P(W_n \leq x) - P(S_x \leq x) &= P(W_n \leq x, W_n = S_x) - P(S_x \leq x) \\ &\quad + P(W_n \leq x, W_n \neq S_x) \\ &\leq P(W_n \leq x, W_n \neq S_x) \\ &\leq P(W_n \neq S_x) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} P(W_n \leq x) - P(S_x \leq x) &= P(W_n \leq x) - P(S_x \leq x, W_n = S_x) \\ &\quad - P(S_x \leq x, W_n \neq S_x) \\ &\geq -P(S_x \leq x, W_n \neq S_x) \\ &\geq -P(W_n \neq S_x). \end{aligned} \tag{3.25}$$

Hence, by (3.24) and (3.25), we have

$$\begin{aligned} |P(W_n \leq x) - \Phi(x)| &- |P(S_x \leq x) - \Phi(x)| \\ &\leq |P(W_n \leq x) - P(S_x \leq x)| \\ &\leq P(W_n \neq S_x) \end{aligned}$$

which implies that

$$|P(W_n \leq x) - \Phi(x)| \leq P(W_n \neq S_x) + |P(S_x \leq x) - \Phi(x)|. \tag{3.26}$$

Note that $W_n = S_x$ if $\max\limits_{1\le i\le n} |X_i| < 1 + x$. Then

$$\begin{aligned} P(W_n \ne S_x) &\le P(\max_{1\le j\le n} |X_i| \ge 1 + x) \\ &\le \sum_{i=1}^{n} P(|X_i| \ge 1 + x) \\ &= \sum_{i=1}^{n} EI(|X_i| \ge 1 + x) \\ &\le \sum_{i=1}^{n} \frac{EX_i^2 I(|X_i| \ge 1 + x)}{(1 + x)^2} \\ &\le \frac{\alpha_x}{(1 + x)^2}. \end{aligned} \tag{3.27}$$

By Chebyshev's inequality, we can show that

$$\begin{aligned} |P(S_x \le x) - \Phi(x)| &= |1 - P(S_x > x) - \Phi(x)| \\ &\le P(S_x > x) + (1 - \Phi(x)) \\ &\le \frac{ES_x^4}{x^4} + (1 - \Phi(x)) \\ &\le \frac{ES_x^4}{x^4} + \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} \end{aligned} \tag{3.28}$$

where we have used (3.6) in the last inequality.

By (3.26), (3.27) and (3.28), we have

$$|P(W_n \le x) - \Phi(x)| \le \frac{\alpha_x}{(1 + x)^2} + |P(S_x \le x) - \Phi(x)| \tag{3.29}$$

$$\le \frac{\alpha_x}{(1 + x)^2} + \frac{ES_x^4}{x^4} + \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}. \tag{3.30}$$

If $0 \le x \le 3.99$, The results obtain immediately from theorem 1.3.

Suppose that $x > 3.99$.

Case1. $x \ge 14$.

Subcase 1.1 $(1 + x)^2\alpha_x + (1 + x)\beta_x \ge \frac{1}{80}$.

By Taylor's formula, we note that for $x \geq 14$,

$$\begin{aligned} e^{\frac{x^2}{2}} &= 1 + \frac{x^2}{2} + \frac{1}{2!}(\frac{x^2}{2})^2 + \frac{1}{3!}(\frac{x^2}{2})^3 + \frac{1}{4!}(\frac{x^2}{2})^4 + ... \\ &= 1 + \frac{x^2}{2} + \frac{14x^3}{2!4} + \frac{14^3x^3}{3!8} + \frac{14^5x^3}{4!16} + ... \\ &\geq 60x^3. \end{aligned}$$

From this fact, proposition 3.4(2), proposition 3.5(2), (3.30) and the fact

$$\delta_x \leq (\frac{1+\frac{x}{4}}{1+x})^2 \delta_{\frac{x}{4}} \leq (0.3)^2 \delta_{\frac{x}{4}},$$

$$\alpha_x = \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1+x) \leq \sum_{i=1}^{n} EX_i^2 = 1,$$

$$\text{and} \quad \frac{1+x}{x} = 1 + \frac{1}{x} \leq 1.072 \quad \text{for} \quad x \geq 14,$$

we have

$$\begin{aligned} &|P(W_n \leq x) - \Phi(x)| \\ &\leq \frac{\alpha_x}{(1+x)^2} + \frac{ES_x^4}{x^4} + \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} \\ &\leq \frac{\alpha_x}{(1+x)^2} + \frac{ES_x^4}{x^4} + \frac{1}{60\sqrt{2\pi}x^4} \\ &\leq \frac{\alpha_x}{(1+x)^2} + \frac{(1+x)\beta_x}{x^4} + \frac{\alpha_x\beta_x}{x^4(1+x)} + \frac{1}{x^4}(\frac{\alpha_x}{1+x})^2 + \frac{1}{x^4}(\frac{\alpha_x}{1+x})^4 + \frac{1.0067}{x^4} \\ &\leq \frac{\alpha_x}{(1+x)^2} + \frac{1+x}{x} \cdot \frac{\beta_x}{x^3} + \frac{1}{x^2} \cdot \frac{\beta_x}{x^2(1+x)} + \frac{\alpha_x}{x^4} \cdot \frac{\alpha_x}{(1+x)^2} \\ &\quad + \frac{\alpha_x^3}{x^4(1+x)^2} \cdot \frac{\alpha_x}{(1+x)^2} + \frac{1.0067}{x^4} \\ &\leq \frac{\alpha_x}{(1+x)^2} + (1.072)(1.072)^3 \frac{\beta_x}{(1+x)^3} + \frac{(1.072)^2}{14^2} \cdot \frac{\beta_x}{(1+x)^3} \\ &\quad + \frac{1}{14^4} \cdot \frac{\alpha_x}{(1+x)^2} + \frac{1}{14^4(1+14)^2} \cdot \frac{\alpha_x}{(1+x)^2} + \frac{(1.0067)(1.072)^4}{(1+x)^4} \\ &\leq 1.327(\frac{\alpha_x}{(1+x)^2}) + \frac{\beta_x}{(1+x)^3}) + \frac{1.33}{(1+x)^4} \\ &\leq 1.327\ \delta_x + \frac{1.33(80)\{(1+x)^2\alpha_x + (1+x)\beta_x\}}{(1+x)^4} \\ &= 107.73\delta_x \\ &\leq 9.7\delta_{\frac{x}{4}}. \end{aligned} \tag{3.31}$$

Subcase 1.2 $(1+x)^2\alpha_x + (1+x)\beta_x < \dfrac{1}{80}$.

