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REGULAR ELEMENTS OF ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

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เราเรียกสมาชิก x ของกึ่งกรุป S ว่า เป็น*สมาชิกปกติ* ถ้ามีสมาชิก $y \in S$ ซึ่ง $x = xyx$ และเรียก S ว่าเป็น*กึ่งกรุปปกติ* ถ้าทุกสมาชิกของ S เป็นสมาชิกปกติ

เรากล่าวว่าการส่ง α จากเซตอันดับบางส่วน X ไปยังเซตอันดับบางส่วน Y เป็น*การส่งที่รักษาอันดับ* ถ้า

สำหรับ $x, x' \in X$ ใด ๆ $x \le x'$ ใน $X \Rightarrow x\alpha \le x'\alpha$ ใน Y

สำหรับเซตอันดับบางส่วน X ให้ $OT(X)$ เป็นกึ่งกรุปการแปลงที่รักษาอันดับของ X ภายใต้การประกอบ ให้ $\mathbb{Z}$ และ $\mathbb{R}$ เป็นเซตอันดับทุกส่วนของจำนวนเต็มและเซตของจำนวนจริง ตามลำดับ ภายใต้อันดับธรรมชาติ เป็นที่รู้กันแล้วว่า $OT(X)$ เป็นกึ่งกรุปปกติสำหรับทุกเซตย่อยไม่ว่าง X ของ $\mathbb{Z}$ และสำหรับช่วง X ใน $\mathbb{R}$, $OT(X)$ เป็นกึ่งกรุปปกติ ก็ต่อเมื่อ X เป็นช่วงปิดที่มีขอบเขต ยิ่งไปกว่านั้น สำหรับช่วง X ในฟิลด์ย่อย F ของ $\mathbb{R}$ ซึ่ง $|X| > 1$, $OT(X)$ เป็นกึ่งกรุปปกติ ก็ต่อเมื่อ $F = \mathbb{R}$ และ X เป็นช่วงปิดที่มีขอบเขต

ในการวิจัยนี้ เราให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับสมาชิกของ $OT(X)$ ที่จะเป็นสมาชิกปกติ เมื่อ X เป็นเซตอันดับทุกส่วนใดๆ เราได้ประยุกต์ความรู้นี้มาพิสูจน์ผลที่ทราบกันแล้วข้างต้นด้วย

สำหรับเซตอันดับทุกส่วน $(X, \le)$ ใด ๆ *เซตอันดับบางส่วนแบบพจนานุกรม* ของ X คือ เซตอันดับทุกส่วน $(X \times X, \le_d)$ โดย $\le_d$ นิยามบน $X \times X$ โดย

$$(a_1, b_1) \le_d (a_2, b_2) \iff \text{(i) } a_1 < a_2 \text{ หรือ } \quad \text{(ii) } a_1 = a_2 \text{ และ } b_1 \le b_2$$

เราประยุกต์การให้ลักษณะของสมาชิกปกติมาศึกษาว่าเมื่อใด $OT(X \times X, \le_d)$ เป็นกึ่งกรุปปกติ เมื่อ X เป็นเซตย่อยไม่ว่างของ $\mathbb{Z}$ ช่วงใน $\mathbb{R}$ หรือ ช่วงในฟิลด์ย่อย F ของ $\mathbb{R}$

ภาควิชา …คณิตศาสตร์…	ลายมือชื่อนิสิต......วินิตา โมรา......
สาขาวิชา …คณิตศาสตร์…	ลายมือชื่ออาจารย์ที่ปรึกษา......ยุพาภรณ์ เข็มประสิทธิ์......
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An element x of a semigroup S is called *regular* if there is an element $y \in S$ such that $x = xyx$ and S is said to be a *regular semigroup* if every element of S is regular.

A mapping α from a partially ordered set X into a partially ordered set Y is said to be *order-preserving* if

$$\text{for any } x, x' \in X,\ x \leq x' \text{ in } X \ \Rightarrow\ x\alpha \leq x'\alpha \text{ in } Y.$$

The semigroup, under composition, of all order-preserving transformations of a partially ordered set X is denoted by $OT(X)$. Let $\mathbb{Z}$ and $\mathbb{R}$ be the chain of integers and the chain of real numbers, respectively, under the natural order. It is known that $OT(X)$ is regular for every nonempty subset X of $\mathbb{Z}$ and for an interval X in $\mathbb{R}$, $OT(X)$ is regular if and only if X is closed and bounded. Moreover, for a nontrivial interval X in a subfield F of $\mathbb{R}$, $OT(X)$ is regular if and only if $F = \mathbb{R}$ and X is closed and bounded.

In this research, we provide necessary and sufficient conditions for the elements of $OT(X)$ to be regular when X is any chain. It is then applied to prove the above known results.

For a chain X, the *dictionary partially ordered set* of X is the chain $(X \times X, \leq_d)$ where $\leq_d$ is defined by

$$(a_1, b_1) \leq_d (a_2, b_2) \ \Leftrightarrow\ \begin{array}{ll}\text{(i)} & a_1 < a_2 \ \text{ or} \\ \text{(ii)} & a_1 = a_2 \ \text{ and } \ b_1 \leq b_2.\end{array}$$

The characterization of regular elements is applied to determine when $OT(X \times X, \leq_d)$ is a regular semigroup where X is a nonempty subset of $\mathbb{Z}$, an interval in $\mathbb{R}$ or an interval in a subfield F of $\mathbb{R}$.

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Field of Study ....Mathematics....	Advisor's Signature.....Yupaporn Kemprasit
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CONTENTS

INTRODUCTION

Let X be a partially ordered set and $OT(X)$ the semigroup, under composition, of all order-preserving transformations $\alpha : X \to X$.

It is known from [3, page 203] that $OT(X)$ is a regular semigroup if X is a finite chain. Kemprasit and Changphas [5] extended this result to any chain which is order-isomorphic to a chain X where $X \subseteq \mathbb{Z}$, the set of integers with their natural order. Equivalently, $OT(X)$ is regular for every nonempty subset of $\mathbb{Z}$ with the usual order. Note that if the partially ordered sets X and Y are order-isomorphic, then the semigroups $OT(X)$ and $OT(Y)$ are isomorphic. It is also proved in [5] that for an interval X in $\mathbb{R}$, the set of real numbers with usual order, $OT(X)$ is a regular semigroup if and only if X is closed and bounded. Rungrattrakoon and Kemprasit [9] extended this fact by showing that for a nontrivial interval X in a subfield F of $\mathbb{R}$, $OT(X)$ is regular if and only if $F = \mathbb{R}$ and X is closed and bounded. Then it follows as a consequence that for a nontrivial interval X in $\mathbb{Q}$, the set of rational number, $OT(X)$ is not a regular semigroup. In fact, the above result in [9] is a consequence of the main theorem in [7].

The regularity of semigroups of order-preserving partial transformations have been also studied. See [1], [2] and [5] for examples.

A standard isomorphism is provided in [8, page 222-223] as follows : For partially ordered sets X and Y, $OT(X) \cong OT(Y)$ if and only if X and Y are order-isomorphic or anti-order-isomorphic. In [6], the authors generalized full order-preserving transformation semigroups by using sandwich multiplication and investigated their regularity and also provided some isomorphism theorems.

For a chain X, let $\leq_d$ denote the dictionary partial order on $X \times X$.

In this research, we extend the above results in [5] and [9]. The regular elements of $OT(X)$ are characterized when X is any chain. Then it is applied to prove those results and to determine the regularity of $OT(X \times X, \leq_d)$ when X is one of the following chains : chains of integers, intervals in $\mathbb{R}$ and intervals in a subfield of $\mathbb{R}$.

Chapter I provides basic definitions and known results which will be used in this research. Also, see [3] and [4] for more details.

In Chapter II, the regular elements of $OT(X)$ are characterized when X is any chain. Then this characterization is applied to prove the above known results of the regularity of $OT(X)$ where X is a nonempty subset of $\mathbb{Z}$, an interval in $\mathbb{R}$ or an interval in a subfield of $\mathbb{R}$.

In Chapter III, the regularity of $OT(X \times X, \leq_d)$ is characterized by using the main result in Chapter II, when X is one of the following chains : chains of integers, intervals in $\mathbb{R}$ and intervals in a subfield of $\mathbb{R}$.

CHAPTER I
PRELIMINARIES

For a set X, let $|X|$ denote the cardinality of X. The identity mapping on a nonempty set A is denoted by 1_A. The set of positive integers, the set of integers, the set of rational numbers and the set of real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. Note that they are chains with the natural order.

The following property of real numbers will be used. If X is an interval in $\mathbb{R}$ and A, B are nonempty subsets of $\mathbb{R}$ such that

$$X = A \dot{\cup} B \quad \text{and} \quad a < b \quad \text{for all } a \in A \text{ and } b \in B,$$

then $\sup(A) = \inf(B)$.

An element a of a semigroup S is called *regular* if $a = aba$ for some $b \in S$, and S is called a *regular semigroup* if every element of S is regular. The set of all regular elements of a semigroup S will be denoted by $\operatorname{Reg} S$, that is,

$$\operatorname{Reg} S = \{a \in S \mid a = aba \text{ for some } b \in S\}.$$

The domain and the range of any mapping α will be denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. For an element x in the domain of a mapping α, the image of α at x is written by $x\alpha$.

Denote by $T(X)$ the full transformation semigroup on a nonempty set X, that is, the semigroup, under composition, of all mappings $\alpha : X \to X$. It is well-known that $T(X)$ is a regular semigroup ([3], page 4 or [4], page 63).

Let X and Y be partially ordered sets. A mapping φ from X into Y is said to be *order-preserving* if

$$\text{for any } x, x' \in X, \quad x \leq x' \text{ in } X \;\Rightarrow\; x\varphi \leq x'\varphi \text{ in } Y.$$

A bijection $\varphi : X \to Y$ is called an *order-isomorphism* if φ and φ^{-1} are order-preserving. It is clear that if both X and Y are chains and $\varphi : X \to Y$ is an order-preserving bijection, then φ is an order-isomorphism from X onto Y. We say that X and Y are *order-isomorphic* if there is an order-isomorphism from X onto Y.

