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COLORABILITY OF GLUED GRAPHS

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ให้ G_1 และ G_2 เป็นกราฟและให้ H_1 และ H_2 เป็นกราฟย่อยเชื่อมโยงที่มีเส้นเชื่อมอย่างน้อยหนึ่งเส้นของ G_1 และ G_2 ตามลำดับ โดยที่ $H_1 \cong H_2$ ด้วยสมสัณฐาน f จะได้ว่า **กราฟปะติดของ** G_1 **และ** G_2 **ที่** H_1 **และ** H_2 **เทียบกับ** f เขียนแทนด้วย $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ คือกราฟที่เกิดจากการรวมกราฟ G_1 และ G_2 โดยการปะติดจุดยอดและเส้นเชื่อมใน H_1 และ H_2 ให้ตรงกับสมสัณฐาน f

เราสนใจการปะติดกราฟระหว่างกราฟชนิดเดียวกัน โดยกราฟที่เราสนใจคือ กราฟป่า กราฟต้นไม้ กราฟสองส่วน กราฟ k ส่วน กราฟมีคอร์ด และกราฟช่วง นอกจากนั้นเราศึกษาสมบัติของกราฟปะติดในการระบายสีจุดยอดและการระบายสีเส้นเชื่อม เราหาขอบเขตของรงคเลขและรงคเลขของเส้นเชื่อมของกราฟปะติด พร้อมทั้งให้กราฟที่รับประกันว่าแต่ละขอบเขตดีที่สุด

ภาควิชา ...คณิตศาสตร์... ลายมือชื่อนิสิต..
สาขาวิชา ...คณิตศาสตร์... ลายมือชื่ออาจารย์ที่ปรึกษา..........................
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Let G_1 and G_2 be any two graphs. Let H_1 and H_2 be non-trivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f, then **the glued graph of G_1 and G_2 at H_1 and H_2 with respect to f**, denoted by $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2.

We investigate the results of the graph obtaining by gluing graphs of the same type where the types we are interested in are forests, trees, bipartite graphs, k-partite graphs, chordal graphs and interval graphs. Furthermore, we study properties of glued graphs involving in their colorability and edge-colorability. We give bounds of the chromatic numbers and the edge-chromatic numbers of glued graphs and also provide graphs to guarantee that each bound is the best possible.

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Field of Study :	...Mathematics...	Advisor's Signature	....
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If I did the wrong thing, I do apologize. It does not happen by intention.

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CHAPTER I

INTRODUCTION

1.1 Introduction

The glue operator is a mathematically operator defined in [6]. C. Uiyyasathain studies about maximal-clique partitions of different sizes whether or not there exists a clique-inseparable graph with n maximal-clique partitions of n different sizes. So she defines the glue operator to solve the problem. The answer is in the form of glued graphs between the line graphs of complete graphs with n different orders. It makes us see how useful of the glued graphs are and motivates us to study properties of glued graphs.

In Section 1.2, we give definitions, examples and also investigate some basic properties of glued graphs.

In Chapter 2, we analyze the results of the graphs obtaining by gluing graphs of the same type where the types we interested in are forests, trees, bipartite graphs, k-partite graphs, chordal graphs and interval graphs. Moreover, we investigate a condition to obtain a glued graph that is the same type as its original graphs.

The colorability of glued graphs is to be considered in Chapter 3. We find bounds of the chromatic numbers of glued graphs and also prove their sharpness.

Lastly, we consider the edge-colorability of glued graphs in Chapter 4. Bounds of the edge-chromatic numbers of glued graphs are provided.

1.2 Definitions and Basic Properties

In this section, we introduce the graph gluing, and give some properties of glued graphs.

Definition 1.2.1. Let G_1 and G_2 be any two graphs. Let H_1 and H_2 be non-trivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f, then **the glued graph of G_1 and G_2 at H_1 and H_2 with respect to f**, denoted by $\underset{H_1\cong_f H_2}{G_1 \lhd\!\!\rhd G_2}$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer H as its **clone** and refer G_1 and G_2 as its **original graphs**.

The glued graph between G_1 and G_2 at the clone H, written $G_1 \underset{H}{\lhd\!\!\rhd} G_2$, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $\underset{H_1\cong_f H_2}{G_1 \lhd\!\!\rhd G_2}$ and H is the copy of H_1 and H_2 in the resulting graph.

We denote $G_1 \lhd\!\!\rhd G_2$ an arbitrary graph resulting from gluing G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism.

Example 1.2.2. Let G_1 and G_2 be graphs as shown in Figure 1.2.1.

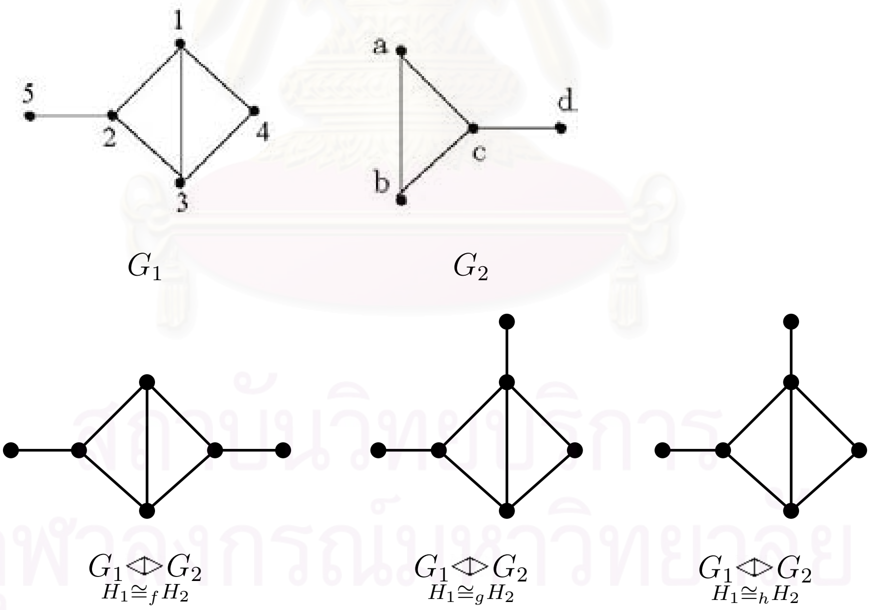


Figure 1.2.1: The results of graph gluing in different isomorphisms

Let $H_1 = K_3(1,3,4) \subseteq G_1$ and $H_2 = K_3(a,b,c) \subseteq G_2$. Consider three isomorphisms between H_1 and H_2, f, g and h, as follows:

$f(1) = a, f(3) = b, f(4) = c,$
$g(1) = b, g(3) = c, g(4) = a$ and
$h(1) = c, h(3) = a, h(4) = b.$

We show glued graphs between G_1 and G_2 with respect to f, g and h in Figure 1.2.1 $\square$

Example 1.2.2 shows that different isomorphisms can give the different or the same result. However, in some cases it is possible that all isomorphisms give the same result as shown in the next example.

Example 1.2.3. Let G_1 and G_2 be graphs as shown in Figure 1.2.2.

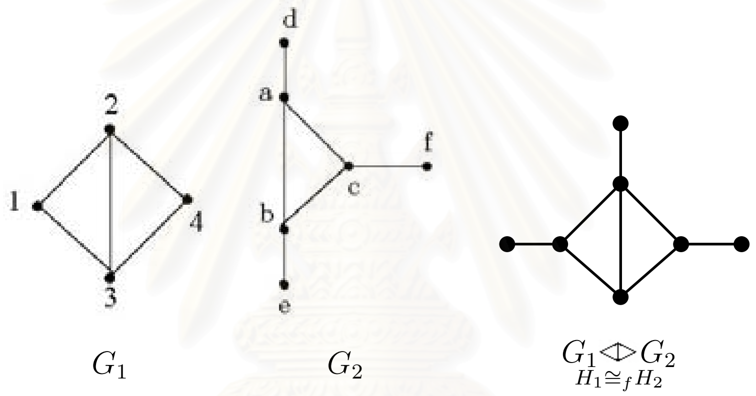


Figure 1.2.2: The same resulting graph for any isomorphism

Let $H_1 = K_3(2, 3, 4) \subseteq G_1$ and $H_2 = K_3(a, b, c) \subseteq G_2$. There are six isomorphisms between H_1 and H_2, but all of them give the same result as shown in a Figure 1.2.2 where f be arbitrary isomorphism between H_1 and H_2. $\square$

We first observe some basic properties of glued graphs in the following remark.

Remark 1.2.4. 1. The original graphs are subgraphs of their glued graph.

2. The graph gluing does not create or destroy an edge.

3. A glued graph between disconnected graphs is also disconnected and a glued graph between connected graphs is also connected.

4. If $u \in V(G_1 \backslash H)$ and $v \in V(G_2 \backslash H)$ where G_1 and G_2 are graphs and H is a clone of $G_1 \underset{H}{\Diamond} G_2$, then u and v are not adjacent in $G_1 \underset{H}{\Diamond} G_2$.

A glued graph could be a simple or not simple graph. Clearly the graph gluing of G_1 and G_2 is not a simple graph if G_1 or G_2 is not a simple graph. If original graphs are simple graphs, it is not necessary that their glued graph is a simple graph. We show in the next example.

Example 1.2.5. Let $G_1 = C_4(u_1, u_2, u_3, u_4)$ and $G_2 = C_4(v_1, v_2, v_3, v_4)$ and let $H_1 = P_4(u_1, u_2, u_3, u_4)$ and $H_2 = P_4(v_1, v_2, v_3, v_4)$. Clearly $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i = 1, 2, 3, 4$. Then we have non-simple glued graph $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$ as shown in Figure 1.2.3.

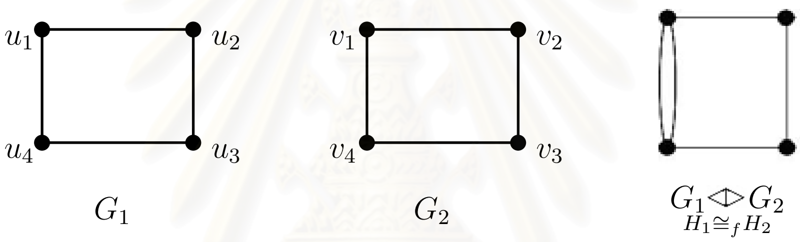


Figure 1.2.3: A glued graph between simple graphs which is not a simple graph $\square$

The following theorem gives a necessary and sufficient condition for glued graphs of simple graphs to be simple.

Theorem 1.2.6. *Let G_1 and G_2 be simple graphs and let H be the clone of a glued graph $G_1 \underset{H}{\Diamond} G_2$. Then $G_1 \underset{H}{\Diamond} G_2$ is a simple graph if and only if there are no verices u and v in H such that there are edges $e_1 \in E(G_1 \backslash H)$ and $e_2 \in E(G_2 \backslash H)$ whose endpoints are u and v.*

Proof. Let G_1 and G_2 be simple graphs and let H be the clone of a $G_1 \underset{H}{\Diamond} G_2$. Consider $G_1 \underset{H}{\Diamond} G_2$ a glued graph of G_1 and G_2 at a clone H. Clearly, if there are verices u and v in H such that there are edges $e_1 \in E(G_1 \backslash H)$ and $e_2 \in E(G_2 \backslash H)$ whose endpoints are u and v, then $G_1 \underset{H}{\Diamond} G_2$ contains multiple edges whose endpoints are u and v. Hence $G_1 \underset{H}{\Diamond} G_2$ is not a simple graph. Conversely, assume that $G_1 \underset{H}{\Diamond} G_2$ is not a simple graph. So $G_1 \underset{H}{\Diamond} G_2$ has a loop or multiple edges. If $G_1 \underset{H}{\Diamond} G_2$ has a loop, then that loop must be in G_1 or G_2 and also G_1 or G_2 is not a simple graph. This is a contradiction. Hence $G_1 \underset{H}{\Diamond} G_2$ contains multiple edges, say e_1 and e_2 with endpoints u and v. Since the graph gluing does not create an edge, we have $e_1 \in E(G_1) \cup E(G_2)$ and $e_2 \in E(G_1) \cup E(G_2)$. Because G_1 and G_2 are simple, so e_1 and e_2 are in different graphs. Without loss of generality, assume $e_1 \in E(G_1 \backslash H)$ and $e_2 \in E(G_2 \backslash H)$. This implies that there are verices u and v in H such that there are edges $e_1 \in E(G_1 \backslash H)$ and $e_2 \in E(G_2 \backslash H)$ whose endpoints are u and v. □

Next, we give the order and size of glued graphs in terms of those of original graphs.

Proposition 1.2.7. *Let G_1 and G_2 be graphs and let H be a clone of $G_1 \underset{H}{\Diamond} G_2$. Then*

1. $\left|V(G_1 \underset{H}{\Diamond} G_2)\right| = |V(G_1)| + |V(G_2)| - |V(H)|$, *and*
2. $\left|E(G_1 \underset{H}{\Diamond} G_2)\right| = |E(G_1)| + |E(G_2)| - |E(H)|$.

Proof. Let G_1 and G_2 be graphs and let H be a clone of $G_1 \underset{H}{\Diamond} G_2$. Because vertices and edges in H are counted twice in the glued graph, so $\left|V(G_1 \underset{H}{\Diamond} G_2)\right| = |V(G_1)| + |V(G_2)| - |V(H)|$ and $\left|E(G_1 \underset{H}{\Diamond} G_2)\right| = |E(G_1)| + |E(G_2)| - |E(H)|$. □

We next give a trivial upper bound of the maximum degree of any glued graph.

Lemma 1.2.8. *Let G_1 and G_2 be graphs and let H be the clone of a glued graph $G_1 \underset{H}{\Diamond} G_2$. Then*

$$\Delta(G_1 \underset{H}{\Diamond} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H).$$

Proof. Let G_1 and G_2 be graphs and let H be the clone of a glued graph $G_1 \underset{H}{\Diamond} G_2$. For convenience, let $G = G_1 \underset{H}{\Diamond} G_2$. Let v be a vertex with maximum degree of G. If v is not in H, then $\deg_G(v) = \max\{\Delta(G_1), \Delta(G_2)\} \leq \Delta(G_1) + \Delta(G_2) - \delta(H)$. Suppose that v is in H. So v is in both G_1 and G_2. Because each edge which is incident to v in H is counted twice, so

$$\deg_G(v) = \deg_{G_1}(v) + \deg_{G_2}(v) - \deg_H(v).$$

Since $v \in H$, we get that $\deg_H(v) \geq \delta(H)$. Hence

$$\deg_G(v) = \deg_{G_1}(v) + \deg_{G_2}(v) - \deg_H(v) \leq \Delta(G_1) + \Delta(G_2) - \delta(H).$$

□

A trivial upper bound of the maximum degree of any glued graph in Lemma 1.2.8 is a useful tool to find the chromatic numbers and the edge-chromatic numbers of glued graphs in Chapter 3 and Chapter 4. In the next chapter, we consider results of the graph gluing when original graphs are particular types of graphs.

CHAPTER II

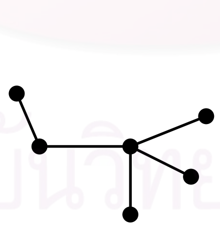
GLUED GRAPHS

Our purpose in this chapter is to study the graph gluing between original graphs which are such as forests, trees, bipartite graphs, chordal graphs and interval graphs. We separate this chapter into two sections. The first section contains the results of a family of bipartite graphs including forests and trees, and k-partite graphs and the other contains the results of chordal graphs and interval graphs.

2.1 The Graph Gluing of Bipartite Graphs and k-partite Graphs

First, we recall definitions and some properties of a forest and a tree.

Definition 2.1.1. A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph.



T_1 T_2

Figure 2.1.1: Examples of trees

To find a result of the graph gluing between trees, we state a well-known characterization of trees in Theorem 2.1.2. Then we give a result of the graph gluing between two trees in Theorem 2.1.3.

Theorem 2.1.2 ([3]). *For any n-vertex graph G with $n \geq 1$, the following are equivalent to definitions of a tree with n vertices:*
A) G is connected and has no cycles.
B) G is connected and has $n-1$ edges.
C) G has $n-1$ edges and no cycles.
D) For $u, v \in V(G)$, G has exactly one u,v-path.

We next show a result of the graph gluing of two trees in Theorem 2.1.3.

Theorem 2.1.3. *Let T_1 and T_2 be graphs.*

A glued graph $T_1 \triangleleft\!\triangleright T_2$ is a tree if and only if T_1 and T_2 are trees.

Proof. Necessity. By contrapositive, suppose that T_1 or T_2 is not a tree. Without loss of generality, we may assume that T_1 is not a tree. Then T_1 contains a cycle or T_1 is disconnected.

Case 1. T_1 contains a cycle : Because $T_1 \subseteq T_1 \triangleleft\!\triangleright T_2$, so that cycle is in $T_1 \triangleleft\!\triangleright T_2$. Therefore $T_1 \triangleleft\!\triangleright T_2$ is not a tree.

Case 2. T_1 is disconnected: By Remark 1.2.4, $T_1 \triangleleft\!\triangleright T_2$ is also disconnected. Hence $T_1 \triangleleft\!\triangleright T_2$ is not a tree.

Sufficiency. Let T_1 and T_2 be trees and let $T_1 \underset{H}{\triangleleft\!\triangleright} T_2$ be a glue graph between T_1 and T_2 at arbitrary clone H. Since a connected subgraph of a tree is a tree, the clone H is also a tree. By Proposition 1.2.7, we have

$$\begin{aligned}
\left|E(T_1 \underset{H}{\triangleleft\!\triangleright} T_2)\right| &= |E(T_1)| + |E(T_2)| - |E(H)| \\
&= |V(T_1)| - 1 + |V(T_2)| - 1 - |V(H)| + 1 \\
&= |V(T_1)| + |V(T_2)| - |V(H)| - 1 \\
&= \left|V(T_1 \underset{H}{\triangleleft\!\triangleright} T_2)\right| - 1.
\end{aligned}$$

Since T_1 and T_2 is connected, so is $T_1 \underset{H}{\triangleleft\!\triangleright} T_2$. By Theorem 2.1.2, $T_1 \underset{H}{\triangleleft\!\triangleright} T_2$ is a tree. $\square$

Theorem 2.1.3 can be restated for connected graphs G_1 and G_2 as follows:

A glued graph $G_1 \triangleleft\!\triangleright G_2$ has a cycle if and only if G_1 or G_2 has a cycle.