Let $K_{i,\frac{x}{4}}(t) = EY_{i,\frac{x}{4}}\{I(0 < t \le Y_{i,\frac{x}{4}}) - I(Y_{i,\frac{x}{4}} \le t < 0)\}$. From Chen and Shao [3], pp.250-251, we set

$$F_n(x) - \Phi(x) = R_1 + R_2 + R_3 + R_4 \tag{3.32}$$

where

$R_1 = \sum_{i=1}^{n} E\{I(|X_i| < 1 + \frac{x}{4}) \int_{|t| \le 1+\frac{x}{4}} (f_x'(W_n^{(i)} + X_i) - f_x'(W_n^{(i)} + t))K_{i,\frac{x}{4}}(t)dt\}$,

$R_2 = \sum_{i=1}^{n} E\{I(|X_i| \ge 1 + \frac{x}{4}) \int_{|t| \le 1+\frac{x}{4}} (f_x'(W_n^{(i)} + X_i) - f_x'(W_n^{(i)} + t))K_{i,\frac{x}{4}}(t)dt\}$,

$R_3 = \alpha_{\frac{x}{4}} Ef_x'(W_n)$,

$R_4 = -\sum_{i=1}^{n} E\{X_i I(|X_i| \ge 1 + \frac{x}{4})(f_x(W_n) - f_x(W_n^{(i)}))\}$.

By (3.4) and the fact that

$$\int_{|t| \le 1+\frac{x}{4}} K_{i,\frac{x}{4}}(t)dt \le \int_{-\infty}^{\infty} K_{i,\frac{x}{4}}(t)dt \le EY_{i,\frac{x}{4}}^2 \le EX_i^2 \le \sum_{i=1}^{n} EX_i^2 = 1, \tag{3.33}$$

we have

$$\begin{aligned} |R_2| &\le \sum_{i=1}^{n} E\{I(|X_i| \ge 1 + \frac{x}{4}) \int_{|t| \le 1+\frac{x}{4}} K_{i,\frac{x}{4}}(t)dt\} \\ &\le \sum_{i=1}^{n} E\{I(|X_i| \ge 1 + \frac{x}{4})\} \\ &\le \sum_{i=1}^{n} \frac{EX_i^2 I(|X_i| \ge 1 + \frac{x}{4})}{(1 + \frac{x}{4})^2} \\ &= \frac{\alpha_{\frac{x}{4}}}{(1 + \frac{x}{4})^2}. \end{aligned} \tag{3.34}$$

By the fact that

$$E|f_x'(W_n)| \le \frac{15}{(1+x)^2} \quad \text{for} \quad x \ge 2$$

(see proposition 2.3 in Neammanee [8], pp.1960), we have

$$\begin{aligned}|R_3| &\le \alpha_{\frac{x}{4}} E|f_x'(W_n)| \\ &\le \frac{15\alpha_{\frac{x}{4}}}{(1+x)^2} \\ &\le \frac{1.35\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} \qquad (3.35)\end{aligned}$$

where we use the fact that $\dfrac{1+\frac{x}{4}}{1+x} \le 0.3$ for $x \ge 14$ in the last inequality.
By (3.5), we have

$$\begin{aligned}|R_4| &\le \sum_{i=1}^{n} E\{|X_i| f_x(W_n) I(|X_i| \ge 1+\frac{x}{4})\} \\ &\le \sum_{i=1}^{n} \frac{E|X_i| I(|X_i| \ge 1+\frac{x}{4})}{x} \\ &\le \sum_{i=1}^{n} \frac{EX_i^2 I(|X_i| \ge 1+\frac{x}{4})}{(1+\frac{x}{4})x} \\ &\le \frac{0.322\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} \qquad (3.36)\end{aligned}$$

where we use the fact that $\dfrac{1+\frac{x}{4}}{x} = \dfrac{1}{x} + \dfrac{1}{4} \le 0.322$ for $x \ge 14$ in the last inequality.
By applying (3.34)-(3.36), we get

$$\begin{aligned}|R_2+R_3+R_4| &\le \frac{\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} + \frac{1.35\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} + \frac{0.322\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} \\ &\le \frac{2.672\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2}. \qquad (3.37)\end{aligned}$$

Note that $|R_1| \le R_{11} + R_{12}$ where

$$R_{11} = \sum_{i=1}^{n} |E\{I(|X_i| < 1+\frac{x}{4}) \int_{|t|\le 1+\frac{x}{4}} K_{i,\frac{x}{4}}(t) \int_t^{X_i} Eg(W_n^{(i)}+u)dudt\}|,$$

$$R_{12} = \sum_{i=1}^{n} E\{I(|X_i| < 1 + \frac{x}{4})$$

$$\int_{|t|\le 1+\frac{x}{4}} P(x - \max(t, X_i) \le W_n^{(i)} \le x - \min(t, X_i)|X_i) K_{i,\frac{x}{4}}(t)dt\}$$

(see Chen and Shao [3], pp.251), and

$$\int_{|t|\le 1+\frac{x}{4}} |t| K_{i,\frac{x}{4}}(t)dt \le \int_{-\infty}^{\infty} |t| K_{i,\frac{x}{4}}(t)dt \le \frac{1}{2} E|Y|^3_{i,\frac{x}{4}} \le \frac{1}{2} E|X_i|^3 \tag{3.38}$$

and

$$E|X_i| E|X_i|^2 \le E^{\frac{1}{3}}|X_i|^3 E^{\frac{2}{3}}|X_i|^3 = E|X_i|^3. \tag{3.39}$$

By (3.33) and (3.38)-(3.39), we can show that

$$\begin{aligned}
&\sum_{i=1}^{n} |E\{I(|X_i| < 1 + \frac{x}{4}) \int_{|t|\le 1+\frac{x}{4}} (|X_i| + |t|) K_{i,\frac{x}{4}}(t)dt\}| \\
&\quad = \sum_{i=1}^{n} \Big| E\Big\{ I(|X_i| < 1 + \frac{x}{4}) \Big[|X_i| \int_{|t|\le 1+\frac{x}{4}} K_{i,\frac{x}{4}}(t)dt + \int_{|t|\le 1+\frac{x}{4}} |t| K_{i,\frac{x}{4}}(t)dt \Big] \Big\} \Big| \\
&\quad \le \sum_{i=1}^{n} E\{(|X_i| E|X_i|^2 + \frac{1}{2} E|X_i|^3) I(|X_i| < 1 + \frac{x}{4})\} \\
&\quad \le 2 \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1 + \frac{x}{4}) \\
&\quad \le 2 \sum_{i=1}^{n} E|Y_{i,\frac{x}{4}}|^3.
\end{aligned} \tag{3.40}$$