For a partially ordered set X, let

$$OT(X) = \{ \alpha \in T(X) \mid \alpha \text{ is order-preserving } \}.$$

It is clear that $OT(X)$ is a subsemigroup of $T(X)$ containing 1_X and all constant mappings. The semigroup $OT(X)$ is called the *full order-preserving transformation semigroup* on X.

Proposition 1.1. *Let X and Y be partially ordered sets. If $\varphi : X \to Y$ is an order-isomorphism, then*
(i) $\varphi^{-1}(OT(X))\varphi \subseteq OT(Y)$ *and* $\varphi(OT(Y))\varphi^{-1} \subseteq OT(X)$.
(ii) $OT(X) \cong OT(Y)$ *through the mapping* $\alpha \mapsto \varphi^{-1}\alpha\varphi$.

Proof. (i) is clearly obtained since $\varphi : X \to Y$ and $\varphi^{-1} : Y \to X$ are order-preserving.

(ii) Define $\theta : OT(X) \to OT(Y)$ by

$$\alpha\theta = \varphi^{-1}\alpha\varphi \ \text{ for all } \ \alpha \in OT(X).$$

If $\alpha, \beta \in OT(X)$, then

$$(\alpha\beta)\theta = \varphi^{-1}(\alpha\beta)\varphi = (\varphi^{-1}\alpha\varphi)(\varphi^{-1}\beta\varphi) = (\alpha\theta)(\beta\theta).$$

Hence θ is a homomorphism. If $\alpha, \beta \in OT(X)$ are such that $\alpha\theta = \beta\theta$, then

$$\alpha = \varphi(\varphi^{-1}\alpha\varphi)\varphi^{-1} = \varphi(\alpha\theta)\varphi^{-1} = \varphi(\beta\theta)\varphi^{-1} = \varphi(\varphi^{-1}\beta\varphi)\varphi^{-1} = \beta.$$

Thus θ is 1-1. If $\lambda \in OT(Y)$, then by (i), $\varphi\lambda\varphi^{-1} \in OT(X)$ and thus

$$(\varphi\lambda\varphi^{-1})\theta = \varphi^{-1}(\varphi\lambda\varphi^{-1})\varphi = \lambda.$$

This proves that θ is an isomorphism from $OT(X)$ onto $OT(Y)$. $\square$

The following result is a direct consequence of Proposition 1.1.

Corollary 1.2. *Let X and Y be partially ordered sets. If X and Y are order-isomorphic, then $OT(X)$ is regular if and only if $OT(Y)$ is regular.*

Intervals in a chain are defined naturally as follows : A nonempty subset Y of a chain X is called an *interval* in X if for $a, b, x \in X$, $a, b \in Y$ and $a \leq x \leq b$ imply that $x \in Y$. We say that an interval Y in X is a *nontrivial interval* if Y contains more than one element. Since every subfield F of $\mathbb{R}$ contains $\mathbb{Q}$, it follows that every nontrivial interval X of F is infinite.

The following results about the semigroup $OT(X)$ are known.

Theorem 1.3 ([5]). *For any nonempty subset X of $\mathbb{Z}$, $OT(X)$ is a regular semigroup.*

Theorem 1.4 ([5]). *For an interval X in $\mathbb{R}$, $OT(X)$ is a regular semigroup if and only if X is closed and bounded.*

Theorem 1.5 ([9]). *If X is a nontrivial interval in a subfield F of $\mathbb{R}$, then $OT(X)$ is regular if and only if $F = \mathbb{R}$ and X is closed and bounded.*

Corollary 1.6. *For every nontrivial interval X in $\mathbb{Q}$, $OT(X)$ is not regular.*

For a chain X, the *dictionary partially ordered set* of X is defined to be the chain $(X \times X, \leq_d)$ where $\leq_d$ is defined on $X \times X$ by

$$(a_1, b_1) \leq_d (a_2, b_2) \;\Leftrightarrow\; \begin{array}{ll} \text{(i)} & a_1 < a_2 \text{ or} \\ \text{(ii)} & a_1 = a_2 \text{ and } b_1 \leq b_2. \end{array}$$

CHAPTER II

REGULAR ELEMENTS OF ORDER-PRESERVING TRANSFORMATION SEMIGROUPS ON CHAINS

The regular elements of $OT(X)$ are characterized in this chapter where X is any chain. Then by this characterization, necessary and sufficient conditions are given for certain chains X so that $OT(X)$ is a regular semigroup.

2.1 Regular Elements

We recall the following result from [5].

Lemma 2.1.1 ([5]). *Let X be a chain. If $\alpha \in OT(X)$ and $a, b \in \operatorname{ran}\alpha$ with $a < b$, then $x < y$ for all $x \in a\alpha^{-1}$ and $y \in b\alpha^{-1}$.*

Also, the following lemma is needed.

Lemma 2.1.2. *If X is a nonempty set and $\alpha, \beta \in T(X)$ are such that $\alpha = \alpha\beta\alpha$, then $X\beta\alpha = (\operatorname{ran}\alpha)\beta\alpha$ and $x\beta\alpha = x$ for all $x \in \operatorname{ran}\alpha$.*

Proof. If $x \in X$, then $x\alpha = x\alpha\beta\alpha = (x\alpha)\beta\alpha$. This implies that $x\beta\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Since $\operatorname{ran}\alpha = X\alpha = (X\alpha)\beta\alpha = (\operatorname{ran}\alpha)\beta\alpha \subseteq X\beta\alpha \subseteq X\alpha = \operatorname{ran}\alpha$, we have that $X\beta\alpha = (\operatorname{ran}\alpha)\beta\alpha$. □

To obtain the main theorem, some necessary conditions for the regular elements of $OT(X)$, where X is any chain, are given as its lemmas.

Lemma 2.1.3. *Let X be a chain and $\alpha \in OT(X)$. If α is a regular element of $OT(X)$ and $\operatorname{ran}\alpha$ has an upper bound in X, then $\max(\operatorname{ran}\alpha)$ exists.*

Proof. Let $\beta \in OT(X)$ be such that $\alpha = \alpha\beta\alpha$, and let $u \in X$ be an upper bound of $\operatorname{ran}\alpha$. Suppose that $\operatorname{ran}\alpha$ has no maximum element in X. Then

$$x < u \quad \text{for all } \; x \in \operatorname{ran}\alpha. \tag{1}$$

From Lemma 2.1.2,

$$X\beta\alpha \;=\; (\operatorname{ran}\alpha)\beta\alpha, \tag{2}$$

$$x\beta\alpha \;=\; x \quad \text{for all } \; x \in \operatorname{ran}\alpha. \tag{3}$$

From (2), there exists an element $a \in \operatorname{ran}\alpha$ such that $u\beta\alpha = a\beta\alpha$. By (3), $a\beta\alpha = a$. Hence $a < u$ by (1) and $u\beta\alpha = a$. Since $a \in \operatorname{ran}\alpha$ and $\max(\operatorname{ran}\alpha)$ does not exist, there exists an element $b \in \operatorname{ran}\alpha$ such that $a < b < u$. Then $b\beta\alpha = b$ by (3). Hence $a = a\beta\alpha \leq b\beta\alpha = b \leq u\beta\alpha = a$ which implies that $a = b$, a contradiction. This proves that $\max(\operatorname{ran}\alpha)$ exists. □

The dual of Lemma 2.1.3 is the following lemma.

Lemma 2.1.4. *Let X be a chain and $\alpha \in OT(X)$. If α is regular in $OT(X)$ and* $\operatorname{ran}\alpha$ *has a lower bound in X, then* $\min(\operatorname{ran}\alpha)$ *exists.*

Lemma 2.1.5. *Let X be a chain and $\alpha \in OT(X)$. If α is regular in $OT(X)$ and* $a \in X \smallsetminus \operatorname{ran}\alpha$ *is neither an upper bound nor a lower bound of* $\operatorname{ran}\alpha$, *then* $\max(\{x \in \operatorname{ran}\alpha \mid x < a\})$ *or* $\min(\{x \in \operatorname{ran}\alpha \mid a < x\})$ *exists.*

Proof. Let $\beta \in OT(X)$ be such that $\alpha = \alpha\beta\alpha$. It follows from the assumption that

$$\{x \in \operatorname{ran}\alpha \mid x < a\} \neq \varnothing,\; \{x \in \operatorname{ran}\alpha \mid a < x\} \neq \varnothing,$$

$$\operatorname{ran}\alpha = \{x \in \operatorname{ran}\alpha \mid x < a\} \,\dot{\cup}\, \{x \in \operatorname{ran}\alpha \mid a < x\}. \tag{1}$$

By Lemma 2.1.2 ,

$$X\beta\alpha \;=\; (\operatorname{ran}\alpha)\beta\alpha, \tag{2}$$

$$x\beta\alpha \;=\; x \quad \text{for all } \; x \in \operatorname{ran}\alpha. \tag{3}$$

By (2), $a\beta\alpha = e\beta\alpha$ for some $e \in \operatorname{ran}\alpha$, and hence $a\beta\alpha = e\beta\alpha = e$ by (3). From (1), either $e < a$ or $a < e$. Suppose that neither $\max(\{x \in \operatorname{ran}\alpha \mid x < a\})$ nor $\min(\{x \in \operatorname{ran}\alpha \mid a < x\})$ exists.

Case 1 : $e < a$. Since $\max(\{x \in \operatorname{ran}\alpha \mid x < a\})$ does not exist, $e < p < a$ for some $p \in \operatorname{ran}\alpha$. By (3), $p\alpha\beta = p$. Then $e = e\beta\alpha \leq p\beta\alpha = p \leq a\beta\alpha = e$, so $e = p$, a contradiction.

Case 2 : $a < e$. Since $\min(\{x \in \operatorname{ran}\alpha \mid a < x\})$ does not exist, there is an element $q \in \operatorname{ran}\alpha$ such that $a < q < e$. Then we have $q\beta\alpha = q$ by (3) and thus $e = a\beta\alpha \leq q\beta\alpha = q \leq e\beta\alpha = e$. Hence $e = q$, a contradiction.