We next consider all cycles in any glued graph. Since G_1 and G_2 are subgraphs of $G_1 \triangleleft\!\triangleright G_2$ where G_1 and G_2 are graphs, all cycles in G_1 and G_2 are in $G_1 \triangleleft\!\triangleright G_2$. However, it is possible that $G_1 \triangleleft\!\triangleright G_2$ contains a new cycle. We illustrate this in the next example.

Example 2.1.4. Let G_1 and G_2 be graphs in Figure 2.1.2.

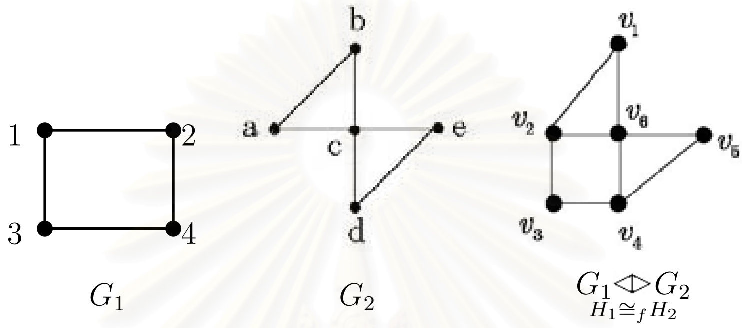


Figure 2.1.2: Created cycles

Let $H_1 = P_3(1, 2, 4) \subseteq G_1$ and $H_2 = P_3(a, c, d) \subseteq G_2$. Define $f : H_1 \to H_2$ be defined by $f(1) = a$, $f(2) = c$ and $f(4) = d$. Then we get $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$ showed in Figure 2.1.2 containing $C_6(v_1, v_2, v_3, v_4, v_5, v_6)$ but $C_6(v_1, v_2, v_3, v_4, v_5, v_6)$ is not a cycle in G_1 and G_2. $\square$

In Example 2.1.4, we can see that the graph gluing can create a new cycle. We call such new cycles as **created cycles** and all cycles in the original graphs as **original cycles**. Theorem 2.1.6 shows a necessary condition to guarantee the existence of created cycles in any glued graph.

Remark 2.1.5. Let C be a created cycle of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ where G_1 and G_2 are graphs and H is a clone of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. There exist non-trivial paths P and P' which are subgraphs of C such that $P \subseteq G_1 \backslash H$ and $P' \subseteq G_2 \backslash H$.

Theorem 2.1.6. *Let G_1 and G_2 be graphs. If $G_1 \triangleleft\!\triangleright G_2$ contains a created cycle, then both G_1 and G_2 are not acyclic.*

Proof. Let G_1 and G_2 be graphs. Without loss of generality, we may assume that G_1 and G_2 are connected. Assume that $G_1 \triangleleft\!\triangleright G_2$ contains a created cycle, say C.

Suppose for a contradiction that G_1 is acyclic. So G_1 is a tree. If G_2 is acyclic, then G_2 is a tree. By Theorem 2.1.3, $G_1 \triangleleft\!\!\triangleright G_2$ is a tree which is acyclic. This is a contradiction. So that G_2 contains a cycle. By Remark 2.1.5, There exists a non-trivial path which is a subgraph of $C \cap (G_1 \backslash H)$ where H is arbitrary clone of $G_1 \triangleleft\!\!\triangleright G_2$. We choose u, v-path P such that $u \neq v$ and $|E(P)|$ is the maximum. Then u and v are vertices in H. Since the clone is connected, there is another u, v-path P' in H. Because $P \subseteq G_1 \backslash H$ but $P' \subseteq H$, so $P' \neq P$. Then for each vertex in $P \cup P'$ has degree two. So $P \cup P'$ contains a cycle. But $P \cup P' \subseteq G_1$, so G_1 contains a cycle, a contradiction. Therefore both G_1 and G_2 are not acyclic. $\square$

The converse of theorem 2.1.6 does not hold. We show in Example 2.1.7.

Example 2.1.7. Let G_1 and G_2 be graphs as shown in Figure 2.1.3.

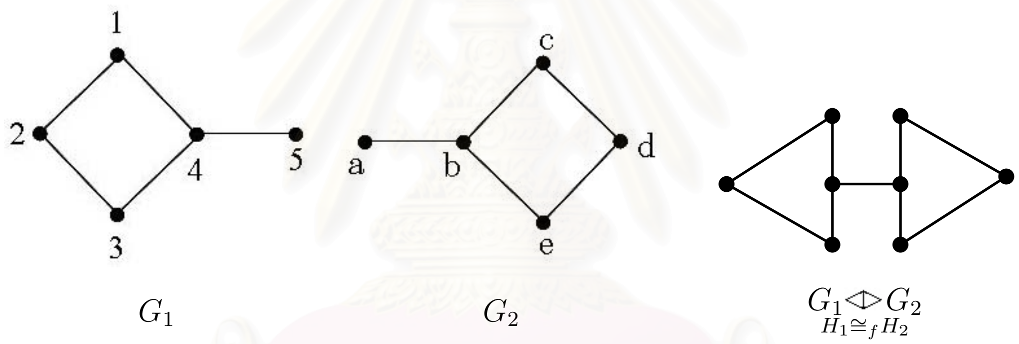


Figure 2.1.3: The converse of Theorem 2.1.6 does not hold.

We glue G_1 and G_2 at $H_1 = P_2(4,5) \subseteq G_1$ and $H_2 = P_2(a,b) \subseteq G_2$ by isomorphism f between H_1 and H_2 such that $f(4) = a$ and $f(5) = b$. So we get $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\!\triangleright G_2}$ which does not contain any new cycle as shown in Figure 2.1.3. $\square$

Now we study a result of the graph gluing between two forests. That result is showed in Corollary 2.1.8.

Corollary 2.1.8. *Let G_1 and G_2 be graphs.*

A glued graph $G_1 \triangleleft\!\!\triangleright G_2$ is a forest if and only if G_1 and G_2 are forests.

Proof. Let G_1 and G_2 be graphs. Necessity. By contrapositive, suppose that G_1 or G_2 is not a forest. Without loss of generality, we may assume that G_1 is not a forest. So G_1 contains a cycle. Since $G_1 \subseteq G_1 \Diamond G_2$, we have that $G_1 \Diamond G_2$ contains that cycle. Hence $G_1 \Diamond G_2$ is not a forest.

Sufficiency. By contrapositive, suppose $G_1 \Diamond G_2$ is not a forest. So $G_1 \Diamond G_2$ contains a cycle, say C. Then C is an original cycle or a created cycle. If C is an original cycle, then it is done. Suppose C is a created cycle. So by Theorem 2.1.6, both G_1 and G_2 are not acyclic. Hence G_1 and G_2 are not forests. $\square$

Next, we consider created cycles in any glued graph obtained by gluing two cycles at a path.

Corollary 2.1.9. *Let C be a created cycle in a glued graph $G_1 \underset{P}{\Diamond} G_2$ where G_1 and G_2 are cycles and P is a clone. Then C is an even cycle if and only if the lengths of G_1 and G_2 have the same parity.*

Proof. Let G_1 and G_2 be cycles and let C be a created cycle in $G_1 \underset{P}{\Diamond} G_2$ where P is a clone. So P is a path because all connected subgraphs of any cycle are paths. We have that $|E(C)| = |E(G_1)| + |E(G_2)| - 2\,|E(P)|$. If $|E(G_1)|$ and $|E(G_2)|$ have the same parity, Then $|E(C)|$ is even and also C is an even cycle. Otherwise, the lengths of G_1 and G_2 have the different parity, then $|E(C)|$ is odd and also C is an odd cycle. $\square$

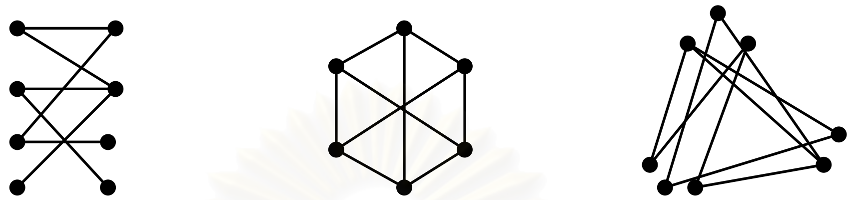
The rest of this section, we investigate results of the graph gluing between two bipartite graphs and k-partite graphs where k is a positive integer such that $k > 2$. First, we recall definitions and a property of bipartite graphs.

Definition 2.1.10. A graph G is **bipartite** if $V(G)$ is the union of two disjoint non-empty independent sets called **partite sets** of G.

A **bipartition** of G is a set of partite sets.

A **complete bipartite graph** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the partite sets have sizes r and s, the complete bipartite graph is denoted as $K_{r,s}$.

Definition 2.1.11. Let k be an interger such that $k \geq 3$. A graph G is k-**partite** if $V(G)$ can be expressed as the union of k disjoint non-empty independent sets called partite sets of G. A k-**partition** of G is a set of partite sets of G.



A bipartite graph A complete bipartite graph A 3-partite graph

Figure 2.1.4: Examples of bipartite graphs, complete bipartite graphs and 3-partite graphs

Theorem 2.1.12 ([3]). *A graph is bipartite if and only if it has no odd cycle.*

Theorem 2.1.12 helps us to characterize a result of the graph gluing of bipartite graphs showed in the next theorem.

Theorem 2.1.13. *Let B_1 and B_2 be graphs.*

A glued graph $B_1 \Diamond B_2$ is a bipartite graph if and only if B_1 and B_2 are bipartite.

Proof. Necessity. By contrapositive, suppose that B_1 or B_2 is not bipartite. Without loss generality, we may assume that B_1 is not bipartite. By Theorem 2.1.12, B_1 contains an odd cycle called C. Since $B_1 \subseteq B_1 \Diamond B_2$, we obtain that $B_1 \Diamond B_2$ contains C. Hence $B_1 \Diamond B_2$ is not a bipartite graph.

Sufficiency. Assume B_1 and B_2 are bipartite. Let $\{X_i, Y_i\}$ be a bipartition of B_i for all $i = 1, 2$. Consider arbitrary glued graph of B_1 and B_2 at a clone H, $B_1 \underset{H}{\Diamond} B_2$. Because H is a subgraph of bipartite graphs, so H is bipartite. Let $\{X_H, Y_H\}$ be a bipartition of H. Without loss of generality, we may assume that X_H is a subset of X_1 and X_2, and Y_H is a subset of Y_1 and Y_2. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. To show that $\{X, Y\}$ is a bipartition of $B_1 \underset{H}{\Diamond} B_2$, let u and v be vertices in $B_1 \underset{H}{\Diamond} B_2$ such that u is adjacent to v. So both u and v are in B_1 or B_2.

We may assume that u and v are in B_1. Because B_1 is a bipartite graph, so u and v are not in the same partite set in B_1. It means that $u \in X_1$ and $v \in Y_1$ or $u \in Y_1$ and $v \in X_1$. Assume that $u \in X_1$ and $v \in Y_1$. So $u \in X$ and $v \in Y$. Clearly, $X \cup Y = V(B_1 \underset{H}{\Diamond} B_2)$. Hence X and Y are partite sets of $B_1 \underset{H}{\Diamond} B_2$. Therefore $B_1 \underset{H}{\Diamond} B_2$ is a bipartite graph. $\square$

In the case of k-partite graphs where $k \geq 3$, it is not necessary that the graph gluing of two k-partite graphs is also k-partite.

Example 2.1.14. Let G_1 and G_2 be graphs as the following figure.

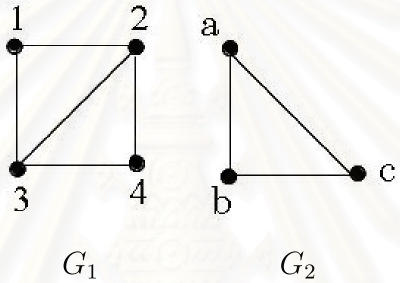


Figure 2.1.5: A glued graph between k-partite graphs which is not k-partite

Let $H_1 = P_3(1, 3, 4) \subseteq G_1$ and $H_2 = P_3(a, b, c) \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(1) = a$, $f(3) = b$ and $f(4) = c$. Clearly, G_1 and G_2 are 3-partite while $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2 = K_4$ which is not 3-partite. $\square$

We next give a condition to obtain a glued graph between two k-partite graphs which is also k-partite.

Theorem 2.1.15. *For an integer $k \geq 3$, let G_1 and G_2 be k-partite graphs and let H be a clone of $G_1 \underset{H}{\Diamond} G_2$. If H is a k-partite graph, then $G_1 \underset{H}{\Diamond} G_2$ is also a k-partite graph.*

Proof. Let G_1 and G_2 be k-partite graphs and let $\{A_1, A_2 \ldots, A_k\}$ and $\{B_1, B_2 \ldots, B_k\}$ be partitions of G_1 and G_2, respectively. let H be a clone of $G_1 \underset{H}{\Diamond} G_2$. Assume that H is a k-partite graph. Let $\{Z_1, Z_2, \ldots, Z_k\}$ be a k-partition of H. Because H is a subgraph of G_1 and G_2, without loss of generality, Z_i is a subset of

A_i and B_i for all $i \in \{1, 2, \ldots, k\}$. Let $M_i = A_i \cup B_i$ for all $i \in \{1, 2, \ldots, k\}$. Clearly, $M_1 \cup M_2 \cup \ldots \cup M_k = V(G_1 \underset{H}{\Diamond} G_2)$. Next, let i be arbitrary and let $u, v \in M_i = A_i \cup B_i$.

Case 1. $u \in V(G_1 \backslash H)$ and $v \in V(G_2 \backslash H)$: Then it is clear that u and v are not adjacent.

Case 2. $u, v \in V(G_1)$: Then $u, v \in A_i$. Because A_i is an independent set of $V(G_1)$, so u and v are not adjacent.

Case 3. $u, v \in V(G_2)$: Similarly to case 2, so u and v are not adjacent.

Hence $\{M_1, M_2, \ldots, M_k\}$ is a k-partition of $G_1 \underset{H}{\Diamond} G_2$ and also $G_1 \underset{H}{\Diamond} G_2$ is a k-partite graph. $\square$

Example 2.1.16. To show that the converse of Theorem 2.1.15 does not hold, let G_1 and G_2 be two copies of triangles K_3. So G_1 and G_2 are 3-partite. Let $H_1 = P_2(u_1, v_1)$ and $H_2 = P_2(u_2, v_2)$ where $u_i, v_i \in V(G_i)$ for all $i = 1, 2$. We glue G_1 and G_2 at H_1 and H_2. We can see that a clone of glued graph of G_1 and G_2 is not a 3-partite graph while $G_1 \underset{H_1 \cong H_2}{\Diamond} G_2$ is isomorphic to $K_4 \backslash \{e\}$ which is 3-partite. $\square$

2.2 The Graph Gluing of Chordal Graphs and Interval Graphs

Unlike the previous section, glued graphs in this section are not necessary to be the same type as their original graphs. So we investigate conditions to obtain the property that glued graphs are the same type as their original graphs.

Definition 2.2.1. A **chord** of a cycle C is an edge not in C whose endpoints lie in C. A **chordless cycle** in G is a cycle of length at least 4 in G that has no chord (that is, the cycle is an induced subgraph). A graph G is **chordal** if it is simple and has no chordless cycle.

Example 2.2.2. Trees are chordal, because trees are acyclic. For all n, K_n is chordal. $\square$

Remark 2.2.3. For all induced subgraphs of any chordal graph are chordal.

Definition 2.2.4. The **join** of simple graphs G_1 and G_2, written $G_1 \vee G_2$, is the graph obtained from the disjoint union between G_1 and G_2 by adding all edges in $\{xy : x \in V(G_1), y \in V(G_2)\}$.

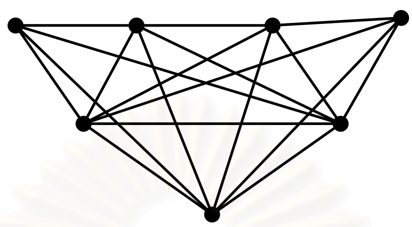


Figure 2.2.1: $K_4 \vee K_3$

If both original graphs of a glued graph are chordal, it is not necessary that their glued graph is chordal. We show this in Example 2.2.5.

Example 2.2.5. Let G_1 and G_2 be graphs as shown in Figure 2.2.2.

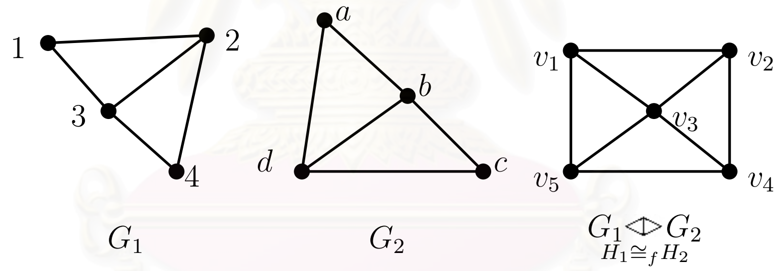


Figure 2.2.2: A glued graph between chordal graphs which is not chordal

Let $H_1 = P_3(1,3,4) \subseteq G_1$ and $H_2 = P_3(a,b,c) \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(1) = a$, $f(3) = b$ and $f(4) = c$. So we get $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2 \cong C_4 \vee K_1$. Because $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ contains $C_4(v_1, v_2, v_4, v_5)$ as an induced subgraph, so $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ is not chordal. $\square$

We observe that if all cycles in a glued graph of two chordal graphs are original cycles, then the glued graph is chordal. Then we use Theorem 2.1.6 to get a condition to guarantee that a glued graph has no created cycles.

Theorem 2.2.6. *For any graphs G_1 and G_2, if G_1 is acyclic and G_2 is chordal, then the glued graph $G_1 \lhd\!\rhd G_2$ is chordal.*

Proof. Let G_1 and G_2 be graphs. Assume that G_1 is acyclic and G_2 is chordal. By Theorem 2.1.6, $G_1 \lhd\!\!\!\rhd G_2$ does not contain a created cycle. So all cycles in $G_1 \lhd\!\!\!\rhd G_2$ are in G_2. Thus they are not chordless. Hence $G_1 \lhd\!\!\!\rhd G_2$ is chordal. $\square$

In [2], Chartrand and Lesniak give a characterization of chordal graphs. We restate and prove it in terms of glued graphs in Theorem 2.2.7.