By proposition 3.8(1) and (3.40), we have

$$\begin{aligned}
R_{11} &\le \Big[\frac{0.458}{(1 + \frac{x}{4})^3} + 0.903(1 + x)\delta_{\frac{x}{4}} \Big] \\
&\quad \sum_{i=1}^{n} |E\Big\{ I(|X_i| < 1 + \frac{x}{4}) \int_{|t|\le 1+\frac{x}{4}} (|X_i| + |t|) K_{i,\frac{x}{4}}(t)dt \Big\}| \\
&\le 2 \Big[\frac{0.458}{(1 + \frac{x}{4})^3} + 0.903(1 + x)\delta_{\frac{x}{4}} \Big] \sum_{i=1}^{n} E|Y_{i,\frac{x}{4}}|^3
\end{aligned}$$

$$\leq \frac{0.916\beta_{\frac{x}{4}}}{(1+\frac{x}{4})^3} + 1.806(1+x)\delta_{\frac{x}{4}}\beta_{\frac{x}{4}}$$

$$\leq \frac{0.916\beta_{\frac{x}{4}}}{(1+\frac{x}{4})^3} + 0.023\delta_{\frac{x}{4}} \tag{3.41}$$

where we have used the fact that

$$\beta_{\frac{x}{4}} = \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1+\frac{x}{4}) \leq \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1+x) = \beta_x \leq \frac{1}{80(1+x)}$$

in the last inequality.

Note that for $x \geq 14$, $|t| < 1+\frac{x}{4}$ and $|X_i| \leq 1+\frac{x}{4}$

$$x - \max(t, X_i) \;\geq\; x - (1+\frac{x}{4}) \;=\; \frac{3x}{4} - 1 \;\geq\; \frac{2x}{3} \geq 9.3. \tag{3.42}$$

From this fact, proposition 3.7(3), (3.33), (3.40), and proposition 3.4(1), we have

$$\begin{aligned}
|R_{12}| &\leq \sum_{i=1}^{n} E\Big\{ I(|X_i| \leq 1+\frac{x}{4}) \\
&\quad \int_{|t|\leq 1+\frac{x}{4}} \Big(\frac{3.522}{(1+\frac{2x}{3})^3}(\max(t,X_i) - \min(t,X_i)) + 3.916\delta_{\frac{2x}{3}}\Big) K_{i,\frac{x}{4}}(t)dt\Big\} \\
&\leq \sum_{i=1}^{n} E\Big\{ I(|X_i| \leq 1+\frac{x}{4}) \int_{|t|\leq 1+\frac{x}{4}} \Big(\frac{3.522}{(1+\frac{2x}{3})^3}(|t| + |X_i|) + 3.916\delta_{\frac{2x}{3}}\Big) K_{i,\frac{x}{4}}(t)dt\Big\} \\
&\leq \frac{3.522}{(1+\frac{2x}{3})^3} \sum_{i=1}^{n} \Big| E\Big(I(|X_i| \leq 1+\frac{x}{4}) \int_{|t|\leq 1+\frac{x}{4}} (|t| + |X_i|) K_{i,\frac{x}{4}}(t)dt\Big)\Big| \\
&\quad + 3.916\delta_{\frac{2x}{3}} \sum_{i=1}^{n} E\Big|\Big(I(|X_i| \leq 1+\frac{x}{4}) \int_{|t|\leq 1+\frac{x}{4}} K_{i,\frac{x}{4}}(t)dt\Big)\Big| \\
&\leq \frac{2(3.522)}{(1+\frac{2x}{3})^3} \sum_{i=1}^{n} |Y_{i,\frac{x}{4}}|^3 + 3.916\delta_{\frac{2x}{3}} \sum_{i=1}^{n} E(EX_i^2 I(|X_i| \leq 1+\frac{x}{4})) \\
&\leq \frac{7.044}{(1+\frac{2x}{3})^3} \sum_{i=1}^{n} |Y_{i,\frac{x}{4}}|^3 + 3.916\delta_{\frac{2x}{3}} \sum_{i=1}^{n} EX_i^2 \\
&\leq \frac{(7.044)(0.436)^3}{(1+\frac{x}{4})^3}\beta_{\frac{x}{4}} + (3.916)(0.190)\delta_{\frac{x}{4}}
\end{aligned}$$

$$\leq \frac{0.584}{(1+\frac{x}{4})^3}\beta_{\frac{x}{4}} + 0.325\delta_{\frac{x}{4}} \tag{3.43}$$

where we have used proposition 3.4(2) and the fact that for $x \geq 14$

$$\frac{1+\frac{x}{4}}{1+\frac{2x}{3}} \leq 0.436 \text{ and } \delta_{\frac{2x}{3}} \leq \frac{(1+\frac{x}{4})^2}{(1+\frac{2x}{3})^2}\delta_{\frac{x}{4}} \leq 0.190\delta_{\frac{x}{4}}$$

in the last inequality.

Hence, by (3.32), (3.37), (3.41), and (3.43)

$$\begin{aligned}
|F_n(x) - \Phi(x)| &= |R_1 + R_2 + R_3 + R_4| \\
&\leq |R_1| + |R_2 + R_3 + R_4| \\
&\leq |R_{11}| + |R_{12}| + |R_2 + R_3 + R_4| \\
&\leq \frac{0.916\beta_{\frac{x}{4}}}{(1+\frac{x}{4})^3} + 0.023\delta_{\frac{x}{4}} + \frac{0.584}{(1+\frac{x}{4})^3}\beta_{\frac{x}{4}} + 0.325\delta_{\frac{x}{4}} + \frac{2.672\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} \\
&\leq 2.672[\frac{\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} + \frac{\beta_{\frac{x}{4}}}{(1+\frac{x}{4})^3}] + 0.348\delta_{\frac{x}{4}} \\
&\leq 3.02\delta_{\frac{x}{4}}. \qquad (3.44)
\end{aligned}$$

By (3.31) and (3.44), we have the result in case $x \geq 14$.

Case2. $7.98 < x < 14$.

We use the same argument as of case 1. by using proposition 3.8(2) and proposition 3.7(2) to bound (3.41) and (3.43), respectively.

Case3. $3.99 < x \leq 7.98$.

We use the same argument as of case 1. by using proposition 3.8(3) and proposition 3.7(1) to bound (3.41) and (3.43), respectively, and replacing inequality (3.42) by the following inequality

$$x - \max(t, X_i) \geq x - (1 + \frac{x}{4}) = \frac{3x}{4} - 1 = 2.$$

□

CHAPTER IV
MAIN RESULT VIA SIGANOV-PADITZ THEOREMS

In this chapter, we use the same notations as in chapter III.

In 1986, Siganov [11] investigated the constant in uniform version of Berry-Esseen theorem. The result is as follows.