Hence the lemma is proved. □

Theorem 2.1.6. *Let X be a chain and $\alpha \in OT(X)$. Then α is regular in $OT(X)$ if and only if the following three conditions hold.*

(i) *If* $\operatorname{ran}\alpha$ *has an upper bound in X, then* $\max(\operatorname{ran}\alpha)$ *exists.*

(ii) *If* $\operatorname{ran}\alpha$ *has a lower bound in X, then* $\min(\operatorname{ran}\alpha)$ *exists.*

(iii) *If $a \in X \setminus \operatorname{ran}\alpha$ is neither an upper bound nor a lower bound of* $\operatorname{ran}\alpha$*, then* $\max(\{x \in \operatorname{ran}\alpha \mid x < a\})$ *or* $\min(\{x \in \operatorname{ran}\alpha \mid a < x\})$ *exists.*

Proof. If α is regular in $OT(X)$, then (i), (ii) and (iii) hold by Lemma 2.1.3, Lemma 2.1.4 and Lemma 2.1.5, respectively.

For the converse, assume that (i), (ii) and (iii) hold. If $\operatorname{ran}\alpha$ has an upper bound, let $u = \max(\operatorname{ran}\alpha)$. If $\operatorname{ran}\alpha$ has a lower bound, let $l = \min(\operatorname{ran}\alpha)$. If $x \in X \setminus \operatorname{ran}\alpha$ is neither an upper bound nor a lower bound of $\operatorname{ran}\alpha$, let

$$m_x = \begin{cases} \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) & \text{if } \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \text{ exists,} \\ \min(\{t \in \operatorname{ran}\alpha \mid x < t\}) & \text{otherwise.} \end{cases}$$

that is,

$$m_x = \begin{cases} \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) & \text{if } \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \text{ exists,} \\ \min(\{t \in \operatorname{ran}\alpha \mid x < t\}) & \text{if } \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \text{ does not exists} \\ & \quad \text{and} \min(\{t \in \operatorname{ran}\alpha \mid x < t\}) \text{ exists.} \end{cases}$$

For each $x \in \operatorname{ran}\alpha$, choose an element $x^{'} \in x\alpha^{-1}$. Then $x^{'}\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Thus $(x\alpha)^{'}\alpha = x\alpha$ for all $x \in X$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x^{'} & \text{if } x \in \operatorname{ran}\alpha, \\ u^{'} & \text{if } x \in X \setminus \operatorname{ran}\alpha \text{ and } x \text{ is an upper bound of } \operatorname{ran}\alpha, \\ l^{'} & \text{if } x \in X \setminus \operatorname{ran}\alpha \text{ and } x \text{ is a lower bound of } \operatorname{ran}\alpha, \\ {m_x}^{'} & \text{if } x \in X \setminus \operatorname{ran}\alpha \text{ and } x \text{ is neither an upper bound nor} \\ & \quad \text{a lower bound of } \operatorname{ran}\alpha. \end{cases}$$

for every $x \in X$. Then $\beta \in T(X)$ and for $x \in X$, $x\alpha \in \operatorname{ran}\alpha$ and thus

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)^{'}\alpha = x\alpha.$$

Hence $\alpha = \alpha\beta\alpha$. It remains to show that β is order-preserving. Let $x, y \in X$ be such that $x < y$.

Case 1 : $x, y \in \operatorname{ran}\alpha$. By Lemma 2.1.1, $s < t$ for all $s \in x\alpha^{-1}$ and $t \in y\alpha^{-1}$. But $x^{'} \in x\alpha^{-1}$ and $y^{'} \in y\alpha^{-1}$, so $x^{'} < y^{'}$. Hence $x\beta = x^{'} < y^{'} = y\beta$.

Case 2 : $x \in \operatorname{ran}\alpha, y \in X \setminus \operatorname{ran}\alpha$ and y is an upper bound of $\operatorname{ran}\alpha$. Since $x \leq u$, by Lemma 2.1.1, $x^{'} \leq u^{'}$, so $x\beta \leq y\beta$.

Case 3 : $x \in X \setminus \operatorname{ran}\alpha$, x is a lower bound of $\operatorname{ran}\alpha$ and $y \in \operatorname{ran}\alpha$. Then $l \leq y$, so by Lemma 2.1.1, $l^{'} \leq y^{'}$. Hence $x\beta \leq y\beta$.

Case 4 : $x, y \in X \setminus \operatorname{ran}\alpha$ and x and y are upper bounds of $\operatorname{ran}\alpha$. Then

$x\beta = u' = y\beta$.

Case 5 : $x, y \in X \smallsetminus \operatorname{ran}\alpha$ and x and y are lower bounds of $\operatorname{ran}\alpha$. Then $x\beta = l' = y\beta$.

Case 6 : $x, y \in X \smallsetminus \operatorname{ran}\alpha$, x is a lower bound of $\operatorname{ran}\alpha$ and y is an upper bound of $\operatorname{ran}\alpha$. Since $l \leq u$, by Lemma 2.1.1, $l' \leq u'$, so $x\beta \leq y\beta$.

Case 7 : $x \in \operatorname{ran}\alpha, y \in X \smallsetminus \operatorname{ran}\alpha$ and y is not an upper bound of $\operatorname{ran}\alpha$. Then $y \in X \smallsetminus \operatorname{ran}\alpha$ and y is neither an upper bound nor a lower bound of $\operatorname{ran}\alpha$.

Subcase 7.1 : $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ exists. Then

$$m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\}).$$

But $x \in \operatorname{ran}\alpha$ and $x < y$, so $x \leq m_y$. Hence $x' \leq {m_y}'$ by Lemma 2.1.1. Thus $x\beta \leq y\beta$.

Subcase 7.2 : $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ does not exist. Then

$$m_y = \min(\{t \in \operatorname{ran}\alpha \mid y < t\}).$$

Thus $x < y < m_y$. Hence $x\beta = x' < {m_y}' = y\beta$, as before.

Case 8 : $x \in X \smallsetminus \operatorname{ran}\alpha, x$ is not a lower bound of $\operatorname{ran}\alpha$ and $y \in \operatorname{ran}\alpha$. Then $x \in X \smallsetminus \operatorname{ran}\alpha$ and x is neither an upper bound nor a lower bound of $\operatorname{ran}\alpha$.

Subcase 8.1 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ exists. Then $m_x < x < y$, so $x\beta = {m_x}' < y' = y\beta$.

Subcase 8.2 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ does not exist. Then $m_x = \min(\{t \in \operatorname{ran}\alpha \mid x < t\})$. Since $y \in \operatorname{ran}\alpha$ and $x < y$, it follows that $m_x \leq y$. Hence $x\beta = {m_x}' \leq y' = y\beta$, as before.

Case 9 : $x, y \in X \smallsetminus \operatorname{ran}\alpha$, x is a lower bound of $\operatorname{ran}\alpha$ and y is neither an upper bound nor a lower bound of $\operatorname{ran}\alpha$.

Subcase 9.1 : $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ exists. Then $l \leq m_y$, so $x\beta = l' \leq {m_y}' = y\beta$.

Subcase 9.2 : $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ does not exist. Then $m_y = \min(\{t \in \operatorname{ran}\alpha \mid y < t\})$, so $l < y < m_y$. Hence $x\beta = l' < {m_y}' = y\beta$.

Case 10 : $x, y \in X \setminus \operatorname{ran}\alpha$, x is neither an upper bound nor a lower bound of $\operatorname{ran}\alpha$ and y is an upper bound of $\operatorname{ran}\alpha$.

Subcase 10.1 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ exists. Then $m_x < x < u$, so $x\beta = {m_x}' < u' = y\beta$.

Subcase 10.2 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ does not exist. Then $m_x = \min(\{t \in \operatorname{ran}\alpha \mid x < t\})$, so $m_x \leq u$. Hence $x\beta = {m_x}' \leq u' = y\beta$.

Case 11 : $x, y \in X \setminus \operatorname{ran}\alpha$ and x and y are neither upper bounds nor lower bounds of $\operatorname{ran}\alpha$.

Subcase 11.1 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ and $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ exist. Then

$$m_x = \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \text{ and } m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\}).$$

Since $x < y$, it follows that $\{t \in \operatorname{ran}\alpha \mid t < x\} \subseteq \{t \in \operatorname{ran}\alpha \mid t < y\}$ which implies that $m_x \leq m_y$. Hence $x\beta = {m_x}' \leq {m_y}' = y\beta$.

Subcase 11.2 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ exists and $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ does not exist. Then

$$m_x = \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \text{ and } m_y = \min(\{t \in \operatorname{ran}\alpha \mid y < t\}).$$

Then $m_x < x < y < m_y$, so $x\beta = {m_x}' < {m_y}' = y\beta$.

Subcase 11.3 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ does not exist and $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ exists. Then

$$m_x = \min(\{t \in \operatorname{ran}\alpha \mid x < t\}) \text{ and } m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\}).$$

If $\{t \in \operatorname{ran}\alpha \mid x < t < y\} = \varnothing$, then $\{t \in \operatorname{ran}\alpha \mid t < y\} = \{t \in \operatorname{ran}\alpha \mid t < x\}$ which is impossible since $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ does not exist but $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ exists. Then there exists an element $c \in \operatorname{ran}\alpha$ such that $x < c < y$. Consequently, $m_x \le c \le m_y$ which implies that $x\beta = {m_x}' \le {m_y}' = y\beta$.

Subcase 11.4 : $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ and $\max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ do not exist. Then

$$m_x = \min(\{t \in \operatorname{ran}\alpha \mid x < t\}) \text{ and } m_y = \min(\{t \in \operatorname{ran}\alpha \mid y < t\}).$$

Since $x < y$, $\{t \in \operatorname{ran}\alpha \mid x < t\} \supseteq \{t \in \operatorname{ran}\alpha \mid y < t\}$. Then $m_x \le m_y$, so $x\beta = {m_x}' \le {m_y}' = y\beta$.

Hence $\beta \in OT(X)$, and the proof is complete. □

The following lemma shows that if X is an interval in $\mathbb{R}$, then every $\alpha \in OT(X)$ satisfies (iii) of Theorem 2.1.6.