Theorem 2.2.7 ([2]). *A graph G is a chordal graph if and only if G is a glued graph of two chordal graphs at a clone which is a complete graph(can be a vertex).*

Proof. Necessity. Let G be a chordal graph. If G is a complete graph, then $G = G \underset{G \cong_I G}{\lhd\!\!\!\rhd} G$ where I is the identity isomorphism. Assume that G is a non-complete chordal graph. Let S be any minimum vertex-cut of G. Let A be the vertex set of one component of $G \backslash S$ and let $B = V(G) \backslash (S \cup A)$. Define the subgraphs G_1 and G_2 of G by $G_1 = G[A \cup S]$ and $G_2 = G[B \cup S]$. We can see that both G_1 and G_2 are induced subgraphs of G. Since G is chordal, both G_1 and G_2 are chordal graphs. We can see that $G = G_1 \underset{G[S] \cong_I G[S]}{\lhd\!\!\!\rhd} G_2$. It remains to show that $G[S]$ is a complete graph. If $|S| = 1$, then $G[S]$ is a complete graph. So we may assume that $|S| \geq 2$. Since S is minimum, each $x \in S$ is adjacent to some vertex of each component of $G \backslash S$. Therefore, for each pair $x, y \in S$, there exist paths $x, a_1, a_2, \ldots, a_r, y$ and $x, b_1, b_2, \ldots, b_t, y$ where each $a_i \in A$ and $b_i \in B$, such that these paths are chosen to be of minimum length. Thus, $C : x, a_1, a_2, \ldots, a_r, y, b_t, b_{t-1}, \ldots, b_1, x$ is a cycle of length at least 4, implying that C has a chord. However, $a_i b_j \notin E(G)$, since S is a vertex-cut and $a_i a_j \notin E(G)$ and $b_i b_j \notin E(G)$ by the minimality of r and t. Thus $xy \in E(G)$. Therefore $G[S]$ is a complete graph.

Sufficiency. Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\lhd\!\!\!\rhd} G_2$ be a glued graph between G_1 and G_2 at a clone H. Assume that G_1 and G_2 are chordal and H is a clique. Let C be a cycle of length at least 4 in $G_1 \underset{H}{\lhd\!\!\!\rhd} G_2$. If C is an original cycle, it is done. Suppose that C is a created cycle. By Remark 2.1.5, There exists a non-trivial path which is a subgraph of $C \cap (G_1 \backslash H)$. We choose u, v-path P such that $u \neq v$ and $|E(P)|$ is the maximum. This implies that u and v are in H. Since H is a clique, there is an edge e incident to u and v in H. If e is in C, then $E(C) = E(P) \cup \{e\}$

and also C is an original cycle, a contradiction. So $e \notin E(C)$. Hence e is a chord of C. Therefore $G_1 \underset{H}{\lhd\!\rhd} G_2$ is a chordal graph. $\square$

Theorem 2.2.7 does not mean that if a glued graph is chordal, then its original graphs are chordal. We illustrate this in the next example.

Example 2.2.8. Let G_1 and G_2 be graphs as shown in Figure 2.2.3.

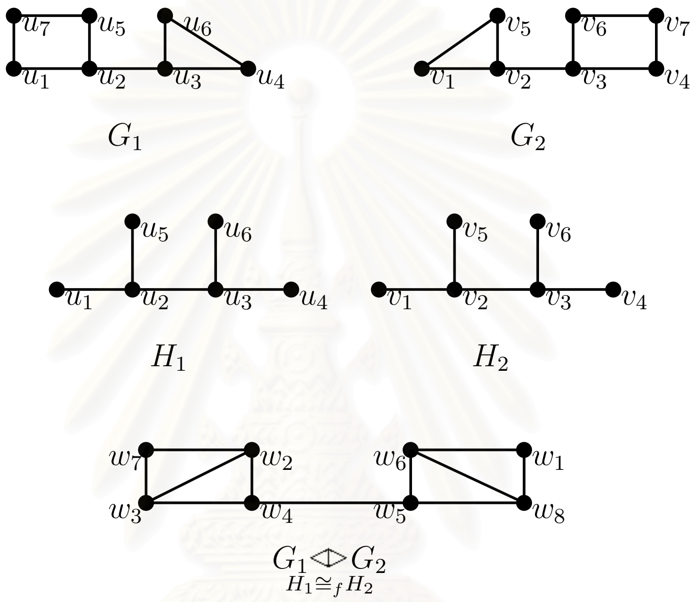


Figure 2.2.3: A glued graph between non-chordal graphs which is chordal

We observe that both G_1 and G_2 are not chordal. Let $f : H_1 \to H_2$ be the isomorphism defined by $f(u_i) = v_i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$. Then the graph $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ is showed in Figure 2.2.3. Clearly, $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ is chordal. However, since $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ is chordal, by Theorem 2.2.7, we can find chordal graphs G_3 and G_4, subgraphs H_3 and H_4 of G_3 and G_4, respectively, which are cliques, and an isomorphism $g : H_3 \to H_4$ such that $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2 = G_3 \underset{H_3 \cong_g H_4}{\lhd\!\rhd} G_4$.

For example, $G_3 = (G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2)[\{w_2, w_3, w_4, w_5, w_7\}]$ and $G_4 = (G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2)[\{w_1, w_4, w_5, w_6, w_8\}]$, $H_3 = P_2(w_4, w_5) = H_4$ with the identity isomorphism g between H_3 and H_4. $\square$

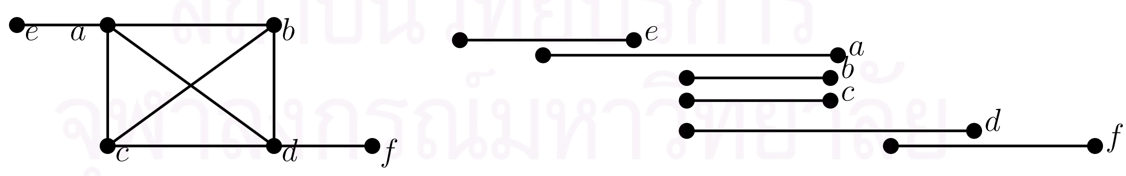
We can see in the previous example that the graph gluing of non-chordal graphs can be chordal. The next lemma gives a condition to make sure that a result of the graph gluing of non-chordal graphs is not chordal.

Lemma 2.2.9. *Let G_1 and G_2 be graphs and H be a clone of $G_1 \underset{H}{\Diamond} G_2$. If H is an induced subgraph of both G_1 and G_2 and $G_1 \underset{H}{\Diamond} G_2$ is chordal, then G_1 and G_2 are chordal.*

Proof. Let G_1 and G_2 be graphs. Then $G_1 \underset{H}{\Diamond} G_2$ is a glued graph between G_1 and G_2 at a clone H. Assume that H is an induced subgraph of both G_1 and G_2 and $G_1 \underset{H}{\Diamond} G_2$ is chordal. Suppose for a contradiction that G_1 is not chordal. So G_1 contains a chordless cycle C of length at least four . Then C is a cycle in $G_1 \underset{H}{\Diamond} G_2$. Because $G_1 \underset{H}{\Diamond} G_2$ is chordal, so C has a chord e which have endpoints u and v. So $u, v \in V(G_1)$. Because C is a chordless cycle in G_1, so $e \in E(G_2) \backslash E(G_1)$ and $u, v \in V(G_2)$. Thus $u, v \in V(H)$ but $e \notin E(H)$. Hence H is not an induced subgraph, a contradiction. Therefore G_1 and G_2 are chordal graphs. $\square$

The converse of Lemma 2.2.9 is not true illustrated by graphs G_1 and G_2 in Example 2.2.5. In the rest of this section, we investigate results of the graph gluing between two interval graphs.

Definition 2.2.10. An **interval representation** of a graph is a family of intervals assigned to the vertices so that vertices are adjacent if and only if the corresponding intervals intersect. A graph having such a representation is an **interval graph**.



G The interval representation of G

Figure 2.2.4: An interval graph

Remark 2.2.11. An induced subgraph of an interval graph is an interval graph.

Lemma 2.2.12 and Theorem 2.2.13 are well-known results about the relation between interval graphs and chordal graphs.

Lemma 2.2.12. [Folklore] *For any integer n such that $n \geq 4$, C_n is not an interval graph.*

Proof. Let n be an integer such that $n \geq 4$. Suppose that C_n is an interval graph. Let $P = C_n \backslash \{v\}$ where v is a vertex in C_n. So P is an induced subgraph of C_n. Hence P is also an interval graph. Because P is a path, so P has an interval representation similarly as Figure 2.2.5 where a and b are endpoints of P.

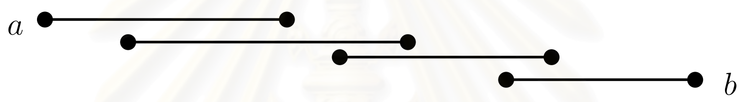


Figure 2.2.5: The interval representation of a path

To add vertex v, v have to intersect a and b but not intersect the other vertices. It is impossible. Hence C_n is not an interval graph. □

Theorem 2.2.13. [Folklore] *Let G be a graph. If G is an interval graph, then G is a chordal graph.*

Proof. Let G be a graph. Suppose that G is not chordal. So G contains a chordless cycle of length at least four, say C. By Lemma 2.2.12, C is not an interval graph. Because C is an induced subgraph of G, so G is not an interval graph. □

Next, we introduce a definition and some theorems about interval graphs.

Definition 2.2.14. Three vertices u, v, w form an **asteroidal-triple** if for each pair of them there is a path connecting that two vertices but not contain a neighborhood of the third vertex. For a graph G, we denote $\mathcal{A}(G)$ for the set of all asteroidal-triples in G.

Remark 2.2.15. Let u, v, w be vertices of a graph G. If u, v, w form an asteroidal-triple in G , then any pair of $\{u, v, w\}$ are not adjacent.

Example 2.2.16. Let G_1^* and G_2^* be graphs as shown in Figure 2.2.6.

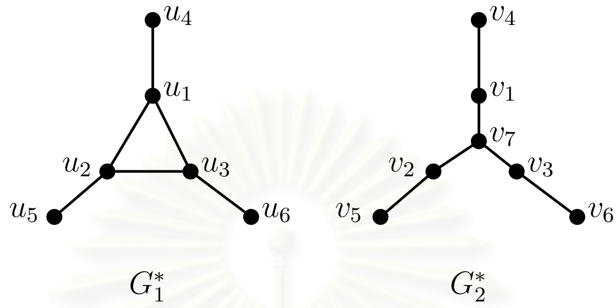


Figure 2.2.6: Examples of graphs containing an asteroidal-triple

We can see that u_4, u_5, u_6 is the only asteroidal-triple in G_1^* and v_4, v_5, v_6 is the only asteroidal-triple in G_2^*. $\square$

Theorem 2.2.17 ([4]). *A graph G is an interval graph if and only if it is chordal and has no asteroidal-triple.*

In Example 2.2.16, G_1 and G_2 are not interval graphs but they are chordal.

A glued graph between two interval graphs may or may not be an interval graph. We show this in Example 2.2.18 and Example 2.2.19.

Example 2.2.18. Let G_1 and G_2 be complete graphs and $G_1 \lhd\!\rhd G_2$ be arbitrary glued graph between G_1 and G_2 with at least 3 vertices. We will show that $G_1 \lhd\!\rhd G_2$ is an interval graph. Clearly, $G_1 \lhd\!\rhd G_2$ is chordal. It remains to prove that $G_1 \lhd\!\rhd G_2$ has no asteroidal-triples. Let u, v and w be distinct vertices in $G_1 \lhd\!\rhd G_2$. By the pigeonhole principle, there are at least two vertices of $\{u, v, w\}$ such that are in the same graph. Without loss of generality, we may assume that u and v are in G_1. Since G_1 is a complete graph, vertex u is adjacent to v. By Remark 2.2.15, u, v, w does not form an asteroidal-triple. Therefore $G_1 \lhd\!\rhd G_2$ is an interval graph. $\square$

Example 2.2.19. Let G_1 and G_2 be graphs as shown in Figure 2.2.7. We can see interval representations of G_1 and G_2 showed in Figure 2.2.7. So G_1 and G_2 are interval graphs.

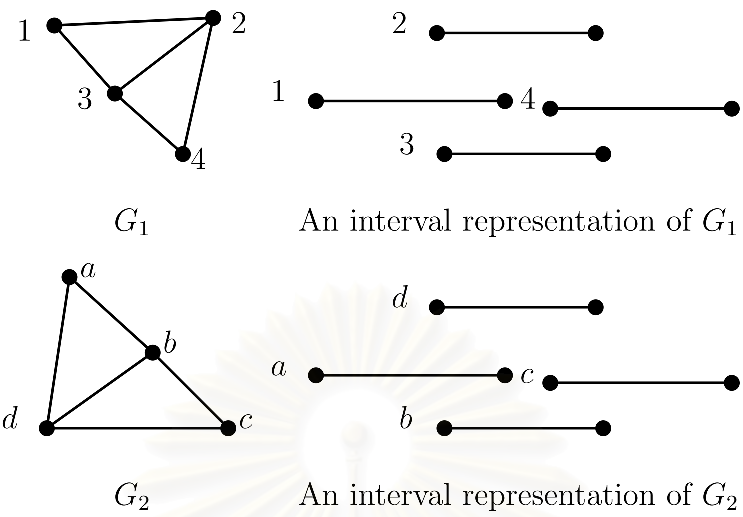


Figure 2.2.7: A glued graph of interval graphs which is not an interval graph

As in Example 2.2.5, $G_1 \underset{H}{\diamondsuit} G_2$ is not chordal. Hence $G_1 \underset{H}{\diamondsuit} G_2$ is not an interval graph. $\square$

The graph gluing can create an asteroidal-triple or destroy an asteroidal-triple in the original graphs. We show this in Example 2.2.20 and Example 2.2.21

Example 2.2.20. Let $G_1 = P_5(u_1, u_2, \ldots, u_5)$ and $G_2 = P_5(v_1, v_2, \ldots, v_5)$ and let $H_1 = P_3(u_1, u_2, u_3)$ and $H_2 = P_3(v_1, v_2, v_3)$. We next define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i = 1, 2, 3$. So $G_1 \underset{H_1 \cong_f H_2}{\diamondsuit} G_2$ is a graph isomorphic to G_2^* in Figure 2.2.6. Hence $G_1 \underset{H_1 \cong_f H_2}{\diamondsuit} G_2$ contains an asteroidal-triple. Thus $G_1 \underset{H_1 \cong_f H_2}{\diamondsuit} G_2$ is not an interval graph. $\square$

Example 2.2.21. Let G_1, G_2, H_1 and H_2 be graphs as shown in Figure 2.2.8. Note that $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$. We can see that u_3, u_{14} and u_{10} form an asteroidal-triple in G_1 and v_3, v_7 and v_{10} form an asteroidal-triple in G_2. Define isomorphism $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i = 1, 2, \ldots, 14$. Then we get $G_1 \underset{H_1 \equiv_f H_2}{\diamondsuit} G_2$ as shown in Figure 2.2.8.

We can see that $G_1 \underset{H_1 \equiv_f H_2}{\diamondsuit} G_2$ does not contain any asteroidal-triple. $\square$

Next, we give a condition to show that all asteroidal-triples in original graphs are still asteroidal-triples in their glued graph.

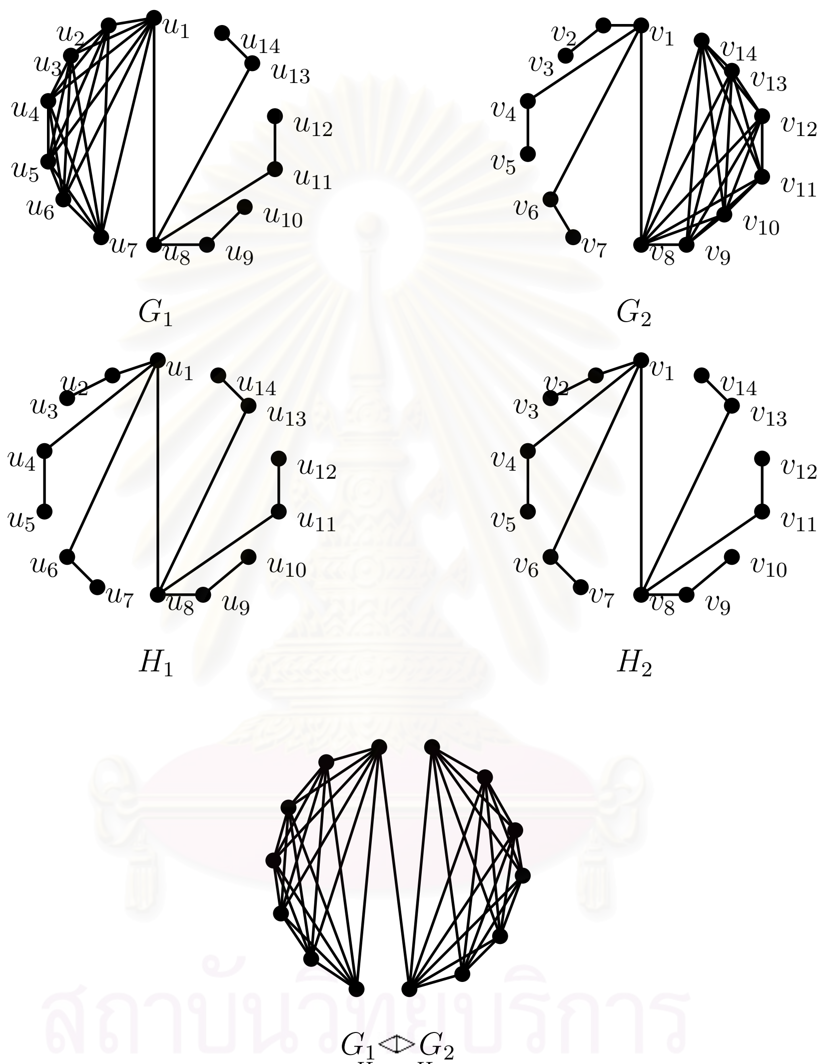


Figure 2.2.8: The graph gluing can destroy an asteroidal-triple.