Theorem 4.1. *(Siganov,1986) Let $X_1, X_2, ..., X_n$ be independent random variables such that $EX_i = 0$ and $E|X_i|^3 < \infty$ for $i = 1, 2, .., n$. Assume that $\sum_{i=1}^{n} EX_i^2 = 1$. Then*

$$\sup_{x\in\mathbb{R}} |P(W_n \le x) - \Phi(x)| \le 0.7915 \sum_{i=1}^{n} E|X_i|^3.$$

In 1989, Paditz [10] gave their constant in non-uniform version. Here is his result.

Theorem 4.2. *(Paditz,1989) Under the assumption of Theorem 4.1, we have for $x \in \mathbb{R}$,*

$$|P(W_n \le x) - \Phi(x)| \le \frac{31.935}{1+|x|^3} \sum_{i=1}^{n} E|X_i|^3.$$

The purpose of this chapter is to improve the constant in Theorem 3.2 by using Siganov and Paditz theorems (i.e., Theorem 4.1 and Theorem 4.2). Here are our main results.

Theorem 4.3. *(Main theorem)*

$$|F_n(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|x|)}{(1+|x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|x|)}{(1+|x|)^3} \right\}$$

where

$$C = \begin{cases} 49.89 & \text{if } 0 \le |x| < 1.3, \\ 59.45 & \text{if } 1.3 \le |x| < 2, \\ 73.52 & \text{if } 2 \le |x| < 3, \\ 76.17 & \text{if } 3 \le |x| < 7.98, \\ 45.80 & \text{if } 7.98 \le |x| < 14, \\ 39.39 & \text{if } |x| \ge 14. \end{cases}$$

Corollary 4.4.

$$|F_n(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^3} \right\}$$

where

$$C = \begin{cases} 9.54 & \text{if } 0 \le |x| < 1.3, \\ 19.74 & \text{if } 1.3 \le |x| < 2, \\ 18.38 & \text{if } 2 \le |x| < 3, \\ 14.63 & \text{if } 3 \le |x| < 7.98, \\ 5.13 & \text{if } 7.98 \le |x| < 14, \\ 3.55 & \text{if } |x| \ge 14. \end{cases}$$

We observe that the result in Corollary 4.4. yields a better bound than that in Theorem 3.3.

Corollary 4.5.

$$|F_n(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |x|)}{1 + |x|^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{1 + |x|^3} \right\}$$

where

$$C = \begin{cases} 13.11 & \text{if } \ 0 \leq |x| < 1.3, \\ 28.54 & \text{if } \ 1.3 \leq |x| < 2, \\ 46.32 & \text{if } \ 2 \leq |x| < 3, \\ 61.40 & \text{if } \ 3 \leq |x| < 7.98, \\ 40.12 & \text{if } \ 7.98 \leq |x| < 14, \\ 39.39 & \text{if } \ |x| \geq 14. \end{cases}$$

We divide this chapter into two parts, auxiliary results and the main theorem.

4.1 Auxiliary Results

Proposition 4.6. *For each* $n \in \mathbb{N}$, *we have*

(1) $\displaystyle\sum_{i=1}^{n} E|Y_{i,x} - EY_{i,x}|^3 \leq \beta_x + \frac{7\alpha_x}{1+x}$,

(2) $1 - 2\alpha_x \leq VarS_x \leq 1$, *and*

(3) if $\alpha_x \leq 0.11$, *then* $0 < \dfrac{1}{\sqrt{VarS_x}} \leq 1 + 1.452\alpha_x$.

Proof. 1. Since $EX_i = 0$,

$$\begin{aligned} |EX_iI(|X_i| < 1+x)| &= |EX_i - EX_iI(|X_i| \geq 1+x)| \\ &= |EX_iI(|X_i| \geq 1+x)|. \end{aligned} \tag{4.1}$$

From this fact and the fact that

$$E|X_i|^2 \leq \sum_{i=1}^{n} EX_i^2 = 1, \tag{4.2}$$

$$E^2X_i \leq EX_i^2, \tag{4.3}$$

we have

$$\begin{aligned} &\sum_{i=1}^{n} E|Y_{i,x} - EY_{i,x}|^3 \\ &\quad = \sum_{i=1}^{n} E|X_iI(|X_i| < 1+x) - EX_iI(|X_i| < 1+x)|^3 \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n}[E|X_i|^3 I(|X_i|<1+x)+3EX_i^2 I(|X_i|<1+x)|EX_iI(|X_i|<1+x)| \\
&\quad + 3E|X_iI(|X_i|<1+x)|E^2X_iI(|X_i|<1+x)| + |EX_iI(|X_i|<1+x)|^3] \\
&\le \sum_{i=1}^{n}E|X_i|^3 I(|X_i|<1+x)+3\sum_{i=1}^{n}|EX_iI(|X_i|<1+x)| \\
&\quad + 3\sum_{i=1}^{n}E|X_i||EX_iI(|X_i|<1+x)||EX_iI(|X_i|<1+x)| \\
&\quad + \sum_{i=1}^{n}E|X_i|^2 I(|X_i|<1+x)|EX_iI(|X_i|<1+x)| \\
&\le \beta_x + 3\sum_{i=1}^{n}|EX_iI(|X_i|\ge 1+x)| + 3\sum_{i=1}^{n}E|X_i|^2|EX_iI(|X_i|\ge 1+x)| \\
&\quad + \sum_{i=1}^{n}|EX_iI(|X_i|\ge 1+x)| \\
&\le \beta_x + 3\sum_{i=1}^{n}E|X_i|I(|X_i|\ge 1+x) + 3\sum_{i=1}^{n}E|X_i|I(|X_i|\ge 1+x) \\
&\quad + \sum_{i=1}^{n}E|X_i|I(|X_i|\ge 1+x) \\
&= \beta_x + 7\sum_{i=1}^{n}E|X_i|I(|X_i|\ge 1+x) \\
&\le \beta_x + 7\sum_{i=1}^{n}\frac{E|X_i|^2 I(|X_i|\ge 1+x)}{(1+x)} \\
&= \beta_x + \frac{7\alpha_x}{(1+x)}.
\end{aligned}
$$

2. By (4.1), we note that

$$
\begin{aligned}
VarS_x &= \sum_{i=1}^{n}VarY_{i,x} \\
&= \sum_{i=1}^{n}(EY_{i,x}^2 - E^2Y_{i,x}) \\
&= \sum_{i=1}^{n}EX_i^2 I(|X_i|<1+x) - \sum_{i=1}^{n}E^2X_iI(|X_i|<1+x)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1+x) - \sum_{i=1}^{n} E^2 X_i I(|X_i| \geq 1+x) \\
&= 1 - \alpha_x - \sum_{i=1}^{n} E^2 X_i I(|X_i| \geq 1+x). \qquad (4.4)
\end{aligned}
$$

From this fact and the fact that $\alpha_x \geq 0$, we have $VarS_x \leq 1$.