Lemma 2.1.7. *Let X be an interval in $\mathbb{R}$ and $\alpha \in OT(X)$. If $a \in X \smallsetminus \operatorname{ran}\alpha$ is neither an upper bound nor a lower bound of $\operatorname{ran}\alpha$, then either $\max(\{x \in \operatorname{ran}\alpha \mid x < a\})$ or $\min(\{x \in \operatorname{ran}\alpha \mid a < x\})$ exists.*

Proof. By assumption, we have that

$$\{x \in \operatorname{ran}\alpha \mid x < a\} \neq \varnothing, \{x \in \operatorname{ran}\alpha \mid a < x\} \neq \varnothing,$$
$$\operatorname{ran}\alpha = \{x \in \operatorname{ran}\alpha \mid x < a\} \mathbin{\dot\cup} \{x \in \operatorname{ran}\alpha \mid a < x\}.$$

It follows that

$$\{x \in \operatorname{ran}\alpha \mid x < a\}\alpha^{-1} \neq \varnothing, \{x \in \operatorname{ran}\alpha \mid a < x\}\alpha^{-1} \neq \varnothing, \tag{1}$$
$$X = \{x \in \operatorname{ran}\alpha \mid x < a\}\alpha^{-1} \mathbin{\dot\cup} \{x \in \operatorname{ran}\alpha \mid a < x\}\alpha^{-1}. \tag{2}$$

By Lemma 2.1.1,

$$\text{for all } s \in \{x \in \operatorname{ran}\alpha \mid x < a\}\alpha^{-1} \text{ and } t \in \{x \in \operatorname{ran}\alpha \mid a < x\}\alpha^{-1},\ s < t. \tag{3}$$

Since X is an interval in $\mathbb{R}$, (1), (2) and (3) yield the fact that

$$\sup\left(\{x \in \operatorname{ran}\alpha \mid x < a\}\alpha^{-1}\right) = \inf\left(\{x \in \operatorname{ran}\alpha \mid a < x\}\alpha^{-1}\right), \text{ say } e.$$

Then either $e = \max\left(\{x \in \operatorname{ran}\alpha \mid x < a\}\alpha^{-1}\right)$ or $e = \min\left(\{x \in \operatorname{ran}\alpha \mid a < x\}\alpha^{-1}\right)$. Since α is order-preserving, we have

$$e = \max\left(\{x \in \operatorname{ran}\alpha \mid x < a\}\alpha^{-1}\right) \Rightarrow e\alpha = \max\left(\{x \in \operatorname{ran}\alpha \mid x < a\}\right),$$
$$e = \min\left(\{x \in \operatorname{ran}\alpha \mid a < x\}\alpha^{-1}\right) \Rightarrow e\alpha = \min\left(\{x \in \operatorname{ran}\alpha \mid a < x\}\right).$$

Hence the lemma is proved. □

The following corollary is obtained directly from Theorem 2.1.6 and Lemma 2.1.7.

Corollary 2.1.8. *Let X be an interval in $\mathbb{R}$ and $\alpha \in OT(X)$. Then α is a regular element of $OT(X)$ if and only if the following two conditions hold.*

(i) *If* $\operatorname{ran}\alpha$ *has an upper bound in X, then* $\max(\operatorname{ran}\alpha)$ *exists.*

(ii) *If* $\operatorname{ran}\alpha$ *has a lower bound in X, then* $\min(\operatorname{ran}\alpha)$ *exists.*

2.2 Regular Semigroups

Throughout this section, the partial order on a nonempty subset of real numbers always means the natural order.

We shall apply Theorem 2.1.6 to prove Theorem 1.3 and Theorem 1.4 given in [5]. In addition, the regularity of $OT(X)$ for some other chains X in $\mathbb{R}$ are determined.

Theorem 2.2.1. *If X is a nonempty subset of $\mathbb{Z}$, then $OT(X)$ is a regular semigroup.*

Proof. Let A be a nonempty subset of X. By the property of subsets of $\mathbb{Z}$, we have that if A is bounded above in X, then $\max(A)$ exists. Also, if A is bounded below in X, then $\min(A)$ exists.

If $c \in X \smallsetminus A$ is neither an upper bound nor a lower bound of A, then $\{x \in A \mid x < c\} \neq \varnothing$ and $\{x \in A \mid c < x\} \neq \varnothing$, so both $\max(\{x \in A \mid x < c\})$ and $\min(\{x \in A \mid c < x\})$ exist.

This shows that for every $\alpha \in OT(X)$, $\operatorname{ran}\alpha$ satisfies (i), (ii) and (iii) of Theorem 2.1.6. By Theorem 2.1.6, every $\alpha \in OT(X)$ is regular in $OT(X)$. Hence $OT(X)$ is a regular semigroup. □

Lemma 2.2.2. *If X is $\mathbb{R}$, $[a, \infty)$ or (a, ∞) where $a \in \mathbb{R}$, then $OT(X)$ is not a regular semigroup.*

Proof. Let $c \in X$ and define $\alpha : X \to \mathbb{R}$ by

$$x\alpha = \begin{cases} c + \dfrac{x-c}{x-c+1} & \text{if } \ x \geq c, \\ c & \text{if } \ x < c. \end{cases}$$

Then $x\alpha = c$ for all $x \in X$ with $x \leq c$, α is continuous on X and the derivative of α at $x > c$ is $\dfrac{1}{(x-c+1)^2} > 0$. These imply that α is a nondecreasing function on X. Also, $\operatorname{ran}\alpha = [c, c+1) \subseteq X$, so $\alpha \in OT(X)$. Since $\operatorname{ran}\alpha$ is bounded in X and $\max(\operatorname{ran}\alpha)$ does not exist, by Theorem 2.1.6, α is not a regular element of $OT(X)$. Hence $OT(X)$ is not a regular semigroup. □

Lemma 2.2.3. *If X is $(-\infty, a]$ or $(-\infty, a)$, then $OT(X)$ is not a regular semigroup.*

Proof. Let $c \in X$ and define $\alpha : X \to \mathbb{R}$ by

$$x\alpha = \begin{cases} c - \dfrac{x-c}{x-c+1} & \text{if } \ x \leq c, \\ c & \text{if } \ x > c. \end{cases}$$

Then $x\alpha = c$ for all $x \geq c$, α is continuous on X and the derivative of α at $x < c$ is $\dfrac{1}{(x-c+1)^2} > 0$. Hence α is a nondecreasing function on X. We also have that

$\operatorname{ran}\alpha = (c-1, c] \subseteq X$. Then $\alpha \in OT(X)$, $\operatorname{ran}\alpha$ is bounded in X and $\min(\operatorname{ran}\alpha)$ does not exist. By Theorem 2.1.6, α is not a regular element of $OT(X)$, hence $OT(X)$ is not a regular semigroup. $\square$

Lemma 2.2.4. *If X is $[a, b)$, $(a, b]$ or (a, b) where $a, b \in \mathbb{R}$ and $a < b$, then the semigroup $OT(X)$ is not regular.*

Proof. Define $\alpha : X \to \mathbb{R}$ by

$$x\alpha = \frac{1}{4}(x-a) + \frac{a+b}{2} \quad \text{for all } x \in X.$$

Then the derivative of α at $x \in X$ is $\frac{1}{4}$. Hence α is a nondecreasing function. Also,

$$\operatorname{ran}\alpha = X\alpha = \begin{cases} [\frac{a+b}{2}, \frac{a+3b}{4}) & \text{if } X = [a,b), \\ (\frac{a+b}{2}, \frac{a+3b}{4}] & \text{if } X = (a,b], \\ (\frac{a+b}{2}, \frac{a+3b}{4}) & \text{if } X = (a,b), \end{cases}$$

$$a < \frac{a+b}{2} < \frac{a+3b}{4} < b.$$

Then we deduce that $\alpha \in OT(X)$. Since $\operatorname{ran}\alpha$ is both bounded above and bounded below in X, $\max(\operatorname{ran}\alpha)$ does not exist if $X = [a, b)$ or $X = (a, b)$ and $\min(\operatorname{ran}\alpha)$ does not exist if $X = (a, b)$ or $X = (a, b]$, it follows from Theorem 2.1.6, α is not a regular element of $OT(X)$. Hence $OT(X)$ is not a regular semigroup. $\square$

Lemma 2.2.5. *For $a, b \in \mathbb{R}$ with $a \leq b$, $OT([a, b])$ is a regular semigroup.*

Proof. To show that every element of $OT([a, b])$ is regular, let $\alpha \in OT([a, b])$. Since α is order-preserving on $[a, b]$, we have that $a\alpha = \min(\operatorname{ran}\alpha)$ and $b\alpha = \max(\operatorname{ran}\alpha)$. By Corollary 2.1.8, α is a regular element of $OT([a, b])$. $\square$

From Lemma 2.2.2, Lemma 2.2.3, Lemma 2.2.4 and Lemma 2.2.5, the following theorem is obtained.

Theorem 2.2.6. *For an interval X in $\mathbb{R}$, $OT(X)$ is a regular semigroup if and only if X is closed and bounded.*

Note that if X is a trivial interval, that is, $|X| = 1$, then $|OT(X)| = 1$, so $OT(X)$ is a regular semigroup.