Lemma 2.2.22. *Let G_1 and G_2 be graphs and H be a clone of $G_1\underset{H}{\Diamond}G_2$. If H is an induced subgraph of G_2, then $\mathcal{A}(G_1) \subseteq \mathcal{A}(G_1\underset{H}{\Diamond}G_2)$.*

Proof. Let G_1 and G_2 be graphs and H be a clone of $G_1\underset{H}{\Diamond}G_2$. Assume that $\mathcal{A}(G_1)\backslash\mathcal{A}(G_1\underset{H}{\Diamond}G_2) \neq \phi$. Let T be an asteroidal triple formed by vertices u, v, w in $\mathcal{A}(G_1)\backslash\mathcal{A}(G_1\underset{H}{\Diamond}G_2)$. Because $T \notin \mathcal{A}(G_1\underset{H}{\Diamond}G_2)$, there are two vertices in $\{u, v, w\}$ such that any path connecting that two vertices in $G_1\underset{H}{\Diamond}G_2$ contains a neighborhood of the third vertex. Without loss of generality, we may assume that such two vertices are u and v. Since $T \in \mathcal{A}(G_1)$, there is a u,v-path $P = P_n(a_1 = u, a_2, \ldots, a_n = v)$ in G_1 that avoids the neighborhood of w. So P is a path in $G_1\underset{H}{\Diamond}G_2$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that a_i is adjacent to w by edge e in $G_1\underset{H}{\Diamond}G_2$. Hence $e \in E(G_2)\backslash E(G_1)$ and also $a_i, w \in V(G_2)$. Since $a_i, w \in V(G_1)$, we can conclude that $a_i, w \in V(H)$. Since $e \notin E(G_1)$, we have that $e \notin E(H)$. Hence H is not an induced subgraph of G_2. □

By applying Lemma 2.2.22, we have a condition to make sure that a result of the graph gluing between non-interval graphs is not an interval graph.

Theorem 2.2.23. *Let G_1 and G_2 be graphs and H be a clone of $G_1\underset{H}{\Diamond}G_2$. If H is an induced subgraph of both G_1 and G_2 and $G_1\underset{H}{\Diamond}G_2$ is an interval graph, then G_1 and G_2 are interval graphs.*

Proof. Let G_1 and G_2 be graphs and H be a clone of $G_1\underset{H}{\Diamond}G_2$. Assume that H is an induced subgraph of both G_1 and G_2 and $G_1\underset{H}{\Diamond}G_2$ is an interval graph. Suppose for a contradiction that G_1 is not an interval graph. By Theorem 2.2.17, G_1 is not chordal or $\mathcal{A}(G_1) \neq \phi$.

Case 1. G_1 is not chordal; By Lemma 2.2.9, we get that $G_1\underset{H}{\Diamond}G_2$ is not chordal. By Lemma 2.2.13, $G_1\underset{H}{\Diamond}G_2$ is not an interval graph.

Case 2. $\mathcal{A}(G_1) \neq \phi$; By Lemma 2.2.22, we have that $\phi \neq \mathcal{A}(G_1) \subseteq \mathcal{A}(G_1\underset{H}{\Diamond}G_2)$. So $G_1\underset{H}{\Diamond}G_2$ contains an asteroidal-triple.

By two cases, we have $G_1\underset{H}{\Diamond}G_2$ is not an interval graph, a contradiction. Hence G_1 and G_2 are interval graphs. □

Example 2.2.19 shows that the converse of Theorem 2.2.23 is not true.

Example 2.2.24. Let T_1 and T_2 be trees. Because a connected subgraph of any tree is an induced subgraph, by Theorem 2.2.23, we have that if T_1 or T_2 is not an interval graph, then $T_1 \lhd\!\!\rhd T_2$ is not an interval graph. $\square$

We have seen that the glued graphs of chordal graphs, interval graphs and k-partite graphs where $k \geq 3$ do not necessary remain the same type as their original graphs. We find conditions to obtain that the glued graphs of chordal graphs are chordal and find a condition to get that the glued graphs between k-partite graphs are also k-partite graphs where $k \geq 3$. It remains an open problem to find other conditions to obtain such property. In the next chapter we consider the colorability of glued graphs.

CHAPTER III

COLORABILITY OF GLUED GRAPHS

In this chapter, we find bounds of the chromatic numbers of glued graphs and show their sharpness. A graph gluing could sometime give a resulting graph with multiple edges. As we focus on graph colorings, we will consider multiple edges as a single edge of any glued graph in this chapter.

3.1 Background

First of all, we recall the definition of the chromatic number of any graph.

Definition 3.1.1. A **k-coloring** of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are **colors**; the vertices of one color form a **color class**. A k-coloring is **proper** if adjacent vertices have different labels. A graph is **k-colorable** if it has a proper k-coloring. The **chromatic number of graph** G, $\chi(G)$, is the least k such that G is k-colorable.

Example 3.1.2. Let G be a nontrivial bipartite graph with a bipartition $\{X, Y\}$. Since G is nontrivial, $\chi(G) \geq 2$. Define $\gamma : V(G) \to \{1, 2\}$ *by*

$$\gamma(v) = \begin{cases} 1 & \textit{if} \quad v \in X, \\ 2 & \textit{if} \quad v \in Y. \end{cases}$$

Since X and Y are independent sets, we have that γ is proper. So $\chi(G) \leq 2$. Hence $\chi(G) = 2$.

Conversely, Let G be a graph such that $\chi(G) = 2$. Let $\gamma : V(G) \to \{1, 2\}$ be a proper 2-coloring of G. Define sets X,$Y \subseteq V(G)$ by

$$X = \{v \in V(G) | \gamma(v) = 1\} \text{ and } \; Y = \{v \in V(G) | \gamma(v) = 2\}.$$

Then $X \cap Y = \phi$ and $X \cup Y = V(G)$. If u and v are in X, then $\gamma(u) = 1 = \gamma(v)$. Since γ is proper, u and v are not adjacent. If u and v are in Y, similarly u and v are not adjacent. So X and Y are independent sets. Hence G is a bipartite graph.

$\square$

Definition 3.1.3. The clique number of a graph G, written $\omega(G)$, is the maximum size of a set of pairwise adjacent vertices(clique) in G.

Remark 3.1.4. For any graph G, we have $\chi(G) \geq \omega(G)$, because vertices of a clique require distinct colors.

Next we state theorems about the chromatic number of any graph that we use to find bounds of the chromatic numbers of glued graphs. Proposition 3.1.5 reveals that the chromatic numbers of graphs are at most their maximum degree plus one and Brooks proved that there are only complete graphs and odd cycles whose chromatic numbers are exactly one more than their maximum degrees showed in Theorem 3.1.6.

Proposition 3.1.5 ([3]). *Let G be a graph.* $\chi(G) \leq \Delta(G) + 1$.

Theorem 3.1.6 ([3]). (Brooks[1941]) *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

3.2 Bounds of the Chromatic Numbers of Glued Graphs

In this section, we investigate bounds of the chromatic numbers of glued graphs and also show their sharpness. First, we give a trivial lower bound of the chromatic numbers of glued graphs.

Remark 3.2.1. Because G_1 and G_2 are subgraphs of $G_1 \triangleleft\!\triangleright G_2$, we have $\chi(G_1) \leq \chi(G_1 \triangleleft\!\triangleright G_2)$ and $\chi(G_2) \leq \chi(G_1 \triangleleft\!\triangleright G_2)$. Hence we get a lower bound of the chromatic number of $G_1 \triangleleft\!\triangleright G_2$ that

$$\chi(G_1 \triangleleft\!\triangleright G_2) \geq \max\{\chi(G_1), \chi(G_2)\}.$$

We apply Theorem 2.1.12 and Example 3.1.2 to prove the next proposition.

Proposition 3.2.2. *Let* G_1 *and* G_2 *be nontrivial graphs. Then* $\chi(G_1 \lhd\!\rhd G_2) \geq 3$ *if and only if* $\chi(G_1) \geq 3$ *or* $\chi(G_2) \geq 3$.

Proof. Let G_1 and G_2 be nontrivial graphs. By contrapositive, the statement in the proposition is equivalent to $\chi(G_1 \lhd\!\rhd G_2) \leq 2$ if and only if $\chi(G_1) \leq 2$ and $\chi(G_2) \leq 2$. Because the chromatic number of any nontrivial graph is at least two, we can prove this proposition by proving the statement $\chi(G_1) = 2 = \chi(G_2)$ if and only if $\chi(G_1 \lhd\!\rhd G_2) = 2$ instead.

Necessity. Assume that $\chi(G_1) = 2 = \chi(G_2)$. By example 3.1.2, G_1 and G_2 are bipartite. So $G_1 \lhd\!\rhd G_2$ is also bipartite by Theorem 2.1.12. Hence $\chi(G_1 \lhd\!\rhd G_2) = 2$.

Sufficiency. Assume that $\chi(G_1 \lhd\!\rhd G_2) = 2$. By example 3.1.2, $G_1 \lhd\!\rhd G_2$ is bipartite. So G_1 and G_2 are bipartite by Theorem 2.1.12. Hence $\chi(G_1) = 2 = \chi(G_2)$. $\square$

Applying Proposition 3.2.2, we get a necessary condition to have that the chromatic numbers of glued graphs are equal to three. This necessary condition is showed in Proposition 3.2.3.

Proposition 3.2.3. *Let* G_1 *and* G_2 *be nontrivial graphs. If* $\chi(G_1 \lhd\!\rhd G_2) = 3$, *then* *max*$\{\chi(G_1), \chi(G_2)\} = 3$.

Proof. Let G_1 and G_2 be nontrivial graphs. Assume that $\chi(G_1 \lhd\!\rhd G_2) = 3$. By Lemma 3.2.2, we have $\chi(G_1) \geq 3$ or $\chi(G_2) \geq 3$. Let $\max\{\chi(G_1), \chi(G_2)\} = \chi(G_1)$. Then $\chi(G_1) \geq 3$. Because $G_1 \subseteq G_1 \lhd\!\rhd G_2$, so $3 \leq \chi(G_1) \leq \chi(G_1 \lhd\!\rhd G_2) = 3$. Hence $\max\{\chi(G_1), \chi(G_2)\} = \chi(G_1) = 3$. $\square$

The converse of the proposition 3.2.3 is not true. We show this in Example 2.1.14, which contains $\chi(G_1) = 3 = \chi(G_2)$ but $\chi(\underset{H_1 \cong_f H_2}{G_1 \lhd\!\rhd G_2}) = 4$.

Remark 3.2.4. By Proposition 3.2.2 and Proposition 3.2.3, we get that if $\chi(G_1 \lhd\!\rhd G_2) \leq 3$, than $\chi(G_1 \lhd\!\rhd G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

Because $\Delta(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1)+\Delta(G_2)-\delta(H)$(in Theorem 1.2.8) where G_1 and G_2 are graphs and H is the clone of a glued graph $G_1 \underset{H}{\lhd\!\rhd} G_2$, so an upper bound in Theorem 3.2.5 follows immediately by using Proposition 3.1.5 and Theorem 3.1.6.

Theorem 3.2.5. *Let G_1 and G_2 be nontrivial connected graphs and let H be a clone of $G_1 \underset{H}{\lhd\!\rhd} G_2$. Then*

$$\chi(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1.$$

Furthermore, if $G_1 \underset{H}{\lhd\!\rhd} G_2$ is not a complete graph or an odd cycle, then

$$\chi(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H).$$

Proof. Let G_1 and G_2 be nontrivial connected graphs and let H be a clone of $G_1 \underset{H}{\lhd\!\rhd} G_2$. If $G_1 \underset{H}{\lhd\!\rhd} G_2$ is a complete graph or an odd cycle, by Proposition 3.1.5, $\chi(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1 \underset{H}{\lhd\!\rhd} G_2)+1 \leq \Delta(G_1)+\Delta(G_2)-\delta(H)+1$. Otherwise, by Brooks' theorem(Theorem 3.1.6), $\chi(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1)+\Delta(G_2)-\delta(H)$. $\square$

Example 3.2.6. To show the sharpness of theorem 3.2.5, let G_1 and G_2 be graphs as shown in Figure 3.2.1.

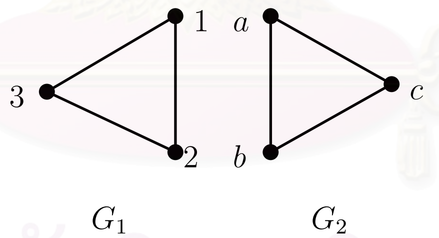


Figure 3.2.1: The sharpness of Theorem 3.2.5.

We glue G_1 and G_2 at $H_1 = P_2(1, 2) \subseteq G_1$ and $H_2 = P_2(a, b) \subseteq G_2$. So $G_1 \underset{H_1 \cong H_2}{\lhd\!\rhd} G_2$ is isomorphic to $K_4 \backslash \{e\}$ where $e \in E(K_4)$ which is not a complete graph or an odd cycle. Consider $\chi(G_1 \underset{H_1 \cong H_2}{\lhd\!\rhd} G_2) = 3 = 2 + 2 - 1 = \Delta(G_1) + \Delta(G_2) - \delta(H)$ where $H \cong H_1 \cong H_2$. $\square$

This upper bound is too large for some graphs as shown in example 3.2.7.

Example 3.2.7. Let n be a positive integer. Define graphs G_1 and G_2 be two copies of $K_{n,1}$. So $\Delta(G_1) = n = \Delta(G_2)$. Let $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$ be such that u_1 and v_1 are vertices with maximum degree of G_1 and G_2, respectively. We glue G_1 and G_2 at $H_1 = P_2(u_1, u_2)$ and $H_2 = P_2(v_1, v_2)$ with isomorphism f defined by $f(u_1) = v_1$ and $f(u_2) = v_2$. So by Theorem 3.2.5, $\chi(G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2) \leq n + n - 1 = 2n - 1$. We know that G_1 and G_2 are trees. So $G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2$ is also a tree and $\chi(G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2) = 2$. If $n \to \infty$, this bound is too large. □

Theorem 3.2.5 shows an upper bound of the chromatic numbers of glued graphs in terms of the maximum degrees of its original graphs. In the next theorem, we introduce another upper bound of the chromatic numbers of glued graphs which is in terms of the chromatic numbers of its original graphs.

Theorem 3.2.8. *Let G_1 and G_2 be graphs. Then*

$$\chi(G_1 \diamond G_2) \leq \chi(G_1)\chi(G_2).$$

Proof. Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\diamond} G_2$ be a glued graph of G_1 and G_2 at an arbitrary clone H. Assume $\chi(G_1) = p$ and $\chi(G_2) = q$. Let $\gamma_1 : V(G_1) \to \{1, 2, \ldots, p\}$ and $\gamma_2 : V(G_2) \to \{1, 2, \ldots, q\}$ be proper colorings of G_1 and G_2, respectively. Define $\beta : V(G_1) \cup V(G_2) \to \{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ by for all $v_i \in V(G_1) \cup V(G_2)$,

$$\beta(v_i) = \begin{cases} (\gamma_1(v_i), 1) & \text{if} \quad v_i \in V(G_1 \backslash H), \\ (\gamma_1(v_i), \gamma_2(v_i)) & \text{if} \quad v_i \in V(H), \\ (1, \gamma_2(v_i)) & \text{if} \quad v_i \in V(G_2 \backslash H). \end{cases}$$

To show that β is proper, let v_i and v_j be vertices in $G_1 \underset{H}{\diamond} G_2$ such that $\beta(v_i) = \beta(v_j)$. We will show that v_i and v_j are not adjacent.

Case 1. $v_i \in V(G_1 \backslash H)$ and $v_j \in V(G_2 \backslash H)$: Then clearly, v_i and v_j are not adjacent in $G_1 \underset{H}{\diamond} G_2$.

Case 2. Both v_i and v_j are in $V(G_1 \backslash H)$(or $V(G_2 \backslash H)$): So $\beta(v_i) = (\gamma_1(v_i), 1) = (\gamma_1(v_j), 1) = \beta(v_j)$ and then $\gamma_1(v_i) = \gamma_1(v_j)$. Hence v_i and v_j are not adjacent in $G_1 \underset{H}{\diamond} G_2$ because γ_1 is proper.

Case 3 v_i **is in** $V(H)$ **but** v_j **is in** $V(G_1\backslash H)$**(or** $V(G_2\backslash H)$**):** So $\beta(v_i) = (\gamma_1(v_i), \gamma_2(v_i)) = (\gamma_1(v_j), 1) = \beta(v_j)$(or $\beta(v_i) = (\gamma_1(v_i), \gamma_2(v_i)) = (1, \gamma_2(v_j)) = \beta(v_j)$). Then $\gamma_1(v_i) = \gamma_1(v_j)$(or $\gamma_2(v_i) = \gamma_2(v_j)$). Because both γ_1 and γ_2 are proper, So v_i and v_j are not adjacent in $G_1 \underset{H}{\oplus} G_2$.

Therefore β is proper and hence $\chi(G_1 \underset{H}{\oplus} G_2) \leq \chi(G_1)\chi(G_2)$. $\square$

We show the sharpness of Theorem 3.2.8 by proving the next theorem.

Theorem 3.2.9. *Let p and q be integers such that $p, q \geq 2$ but $pq \neq 4$. Then there exist G_1 and G_2 with a glue graph $G_1 \underset{H}{\oplus} G_2$ where H is a clone such that $\chi(G_1) = p$, $\chi(G_2) = q$ and $\chi(G_1 \underset{H}{\oplus} G_2) = pq = \chi(G_1)\chi(G_2)$.*

Proof. Let p and q be integers such that $p, q \geq 2$ but $pq \neq 4$.