By (4.3) and (4.4), we have

$$
\begin{aligned}
VarS_x &= 1 - \alpha_x - \sum_{i=1}^{n} E^2 X_i I(|X_i| \geq 1+x) \\
&\geq 1 - \alpha_x - \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1+x) \\
&\geq 1 - 2\alpha_x.
\end{aligned}
$$

From this fact and (4.4), we have $1 - 2\alpha_x \leq VarS_x \leq 1$.

3. For $0 < t \leq 0.11$, by using Taylor's formula, we have

$$
\begin{aligned}
\frac{1}{\sqrt{1-2t}} &= 1 + \frac{t}{(1-2c)^{\frac{3}{2}}} \quad \text{for some } \; c \in (0, 0.11] \\
&\leq 1 + \frac{t}{(1-2(0.11))^{\frac{3}{2}}} \\
&\leq 1 + 1.452t.
\end{aligned}
$$

From this fact and 2., we have

$$
0 < \frac{1}{\sqrt{VarS_x}} \leq \frac{1}{\sqrt{1-2\alpha_x}} \leq 1 + 1.452\alpha_x
$$

for $\alpha_x \leq 0.11$. □

Proposition 4.7.

For each $x > 0$, let $\bar{Y}_{i,x} = \dfrac{Y_{i,x} - EY_{i,x}}{\sqrt{VarS_x}}$ and $\bar{S}_x = \sum_{i=1}^{n} \bar{Y}_{i,x}$.

1. If $\alpha_x \leq 0.099$ and $1.3 \leq x \leq 2$, then

$$
|P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| = \frac{54.513\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3}.
$$

2. If $(1+x)^2\alpha_x < \frac{1}{5}$, *then*

$$|P(\bar{S}_x \leq \frac{x-ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x-ES_x}{\sqrt{VarS_x}})| = \frac{C_1\alpha_x}{(1+x)^2} + \frac{C_2\beta_x}{(1+x)^3}$$

where $C_1 = 57.186 \quad C_2 = 73.515 \quad$ *for* $\quad 2 \leq x < 3,$

$C_1 = 33.318 \quad C_2 = 76.17 \quad$ *for* $\quad 3 \leq x < 7.98,$

$C_1 = 3.976 \quad C_2 = 45.8 \quad$ *for* $\quad 7.98 \leq x < 14,$ *and*

$C_1 = 1.226 \quad C_2 = 39.382 \quad$ *for* $\quad x \geq 14.$

Proof.

1. By proposition 3.4(1) and proposition 4.6(2), we have

$$|ES_x| \leq \frac{\alpha_x}{1+x} \leq \frac{0.099}{1+1.3} = 0.043 \tag{4.5}$$

$$\text{and} \quad 1 \geq VarS_x \geq 1 - 2(\alpha_x) \geq 1 - 2(0.099) = 0.802 \tag{4.6}$$

which imply

$$0 \leq \frac{x-ES_x}{\sqrt{VarS_x}} \leq \frac{2+0.043}{\sqrt{0.802}} = 2.2813. \tag{4.7}$$

By proposition 4.6(1) and 4.6,

$$\begin{aligned}
\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3 &= \sum_{i=1}^{n} E|\frac{Y_{i,x}-EY_{i,x}}{\sqrt{VarS_x}}|^3 \\
&= \frac{1}{(VarS_x)^{\frac{3}{2}}} \sum_{i=1}^{n} E|Y_{i,x}-EY_{i,x}|^3 \\
&\leq \frac{1}{(VarS_x)^{\frac{3}{2}}}(\beta_x + \frac{7\alpha_x}{1+x}) \\
&\leq \frac{1}{(0.802)^{\frac{3}{2}}}(\beta_x + \frac{7\alpha_x}{(1+1.3)}) \\
&= 1.3923\beta_x + 4.2375\alpha_x.
\end{aligned} \tag{4.8}$$

Note that $\bar{S}_x = \sum_{i=1}^{n} \bar{Y}_{i,x}$ is the sum of independent random variables whose

$$E\bar{Y}_{i,x} = E(\frac{Y_{i,x}-EY_{i,x}}{\sqrt{VarS_x}}) = \frac{EY_{i,x}}{\sqrt{VarS_x}} - \frac{EY_{i,x}}{\sqrt{VarS_x}} = 0 \text{ and}$$

$$Var\bar{S}_x = Var(\sum_{i=1}^{n}\bar{Y}_{i,x}) = Var(\sum_{i=1}^{n}\frac{Y_{i,x}-EY_{i,x}}{\sqrt{VarS_x}}) = \frac{1}{VarS_x}Var(\sum_{i=1}^{n}Y_{i,x}) = 1.$$

By (4.8) and Theorem 4.1(Siganov),

$$\begin{aligned}|P(\bar{S}_x \le z) - \Phi(z)| &\le 0.7915\sum_{i=1}^{n}E|\bar{Y}_{i,x}|^3\\ &\le 0.7915(1.3923\beta_x + 4.2375\alpha_x)\\ &\le 1.102\beta_x + 3.354\alpha_x\end{aligned}$$

for all $z \in \mathbb{R}$. From this fact, (4.5)- (4.7), we have

$$\begin{aligned}&|P(\bar{S}_x \le \frac{x-ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x-ES_x}{\sqrt{VarS_x}})|\\ &= \frac{(1+(\frac{x-ES_x}{\sqrt{VarS_x}}))^3(1.102\beta_x + 3.354\alpha_x)}{(1+(\frac{x-ES_x}{\sqrt{VarS_x}}))^3}\\ &\le \frac{(1+2.2813)^3(1.102\beta_x + 3.354\alpha_x)}{(1+(\frac{x-ES_x}{\sqrt{VarS_x}}))^3}\\ &\le \frac{(3.2813)^3(1.102\beta_x + 3.354\alpha_x)}{(1+(x-0.043))^3}\\ &\le \frac{38.933\beta_x + 118.495\alpha_x}{(0.957+x)^3}\\ &\le \frac{38.933(1.019)^3\beta_x}{(1+x)^3} + \frac{118.495(1.019)^3\alpha_x}{(1+x)^3}\\ &\le \frac{41.195\beta_x}{(1+x)^3} + \frac{125.379\alpha_x}{(1+x)^3}\\ &\le \frac{41.195\beta_x}{(1+x)^3} + \frac{54.513\alpha_x}{(1+x)^2}\end{aligned}$$

where we use the fact that

$$\frac{1+x}{0.957+x} \le 1.019 \text{ for all } 1.3 < x < 2$$

in the forth inequality.