Theorem 2.2.7. *If X is a nontrivial interval of a proper subfield F of $\mathbb{R}$, then $OT(X)$ is not a regular semigroup.*

Proof. We first note that $\mathbb{Q} \subseteq F \subsetneq \mathbb{R}$. Then there is an irrational number $c \in \mathbb{R} \setminus F$. Let $a, b \in X$ be such that $a < b$. Thus $a-c < b-c$, so $a-c < d < b-c$ for some $d \in \mathbb{Q}$. Hence $a < c+d < b$. Since $c \in \mathbb{R} \setminus F$ and $d \in \mathbb{Q} \subseteq F$, it follows that $c+d \in \mathbb{R} \setminus F$ and $c+d$ is an irrational number. Let $e = c+d$. Consequently,

$$X = \big((-\infty, a) \cap X\big) \cup \big([a, e) \cap X\big) \cup \big((e, \infty) \cap X\big). \tag{1}$$

Define $\mu : \mathbb{R} \to F$ by

$$x\mu = \begin{cases} x & \text{if } x \in (-\infty, a), \\ \dfrac{a+x}{2} & \text{if } x \in [a, e), \\ x & \text{if } x \in (e, \infty). \end{cases} \tag{2}$$

Then $a\mu = a < e, \alpha$ is continuous on $(-\infty, e)$ and the derivative of μ at $x \in (a, e)$ is $\dfrac{1}{2}$. Consequently, μ is an order-preserving function on $\mathbb{R}$. Let $\alpha = \mu|_X : X \to F$. Then α is order-preserving. We claim that

$$([a, e) \cap X)\,\alpha = [a, \frac{a+e}{2}) \cap X. \tag{3}$$

Let $x \in [a, e) \cap X$. Then $a \le x < e < b$ and $x \in X \subseteq F$, so

$$a \le \frac{a+x}{2} = x\alpha < \frac{a+e}{2} < \frac{a+b}{2} < b \quad \text{and} \quad \frac{a+x}{2} \in F.$$

This implies that $x\alpha \in [a, \dfrac{a+e}{2}) \cap X$ since X is an interval in F and $a, b \in X$ with $a < b$. For the reverse inclusion, let $y \in [a, \dfrac{a+e}{2}) \cap X$. Then $a \le y < \dfrac{a+e}{2}$ and $y \in X \subseteq F$. Hence

$$a \le 2y - a < e < b \quad \text{and} \quad 2y - a \in F.$$

Then $2y-a \in [a,e) \cap X$ since $a, b \in X$ and X is an interval in F and $(2y-a)\alpha = \dfrac{a+(2y-a)}{2} = y$. Therefore (3) holds. From (1), (2) and (3), we have

$$\begin{aligned} \operatorname{ran}\alpha = X\alpha &= \big((-\infty,a)\cap X\big) \cup \big([a,\tfrac{a+e}{2})\cap X\big) \cup \big((e,\infty)\cap X\big) \\ &= \big((-\infty,\tfrac{a+e}{2})\cap X\big) \cup \big((e,\infty)\cap X\big) \subseteq X. \end{aligned} \tag{4}$$

Hence $\alpha \in OT(X)$. Let $q \in \mathbb{Q}$ be such that $\dfrac{a+e}{2} < q < e$. But

$$a < \frac{a+e}{2} < q < e < b,$$

$q \in \mathbb{Q} \subseteq F$, $a, b \in X$ and X is an interval in F, thus by (4), $q \in X \setminus \operatorname{ran}\alpha$, $\{x \in \operatorname{ran}\alpha \mid x < q\} = (-\infty, \dfrac{a+e}{2}) \cap X$ and $\{x \in \operatorname{ran}\alpha \mid q < x\} = (e,\infty) \cap X$. If $\max\big((-\infty,\dfrac{a+e}{2})\cap X\big)$ exists, say m, then

$$a \leq m < \frac{a+e}{2} < b \text{ and } m \in X.$$

Let $p \in \mathbb{Q}$ be such that $m < p < \dfrac{a+e}{2}$. Then $p \in F$ and $a < p < b$ which imply that $m < p \in (-\infty, \dfrac{a+e}{2}) \cap X$, a contradiction. Then $\max\big((-\infty,\dfrac{a+e}{2})\cap X\big)$ does not exist. We can show similarly that $\min\big((e,\infty)\cap X\big)$ does not exist. By Theorem 2.1.6, α is not a regular element of $OT(X)$. This proves that $OT(X)$ is not a regular semigroup, as desired. □

The following corollary is a direct consequence of Theorem 2.2.7.

Corollary 2.2.8. *If X is a nontrivial interval in $\mathbb{Q}$, then $OT(X)$ is not a regular semigroup.*

Example 2.2.9. Under the usual order, $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is order-isomorphic to $\{-1,-2,-3,\ldots\}$ through $\frac{1}{n} \mapsto -n$ for $n \in \mathbb{N}$. Then $OT(X) \cong OT(\{-1,-2,-3,\ldots\})$ by Proposition 1.1. Since $OT(\{-1,-2,-3,\ldots\})$ is a regular semigroup by Theorem 2.2.1, it follows that $OT(X)$ is a regular semigroup.

It is natural to ask that whether $OT(X \cup \{0\})$ is a regular semigroup. Note that 1 and 0 are the maximum element and the minimum element of $X \cup \{0\}$, respectively. Since an infinite subset of $\mathbb{Z}$ cannot have both a maximum element and a minimum element, it follows that $X \cup \{0\}$ is not order-isomorphic to any chain of integers. However, we can show by Theorem 2.1.6 that $OT(X \cup \{0\})$ is a regular semigroup. To prove this, let $\alpha \in OT(X \cup \{0\})$. Then $1\alpha = \max(\operatorname{ran}\alpha)$ and $0\alpha = \min(\operatorname{ran}\alpha)$. Let $m \in \mathbb{N} \smallsetminus \{1\}$ be such that $\frac{1}{m} \notin \operatorname{ran}\alpha$, $\{x \in \operatorname{ran}\alpha \mid x < \frac{1}{m}\} \neq \varnothing$ and $\{x \in \operatorname{ran}\alpha \mid \frac{1}{m} < x\} \neq \varnothing$. Since

$$\varnothing \neq \{x \in \operatorname{ran}\alpha \mid x < \frac{1}{m}\} \subseteq \{\frac{1}{m+1}, \frac{1}{m+2}, \ldots\} \cup \{0\},$$
$$\varnothing \neq \{x \in \operatorname{ran}\alpha \mid \frac{1}{m} < x\} \subseteq \{1, \frac{1}{2}, \ldots, \frac{1}{m-1}\},$$

it follows clearly both $\max(\{x \in \operatorname{ran}\alpha \mid x < \frac{1}{m}\})$ and $\min(\{x \in \operatorname{ran}\alpha \mid \frac{1}{m} < x\})$ exist. Hence by Theorem 2.1.6, α is a regular element of $OT(X \cup \{0\})$.

Example 2.2.10. Let $X = [0,1) \cup (2,3]$ with the natural order. Then $OT(X)$ is not regular. To prove this, define $\alpha \in OT([0,1))$ be as in Lemma 2.2.4. Then $\operatorname{ran}\alpha = [\frac{0+1}{2}, \frac{0+3}{4}) = [\frac{1}{2}, \frac{3}{4})$. Define $\bar{\alpha} : X \to \mathbb{R}$ by

$$x\bar{\alpha} = \begin{cases} x\alpha & \text{if} \quad x \in [0,1), \\ x & \text{if} \quad x \in (2,3]. \end{cases}$$

Thus, $\bar{\alpha} \in OT(X)$ and $\operatorname{ran}\bar{\alpha} = \operatorname{ran}\alpha \cup (2,3] = [\frac{1}{2}, \frac{3}{4}) \cup (2,3]$. Since $\frac{4}{5} \in X \smallsetminus \operatorname{ran}\bar{\alpha}$,

$$\{x \in \operatorname{ran}\bar{\alpha} \mid x < \frac{4}{5}\} = [\frac{1}{2}, \frac{3}{4})$$

and

$$\{x \in \operatorname{ran}\bar{\alpha} \mid \frac{4}{5} < x\} = (2,3],$$

it follows that neither $\max(\{x \in \operatorname{ran}\bar{\alpha} \mid x < \frac{4}{5}\})$ nor $\min(\{x \in \operatorname{ran}\bar{\alpha} \mid \frac{4}{5} < x\})$ exists. By Theorem 2.1.6, $\bar{\alpha}$ is not a regular element of $OT(X)$.

A natural question arises. If $X = [0,1) \cup [2,3]$ or $[0,1] \cup (2,3]$, is $OT(X)$ a regular semigroup? The following theorem gives a general result. This result indicates that this semigroup $OT(X)$ is a regular semigroup.

Theorem 2.2.11. *Let* $X = I_1 \cup I_2 \cup \ldots \cup I_n$ *where* $n > 1$,

$$\begin{aligned} &I_i \text{ is an interval in } \mathbb{R} \text{ for all } i \in \{1, 2, \ldots, n\}, \\ &\text{for } i \in \{1, 2, \cdots, n-1\},\ x < y \text{ for all } x \in I_i \text{ and } y \in I_{i+1}, \\ &I_i \cup I_{i+1} \text{ is not an interval in } \mathbb{R}, \end{aligned} \qquad (1)$$

then $OT(X)$ *is regular if and only if the following three conditions hold.*

(i) $\min(I_1)$ *exists.*

(ii) $\max(I_n)$ *exists.*

(iii) *For each* $i \in \{1, 2, \ldots, n-1\}, \max(I_i)$ *or* $\min(I_{i+1})$ *exists.*

Proof. We shall show by contrapositive that if $OT(X)$ is regular, then (i), (ii) and (iii) hold. Assume that at least one of (i), (ii) and (iii) is not true.

Case 1 : $\min(I_1)$ does not exist. By the proofs of Lemma 2.2.3 and Lemma 2.2.4, there exists an element $\alpha \in OT(I_1)$ such that

$$\operatorname{ran}\alpha \text{ has a lower bound in } I_1 \text{ and } \min(\operatorname{ran}\alpha) \text{ does not exist.} \qquad (2)$$

Define $\overline{\alpha} : X \to X$ by

$$x\overline{\alpha} = \begin{cases} x\alpha & \text{if } x \in I_1, \\ x & \text{if } x \in I_2 \cup \ldots \cup I_n. \end{cases}$$

Since $\alpha \in OT(I_1)$, by (1), $\overline{\alpha} \in OT(X)$. Also, $\operatorname{ran}\overline{\alpha} = \operatorname{ran}\alpha \cup I_2 \cup \ldots \cup I_n$. By (1) and (2), $\operatorname{ran}\overline{\alpha}$ has a lower bound and $\min(\operatorname{ran}\overline{\alpha})$ does not exist. By Theorem 2.1.6, $\overline{\alpha}$ is not regular in $OT(X)$.