Case 1. $p = q$**:** Let G_1 be a graph such that $V(G_1) = \{u_1, u_2, \ldots, u_{pq}\}$ and u_i and u_j are adjacent if and only if $i \not\equiv j \pmod p$. Any $i \in \{1, 2, 3, \ldots, pq\}$, there are q numbers which are equivalent to i mod p. So there are $pq - q$ vertices which are adjacent to v_i. Hence G_1 is $(pq-q)$-regular. Next, let $\gamma_1 : V(G_1) \to \{1, 2, 3, \ldots, p\}$ be a coloring of G_1 defined by for all $u_i \in V(G_1)$

$$\gamma_1(u_i) = l \text{ where } l \equiv i \pmod p \text{ and } l \in \{1, 2, \ldots, p\}.$$

To show γ_1 is proper, let u_i and u_j be vertices in G_1 such that u_i and u_j are adjacent. Then $i \not\equiv j \pmod p$. Assume $i \equiv l \pmod p$ and $j \equiv k \pmod p$ where $l, k \in \{1, 2, \ldots, p\}$. So $\gamma_1(u_i) = l \equiv i \not\equiv j \equiv k = \gamma_1(u_j)$. Hence γ_1 is proper and also $\chi(G_1) \leq p$. We can see that the set of vertex $\{u_1, u_2, \ldots, u_p\}$ forms a p-clique. So $\chi(G_1) \geq p$. Hence $\chi(G_1) = p$.

We next define graph G_2 by $V(G_2) = \{v_1, v_2, \ldots, v_{pq}\}$ and v_i and v_j are adjacent if and only if $i = j + 1$ or $i \equiv j \pmod p$. Any $i \in \{1, 2, 3, \ldots, pq\}$, there are $q - 1$ numbers in $\{1, 2, 3, \ldots, i-1, i+1, \ldots, pq\}$ which are equivalent to i mod p. So there are at least $q-1$ vertices which are adjacent to v_i. Since vertices v_{i-1} and v_{i+1} are adjacent to v_i, we obtain that $deg(v_i) = q - 1 + 2 = q + 1$. Hence $\Delta(G_2) = (q + 1)$. Let $\gamma_2 : V(G_2) \to \{1, 2, 3, \ldots, q\}$ be a coloring of G_2. For each $i \in \{1, 2, \ldots, pq\}$, we write $i = ap + b$ where $a, b \in \mathbb{Z}$, $a \geq 0$ and $0 < b \leq p$, defined $\gamma_2(v_i)$ by

$$\gamma_2(v_i) = l \text{ where } l \equiv a + b \pmod q \text{ and } l \in \{1, 2, \ldots, q\}.$$

To show that γ_2 is proper, let v_i, $v_j \in V(G_2)$ be such that v_i and v_j are adjacent. Then $i = j + 1$ or $i \equiv j \pmod{p}$. Let $j = ap + b$ where $a, b \in \mathbb{Z}, a \geq 0$ and $0 < b \leq p$. So $\gamma_2(v_j) \equiv a + b \pmod{q}$.

Case 1.1. $i = j + 1 = ap + b + 1$**:** If $b < p$, then $b + 1 \leq p$, consequently, $\gamma_2(v_i) \equiv a + b + 1 \pmod{q} \not\equiv a + b \pmod{q} \equiv \gamma_2(v_j)$. Assume that $b = p$, then $i = ap + p + 1 = p(a + 1) + 1$ and $\gamma_2(v_i) \equiv a + 2 \pmod{q}$. Because $pq \neq 4$, so $p \neq 2$. Hence $\gamma_2(v_i) \equiv a + 2 \pmod{q} \not\equiv a + p \pmod{q} \equiv \gamma_2(v_j)$.

Case 1.2. $i \equiv j \pmod{p}$**:** Without loss of generality, we assume $i > j$. Then $i = j + np = ap + b + np = p(a + n) + b$ where $n \in \mathbb{N}$. Since $1 \leq i \leq pq$, we get that $1 \leq n \leq q - 1$. Hence $\gamma_2(v_i) \equiv a + n + b \pmod{q} \not\equiv a + b \pmod{q} \equiv \gamma_2(v_j)$.

Therefore, by both cases, we have $\gamma_2(v_i) \neq \gamma_2(v_j)$. So γ_2 is proper and hence $\chi(G_2) \leq q$. Since the set of vertex $\{v_1, v_{1+p}, v_{1+2p}, \ldots, v_{1+(q-1)p}\}$ forms a q-clique, we have $\chi(G_2) \geq q$. Hence $\chi(G_2) = q$.

Now consider $H_1 = P_{pq}(u_1, u_2, \ldots, u_{pq}) \subseteq G_1$ and $H_2 = P_{pq}(v_1, v_2, \ldots, v_{pq}) \subseteq G_2$. We next define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i \in \{1, 2, 3, \ldots, pq\}$. Then we obtain the glued graph of G_1 and G_2 at H_1 and H_2 with respect to f, written as $\underset{H_1 \cong_f H_2}{G_1 \diamondsuit G_2}$. Let $G = \underset{H_1 \cong_f H_2}{G_1 \diamondsuit G_2}$ and $V(G) = \{w_i : i = 1, 2, \ldots, pq$ where w_i corresponds to u_i and $v_i\}$. Let w_i, $w_j \in V(G)$. If $i \equiv j \pmod{p}$, then w_i and w_j are adjacent in G_2. Otherwise, w_i and w_j are adjacent in G_1. It follows that w_i and w_j are adjacent in G because $G_1, G_2 \subseteq G$. Therefore $G = K_{pq}$ and also $\chi(G) = \chi(\underset{H_1 \cong_f H_2}{G_1 \diamondsuit G_2}) = pq = \chi(G_1)\chi(G_2)$.

Case 2. $p < q$**:** Define G_1, G_2 and γ_1 similarly as case 1. Then $\chi(G_1) = p$. We next define $\gamma_2 : V(G_2) \to \{1, 2, \ldots, q\}$ as follows: For each by for $i \in \{1, 2, \ldots, pq\}$, we write $i = ap + b$ where $a, b \in \mathbb{Z}$, $a \geq 0$ and $0 < b \leq p$,

$$\gamma_2(v_i) = l \text{ where } l \equiv 2 + a - b \pmod{q} \text{ and } l \in \{1, 2, \ldots, q\}.$$

To show that γ_2 is proper, let v_i, $v_j \in V(G_2)$ be such that v_i and v_j are adjacent. Then $i = j + 1$ or $i \equiv j \pmod{p}$. Let $j = ap + b$ where $a, b \in \mathbb{Z}, a \geq 0$ and $0 < b \leq p$. So $\gamma_2(v_j) \equiv 2 + a - b \pmod{q}$.

Case 2.1. $i = j + 1 = ap + b + 1$**:** If $b < p$, then $b + 1 \leq p$ and also $\gamma_2(v_i) \equiv 2 + a - b - 1 \pmod{q} \not\equiv 2 + a - b \pmod{q} \equiv \gamma_2(v_j)$. If $b = p$, $i =$

$ap + p + 1 = p(a+1) + 1$ and $\gamma_2(v_i) \equiv 2 + a \pmod{q}$. Because $p < q$, so $\gamma_2(v_i) \equiv 2 + a \pmod{q} \not\equiv 2 + a - p \pmod{q} \equiv \gamma_2(v_j)$.

Case 2.2. $i \equiv j \pmod{p}$**:** We assume $i > j$. So $i = j + np = ap + b + np = p(a+n) + b$ where $n \in \mathbb{N}$. Since $1 \leq i \leq pq$, we get that $1 \leq n \leq q - 1$. So $\gamma_2(v_i) \equiv 2 + a + n - b \pmod{q} \not\equiv 2 + a - b \pmod{q} \equiv \gamma_2(v_j)$.

Hence, by both cases, γ_2 is proper and also $\chi(G_2) \leq q$. We can see that the set of vertex $\{v_1, v_{1+p}, v_{1+2p}, \ldots, v_{1+(q-1)p}\}$ forms a q-clique. So $\chi(G_2) \geq q$. Hence $\chi(G_2) = q$.

We define H_1, H_2 and f similarly to case 1. So $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2 = K_{pq}$ and hence $\chi(G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2) = pq$. $\square$

Example 3.2.10. An example of graphs constracted in the Theorem 3.2.9 is illustrated here. For $n = 9$ and $p = 3 = q$, we have that G_1 and G_2 are graphs in Figure 3.2.2

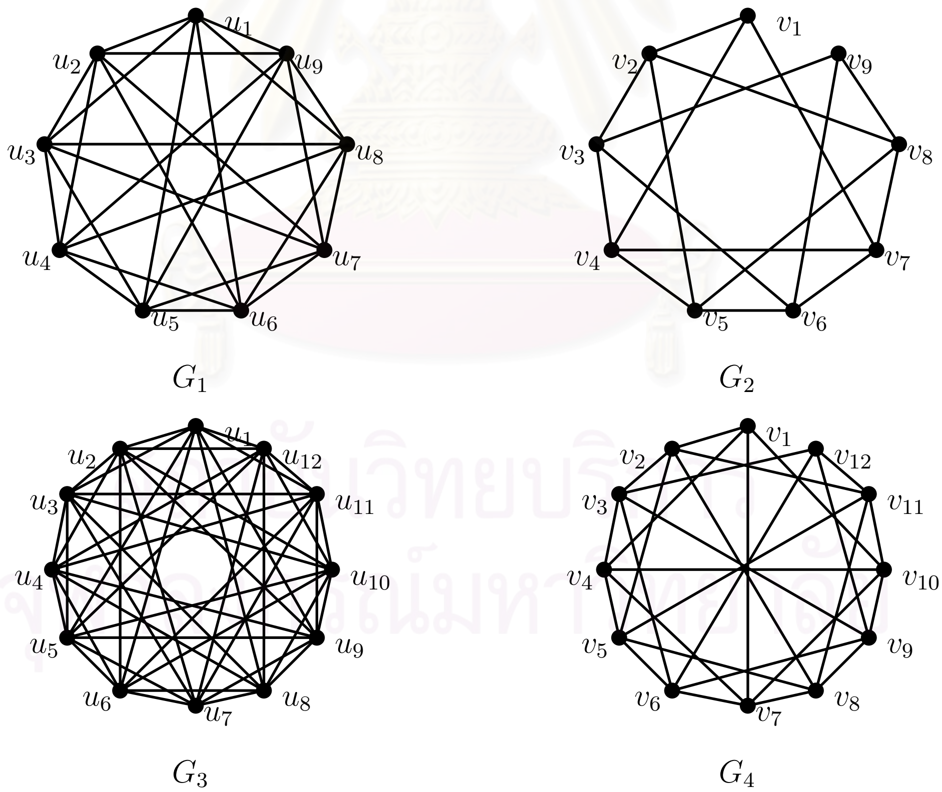


Figure 3.2.2: Graphs with their glued graphs isomorphic to K_9 and K_{12}.

We glue G_1 and G_2 at the clones $H_1 = P_9(u_1, u_2, \ldots, u_9)$ and $H_2 = P_9(v_1, v_2, \ldots, v_9)$ with the isomorphism $f : H_1 \to H_2$ defined by $f(u_i) = v_i$ for all $i = 1, 2, \ldots, 9$. Then $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2 = K_9$. So $\chi(G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2) = \chi(K_9) = 9 = 3 \times 3 = \chi(G_1)\chi(G_2)$.

For $p = 3$ and $q = 4$, we have that G_3 and G_4 are graphs in Figure 3.2.2. Let $H_3 = P_{12}(u_1, u_2, \ldots, u_{12})$ and $H_4 = P_{12}(v_1, v_2, \ldots, v_{12})$. Define $g : H_3 \to H_4$ by $g(u_i) = v_i$ for all $i = 1, 2, \ldots, 12$. Then $G_3 \underset{H_3 \cong_g H_4}{\Diamond} G_4 = K_{12}$. So $\chi(G_3 \underset{H_3 \cong_g H_4}{\Diamond} G_4) = \chi(K_{12}) = 12 = 3 \times 4 = \chi(G_3)\chi(G_4)$. $\square$

The chromatic numbers of graphs defined in Theorem 3.2.9 also satisfy the condition in Theorem 3.2.5 namely $\chi(G_1 \underset{H}{\Diamond} G_2) = \chi(K_{pq}) = pq = pq - q + q + 1 - 2 + 1 = \Delta(G_1) + \Delta(G_2) - \delta(H) + 1$ where $H \cong H_1 \cong H_2$ is a clone of $G_1 \underset{H}{\Diamond} G_2$.

Graphs G_1 and G_2 in Theorem 3.2.9 such that $G_1 \underset{H}{\Diamond} G_2 = K_n$ satisfy $n = \chi(G_1)\chi(G_2)$. Does there exist graphs G_1 and G_2 such that $G_1 \underset{H}{\Diamond} G_2 = K_n$ but $n \neq \chi(G_1)\chi(G_2)$? Lemma 3.2.11 and Lemma 3.2.12 answer this question.

Lemma 3.2.11. *Let G_1 and G_2 be graphs such that $\chi(G_1) = p$ and $\chi(G_2) = q$. If $G_1 \underset{H}{\Diamond} G_2 = K_n$ at some clone H, then $pq \geq n$.*

Proof. Let G_1 and G_2 be graphs such that $\chi(G_1) = p$ and $\chi(G_2) = q$. Assume that $G_1 \underset{H}{\Diamond} G_2 = K_n$ at a clone H. If there are $u \in V(G_1 \backslash H)$ and $v \in V(G_2 \backslash H)$, then u and v are not adjacent in $G_1 \underset{H}{\Diamond} G_2 = K_n$, a contradiction. So $V(G_1) = V(H)$ or $V(G_2) = V(H)$. Without loss of generality, we may assume that $V(G_1) = V(H)$. So $V(G_1 \underset{H}{\Diamond} G_2) = V(G_2)$ and also $n = \left|V(G_1 \underset{H}{\Diamond} G_2)\right| = |V(G_2)|$. Let $\left\lceil \frac{n}{q} \right\rceil = d$. Since $\chi(G_2) = q$, by the pigeonhole principle, there exist at least $\left\lceil \frac{n}{q} \right\rceil = d$ vertices in G_2 which are labeled as the same color, say $v_1, v_2, \ldots v_d$. So edge $v_i v_j \notin E(G_2)$ for all $i, j = 1, 2, \ldots, d$. Then $v_i v_j \in E(G_1)$ for all $i, j = 1, 2, \ldots, d$. So G_1 contains K_d and also $p = \chi(G_1) \geq d = \left\lceil \frac{n}{q} \right\rceil \geq \frac{n}{q}$. Hence $pq \geq n$. $\square$

Lemma 3.2.12. *Let $n \in \mathbb{N}$ be such that $n \geq 5$. For any integer p and q such that $p, q \leq n$ and $pq \geq n$, there exist graphs G_1 and G_2 such that $\chi(G_1) = p$, $\chi(G_2) = q$ and $G_1 \underset{H}{\Diamond} G_2 = K_n$ at some clone H. Moreover, for any two graph A_1 and A_2 , if $A_1 \underset{B}{\Diamond} A_2 = K_n$ where B is a clone, then max$\{\chi(A_1), \chi(A_2)\} \geq \lceil \sqrt{n} \rceil$.*

Proof. Let $n \in \mathbb{N}$ be such that $n \geq 5$ and let p and q be integers such that $p, q \leq n$ and $pq \geq n$. If $pq = n$, by Theorem 3.2.9, we get G_1 and G_2 such that $G_1 \underset{H}{\diamond} G_2 = K_{pq} = K_n$ at some clone H, $\chi(G_1) = p$ and $\chi(G_2) = q$. It is done. Suppose that $pq > n$. Assume that $p \leq q$. By Theorem 3.2.9, we get G_1 and G_2 such that $G_1 \underset{H}{\diamond} G_2 = K_{pq}$ where H is a clone, $\chi(G_1) = p$ and $\chi(G_2) = q$ with proper colorings γ_1 and γ_2 of G_1 and G_2, respectively. Below are properties of G_1 and G_2 defined in Theorem 3.2.9.

$V(G_1) = \{u_1, u_2, \ldots, u_{pq}\}$, u_i is adjacent to u_j if and only if $i \not\equiv j \pmod p$, $\gamma_1(u_i) = l$ where $l \equiv i \pmod p$ and $l \in \{1, 2, \ldots, p\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{pq}\}$, v_i and v_j are adjacent if and only if $i = j + 1$ or $i \equiv j \pmod p$. If $p < q$, then for each $i = ap + b \in \{1, 2, \ldots, pq\}$ where $a, b \in \mathbb{Z}$, $a \geq 0$ and $0 < b \leq p$, define $\gamma_2(v_i) = l$ where $l \equiv 2 + a - b \pmod q$ and $l \in \{1, 2, \ldots, q\}$. If $p = q$, then for each $i = ap + b \in \{1, 2, \ldots, pq\}$ where $a, b \in \mathbb{Z}$, $a \geq 0$ and $0 < b \leq p$, define $\gamma_2(v_i) = l$ where $l \equiv a + b \pmod q$ and $l \in \{1, 2, \ldots, q\}$.

Then we constuct G_1^* and G_2^* as follows: Construct G_1^* by deleting vertices $u_{n+1}, u_{n+2}, \ldots, u_{pq}$ in G_1. Since G_1^* is an induced subgraph of G_1, we get $\chi(G_1^*) \leq p$. Because $p \leq n$, so G_1^* contains a p-clique which is $K_p(u_1, u_2, \ldots, u_p)$. So $\chi(G_1^*) \geq p$. Hence $\chi(G_1^*) = p$.

We next construct G_2^* by deleting vertices $v_{n+1}, v_{n+2}, \ldots, v_{pq}$ in G_2. Since G_2^* is an induced subgraph of G_2, we get $\chi(G_2^*) \leq q$. If $n \geq 1 + (q-1)p$, then $K_q(v_1, v_{1+p}, \ldots, v_{1+(q-1)p})$ is in G_2^* and also $\chi(G_2^*) \geq q$. Hence $\chi(G_2^*) = q$ and γ_2 is still a proper coloring of G_2^*. If $n < 1 + (q-1)p$, we will construct a q-clique. Since $\chi(G_2^*) \leq q \leq n$, there exists a proper q-coloring $f : V(G_2^*) \to \{1, 2, \ldots, q\}$ of G_2^* such that for any $j \in \{1, 2, \ldots, q\}$, there is $v \in V(G_2^*)$ such that $f(v) = j$. Without loss of generality, we may assume that $f(v_i) = i$ for all $i \in \{1, 2, \ldots, q\}$. Let G_2^{**} be graph constructed from G_2^* by adding edges between v_i and v_j where $i, j \in \{1, 2, \ldots, q\}$ and $i \neq j$. Let $E = \{v_iv_j \text{ for all } i, j = 1, 2, \ldots, q \text{ and } i \neq j\}$. Hence G_2^{**} contains a q-clique $K_q(v_1, v_2, \ldots, v_q)$ and then $\chi(G_2^{**}) \geq q$. It is easy to see that f is still proper in G_2^{**}. Then $\chi(G_2^{**}) \leq q$. Hence $\chi(G_2^{**}) = q$.