2. **case** $2 \le x < 3$**.**

We can proof the result of this case by using the same argument of 1. and the fact that

$$0 \leq \alpha_x \leq \frac{1}{5(1+x)^2} \leq \frac{1}{5(1+2)^2} \leq 0.023,$$

$$|ES_x| \leq \frac{\alpha_x}{1+x} \leq \frac{0.023}{1+2} \leq 0.008,$$

$$1 \geq VarS_x \geq 1 - 2\alpha_x \geq 1 - 2(0.023) = 0.954,$$

$$\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3 \ \leq \ 1.073\beta_x + 2.504\alpha_x,$$

$$\text{and} \quad \frac{1+x}{0.992+x} \leq 1.003 \text{ for all } 2 \leq x < 3.$$

case $3 \leq x < 7.98$.

We note that

$$0 \leq \alpha_x \leq \frac{1}{5(1+x)^2} \leq \frac{1}{5(1+3)^2} \leq 0.0125.$$

$$|ES_x| \leq \frac{\alpha_x}{1+x} \leq \frac{0.0125}{(1+3)} \leq 0.00313,$$

$$1 \geq VarS_x \geq 1 - 2\alpha_x \geq 1 - 2(0.0125) \geq 0.975,$$

$$\text{and} \quad \sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3 \leq \ 1.039\beta_x + 1.819\alpha_x,$$

From these facts , we have

$$\frac{x - ES_x}{\sqrt{VarS_x}} \geq 3 - 0.00313 = 2.997. \tag{4.9}$$

To bound $|P(\bar{S}_x \leq \frac{x-ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x-ES_x}{\sqrt{VarS_x}})|$ in 1., we use Theorem 4.1(Siganov).

But in this case, we will use Theorem 4.2(Paditz).

Since $f(x) = 2.29(1+x^3) - (1+x)^3$ is increasing on $[2.997, \infty)$ and $f(2.997) \geq 0$,

$$f(x) = 2.29(1+x^3) - (1+x)^3 \geq 0 \quad \text{for} \quad x \geq 2.997.$$

From this fact and (4.9), we have

$$\frac{1}{1 + (\frac{x-ES_x}{\sqrt{VarS_x}})^3} \leq \frac{2.29}{(1 + \frac{x-ES_x}{\sqrt{VarS_x}})^3}. \tag{4.10}$$

By Theorem 4.2(Paditz) and (4.10), we have

$$
\begin{aligned}
|P(\bar{S}_x \leq \frac{x-ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x-ES_x}{\sqrt{VarS_x}})| \\
&\leq \frac{(31.935)\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3}{1+(\frac{x-ES_x}{\sqrt{VarS_x}})^3} \\
&\leq \frac{(31.935)(2.29)\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3}{(1+(\frac{x-ES_x}{\sqrt{VarS_x}}))^3} \\
&\leq \frac{73.131(1.039\beta_x+1.818\alpha_x)}{(1+(x-0.00313))^3} \\
&\leq \frac{73.131(1.039\beta_x+1.818\alpha_x)}{(0.99687+x)^3} \\
&\leq \frac{(1.0008)^3(75.983\beta_x+132.952\alpha_x)}{(1+x)^3} \\
&\leq \frac{76.17\beta_x}{(1+x)^3}+\frac{133.27\alpha_x}{(1+x)^3} \\
&\leq \frac{76.17\beta_x}{(1+x)^3}+\frac{33.318\alpha_x}{(1+x)^2}
\end{aligned}
$$

where we use the fact that

$$\frac{1+x}{0.99687+x} \leq 1.0008 \text{ for all } 3 \leq x < 7.98$$

in the fifth inequality.

case $7.98 \leq x < 14$.

We can proof the result of this case by using the same argument of case $3 \leq x < 7.98$, and the fact that

$$0 \leq \alpha_x \leq \frac{1}{5(1+x)^2} \leq \frac{1}{5(1+7.98)^2} \leq 0.0025,$$

$$|ES_x| \leq \frac{\alpha_x}{1+x} \leq \frac{0.0025}{(1+7.98)} \leq 0.000279$$

$$1 \geq VarS_x \geq 1-2\alpha_x \geq 1-2(0.0025) = 0.995,$$

$$\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3 \leq \ 1.0076\beta_x + 0.7854\alpha_x,$$

$$\frac{x - ES_x}{\sqrt{VarS_x}} \geq 7.98 - 0.000279 \geq 7.9797,$$

$$\frac{1}{1+x^3} \leq \frac{1.423}{(1+x)^3} \text{ on } [7.9797, \infty),$$

$$\text{and} \quad \frac{1+x}{0.999+x} = 1.0001.$$

case $x \geq 14$.

We can proof the result of this case by using the same argument of case $3 \leq x < 7.98$, and the fact that

$$0 \leq \alpha_x \leq \frac{1}{5(1+x)^2} \leq \frac{1}{5(1+14)^2} \leq 0.00089,$$

$$|ES_x| \leq \frac{\alpha_x}{1+x} \leq \frac{0.00089}{(1+14)} \leq 0.00006,$$

$$1 \geq VarS_x \geq 1 - 2\alpha_x \geq 1 - 2(0.00089) = 0.9983$$

$$\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3 \leq \ 1.0026\beta_x + 0.4679\alpha_x,$$

$$\frac{x - ES_x}{\sqrt{VarS_x}} \geq 14 - 0.00006 \approx 14,$$

$$\frac{1}{1+x^3} \leq \frac{1.23}{(1+x)^3} \text{ on } [14, \infty),$$

$$\text{and} \quad \frac{1+x}{0.9999+x} \approx 1 \text{ for } x \geq 14.$$

□

We are now ready to prove the main theorem.

4.2 Proof of the main theorem

It suffices to consider only $x \geq 0$ as we can simply apply the results to $-W_n$ when $x < 0$.

Case 1 $0 \leq x < 1.3$.

Note that for $x \geq 0$,

$$\begin{aligned} &EX_i^2 I(|X_i| \geq 1) + E|X_i|^3 I(|X_i| < 1) \\ &\qquad \leq EX_i^2 I(|X_i| \geq 1+x) + EX_i^2 I(1 \leq |X_i| < 1+x) \\ &\qquad\quad + E|X_i|^3 I(|X_i| < 1+x) - E|X_i|^3 I(1 \leq |X_i| < 1+x) \\ &\qquad \leq EX_i^2 I(|X_i| \geq 1+x) + E|X_i|^3 I(|X_i| < 1+x) \end{aligned}$$

and for $0 \leq x \leq 1.3$,

$$(1+x)^3 \leq (1+1.3)^3 \leq 12.167.$$

From this fact and (1.3), we have

$$\begin{aligned} &|F_n(x) - \Phi(x)| \\ &\leq 4.1 \sum_{i=1}^n \Big\{ EX_i^2 I(|X_i| \geq 1) + E|X_i|^3 I(|X_i| < 1) \Big\} \\ &\leq 4.1 \sum_{i=1}^n \Big\{ EX_i^2 I(|X_i| \geq 1+x) + E|X_i|^3 I(|X_i| < 1+x) \Big\} \\ &\leq \frac{4.1(12.167)}{(1+x)^3} \sum_{i=1}^n \Big\{ EX_i^2 I(|X_i| \geq 1+x) + E|X_i|^3 I(|X_i| < 1+x) \Big\} \\ &\leq 49.89 \sum_{i=1}^n \Big\{ \frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^3} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3} \Big\} \\ &\leq 49.89 \sum_{i=1}^n \Big\{ \frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3} \Big\}. \qquad (4.11) \end{aligned}$$

Before proving another cases, we need the equation

$$|F_n(x) - \Phi(x)| \leq \frac{4.931\alpha_x}{(1+x)^2} + |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| \qquad (4.12)$$

for $\alpha_x \leq 0.11$ and $x \geq 1.3$.