Case 2 : $\max(I_n)$ does not exist. By the proofs of Lemma 2.2.2 and Lemma 2.2.4, there is an element $\beta \in OT(I_n)$ such that

$$\operatorname{ran}\beta \text{ has an upper bound in } I_n \text{ and } \max(\operatorname{ran}\beta) \text{ does not exist.} \qquad (3)$$

Define $\overline{\beta} : X \to X$ by

$$x\overline{\beta} = \begin{cases} x & \text{if } x \in I_1 \cup \ldots \cup I_{n-1}, \\ x\beta & \text{if } x \in I_n. \end{cases}$$

Since $\beta \in OT(I_n)$, by (1), $\overline{\beta} \in OT(X)$. We also have $\operatorname{ran}\overline{\beta} = I_1 \cup \ldots \cup I_{n-1} \cup \operatorname{ran}\beta$. It follows from (1) and (3) that $\operatorname{ran}\overline{\beta}$ has an upper bound and $\max(\operatorname{ran}\overline{\beta})$ does not exist. By Theorem 2.1.6, $\overline{\beta}$ is not regular in $OT(X)$.

Case 3 : $\min(I_1)$ exists, $\max(I_n)$ exists and there exists $j \in \{1, 2, \ldots, n-1\}$ such that neither $\max(I_j)$ nor $\min(I_{j+1})$ exists. By the proof of Lemma 2.3.4, there are elements $\gamma_1 \in OT(I_j)$ and $\gamma_2 \in OT(I_{j+1})$ such that

$$\operatorname{ran}\gamma_1 \text{ has an upper bound in } I_j \text{ and } \max(\operatorname{ran}\gamma_1) \text{ does not exist.} \tag{4}$$

and

$$\operatorname{ran}\gamma_2 \text{ has a lower bound in } I_{j+1} \text{ and } \min(\operatorname{ran}\gamma_2) \text{ does not exist.} \tag{5}$$

Define $\overline{\gamma} : X \to X$ by

$$x\overline{\gamma} = \begin{cases} x\gamma_1 & \text{if } x \in I_j, \\ x\gamma_2 & \text{if } x \in I_{j+1} \\ x & \text{if } x \in X \setminus (I_j \cup I_{j+1}). \end{cases}$$

Since $\gamma_1 \in OT(I_j)$ and $\gamma_2 \in OT(I_{j+1})$, it follows from (1) that $\overline{\gamma} \in OT(X)$. Moreover,

$$\operatorname{ran}\overline{\gamma} = I_1 \cup \ldots I_{j-1} \cup \operatorname{ran}\gamma_1 \cup \operatorname{ran}\gamma_2 \cup I_{j+2} \cup \ldots \cup I_n.$$

Let $a \in I_j$ be an upper bound of $\operatorname{ran}\gamma_1$. By (4), $a \in I_1 \setminus \operatorname{ran}\gamma_1$. Then $a \in X \setminus \operatorname{ran}\overline{\gamma}$,

$$\operatorname{ran}\overline{\gamma} = \{x \in \operatorname{ran}\overline{\gamma} \mid x < a\} \ \dot{\cup}\ \{x \in \operatorname{ran}\overline{\gamma} \mid a < x\},$$

$$\{x \in \operatorname{ran}\overline{\gamma} \mid x < a\} \ = \ I_1 \cup \ldots I_{j-1} \cup \operatorname{ran}\gamma_1, \tag{6}$$

$$\{x \in \operatorname{ran}\overline{\gamma} \mid a < x\} \ = \ \operatorname{ran}\gamma_2 \cup I_{j+2} \cup \ldots I_n. \tag{7}$$

By (1), (4) and (6), $\max\{x \in \operatorname{ran}\overline{\gamma} \mid x < a\}$ does not exist. Also, by (1), (5) and (7), $\min\{x \in \operatorname{ran}\overline{\gamma} \mid a < x\}$ does not exist. Hence by Theorem 2.1.6, $\overline{\gamma}$ is not regular in $OT(X)$.

For the converse, assume that (i), (ii) and (iii) hold. Note that by (1),

$\min(X) = \min(I_1)$ and $\max(X) = \max(I_n)$. Let $\alpha \in OT(X)$. Since α is order-preserving, $\min(\operatorname{ran}\alpha) = (\min(X))\alpha$ and $\max(\operatorname{ran}\alpha) = (\max(X))\alpha$. Let $c \in X \smallsetminus \operatorname{ran}\alpha$ be such that $\{x \in \operatorname{ran}\alpha \mid x < c\} \neq \varnothing$ and $\{x \in \operatorname{ran}\alpha \mid c < x\} \neq \varnothing$. Then

$$X = \{x \in \operatorname{ran}\alpha \mid x < c\}\alpha^{-1} \,\dot{\cup}\, \{x \in \operatorname{ran}\alpha \mid c < x\}\alpha^{-1}, \tag{8}$$

and by Lemma 2.2.1,

$$\text{for all } \ s \in \{x \in \operatorname{ran}\alpha \mid x < c\}\alpha^{-1} \ \text{ and } \ t \in \{x \in \operatorname{ran}\alpha \mid c < x\}\alpha^{-1},\ s < t. \tag{9}$$

From (9) and (10), we have that

either $\{x \in \operatorname{ran}\alpha \mid x < c\}\alpha^{-1} = I_1 \cup I_2 \ldots \cup I_k$ and
$\{x \in \operatorname{ran}\alpha \mid c < x\}\alpha^{-1} = I_{k+1} \cup \ldots \cup I_n$ for some $k \in \{1, 2, \ldots, n-1\}$

or there exists $k \in \{1, 2, \ldots, n\}$ such that $I_k = A \dot{\cup} B$, A and B are nonempty interval , $a < b$ for all $a \in A$ and $b \in B$,
$\{x \in \operatorname{ran}\alpha \mid x < c\}\alpha^{-1} = I_1 \cup I_2 \ldots \cup I_{k-1} \cup A$ and
$\{x \in \operatorname{ran}\alpha \mid c < x\}\alpha^{-1} = B \cup I_{k+1} \cup \ldots \cup I_n$.

By this fact, the assumption and the property of interval in $\mathbb{R}$, either $\max(\{x \in \operatorname{ran}\alpha \mid x < c\}\alpha^{-1})$ or $\min(\{x \in \operatorname{ran}\alpha \mid c < x\}\alpha^{-1})$ exists. Since $\{x \in \operatorname{ran}\alpha \mid x < c\} = (\{x \in \operatorname{ran}\alpha \mid x < c\}\alpha^{-1})\alpha$ and $\{x \in \operatorname{ran}\alpha \mid c < x\} = (\{x \in \operatorname{ran}\alpha \mid c < x\}\alpha^{-1})\alpha$ and α is order-preserving, it follows that either $\max(\{x \in \operatorname{ran}\alpha \mid x < c\})$ or $\min(\{x \in \operatorname{ran}\alpha \mid c < x\})$ exists. □

From obove Theorem, we can determine the regularity of $OT(X)$ for various kinds of $X \subseteq \mathbb{R}$, for examples, $OT([0,1) \cup [2,3) \cup [4,5])$ is a regular semigroup and $OT((0,1) \cup [2,3) \cup [4,5])$ is not a regular semigroup.

CHAPTER III

REGULAR ORDER-PRESERVING TRANSFORMATION SEMIGROUPS ON DICTIONARIES PARTIALLY ORDERED SETS OF CHAINS

In this chapter, we characterize the regularity of $OT(X \times X, \leq_d)$ when X is one of the following chains : chains of integers, intervals in $\mathbb{R}$ and intervals in a subfield of $\mathbb{R}$. Theorem 2.1.6 is a main tool for these characterizations.

3.1 Chains of integers

The following lemma gives an important necessary condition for $OT(X\times X, \leq_d)$ to be regular when X is any chain.

Lemma 3.1.1. *Let X be a chain. If $OT(X \times X, \leq_d)$ is a regular semigroup, then X has a maximum and a minimum.*

Proof. Suppose that $OT(X\times X, \leq_d)$ is regular. If $|X| = 1$, then we are done. Next, assume that $|X| > 1$. Let $u, v \in X$ be such that $u < v$. Define $\alpha : X\times X \to X\times X$ by

$$(x, y)\alpha = (u, x) \quad \text{for all } x, y \in X. \tag{1}$$

Then

$$(\{x\} \times X)\alpha = \{(u, x)\} \quad \text{for all } x \in X$$

and so

$$\operatorname{ran} \alpha = \{u\} \times X. \tag{2}$$

We have that for $x, y \in X$,

$$x \leq y \;\Rightarrow\; (u, x) \leq_d (u, y). \tag{3}$$

Then (1) and (3) give the fact that α is order-preserving on $(X \times X, \leq_d)$. Hence $\alpha \in OT(X \times X, \leq_d)$. Since $OT(X \times X, \leq_d)$ is regular, we have that $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(X \times X, \leq_d)$. By Lemma 2.1.2, $(\beta\alpha)|_{\operatorname{ran}\alpha}$ is the identity map on $\operatorname{ran}\alpha$ which implies from (2) that

$$(u, x)\beta\alpha = (u, x) \quad \text{for all } x \in X. \tag{4}$$

Since $u < v$, it follows that

$$(u, x) <_d (v, v) \quad \text{for all } x \in X.$$

Thus $(u, x)\beta\alpha \leq_d (v, v)\beta\alpha$ for all $x \in X$. This implies by (4) that

$$(u, x) \leq_d (v, v)\beta\alpha \quad \text{for all } x \in X. \tag{5}$$

Since $(v, v)\beta\alpha \in \operatorname{ran}\alpha$, by (2), $(v, v)\beta\alpha = (u, f)$ for some $f \in X$. Hence from (5),

$$(u, x) \leq_d (u, f) \quad \text{for all } x \in X$$

which implies that $x \leq f$ for all $x \in X$. This shows that f is the maximum of X.