Let $H_1 = P_n(u_1, u_2, \ldots, u_n) \subseteq G_1^*$ and $H_2 = P_n(v_1, v_2, \ldots, v_n) \subseteq G_2^*$. Define $g : H_1 \to H_2$ by $g(u_i) = v_i$ for all $i \in \{1, 2, 3, \ldots, n\}$. We obtain the glued graph of G_1^* and G_2^* at H_1 and H_2 with respect to g, denoted by $G_1^* \underset{H_1 \cong_g H_2}{\Diamond} G_2^*$. Let w_i, $w_j \in V(G_1^* \underset{H_1 \cong_g H_2}{\Diamond} G_2^*)$. If $i \equiv j \pmod p$, then edge $w_i w_j \in E(G_2^*) \subseteq E(G_1^* \underset{H_1 \cong_g H_2}{\Diamond} G_2^*)$. Otherwise, edge $w_i w_j \in E(G_1^*) \subseteq G_1^* \underset{H_1 \cong_g H_2}{\Diamond} G_2^*$. So $w_i w_j \in E(G_1^* \underset{H_1 \cong_g H_2}{\Diamond} G_2^*)$. Hence $G_1^* \underset{H_1 \cong_g H_2}{\Diamond} G_2^* = K_n$. In the case of G_2^{**}, we know that $E \subseteq E(G_1^*)$. Let H_1^* and H_2^* be graphs such that $V(H_i^*) = V(H_i)$ and $E(H_i^*) = E(H_i) \cup E$ for all $i = 1, 2$ where $H_1 = P_n(u_1, u_2, \ldots, u_n)$ and $H_2 = P_n(v_1, v_2, \ldots, v_n)$. Clearly, $H_1^* \subseteq G_1^*$ and $H_2^* \subseteq G_2^{**}$. Hence $G_1^* \underset{H_1^* \cong_g H_2^*}{\Diamond} G_2^{**} = K_n$.

To prove the last statement, let A_1 and A_2 be graphs such that $A_1 \underset{B}{\Diamond} A_2 = K_n$ where B is a clone. Assume that $\chi(A_1) \geq \chi(A_2)$. By Lemma 3.2.11, we get that $\chi(A_1)\chi(A_2) \geq n$. Then $\chi(A_1)^2 \geq \chi(A_1)\chi(A_2) \geq n$. So $max\{\chi(A_1), \chi(A_2)\} = \chi(A_1) \geq \lceil \sqrt{n} \rceil$. $\square$

Graphs G_1 and G_2 in theorem 3.2.9 have a property that $\chi(G_1 \underset{H}{\Diamond} G_2) = \chi(G_1)\chi(G_2) = \omega(G_1 \underset{H}{\Diamond} G_2)$. Do there exist graphs G_1 and G_2 such that $\chi(G_1)\chi(G_2) = \chi(G_1 \underset{H}{\Diamond} G_2) \neq \omega(G_1 \underset{H}{\Diamond} G_2)$? Since the chromatic number of a graph is always at least the clique number of such graph, we look for G_1, G_2 and H such that $\chi(G_1)\chi(G_2) = \chi(G_1 \underset{H}{\Diamond} G_2) > \omega(G_1 \underset{H}{\Diamond} G_2)$. Before we answer such question, we next provide a graph with the property that its chromatic number is strictly more than its clique number. Such graph is constructed by joining two specified graphs.

Recall that the **join** of simple graphs G_1 and G_2, $G_1 \vee G_2$, is the graph that

$$V(G_1 \vee G_2) = V(G_1) \cup V(G_2) \text{ and}$$
$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}.$$

Theorem 3.2.13. *Let G_1 and G_2 be graphs. Then $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$.*

Proof. Let G_1 and G_2 be graphs. Let f and g be proper colorings of G_1 and G_2, respectively. Define $\alpha : V(G_1) \cup V(G_2) \to \{1, 2 \ldots, \chi(G_1) + \chi(G_2)\}$ by for all $v \in V(G_1) \cup V(G_2)$

$$\alpha(v) = \begin{cases} f(v) & \text{if } v \in V(G_1), \\ \chi(G_1) + g(v) & \text{if } v \in V(G_2). \end{cases}$$

It is easy to see that α is proper. So $\chi(G_1 \vee G_2) \leq \chi(G_1) + \chi(G_2)$. Suppose for a contradiction that $\chi(G_1 \vee G_2) < \chi(G_1) + \chi(G_2)$. There exist $u \in V(G_1)$ and

$v \in V(G_2)$ such that $\alpha(u) = \alpha(v)$. So u and v are not adjacent in $G_1 \vee G_2$. This contradicts to the definition of the join graphs. Hence $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$. $\square$

Example 3.2.14. Let $W = C_5 \vee K_1$. So $\chi(W) = \chi(C_5) + \chi(K_1) = 3 + 1 = 4$ and also $\chi(W \vee K_n) = 4 + n$ for all $n \in \mathbb{N}$. Because $\omega(W) = 3$, so $\omega(W \vee K_n) = n + 3 < n + 4 = \chi(W \vee K_n)$. $\square$

Now, we give graphs G_1 and G_2 with the property $\chi(G_1)\chi(G_2) = \chi(G_1 \underset{H}{\diamondsuit} G_2) > \omega(G_1 \underset{H}{\diamondsuit} G_2)$ where H is a clone in the next theorem.

Theorem 3.2.15. *For all $p, q \geq 3$, there exist graphs G_1, G_2 and $G_1 \underset{H}{\diamondsuit} G_2$ at some clone H such that $\chi(G_1) = p$, $\chi(G_2) = q$ and $pq = \chi(G_1)\chi(G_2) = \chi(G_1 \underset{H}{\diamondsuit} G_2) > \omega(G_1 \underset{H}{\diamondsuit} G_2)$.*

Proof. Let p and q be integers such that $p, q \geq 3$.

Case 1. $p = q$: By Lemma 3.2.12, we have graphs G_1 and G_2 such that $G_1 \underset{H}{\diamondsuit} G_2 = K_{pq-1}$ at some clone H and $\chi(G_1) = p = q = \chi(G_2)$. Following from the proof of Lemma 3.2.12, since $pq - 1 > 1 + (q-1)p$, we obtain that γ_1 and γ_2 are proper colorings of G_1 and G_2, respectively. Below are properties of G_1 and G_2 defined in Lemma 3.2.12.

$V(G_1) = \{u_1, u_2, \ldots, u_{pq-1}\}$, u_i is adjacent to u_j if and only if $i \not\equiv j \pmod p$, $\gamma_1(u_i) = l$ where $l \equiv i \pmod p$ and $l \in \{1, 2, \ldots, p\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{pq-1}\}$, v_i and v_j are adjacent if and only if $i = j + 1$ or $i \equiv j \pmod p$. For each $i = ap + b \in \{1, 2, \ldots, pq - 1\}$ where $a, b \in \mathbb{Z}$, $a \geq 0$ and $0 < b \leq p$, define $\gamma_2(v_i) = l$ where $l \equiv a + b \pmod q$ and $l \in \{1, 2, \ldots, q\}$.

Let G_1^* be a graph such that $V(G_1^*) = V(G_1) \cup \{u_{pq}, u_{pq+1}, u_{pq+2}\}$ and $E(G_1^*) = E(G_1) \cup \{u_{pq}u_i, u_{pq+1}u_i$ where $i = 3, 4, 5, \ldots, pq-2$ and $i \not\equiv 0 \pmod p\} \cup \{u_{pq+2}u_i$ where $i = 3, 4, 5, \ldots, pq-2$ and $i \not\equiv p-1 \pmod p\} \cup \{u_{pq}u_1, u_{pq}u_{pq-1}\} \cup \{u_{pq+1}u_1, u_{pq+1}u_2\} \cup \{u_{pq+2}u_i$ where $i = 1, pq, pq+1\}$.

Clearly, $G_1 \subseteq G_1^*$. So $\chi(G_1^*) \geq \chi(G_1) = p$. Define $f_1 : V(G_1^*) \to \{1, 2, \ldots, p\}$

by for all $u_i \in V(G_1^*)$

$$f_1(u_i) = \begin{cases} \gamma_1(u_i) & \text{if } i = 1, 2, \ldots, pq-1, \\ p & \text{if } i = pq, pq+1, \\ p-1 & \text{if } i = pq+2. \end{cases}$$

It is easy to check that f_1 is proper. So $\chi(G_1^*) \le p$. Hence $\chi(G_1^*) = p$.

Next, let G_2^* be a graph such that $V(G_2^*) = V(G_2) \cup \{v_{pq}, v_{pq+1}, v_{pq+2}\}$ and $E(G_2^*) = E(G_2) \cup \{v_{pq}v_i, v_{pq+1}v_i$ where $i = 3, 4, 5, \ldots, pq-2$ and $i \equiv 0 \pmod p\} \cup \{v_{pq+2}v_i$ where $i = 3, 4, 5, \ldots, pq-2$ and $i \equiv p-1 \pmod p\} \cup \{v_{pq}v_1\} \cup \{v_{pq+2}v_i$ where $i = pq, pq+1\}$.

Clearly, $G_2 \subseteq G_2^*$. So $\chi(G_2^*) \ge q$. Define $f_2 : V(G_2^*) \to \{1, 2, \ldots, q\}$ by for all $v_i \in V(G_2^*)$

$$f_2(v_i) = \begin{cases} \gamma_2(v_i) & \text{if } i = 1, 2, \ldots, pq-1, \\ q-1 & \text{if } i = pq, pq+1, \\ q-2 & \text{if } i = pq+2. \end{cases}$$

To show that f_2 is proper, it suffices to show that for all $s \in \{3, 4, \ldots, pq-2\}$ and $t \in \{pq, pq+2\}$, if v_s is adjacent to v_t, then $f_2(v_s) \ne f_2(v_t)$. Let $s \in \{3, 4, \ldots, pq-2\}$ and $t \in \{pq, pq+2\}$ such that v_s is adjacent to v_t.

Case 1.1. $t = pq+2$: So $s \equiv p-1 \pmod p$. Then $s \ge p-1$ and also $1 \le s = kp + (p-1) \le pq-2$ for some $k \in \mathbb{N} \cup \{0\}$. Then $0 \le k \le q-2$. So $-2 < -1 \le k-1 \le q-3 < q-2$. Hence $f_2(v_s) \equiv k+p-1 \pmod q \equiv k+q-1 \pmod q \equiv k-1 \pmod q \not\equiv q-2 \pmod q = q-2 = f_2(v_{pq+2}) = f_2(v_t)$. Therefore $f_2(v_s) \ne f_2(v_t)$.

Case 1.2. $t = pq$: We get that $s \equiv 0 \pmod p$. So $s = kp = (k-1)p + p \le pq-2$ for some $k \in \mathbb{N}$. Then $1 \le k \le q-1$. So $-1 < 0 \le k-1 \le q-2 < q-1$. Hence $f_2(v_s) \equiv k-1+p \pmod q \equiv k-1+q \pmod q \equiv k-1 \pmod q \not\equiv q-1 \pmod q = q-1 = f_2(v_{pq}) = f_2(v_t)$. Therefore $f_2(v_s) \ne f_2(v_t)$.

By both cases, we can conclude that f_2 is proper. So $\chi(G_2^*) \le q$. Hence $\chi(G_2^*) = q$.

Let $H_1 = P_{pq+2}(u_{pq+1}, u_{pq+2}, u_{pq}, u_1, u_2, \ldots, u_{pq-1})$ and $H_2 = P_{pq+2}(v_{pq+1}, v_{pq+2}, v_{pq}, v_1, v_2, \ldots, v_{pq-1})$. Clearly, $H_1 \subseteq G_1^*$ and $H_2 \subseteq G_2^*$. Define $f : H_1 \rightarrow H_2$ by $f(u_i) = v_i$ for all $i \in \{1, 2, 3, \ldots, pq+2\}$. We obtain a glued graph of G_1^* and G_2^* at H_1 and H_2 with respect to f, denoted by $G_1^* \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2^*$. Let $V(G_1^* \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2^*) = \{w_i : i = 1, 2, \ldots, pq+2$ where w_i corresponds to u_i and $v_i\}$. It is easy to check that $G_1^* \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2^* = W \vee K_{pq-4}(w_3, w_4, \ldots, w_{pq-2})$ where $W = C_5(w_2, w_{pq+1}, w_{pq+2}, w_{pq}, w_{pq-1}) \vee K_1(w_1)$. By Example 3.2.14, we obtain that $\chi(G_1^* \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2^*) = 4 + pq - 4 = pq > pq - 1 = 3 + pq - 4 = \omega(G_1^* \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2^*)$.

Case 2. $p < q$: By Lemma 3.2.12, we get graphs G_1 and G_2 such that $G_1 \triangleleft\!\triangleright G_2 = K_{pq-1}$ and $\chi(G_1) = p = \chi(G_2)$. Following from the proof of Lemma 3.2.12, since $pq - 1 > 1 + (q-1)p$, we have that γ_1 and γ_2 are proper colorings of G_1 and G_2, respectively. Below are properties of G_1 and G_2.

$V(G_1) = \{u_1, u_2, \ldots, u_{pq-1}\}$, u_i is adjacent to u_j if and only if $i \not\equiv j \pmod p$, $\gamma_1(u_i) = l$ where $l \equiv i \pmod p$ and $l \in \{1, 2, \ldots, p\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{pq-1}\}$, v_i and v_j are adjacent if and only if $i = j + 1$ or $i \equiv j \pmod p$. For each $i = ap + b \in \{1, 2, \ldots, pq-1\}$ where $a, b \in \mathbb{Z}$, $a \geq 0$ and $0 < b \leq p$, define $\gamma_2(v_i) = l$ where $l \equiv 2 + a - b \pmod q$ and $l \in \{1, 2, \ldots, q\}$.

Define G_1^*, G_2^* and f_1 similarly to case 1. So $\chi(G_1^*) = p$. Next, let f_2 be a coloring of G_2^* defined by

$$f_2(v_i) = \begin{cases} \gamma_2(v_i) & \text{if } i = 1, 2, \ldots, pq-1, \\ q - p + 1 & \text{if } i = pq, pq+1, \\ q - p + 2 & \text{if } i = pq + 2. \end{cases}$$

Note that $f_2(v_1) \equiv 2 + 0 - 1 \pmod q = 1 \neq q - p + 1 = f_2(v_{pq})$ because $1 = 0(p) + 1$ and $p \neq q$. To show that f_2 is proper, it suffice to prove that for all $s \in \{3, 4, \ldots, pq-2\}$ and $t \in \{pq, pq+2\}$, if v_s and v_t are adjacent, then $f_2(v_s) \neq f_2(v_t)$. Let $s \in \{3, 4, \ldots, pq-2\}$ and $t \in \{pq, pq+2\}$. Assume that v_s and v_t are adjacent.

Case 2.1. $t = pq$: So $s \equiv 0 \pmod p$ and hence $s = kp = (k-1)p + p$ for some $k \in \mathbb{N}$. Since $s \leq pq - 2$, we obtain that $1 \leq k \leq q - 1$. Because $1 - p < 2 - p \leq 2 + (k-1) - p = 1 + k - p \leq q - p < q - p + 1$, so $f_2(v_s) \equiv 2 + (k-1) - p$

$\pmod q = 1 + k - p \not\equiv q - p + 1 \pmod q = q - p + 1 = f_2(v_{pq}) = f_2(v_t)$. Hence $f_2(v_s) \neq f_2(v_t)$.

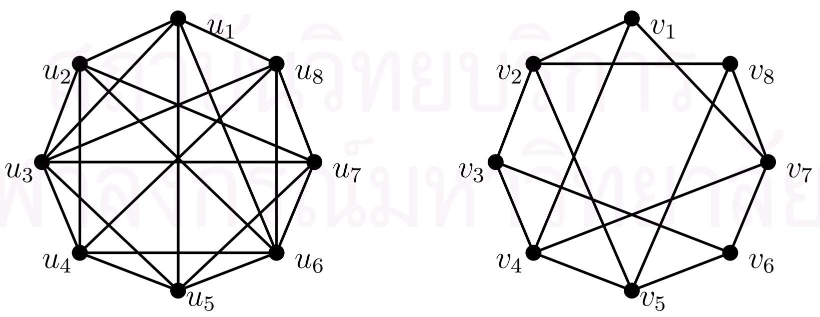
Case 2.2. $t = pq + 2$: Then $s \equiv p - 1 \pmod p$. So $s \geq p - 1$ and then $s = kp + p - 1$ for some $k \in \mathbb{N}^0$. Since $s \leq pq - 2$, we obtain that $0 \leq k \leq q - 2$. Because $2 - p < 3 - p \leq 2 + k - p + 1 = 3 + k - p \leq q - p + 1 < q - p + 2$, so $f_2(v_s) \equiv 2 + k - p + 1 \pmod q = 3 + k - p \not\equiv q - p + 2 \pmod q = q - p + 2 = f_2(v_{pq+2}) = f_2(v_t)$. Hence $f_2(v_s) \neq f_2(v_t)$.

By both cases, we can conclude that f_2 is proper and also $\chi(G_2^*) \leq q$. Because G_2 is a subgraph of G_2^*, so $\chi(G_2^* \geq \chi(G_2) = q$. Hence $\chi(G_2^*) = q$.

We define graphs H_1, H_2 and the isomorphism f similarly to case 1. So we have $G_1^* \underset{H_1 \cong_f H_2}{\Diamond} G_2^* = W \vee K_{pq-4}(w_3, w_4, \ldots, w_{pq-2})$ where $W = C_5(w_2, w_{pq+1}, w_{pq+2}, w_{pq}, w_{pq-1}) \vee K_1(w_1)$. and also $\chi(G_1^* \underset{H_1 \cong_f H_2}{\Diamond} G_2^*) = 4 + pq - 4 = pq > pq - 1 = 3 + pq - 4 = \omega(G_1^* \underset{H_1 \cong_f H_2}{\Diamond} G_2^*)$. $\square$

In the next example, we illustrate an example of graphs in Theorem 3.2.15. We construct graphs G_1^* and G_2^* such that $9 = \chi(G_1^*)\chi(G_2^*) = \chi(G_1^* \underset{H}{\Diamond} G_2^*) > \omega(G_1^* \underset{H}{\Diamond} G_2^*) = 8$ where H is the clone of a glued graph between G_1^* and G_2^*.