Firstly, we will show that for $\alpha_x \leq 0.11$ and $x \geq 1.3$, we have

$$|P(S_x \leq x) - \Phi(x)| \leq \frac{3.319\alpha_x}{(1+x)^2} + |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})|.$$

By proposition 3.4(1) and proposition 4.6(2), we have

$$\frac{x - ES_x}{\sqrt{VarS_x}} \geq x - ES_x \geq x - \frac{\alpha_x}{(1+x)}$$

which implies

$$\min\{x, \frac{x - ES_x}{\sqrt{VarS_x}}\} \geq x - \frac{\alpha_x}{1+x}.$$

From this fact and the fact that

$$\Phi(b) - \Phi(a) \;=\; \frac{1}{\sqrt{2\pi}}\int_a^b e^{\frac{-t^2}{2}}dt \leq \frac{1}{\sqrt{2\pi}e^{\frac{a^2}{2}}}\int_a^b 1dt = \frac{(b-a)}{\sqrt{2\pi}e^{\frac{a^2}{2}}} \tag{4.13}$$

for $0 < a < b$, we have

$$\begin{aligned}
&|P(S_x \leq x) - \Phi(x)| \\
&= |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}}) + \Phi(\frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(x)| \\
&\leq |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| + |\Phi(\frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(x)| \\
&\leq |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| \qquad (4.14) \\
&\quad + \frac{1}{\sqrt{2\pi}e^{\frac{1}{2}[\min(x, \frac{x - ES_x}{\sqrt{VarS_x}})]^2}}|\frac{x}{\sqrt{VarS_x}} - x - \frac{ES_x}{\sqrt{VarS_x}}| \\
&\leq |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| + \frac{1}{\sqrt{2\pi}e^{\frac{1}{2}(x - \frac{\alpha_x}{1+x})^2}}|\frac{x}{\sqrt{VarS_x}} - x - \frac{ES_x}{\sqrt{VarS_x}}| \\
&\leq |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| + \frac{1}{\sqrt{2\pi}(0.89e^{\frac{x^2}{2}})}|\frac{x}{\sqrt{VarS_x}} - x - \frac{ES_x}{\sqrt{VarS_x}}|.
\end{aligned}$$

(4.15)

where we used the fact that

$$e^{\frac{1}{2}(x - \frac{\alpha_x}{1+x})^2} \geq e^{\frac{x^2}{2} - (\frac{x}{1+x})\alpha_x} \geq \frac{e^{\frac{x^2}{2}}}{e^{0.11}} \geq 0.89e^{\frac{x^2}{2}}$$

in the last inequality.

Since $f(x) = e^{\frac{x^2}{2}} - 0.933(1+x)$ is increasing and $f(1.3) \geq 0$,

$$e^{\frac{x^2}{2}} \geq 0.933(1+x) \quad \text{for} \;\; x \geq 1.3. \tag{4.16}$$

Using the same argument as in (4.16), we also can show that

$$e^{\frac{x^2}{2}} \geq 0.193(1+x)^3 \quad \text{for} \;\; x \geq 1.3.$$

From these two facts, proposition 3.4(1) and proposition 4.6(3), we have

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi}(0.89e^{\frac{x^2}{2}})}\left|\frac{x}{\sqrt{VarS_x}} - x - \frac{ES_x}{\sqrt{VarS_x}}\right| \\
&\qquad \leq \frac{1}{\sqrt{2\pi}(0.89e^{\frac{x^2}{2}})}\left|\frac{x}{\sqrt{VarS_x}} - x\right| + \frac{1}{\sqrt{2\pi}(0.89e^{\frac{x^2}{2}})}\left|\frac{ES_x}{\sqrt{VarS_x}}\right| \\
&\qquad \leq \frac{1.452\alpha_x x}{\sqrt{2\pi}(0.89)(0.193)(1+x)^3} + \frac{\alpha_x}{(1+x)}\frac{(1+1.452\alpha_x)}{\sqrt{2\pi}(0.89)(0.933)(1+x)} \\
&\qquad \leq \frac{3.373\alpha_x x}{(1+x)^3} + \frac{(1+1.452(0.11))\alpha_x}{\sqrt{2\pi}(0.89)(0.933)(1+x)^2} \\
&\qquad \leq \frac{3.373\alpha_x}{(1+x)^2} + \frac{0.558\alpha_x}{(1+x)^2} \\
&\qquad \leq \frac{3.931\alpha_x}{(1+x)^2}. \qquad (4.17)
\end{aligned}
$$

From this fact, (3.29) and (4.15), we have (4.12)

Case 2 $1.3 \leq x < 2$.

If $\frac{\alpha_x}{(1+x)^2} \geq 0.011$, then

$$
\begin{aligned}
&59.45\sum_{i=1}^{n}\left\{\frac{EX_i^2 I(|X_i| \geq 1+|x|)}{(1+|x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|x|)}{(1+|x|)^3}\right\} \\
&\geq 59.45(0.011) \\
&= 0.654.
\end{aligned}
$$

From this fact and the fact that

$$|F_n(x) - \Phi(x)| \leq 0.55$$

(see Chen and Shao [3], pp.246), we can assume $\frac{\alpha_x}{(1+x)^2} \leq 0.011$. Hence

$$\alpha_x \leq 0.011(1+2)^2 \leq 0.099.$$

From this fact, (4.12) and proposition 4.7(1), we have

$$
\begin{aligned}
|F_n(x) - \Phi(x)| &\leq \frac{4.931\alpha_x}{(1+x)^2} + |P(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| \\
&\leq \frac{4.931\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3} + \frac{54.513\alpha_x}{(1+x)^2} \\
&= \frac{59.444\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3} \\
&\leq 59.444\delta_x. \qquad (4.18)
\end{aligned}
$$

Case 3 $2 \le x \le 14$.

Subcase 3.1. $(1+x)^2\alpha_x \ge \dfrac{1}{5}$.