To show that X also has a minimum, let $\gamma : X \times X \to X \times X$ be defined by

$$(x, y)\gamma = (v, x) \quad \text{for all } x, y \in X. \tag{6}$$

Then

$$(\{x\} \times X)\gamma = \{(v, x)\} \quad \text{for all } x \in X$$

and thus

$$\operatorname{ran}\gamma = \{v\} \times X. \tag{7}$$

Since for $x, y \in X$,

$$x < y \;\Rightarrow\; (v, x) <_d (v, y), \tag{8}$$

we deduce from (6) and (8) that $\gamma \in OT(X \times X, \leq_d)$. Since $OT(X \times X, \leq_d)$ is regular, we have that $\gamma = \gamma\lambda\gamma$ for some $\lambda \in OT(X \times X, \leq_d)$. By Lemma 2.1.2, $(\lambda\gamma)|_{\operatorname{ran}\gamma} = 1|_{\operatorname{ran}\gamma}$, so by (7), we have

$$(v, x)\lambda\gamma = (v, x) \quad \text{for all } x \in X. \tag{9}$$

Since $u < v$, it follows that

$$(u,u) <_d (v,x) \quad \text{for all } x \in X,$$

and so $(u,u)\lambda\gamma \leq_d (v,x)\lambda\gamma$ for all $x \in X$. This implies by (9) that

$$(u,u)\lambda\gamma \leq_d (v,x) \quad \text{for all } x \in X. \tag{10}$$

But $(u,u)\lambda\gamma \in \operatorname{ran}\gamma$, so $(u,u)\lambda\gamma = (v,e)$ for some $e \in X$ by (7). Hence from (10),

$$(v,e) \leq_d (v,x) \quad \text{for all } x \in X$$

which implies that $e \leq x$ for all $x \in X$. Hence e is the minimum of X.

Hence X has a maximum and a minimum, and the proof is complete. □

Theorem 3.1.2. *For $\varnothing \neq X \subseteq \mathbb{Z}$, $OT(X \times X, \leq_d)$ is a regular semigroup if and only if X is finite.*

Proof. If $OT(X \times X, \leq_d)$ is regular, then by Lemma 3.1.1, $\max(X)$ and $\min(X)$ exist. But X is a nonempty subset of $\mathbb{Z}$, so we have that X must be finite.

Conversely, if X is a finite set, then $(X \times X, \leq_d)$ is a finite chain. It follows that $(X \times X, \leq_d)$ is order-isomorphic to a (finite) chain of integers. Hence by Theorem 2.2.1, $OT(X \times X, \leq_d)$ is regular. □

Remark 3.1.3. By Theorem 2.2.1 and Theorem 3.1.2, $OT(\mathbb{Z})$ is regular and $OT(\mathbb{Z} \times \mathbb{Z}, \leq_d)$ is not regular, respectively. In addition, $OT(\mathbb{Z} \times \mathbb{Z}, \leq_d)$ contains an infinitely many nonregular element. To see this, let $c \in \mathbb{Z}$ and define $\alpha_c : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by

$$(x,y)\alpha_c = (c,x) \quad \text{for all } x,y \in \mathbb{Z}.$$

From the proof of Lemma 3.1.1, $\alpha_c \in OT(\mathbb{Z}\times\mathbb{Z}, \leq_d)$ and $\operatorname{ran}(\alpha_c) = \{c\}\times\mathbb{Z}$. Since

$$(c,x) <_d (c+1,0) \quad \text{for all } x \in \mathbb{Z},$$

we deduce that $(c+1,0)$ is an upper bound of $\operatorname{ran}(\alpha_c)$. But $\{c\} \times \mathbb{Z}$ has no maximum, so by Theorem 2.1.6, α_c is not a regular element of $OT(\mathbb{Z} \times \mathbb{Z}, \leq_d)$.

Hence

$$\{\alpha_c \mid c \in \mathbb{Z}\} \subseteq OT(\mathbb{Z}\times\mathbb{Z}, \leq_d) \smallsetminus \mathrm{Reg}(OT(\mathbb{Z}\times\mathbb{Z}, \leq_d)).$$

If $c_1 \neq c_2$ in $\mathbb{Z}$, then $\mathrm{ran}(\alpha_{c_1}) = \{c_1\}\times\mathbb{Z} \neq \{c_2\}\times\mathbb{Z} = \mathrm{ran}(\alpha_{c_2})$ which implies that $\alpha_{c_1} \neq \alpha_{c_2}$. Hence $\{\alpha_c \mid c \in \mathbb{Z}\}$ is an infinite subset of $OT(\mathbb{Z}\times\mathbb{Z}, \leq_d) \smallsetminus \mathrm{Reg}(OT(\mathbb{Z}\times\mathbb{Z}, \leq_d))$. Therefore, we deduce that $OT(\mathbb{Z}\times\mathbb{Z}, \leq_d)$ contains an infinitely many nonregular elements. Since every constant map in $OT(\mathbb{Z}\times\mathbb{Z}, \leq_d)$ is a regular element, it follows that $OT(\mathbb{Z}\times\mathbb{Z}, \leq_d)$ also contains an infinitely many regular elements.

From the above proof, we can show similarly by Theorem 2.1.6 that if X is an infinite subset of $\mathbb{Z}$, then $OT(X\times X, \leq_d)$ contains an infinitely many nonregular elements and an infinitely many regular elements.

3.2 Intervals in $\mathbb{R}$

We shall show that for an interval X in $\mathbb{R}$, $OT(X\times X, \leq_d)$ is regular if and only if X is closed and bounded.

Lemma 3.2.1. *Let $a, b \in \mathbb{R}$ be such that $a < b$. If A and B are nonempty subsets of $[a,b]\times[a,b]$ such that*

$$[a,b]\times[a,b] = A \mathbin{\dot\cup} B \tag{1}$$

and

$$\text{for all } (x,y) \in A \text{ and } (x',y') \in B,\ (x,y) <_d (x',y'), \tag{2}$$

then $\sup(A) = \inf(B)$*, hence either* $\sup(A) = \max(A)$ *or* $\inf(B) = \min(B)$.

Proof. Since $(a,a) = \min([a,b]\times[a,b], \leq_d)$ and $(b,b) = \max([a,b]\times[a,b], \leq_d)$, we have $(a,a) \in A$ and $(b,b) \in B$. Let

$$\begin{aligned} A_1 &= \{x \in [a,b] \mid (x,a) \in A\},\\ B_1 &= \{x \in [a,b] \mid (x,a) \in B\}. \end{aligned} \tag{3}$$

By (1),

$$\begin{aligned}[a,b]\times\{a\} &= (A\,\dot\cup\, B)\cap([a,b]\times\{a\})\\ &= \big(A\cap([a,b]\times\{a\})\big)\,\dot\cup\,\big(B\cap([a,b]\times\{a\})\big).\end{aligned}$$

It follows that

$$[a,b] = A_1\,\dot\cup\, B_1. \tag{4}$$

If $x\in A_1$ and $y\in B_1$, then by (3), $(x,a)\in A$ and $(y,a)\in B$. Hence $(x,a) <_d (y,a)$ by (2) which implies that $x<y$. Therefore we have that

$$\text{for all } x\in A_1 \text{ and } y\in B_1,\ \ x<y. \tag{5}$$

Since $(a,a)\in A$, we have by (3) that $a\in A_1$.

Case 1 : $B_1=\varnothing$. By (3), $(b,a)\notin B$. Then $(b,a)\in A$ by (1). By the definition of $\leq_d$, we have

$$\text{for all } (x,y)\in[a,b)\times[a,b],\ \ (x,y) <_d (b,a)\notin B.$$

This fact, (1) and (2) imply that $B\subseteq\{b\}\times(a,b]$. Let

$$A_2=\{y\in[a,b]\mid(b,y)\in A\} \text{ and } B_2=\{y\in[a,b]\mid(b,y)\in B\}.$$

Then $a\in A_2$ and $b\in B_2$ since $(b,a)\in A$ and $(b,b)\in A$. From (1) and (2), we respectively have

$$[a,b]=A_2\,\dot\cup\, B_2$$

and

$$\text{for all } x\in A_2 \text{ and } y\in B_2,\ x<y.$$

These imply that $\sup(A_2)=\inf(B_2)$, say c. Since $B\subseteq\{b\}\times(a,b]$, it follows from (2) that either $B=\{b\}\times(c,b]$ or $B=\{b\}\times[c,b]$. Then we deduce from (1) that

$$\begin{aligned}&B=\{b\}\times(c,d]\Rightarrow A=([a,b)\times[a,b])\cup(\{b\}\times[a,c]),\\ &B=\{b\}\times[c,d]\Rightarrow A=([a,b)\times[a,b])\cup(\{b\}\times[a,c)).\end{aligned}$$

Consequently, $\max(A) = (b, c) = \inf(B)$.

Case 2 : $B_1 \neq \varnothing$. Then $b \in B_1$ by (4) and (5). It follows that $\sup(A_1) = \inf(B_1)$, say e. Let

$$A_3 = \{y \in [a, b] \mid (e, y) \in A\} \text{ and } B_3 = \{y \in [a, b] \mid (e, y) \in B\}. \tag{6}$$

By (1) and (2), we have respectively that

$$[a, b] = A_3 \,\dot{\cup}\, B_3 \tag{7}$$

and

$$\text{for all } x \in A_3 \text{ and } y \in B_3,\ x < y. \tag{8}$$

Subcase 2.1 : $A_3 = \varnothing$. By (6) and (7), we have $(e, a) \notin A$ and $(e, a) \in B$. Since $(a, a) \in A$, we have $a < e$. By the definition of $\leq_d$, (1) and (2), we have

$$A = [a, e) \times [a, b] \text{ and } B = [e, b] \times [a, b],$$

and thus $\min(B) = (e, a)$ which is an upper bound of A. If $(u, v) <_d (e, a)$, then $u < e$. But $u < e$ implies that $(u, v) <_d (\frac{u+e}{2}, v)$ and both belong to $[a, e) \times [a, b]$, so (u, v) is not an upper bound of A. This shows that $\sup(A) = (e, a)$. Hence $\sup(A) = (e, a) = \inf(B)$.

Subcase 2.2 : $B_3 = \varnothing$. Then by (6) and (7), $(e, b) \notin B$ and $(e, b) \in A$. Thus by (1) and (2),

$$A = [a, e] \times [a, b] \text{ and } B = (e, b] \times [a, b].$$

Hence $\max(A) = (e, b)$ and we can show similarly that $\inf(B) = (e, b)$.

Subcase 2.3 : $A_3 \neq \varnothing$ and $B_3 \neq \varnothing$. From (7) and (8), we have $\sup(A_3) = \inf(B_3)$, say f.