Example 3.2.16. We illustrate an example of graphs in Theorem 3.2.15 here. Let $p = 3 = q$. First, we construct graphs G_1 and G_2 such that their glued graph at some clone H is isomorphic to K_8 by using Lemma 3.2.12. That graphs G_1 and G_2 are showed in Figure 3.2.3.



G_1 G_2

Figure 3.2.3: Graphs with their glued graph isomorphic to K_8.

Moreover, we obtain proper colorings, γ_1 and γ_2, of G_1 and G_2, respectively, defined by

$$\gamma_1(v_i) = \begin{cases} 1 & \textit{if } i = 1, 4, 7, \\ 2 & \textit{if } i = 2, 5, 8, \\ 3 & \textit{if } i = 3, 6. \end{cases} \qquad \gamma_2(v_i) = \begin{cases} 1 & \textit{if } i = 1, 6, 8, \\ 2 & \textit{if } i = 2, 4, \\ 3 & \textit{if } i = 3, 5, 7. \end{cases}$$

Next, we add vertices $\{u_9, u_{10}, u_{11}\}$ into $V(G_1)$ and edges $\{u_9u_i, u_{10}u_i, u_{11}u_j$ where $i, j = 3, 4, 5, \ldots, 7$ and $i \not\equiv 0 \pmod 3$ and $j \not\equiv 2 \pmod 3\} \cup \{u_9u_1, u_9u_8, u_{10}u_1, u_{10}u_2, u_{11}u_1, u_{11}u_9, u_{11}u_{10}\}$ into $E(G_1)$ to obtain G_1^*. We also add vertices $\{v_9, v_{10}, v_{11}\}$ into $V(G_2)$ and edges $\{v_9v_k, v_{10}v_k, v_{11}v_l$ where $k, l = 3, 4, 5, \ldots, 7$, $k \equiv 0 \pmod 3$ and $l \equiv 2 \pmod 3\} \cup \{v_9v_1, v_{11}v_9, v_{11}v_{10}\}$ into $E(G_2)$ to obtain G_2^*. Figure 3.2.4 shows G_1^* and G_2^*.

We next define proper colorings $f_1 : V(G_1^*) \to \{1, 2, 3\}$ and $f_2 : V(G_2^*) \to \{1, 2, 3\}$ of G_1^* and G_2^*, respectively, by for all $u_i \in V(G_1^*)$ and for all $v_i \in V(G_2^*)$.

$$f_1(u_i) = \begin{cases} \gamma(v_i) & \textit{if } i = 1, 2, \ldots, 8, \\ 3 & \textit{if } i = 9, 10, \\ 2 & \textit{if } i = 11. \end{cases} \qquad f_2(v_i) = \begin{cases} \gamma(v_i) & \textit{if } i = 1, 2, \ldots, 8, \\ 2 & \textit{if } i = 9, 10, \\ 1 & \textit{if } i = 11. \end{cases}$$

We glue G_1^* and G_2^* at $H_1 = P_{11}(u_{10}, u_{11}, u_9, u_1, u_2, \ldots, u_8) \subseteq G_1^*$ and $H_2 = P_{11}(v_{10}, v_{11}, v_9, v_1, v_2, \ldots, v_8) \subseteq G_2^*$ by isomorphism g defined by $g(u_i) = v_i$ for all $i = 1, 2, \ldots, 11$. Then we obtain that $G_1^* \underset{H_1 \cong_g H_2}{\triangleleft\!\triangleright} G_2^*$ as shown in Figure 3.2.4. We observe that $G_1^* \underset{H_1 \cong_g H_2}{\triangleleft\!\triangleright} G_2^* = K_5(w_3, w_4, \ldots, w_7) \vee W$ where $W = C_5(w_2, w_8, w_9, w_{11}, w_{10}) \vee K_1(w_1)$. So $\chi(G_1^* \underset{H_1 \cong_g H_2}{\triangleleft\!\triangleright} G_2^*) = 5 + 4 = 9 = \chi(G_1^*)\chi(G_2^*)$. Moreover, $\chi(G_1^*)\chi(G_2^*) = 9 > 5 + 3 = 8 = \omega(G_1^* \underset{H_1 \cong_g H_2}{\triangleleft\!\triangleright} G_2^*)$. $\square$

Though the upper bound in Theorem 3.2.8 is sharp, under a specified circumstance we can reduce it down.

Theorem 3.2.17. *Let G_1 and G_2 be graphs and H be the clone of a glued graph $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. If H is an induced subgraph of both G_1 and G_2, then $\chi(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) \leq \chi(G_1) + \chi(G_2)$.*

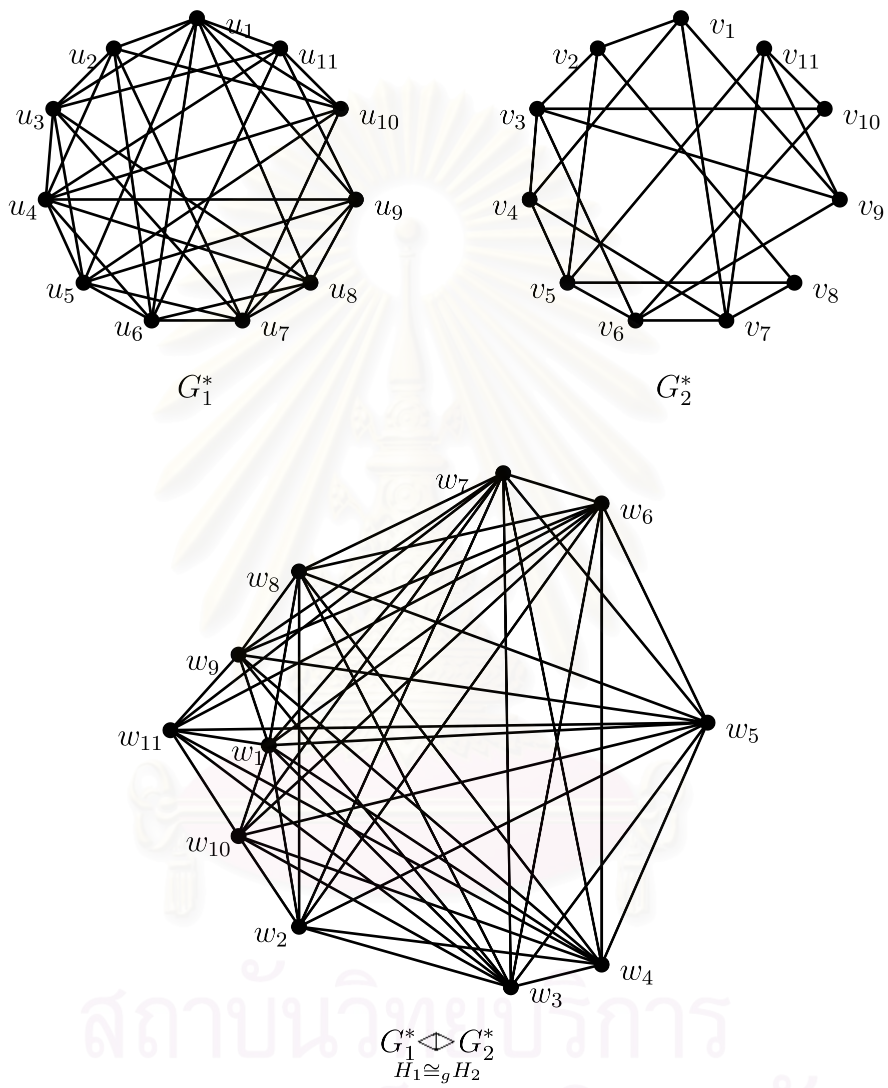


Figure 3.2.4: A glued graph whose chromatic number is larger than its clique number

Proof. Let G_1 and G_2 be graphs and H be a clone of $G_1 \underset{H}{\diamondsuit} G_2$. Assume H is an induced subgraph of both G_1 and G_2. There are proper colorings $f : V(G_1) \to S_1$ and $g : V(G_2) \to S_2$ of G_1 and G_2, respectively where S_1 and S_2 are sets such that $|S_1| = \chi'(G_1)$, $|S_2| = \chi'(G_2)$ and $S_1 \cap S_2 = \phi$.
Define $\alpha : V(G_1 \underset{H}{\diamondsuit} G_2) \to S_1 \cup S_2$ by for each $u \in V(G_1 \underset{H}{\diamondsuit} G_2)$,

$$\alpha(u) = \begin{cases} f(u) & \text{if } u \in V(G_1), \\ g(u) & \text{if } v \in V(G_2 \backslash H). \end{cases}$$

To show that α is proper, let u and v be vertices in $G_1 \underset{H}{\diamondsuit} G_2$ such that u and v are adjacent by an edge e in $G_1 \underset{H}{\diamondsuit} G_2$.

Case 1. $u \in V(G_1)$ and $v \in V(G_2 \backslash H)$: So $\alpha(u) = f(u) \neq g(u) = \alpha(v)$ because $S_1 \cap S_2 = \phi$.

Case 2. u and v are in G_1: If u is not adjacent to v in G_1, then $e \in E(G_2) \backslash E(G_1)$ and also $u, v \in V(G_2)$. Hence $u, v \in V(H)$ but $e \notin E(H)$. Therefore H is not an induced subgraph, a contradiction. So u is adjacent to v in G_1 and $\alpha(u) = f(u) \neq f(v) = \alpha(v)$.

Case 3. u and v are in $G_2 \backslash H$: Similarly to case 1, u is adjacent to v in G_2 and $\alpha(u) = g(u) \neq g(v) = \alpha(v)$.

By all cases, we can conclude that α is proper. Hence $\chi(G_1 \underset{H}{\diamondsuit} G_2) \leq \chi(G_1) + \chi(G_2)$. $\square$

Example 2.1.14 reveals that the converse of Theorem 3.2.17 is not true.

We investigate the chromatic numbers of glued graphs and obtain two upper bounds along with their sharpness. In next chapter, we consider the edge-chromatic numbers of glued graphs.

CHAPTER IV

EDGE-COLORABILITY OF GLUED GRAPHS

Similarly to the previous chapter, we find bounds of the edge-chromatic numbers of glued graphs. Graphs in this chapter are not necessary simple. We separate this chapter into two sections. In the first section, we give background of the edge-chromatic numbers of any graphs. We next find bounds of the edge-chromatic numbers of glued graphs in the other section.

4.1 Background

First, we recall the definition and some bounds of the edge-chromatic number of any graph.

Definition 4.1.1. A **k-edge-coloring** of a graph G is a labeling $f : E(G) \to S$, where $|S| = k$. The labels are **colors**; the edges of one color form a color class. A k-edge-coloring is **proper** if incident edges have different labels; that is, if each color class is a matching. A graph is **k-edge-colorable** if it has a proper k-edge-coloring. The **edge-chromatic number** $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.

Remark 4.1.2. Let G be a graph. Clearly, $\chi'(G) \geq \Delta(G)$. Because no edge in G is incident to more than $2\Delta(G) - 1$ other edges, so $2\Delta(G) - 1 \geq \chi'(G) \geq \Delta(G)$.

In [1], there is a theorem showing the edge-chromatic number of complete graphs. We state such theorem without prove here.

Theorem 4.1.3. *The edge-chromatic number of a complete graph K_n is*

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Vizing and Gupta([3]) can prove that $\Delta(G)+1$ colors suffice when G is a simple graph. We show that in the next theorem.

Theorem 4.1.4 ([3]). (Vizing [1964, 1965], Gupta [1966])
If G is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$.

By Theorem 4.1.4 and Remark 4.1.2, we can conclude that for a simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. We can see the sharpness of that bounds in Theorem 4.1.3. In non-simple graphs, their edge-chromatic numbers can be more than their maximum degrees because of their multiple edges. Shannon showed an upper bound of the edge-chromatic number for any graph in Theorem 4.1.5.

Theorem 4.1.5 ([3]). (Shannon [1949])
If G is a graph, then $\chi'(G) \leq \frac{3}{2}\Delta(G)$.

Example 4.1.6. We introduce a graph G such that $\chi'(G) = \frac{3}{2}\Delta(G)$. The **fat triangles**, loopless triangles with multiple edges, are graphs similar to Figure 4.1.1.

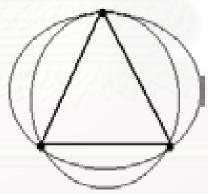


Figure 4.1.1: A fat triangle

The edges are pairwisely intersecting and hence require distinct colors. Thus the edge-chromatic number of a fat triangle G is $\frac{3}{2}\Delta(G)$. $\square$

4.2 Bounds of the Edge-Chromatic Numbers of Glued Graphs

This section, we investigate bounds of the edge-chromatic numbers of glued graphs including non-simple glued graphs. We also study the line graphs of glued graphs in order to obtain a bound of the chromatic number of any glued graph. We begin this section by giving a trivial lower bound of the edge-chromatic number of any glued graph.

Remark 4.2.1. Let G_1 and G_2 be graphs. Because G_1 and G_2 are subgraphs of $G_1 \diamond G_2$, we have that $\chi'(G_1), \chi'(G_2) \leq \chi'(G_1 \diamond G_2)$. Hence

$$\chi'(G_1 \diamond G_2) \geq \max\{\chi'(G_1), \chi'(G_2)\}.$$

By applying Theorem 4.1.4, Theorem 4.1.5 and Lemma 1.2.8, we obtain upper bounds of the edge-chromatic numbers of glued graphs as the following theorem.

Theorem 4.2.2. *Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\diamond} G_2$ be a glued graph of G_1 and G_2 at a clone H. Then*

$$\chi'(G_1 \underset{H}{\diamond} G_2) \leq \frac{3}{2}(\Delta(G_1) + \Delta(G_2) - \delta(H)).$$

In particular, if $G_1 \underset{H}{\diamond} G_2$ is a simple graph, then

$$\chi'(G_1 \underset{H}{\diamond} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1.$$

Proof. Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\diamond} G_2$ be a glued graph of G_1 and G_2 at a clone H. Following from Theorem 4.1.5 and Lemma 1.2.8, we have that $\chi'(G_1 \underset{H}{\diamond} G_2) \leq \frac{3}{2}(\Delta(G_1) + \Delta(G_2) - \delta(H))$. If $G_1 \underset{H}{\diamond} G_2$ is a simple graph, by Theorem 4.1.4 and Lemma 1.2.8, we have that $\chi'(G_1 \underset{H}{\diamond} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1$. □

We show the sharpness of Theorem 4.2.2 in Example 4.2.3 and Example 4.2.4.

Example 4.2.3. Let G_1 and G_2 be graphs in Figure 4.2.1.

Let $H_1 = C_9(u_1, u_2, \ldots, u_9)$ and $H_2 = C_9(v_1, v_2, \ldots, v_9)$. We glue G_1 and G_2 at H_1 and H_2 by isomorphism f defined by $f(u_i) = v_i$ for all $i = 1, 2, \ldots, 9$. So we have $G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2$ which is isomorphic to K_9. Hence $\chi'(G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2) = 9 = 6+4-2+1 = \Delta(G_1) + \Delta(G_2) - \delta(H) + 1$.

□

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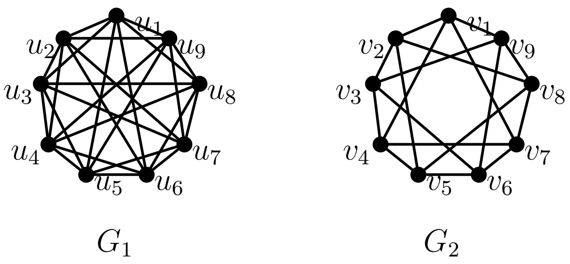


Figure 4.2.1: A simple glued graph showing the sharpness of Theorem 4.2.2

The next example, we reveal the sharpness of Theorem 4.2.2 when the glued graph is a non-simple graph.

Example 4.2.4. Let G_1 and G_2 be graphs as shown in Figures 4.2.2.

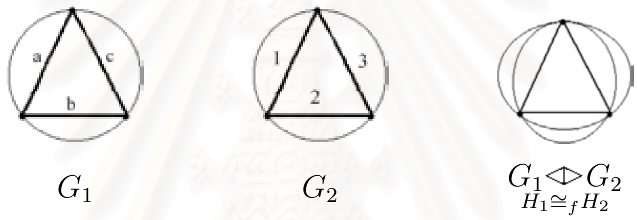


Figure 4.2.2: A non-simple glued graph showing the sharpness of Theorem 4.2.2

Clearly, $\Delta(G_1) = 4 = \Delta(G_2)$. We glue G_1 and G_2 at edge sets $\{a, b, c\}$ and $\{1, 2, 3\}$ with isomorphism f such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. Then we have $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$ as shown in Figure 4.2.2. Because $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$ is a fat triangle, so $\chi'(\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}) = \frac{3}{2}(6) = 9$. Hence $\chi'(\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}) = 9 = \frac{3}{2}(4 + 4 - 2) = \frac{3}{2}(\Delta(G_1) + \Delta(G_2) - \delta(H))$. $\square$

For any graphs G_1 and G_2, we can prove that $\chi'(G_1 \triangleleft\!\triangleright G_2) \leq \chi'(G_1) + \chi'(G_2)$ in Theorem 4.2.5. After that, we show the sharpness of this upper bound in Example 4.2.6.

Theorem 4.2.5. *For any graph G_1 and G_2,*

$$\chi'(G_1 \triangleleft\!\triangleright G_2) \leq \chi'(G_1) + \chi'(G_2).$$

Proof. Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ be a glued graph of G_1 and G_2 at arbitrary clone H. There are proper edge-colorings $f : E(G_1) \to S_1$ and $g : E(G_2) \to S_2$ of G_1 and G_2, respectively, where S_1 and S_2 are sets such that $|S_1| = \chi'(G_1)$, $|S_2| = \chi'(G_2)$ and $S_1 \cap S_2 = \phi$. Define $\alpha : E(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) \to S_1 \cup S_2$ by for all $e \in E(G_1 \underset{H}{\triangleleft\!\triangleright} G_2)$

$$\alpha(e) = \begin{cases} f(e) & \text{if } e \in E(G_1), \\ g(e) & \text{if } e \in E(G_2 \backslash H). \end{cases}$$

To prove that α is proper, let e_1 and e_2 be edges in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ such that e_1 and e_2 are incident in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$.