Using the same argument as in (4.16), we can show that

$$e^{\frac{x^2}{2}} \ge 0.92x^3 \ \text{ for } \ 2 \le x \le 14. \tag{4.19}$$

By using the same argument of subcase 1.1 in theorem 3.3 with (4.19) and the fact that

$$\frac{1+x}{x} = 1 + \frac{1}{x} \le 1.5, \tag{4.20}$$

we can show that

$$|F_n(x) - \Phi(x)| \le 37.408\delta_x. \tag{4.21}$$

Subcase 3.2. $(1+x)^2\alpha_x < \dfrac{1}{5}$.

Note that for $x \ge 2$, we have

$$0 \le \alpha_x \le \frac{1}{5(1+x)^2} \le \frac{1}{5(1+2)^2} \le 0.023 \le 0.11.$$

If $2 \le x \le 3$, then by proposition 4.7(2) and (4.12), we have

$$\begin{aligned} |F_n(x) - \Phi(x)| &\le \frac{4.931\alpha_x}{(1+x)^2} + |P(\bar{S}_x \le \frac{x - ES_x}{\sqrt{VarS_x}}) - \Phi(\frac{x - ES_x}{\sqrt{VarS_x}})| \\ &\le \frac{4.931\alpha_x}{(1+x)^2} + \frac{73.515\beta_x}{(1+x)^3} + \frac{57.186\alpha_x}{(1+x)^2} \\ &\le \frac{56.117\alpha_x}{(1+x)^2} + \frac{73.515\beta_x}{(1+x)^3} \\ &\le 73.515\delta_x. \end{aligned} \tag{4.22}$$

For $3 \le x < 7.98$ and $7.98 \le x < 14$ we can use the same argument of the case $2 \le x \le 3$.

Case4 $x > 14$.

Follows the argument of case 3 by using the inequality

$$e^{\frac{x^2}{2}} \ge 60x^3 \ \text{ and } \ \frac{1+x}{x} = 1 + \frac{1}{x} \le 1.071$$

instead of (4.19) and (4.20), respectively. □

Proof of corollary 4.4

If $0 \leq x < 1.3$.

We used the same argument of case 1 of theorem 4.3 and the fact that $(1+\frac{x}{4})^3 \leq 2.327$ to get $C = 9.54$.

Suppose that $x \geq 1.3$. By proposition 3.4(2)

$$\delta_x \leq (\frac{1+\frac{x}{4}}{1+x})^2 \delta_{\frac{x}{4}},$$

we use the same argument as in (4.16) to show that

$$\delta_x \leq \begin{cases} 0.332\delta_{\frac{x}{4}} & \text{if} \quad 1.3 \leq x < 2, \\ 0.250\delta_{\frac{x}{4}} & \text{if} \quad 2 \leq x < 3, \\ 0.192\delta_{\frac{x}{4}} & \text{if} \quad 3 \leq x < 7.98, \\ 0.112\delta_{\frac{x}{4}} & \text{if} \quad 7.98 \leq x < 14, \\ 0.090\delta_{\frac{x}{4}} & \text{if} \quad x \geq 14. \end{cases}$$

From this fact and theorem 4.3, we have

$$\begin{aligned} |F_n(x) - \Phi(x)| &\leq \begin{cases} 0.332(59.45)\delta_{\frac{x}{4}} & \text{if} \quad 1.3 \leq x < 2, \\ 0.250(73.52)\delta_{\frac{x}{4}} & \text{if} \quad 2 \leq x < 3, \\ 0.192(76.17)\delta_{\frac{x}{4}} & \text{if} \quad 3 \leq x < 7.98, \\ 0.112(45.80)\delta_{\frac{x}{4}} & \text{if} \quad 7.98 \leq x < 14, \\ 0.090(39.39)\delta_{\frac{x}{4}} & \text{if} \quad x \geq 14, \end{cases} \\ &= \begin{cases} 19.74\delta_{\frac{x}{4}} & \text{if} \quad 1.3 \leq x < 2, \\ 18.38\delta_{\frac{x}{4}} & \text{if} \quad 2 \leq x < 3, \\ 14.63\delta_{\frac{x}{4}} & \text{if} \quad 3 \leq x < 7.98, \\ 5.13\delta_{\frac{x}{4}} & \text{if} \quad 7.98 \leq x < 14, \\ 3.55\delta_{\frac{x}{4}} & \text{if} \quad x \geq 14. \end{cases} \end{aligned}$$

□

Proof of corollary 4.5

By the same reason as of Theorem 4.3, we can assume that $x \geq 0$.

If $0 \leq x < 1.3$.

We use the same argument of case 1 in Theorem 4.3 and the fact that $1 + x^2 \leq 1 + x^3 \leq 3.197$, to get $C = 13.11$.

Assume that $x \geq 1.3$.

Case 1 $1.3 \leq x < 2$.

Let $f(x) = 0.48(1+x)^3 - (1+x^3)$.

Note that $f(1.3) \geq 0$, $f(2) \geq 0$ and $f'(x) \neq 0$ for $1.3 \leq x < 2$.

Then f is non-negative function for $1.3 \leq x < 2$ which implies that

$$\frac{1}{(1+x)^3} \leq \frac{0.48}{1+x^3}. \tag{4.23}$$

Using the same argument as in (4.23), we can show that

$$\frac{1}{(1+x)^2} \leq \frac{0.48}{1+x^2}. \tag{4.24}$$

By (4.23) and (4.24), we have

$$\frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \leq 0.48\left(\frac{\alpha_x}{1+x^2} + \frac{\beta_x}{1+x^3}\right).$$

From this fact and Theorem 4.3, we have

$$\begin{aligned} |F_n(x) - \Phi(x)| &\leq 59.45\left(\frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3}\right) \\ &\leq 59.45(0.48)\left(\frac{\alpha_x}{1+x^2} + \frac{\beta_x}{1+x^3}\right) \\ &\leq 28.54\left(\frac{\alpha_x}{1+x^2} + \frac{\beta_x}{1+x^3}\right) \end{aligned}$$

Case 2 $x \geq 2$.

Follows from case 1 by using the same argument as in (4.23) to show that

$$\frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \leq 0.63\left(\frac{\alpha_x}{1+x^2} + \frac{\beta_x}{1+x^3}\right),$$

$$\frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \leq 0.806\left(\frac{\alpha_x}{1+x^2} + \frac{\beta_x}{1+x^3}\right),$$

$$\frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \leq 0.876\left(\frac{\alpha_x}{1+x^2} + \frac{\beta_x}{1+x^3}\right), \quad \text{and}$$

$$\frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \ \leq \ \frac{\alpha_x}{1+x^2} + \frac{\beta_x}{1+x^3}$$

in case of $2 \leq x < 3$, $3 \leq x < 7.98$, $7.98 \leq x < 14$ and $x \geq 14$, respectively. $\square$

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VITA

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