If $f \in A_3$, then $(e, f) \in A$ and $(e, f) \notin B$ by (6) and (7), so from (1) and (2), we have

$$A = ([a, e) \times [a, b]) \cup (\{e\} \times [a, f]),$$

$$B = ((e, b] \times [a, b]) \cup (\{e\} \times (f, b])$$

which implies that $\max(A) = (e, f)$. We can see that (e, f) is a lower bound of B. If $(u, v) >_d (e, f)$, then $u > e$ or $u = e$ and $v > f$. Hence

$$\begin{aligned} u > e \;&\Rightarrow\; (u,v), (\frac{u+e}{2}, v) \in (e,b] \times [a,b] \subseteq B \\ &\quad\;\text{and } (\frac{u+e}{2}, v) <_d (u,v), \\ u = e \text{ and } v > f \;&\Rightarrow\; (u,v), (u, \frac{v+f}{2}) \in \{e\} \times (f,b] \subseteq B \\ &\quad\;\text{and } (u, \frac{v+f}{2}) <_d (u,v). \end{aligned}$$

Consequently, $\inf(B) = (e, f)$. Hence $\sup(A) = (e, f) = \inf(B)$.

If $f \in B_3$, then $(e, f) \in B$ and $(e, f) \notin A$, by (6) and (7), so

$$A = ([a, e) \times [a, b]) \cup (\{e\} \times [a, f)),$$
$$B = ([e, b] \times [a, b]) \cup (\{e\} \times [f, b])$$

by (1) and (2). Thus $\min(B) = (e, f)$. We can show similarly that $\sup(A) = (e, f)$.

Hence $\sup(A) = (e, f) = \inf(B)$.

Therefore the proof is complete. □

Theorem 3.2.2. *For an interval X in $\mathbb{R}$, $OT(X \times X, \leq_d)$ is a regular semigroup if and only if X is closed and bounded.*

Proof. Assume that the semigroup $OT(X \times X, \leq_d)$ is regular. By Lemma 3.1.1, X has a maximum and a minimum, say a and b, respectively. Hence $X = [a, b]$.

For the converse, assume that $X = [a, b]$ where $a, b \in \mathbb{R}$ and $a < b$. We shall prove that $OT(X \times X, \leq_d)$ is a regular semigroup by Theorem 2.1.6 and Lemma 3.2.1. Let $\alpha \in OT(X \times X, \leq_d)$. Since α is order-preserving, $(a, a) = \min(X \times X, \leq_d)$ and $(b, b) = \max(X \times X, \leq_d)$, it following that $(a, a)\alpha = \min(\operatorname{ran} \alpha)$ and $(b, b)\alpha = \max(\operatorname{ran} \alpha)$. Next, let $(e, f) \in (X \times X) \smallsetminus \operatorname{ran} \alpha$ be such that

$$A = \{(x, y) \in \operatorname{ran} \alpha \mid (x, y) <_d (e, f)\} \neq \varnothing$$

and

$$B = \{(x, y) \in \operatorname{ran} \alpha \mid (e, f) <_d (x, y)\} \neq \varnothing.$$

This implies that

$$A\alpha^{-1} \neq \varnothing,\ B\alpha^{-1} \neq \varnothing,$$
$$[a,b] \times [a,b] = A\alpha^{-1} \,\dot{\cup}\, B\alpha^{-1},$$

and by Lemma 2.1.1,

$$\text{for all } x \in A\alpha^{-1} \text{ and } y \in B\alpha^{-1},\ x < y.$$

From these facts and Lemma 3.2.1, $\sup(A\alpha^{-1}) = \inf(B\alpha^{-1})$. If $\sup(A\alpha^{-1}) = \max(A\alpha^{-1})$, then $\big(\max(A\alpha^{-1})\big)\alpha = \max(A)$ since α is order-preserving. Also, if $\inf(B\alpha^{-1}) = \min(B\alpha^{-1})$, then $\big(\min(B\alpha^{-1})\big)\alpha = \min(B)$. Hence by Theorem 2.1.6, α is a regular element of $OT(X \times X, \leq_d)$, as desired. $\square$

As a direct consequence of Theorem 2.2.6 and Theorem 3.2.2, we have

Corollary 3.2.3. *Let X be an interval in $\mathbb{R}$. Then the following statements are equivalent.*

(i) $OT(X \times X, \leq_d)$ *is a regular semigroup.*

(ii) $OT(X)$ *is a regular semigroup.*

(iii) X *is closed and bounded.*

Remark 3.2.4. We define $\leq_d$ on $[a,b] \times \{1,2,...,n\}$, where $a < b$ in $\mathbb{R}$ and $n \in \mathbb{N}$, as before, that is,

$$(x,k) \leq_d (y,l) \;\Leftrightarrow\; \text{either (i) } x < y \text{ or (ii) } x = y \text{ and } k \leq l.$$

Then $([a,b] \times \{1,2,...,n\}, \leq_d)$ is a chain. It can be easily seen that

$$([a,b] \times \{1,2,...,n\}, \leq_d) \text{ and } \Big(\bigcup_{i=0}^{n-1} [a,b] + 2i(b-a), \leq\Big)$$

are order-isomorphic through the map $(x,k) \mapsto x + 2(k-1)(b-a)$ where $\leq$ is the natural order of real numbers. For an example,

$$([1,2] \times \{1,2,3,4\}, \leq_d) \;\cong\; ([1,2] \cup [3,4] \cup [5,6] \cup [7,8], \leq).$$

By Theorem 2.2.6, $OT(\bigcup_{i=0}^{n-1} [a,b] + 2i(b-a), \leq)$ is regular. Hence $OT([a,b] \times \{1,2,...,n\}, \leq_d)$ is a regular semigroup.

3.3 Intervals in Subfields of $\mathbb{R}$

We shall show in this section that if X is a nontrivial interval in a subfield F of $\mathbb{R}$, then $OT(X \times X, \leq_d)$ is regular only the case that $F = \mathbb{R}$ and X is closed and bounded.

Lemma 3.3.1. *If X is a nontrivial interval in a proper subfield F of $\mathbb{R}$, then $OT(X \times X, \leq_d)$ is not a regular semigroup.*

Proof. Let $a, b \in X$ be such that $a < b$. Then there is an irrational number $e \in \mathbb{R} \smallsetminus F$ such that $a < e < b$ (see the proof of Theorem 2.2.7). Thus

$$X = \big((-\infty, a) \cap X\big) \cup \big([a, e) \cap X\big) \cup \big((e, \infty) \cap X\big).$$

Hence

$$X \times X = \big(((-\infty, a) \cap X) \times X\big) \cup \big(([a, e) \cap X) \times X\big) \cup \big(((e, \infty) \cap X) \times X\big).$$

Define $\alpha : X \times X \to X \times X$ by

$$(x,y)\alpha = \begin{cases} (x, a) & \text{if } x \in (-\infty, a) \cap X \text{ and } y \in X, \\ (\frac{a+x}{2}, a) & \text{if } x \in [a, e) \cap X \text{ and } y \in X, \\ (x, a) & \text{if } x \in (e, \infty) \cap X \text{ and } y \in X. \end{cases}$$

We can see from the proof of Theorem 2.2.7 that $\alpha \in OT(X \times X, \leq_d)$ and

$$\operatorname{ran} \alpha = \big(((-\infty, \frac{a+e}{2}) \cap X) \,\dot{\cup}\, ((e, \infty) \cap X)\big) \times \{a\}.$$

Let $q \in (\frac{a+e}{2}, e) \cap X$. Then $(q, a) \in (X \times X) \smallsetminus \operatorname{ran} \alpha$. We also have from the definition of α that

$$\{(x, y) \in \operatorname{ran} \alpha \mid (x, y) <_d (q, a)\} = \big((-\infty, \frac{a+e}{2}) \cap X\big) \times \{a\}$$

and

$$\{(x, y) \in \operatorname{ran}\alpha \mid (q, a) <_d (x, y)\} = \big((e, \infty) \cap X\big) \times \{a\}$$

It can be seen from the proof of Theorem 2.2.7 that none of $\max\big(((-\infty, \frac{a+e}{2}) \cap X) \times \{a\}\big)$ and $\min\big(((e, \infty) \cap X) \times \{a\}\big)$ exists. By Theorem 2.1.6, α is not a regular element of $OT(X \times X, \leq_d)$. □

As a direct consequence of Lemma 3.3.1, we have

Corollary 3.3.2. *It X is a nontrivial interval in $\mathbb{Q}$, then $OT(X \times X, \leq_d)$ is not a regular semigroup.*

Remark 3.3.3. Notice that the converse of Lemma 3.1.1 is true under the assumption that $\varnothing \neq X \subseteq \mathbb{Z}$ or X is an interval in $\mathbb{R}$. This follows from Theorem 3.1.2 and Theorem 3.2.2. However, the converse of Lemma 3.1.1 is not generally true. To see this, let $a, b \in \mathbb{Q}$ be such that $a < b$. Then $[a, b] \cap \mathbb{Q}$ is a nontrivial interval in $\mathbb{Q}$. By Corollary 3.3.2, $OT\big(([a, b] \cap \mathbb{Q}) \times ([a, b] \cap \mathbb{Q}), \leq_d\big)$ is not a regular semigroup. However, $b = \max([a, b] \cap \mathbb{Q})$ and $a = \min([a, b] \cap \mathbb{Q})$.

Theorem 3.3.4. *Let X be a nontrivial interval in a subfield F of $\mathbb{R}$. Then $OT(X \times X, \leq_d)$ is a regular semigroup if and only if $F = \mathbb{R}$ and X is closed and bounded.*

Proof. If $F \neq \mathbb{R}$, then by Lemma 3.3.1, $OT(X \times X, \leq_d)$ is not regular. Therefore if $OT(X \times X, \leq_d)$ is regular, then $F = \mathbb{R}$, and hence by Theorem 3.2.2, X is closed and bounded.

The converse holds by Theorem 3.2.2. □

The following corollary is obtained from Theorem 2.2.7 and Theorem 3.3.4.

Corollary 3.3.5. *Let X be a nontrivial interval in a subfield F of $\mathbb{R}$. Then the following statements are equivalent.*

(i) *$OT(X \times X, \leq_d)$ is a regular semigroup.*

(ii) *$OT(X)$ is a regular semigroup.*

(iii) *$F = \mathbb{R}$ and X is closed and bounded.*

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