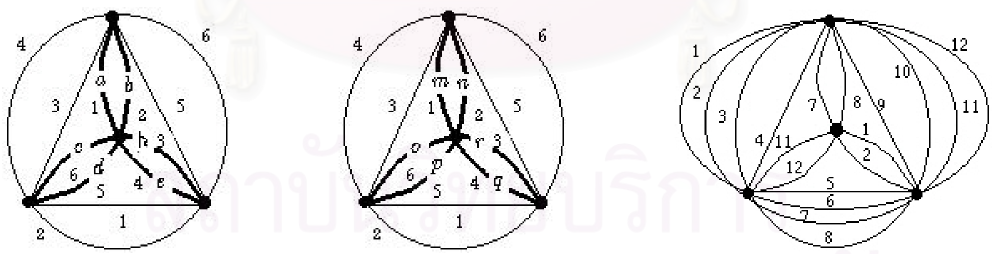
Case 1. $e_1 \in E(G_1)$ and $e_2 \in E(G_2 \backslash H)$: Because $S_1 \cap S_2 = \phi$, so $\alpha(e_1) \neq \alpha(e_2)$.

Case 2. e_1 and e_2 are edges in G_1: Then e_1 and e_2 are incident in G_1 and also $\alpha(e_1) = f(e_1) \neq f(e_2) = \alpha(e_2)$.

Case 3. e_1 and e_2 are edges in $G_2 \backslash H$: Similarly to case 2., we have $\alpha(e_1) = g(e_1) \neq g(e_2) = \alpha(e_2)$.

By all cases, we have that α is proper and hence $\chi'(G_1 \triangleleft\!\triangleright G_2) \leq \chi'(G_1) + \chi'(G_2)$. $\square$

Example 4.2.6. Let G_1 and G_2 be graphs as shown in Figure 4.2.3.



G_1 $\quad$ G_2 $\quad$ $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$

Figure 4.2.3: The sharpness of Theorem 4.2.5

Since $\Delta(G_1) = 6$, we have that $\chi(G_1) \geq 6$. In Figure 4.2.3, labels are colors. We can see that the edge-coloring of G_1 in Figure 4.2.3 is proper and $G_1 \cong G_2$. So $\chi(G_1), \chi(G_2) \leq 6$. Hence $\chi'(G_1) = 6 = \chi'(G_2)$. We glue G_1 and G_2 with the isomorphism f defined by $f(a) = m$, $f(b) = n$, $f(c) = o$, $f(d) = p$, $f(e) = q$ and $f(h) = r$. So we have $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$ as in Figure 4.2.3. Because a fat triangle with the maximum degree 8 is subgraph of $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$, *so* $\chi'(\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}) \geq \frac{3}{2}(8) = 12$. Next, let g be an edge-coloring of $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}$ as in Figure 4.2.3. Clearly, g is proper. So $\chi'(\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}) \leq 12$. Hence $\chi'(\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}) = 12$. Consider $\chi'(\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2}) = 12 = 6 + 6 = \chi'(G_1) + \chi'(G_2)$. Hence the upper bound of the edge-chromatic number in Theorem 4.2.5 is sharp. $\square$

Because of $\chi'(G) = \chi(L(G))$([2]), it is our motivation to study the line graphs of glued graphs.

Definition 4.2.7. Let G be a connected graph. The **line graph** $L(G)$ **of G** is the graph generated from G by $V(L(G)) = E(G)$ and for any two vertices $e, f \in V(L(G))$, vertex e and vertex f are adjacent in $L(G)$ if and only if edge e and edge f share a common vertex in G. If H is the line graph of G, we call G the **root graph** of H.

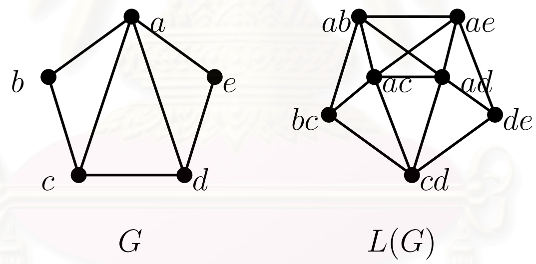


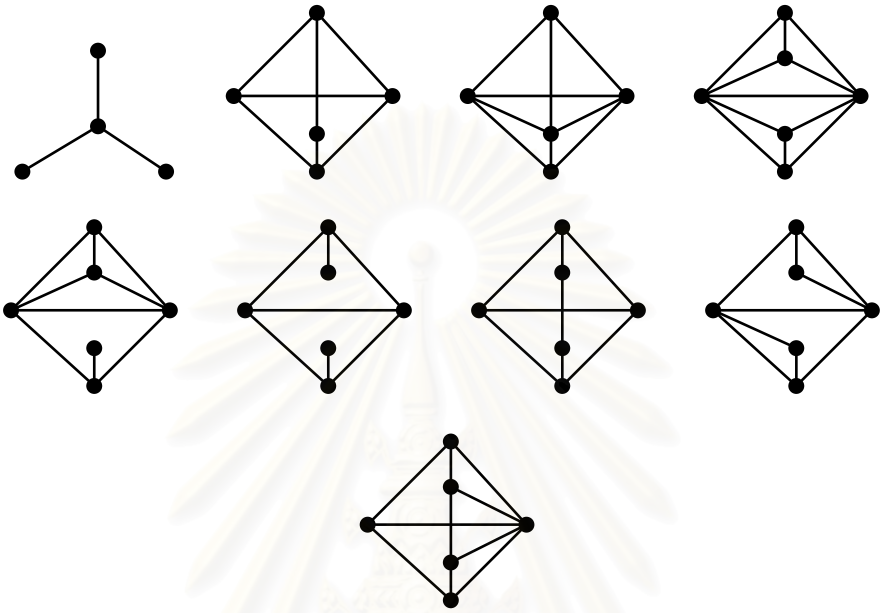
Figure 4.2.4: A line graph

Remark 4.2.8. For any subgraph H of a graph G, $L(H) \subseteq L(G)$.

All graphs have their line graphs, but not all graphs are line graphs. For example, there is no graph G such that $L(G) = K_{1,3}$. So the $K_{1,3}$ is not a line graph. The next two theorems are characterization of the line graphs.

Theorem 4.2.9 ([3]). (Krausz [1943])
For a simple graph G, there is a solution to $L(H) = G$ if and only if G decomposes into complete subgraphs, with each vertex of G appearing in at most two in the list.

Theorem 4.2.10 ([3]). (Beineke [1968])
A simple graph G is the line graph of simple graph if and only if G does not have any of the nine graphs below as an induced subgraph.



Lemma 4.2.11 shows the relationship between $L(G_1)\triangleleft\!\triangleright L(G_2)$ and $L(G_1\triangleleft\!\triangleright G_2)$ where G_1 and G_2 are graphs. This result helps us to find a condition to obtain a smaller upper bound of the chromatic numbers of glued graphs showed in Theorem 4.2.12.

Lemma 4.2.11. *Let G_1 and G_2 be graphs. $L(G_1)\triangleleft\!\triangleright L(G_2) \subseteq L(G_1\triangleleft\!\triangleright G_2)$.*

Proof. Since G_1 and G_2 are subgraphs of $G_1\triangleleft\!\triangleright G_2$, we have that $L(G_1)$ and $L(G_2)$ are subgraphs of $L(G_1\triangleleft\!\triangleright G_2)$. So $L(G_1) \cup L(G_2) \subseteq L(G_1\triangleleft\!\triangleright G_2)$. Because for each vertex and edge in $L(G_1)\triangleleft\!\triangleright L(G_2)$ are in $L(G_1) \cup L(G_2)$ which is a subgraph of $L(G_1\triangleleft\!\triangleright G_2)$, so $L(G_1)\triangleleft\!\triangleright L(G_2) \subseteq L(G_1\triangleleft\!\triangleright G_2)$. □

Theorem 3.2.17 gives a condition to reduce an upper bound of the chromatic numbers of glued graphs into the sum of the chromatic numbers of its original graphs. This is another condition to get a smaller upper bound of the chromatic numbers of glued graphs.

Theorem 4.2.12. *Let G_1 and G_2 be graphs. If G_1 and G_2 are line graphs, then* $\chi(G_1 \triangleleft\!\triangleright G_2) \leq \chi(G_1) + \chi(G_2)$.

Proof. Let G_1 and G_2 be graphs. Assume that G_1 and G_2 are line graphs. So there are graphs G_1^* and G_2^* such that $L(G_1^*) = G_1$ and $L(G_2^*) = G_2$. By lemma 4.2.11, we have that $L(G_1^*) \triangleleft\!\triangleright L(G_2^*) \subseteq L(G_1^* \triangleleft\!\triangleright G_2^*)$. So $\chi(L(G_1^*) \triangleleft\!\triangleright L(G_2^*)) \leq \chi(L(G_1^* \triangleleft\!\triangleright G_2^*))$. Hence

$$
\begin{aligned}
\chi(G_1 \triangleleft\!\triangleright G_2) &= \chi(L(G_1^*) \triangleleft\!\triangleright L(G_2^*)) \\
&\leq \chi(L(G_1^* \triangleleft\!\triangleright G_2^*)) \\
&= \chi'(G_1^* \triangleleft\!\triangleright G_2^*) \\
&\leq \chi'(G_1^*) + \chi'(G_2^*) \qquad \text{by Theorem 4.2.5} \\
&= \chi(L(G_1^*)) + \chi(L(G_2^*)) \\
&= \chi(G_1) + \chi(G_2).
\end{aligned}
$$

$\square$

The next example, we show that the converse of Theorem 4.2.12 does not hold.

Example 4.2.13. Let G_1 and G_2 be graphs as shown in Figure 4.2.5.

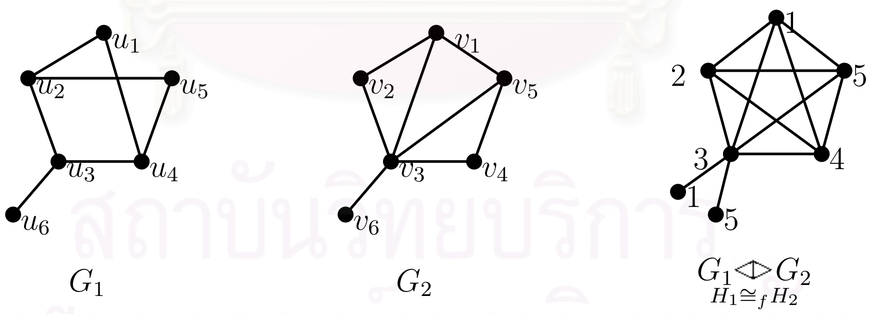


Figure 4.2.5: The converse of Theorem 4.2.12 does not hold

Because $\omega(G_1) = 2$ and $\omega(G_2) = 3$ where $\omega(G)$ is the maximum size of a clique of G, so $\chi(G_1) \geq 2$ and $\chi(G_2) \geq 3$. Next define colorings $g_1 : V(G_1) \to \{1, 2\}$ and $g_2 : V(G_2) \to \{1, 2, 3\}$ of G_1 and G_2, respectively, as follows:

$$g_1(u_i) = \begin{cases} 1 & \textit{if } i = 1, 3, 5, \\ 2 & \textit{if } i = 2, 4, 6. \end{cases} \qquad g_2(v_i) = \begin{cases} 1 & \textit{if } i = 1, 4, 6, \\ 2 & \textit{if } i = 2, 5, \\ 3 & \textit{if } i = 3. \end{cases}$$

We obvious that g_1 and g_2 are proper. So $\chi(G_1) \leq 2$ and $\chi(G_2) \leq 3$. Hence $\chi(G_1) = 2$ and $\chi(G_2) = 3$. We can see that both G_1 and G_2 contain a copy of $K_{1,3}$(vertices u_2, u_3, u_4, u_6 in G_1 and vertices v_2, v_3, v_4, v_6 in G_2) which is one of the nine graphs in Theorem 4.2.10. By Theorem 4.2.10, both G_1 and G_2 are not line graphs. Let $H_1 = P_5(u_1, u_2, \ldots, u_5) \subseteq G_1$ and $H_1 = P_5(v_1, v_2, \ldots, v_5) \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i = 1, 2, \ldots, 5$. So we have $G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ as shown in Figure 4.2.5. Labels of vertices in Figure 4.2.5 are colors. We can see that $G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ is 5-colorable. So $\chi(G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2) \leq 5$. Since $G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ contains K_5, we have that $\chi(G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2) \geq 5$. Hence $\chi(G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2) = 5 = 2 + 3 = \chi(G_1) + \chi(G_2)$.

□

We have obtained a lower bound and upper bounds of the edge-chromatic numbers of glued graphs. Together with the result about the line graphs of glued graphs, we find a condition to get a smaller upper bound of the chromatic numbers of glued graphs.

CHAPTER V

CONCLUSION AND OPEN PROBLEMS

5.1 Conclusion

We have introduced the glue operation and investigated properties of glued graphs emphasizing to their colorability. As follows, there are results in this thesis:
Let G_1 and G_2 be graphs.
Characterization:

1. A glued graph $G_1 \triangleleft\!\triangleright G_2$ is a tree if and only if G_1 and G_2 are trees.

2. A glued graph $G_1 \triangleleft\!\triangleright G_2$ is a forest if and only G_1 and G_2 are forests.

3. A glued graph $G_1 \triangleleft\!\triangleright G_2$ is a bipartite graph if and only if G_1 and G_2 are bipartite.

4. Let H be a clone of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. If G_1, G_2 and H are k-partite graphs, then $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is also a k-partite graph.

5. If G_1 is acyclic and G_2 is chordal, then $G_1 \triangleleft\!\triangleright G_2$ is chordal.

6. Let H be the clone of a glued graph $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. If H is an induced subgraph of both G_1 and G_2 and $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is chordal, then G_1 and G_2 are chordal graphs.

7. Let H be the clone of a glued graph $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. If H is an induced subgraph of both G_1 and G_2 and $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is an interval graph, then G_1 and G_2 are interval graphs.

8. $L(G_1) \triangleleft\!\triangleright L(G_2) \subseteq L(G_1 \triangleleft\!\triangleright G_2)$.

The chromatic numbers of glued graphs:

1. $\chi(G_1 \lhd\!\rhd G_2) \geq \max\{\chi(G_1), \chi(G_2)\}$.

2. Let H be the clone of a glued graph $G_1 \underset{H}{\lhd\!\rhd} G_2$. Then $\chi(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1$. In particular, if $G_1 \underset{H}{\lhd\!\rhd} G_2$ is not a complete graph or an odd cycle, then $\chi(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H)$.

3. $\chi(G_1 \lhd\!\rhd G_2) \leq \chi(G_1)\chi(G_2)$.

4. For all positive integer n which is not prime, K_n is a glued graph such that the product of the chromatic numbers of the original graphs is n. Hence the bound $\chi(G_1 \lhd\!\rhd G_2) \leq \chi(G_1)\chi(G_2)$ is sharp

5. Let H be a clone of $G_1 \underset{H}{\lhd\!\rhd} G_2$. If H is an induced subgraph of both G_1 and G_2, then $\chi(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \chi(G_1) + \chi(G_2)$.

6. If G_1 and G_2 are line graphs, then $\chi(G_1 \lhd\!\rhd G_2) \leq \chi(G_1) + \chi(G_2)$.

The edge-chromatic numbers of glued graphs:

1. $\chi'(G_1 \lhd\!\rhd G_2) \geq \max\{\chi'(G_1), \chi'(G_2)\}$.

2. $\chi'(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \frac{3}{2}(\Delta(G_1) + \Delta(G_2) - \delta(H))$. In particular, if $G_1 \underset{H}{\lhd\!\rhd} G_2$ is a simple graph, then $\chi'(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1$ where H is a clone of a glued graph between G_1 and G_2.

3. $\chi'(G_1 \lhd\!\rhd G_2) \leq \chi'(G_1) + \chi'(G_2)$.

5.2 Open Problems

This thesis brings some open problems for future work as follows:

1. In Section 2.2, we show that a glued graph between two interval graphs may not be an interval graph while a glued graph between two non-interval graphs may be an interval graph. Moreover, we give a condition to make sure that a glued graph of two non-interval graphs is not an interval graph in Theorem

2.2.23. An open problem is to investigate conditions to obtain that a glued graph between two interval graphs is an interval graph.

2. By Theorem 3.2.8, we have that the chromatic number of any glued graph is at most the product of the chromatic numbers of its original graphs. Since the chromatic number of any graph is at least its clique number, we get that $\omega(G_1 \Diamond G_2) \leq \chi(G_1 \Diamond G_2) \leq \chi(G_1)\chi(G_2)$ where G_1 and G_2 are graphs. What is a relation of $\omega(G_1)\omega(G_2)$ and above parameters? Whether or not $\omega(G_1 \Diamond G_2) \geq \omega(G_1)\omega(G_2)$? Analyze the relation between the clique numbers and the chromatic numbers of glued graphs.

   We had investigated the following two statements. Let G_1^* and G_2^* be graphs.

   - If $\omega(G_1^*) < \chi(G_1^*)$ and $\omega(G_2^*) < \chi(G_2^*)$, then $\omega(G_1^* \Diamond G_2^*) < \chi(G_1^* \Diamond G_2^*)$.
   - If $\omega(G_1^*) = \chi(G_1^*)$ and $\omega(G_2^*) = \chi(G_2^*)$, then $\omega(G_1^* \Diamond G_2^*) = \chi(G_1^* \Diamond G_2^*)$.

   We found that these two statements do not hold showed in the following example. Let G_1, G_2, G_3 and G_4 be graphs as shown in Figure 5.2.1.

   We can see that $\chi(G_1) = 4 > 3 = \omega(G_1)$. It is easy to see that $G_1 \cong G_2$. So $\chi(G_2) = 4 > 3 = \omega(G_2)$, $\chi(G_3) = 3 = \omega(G_3)$, and $\chi(G_4) = 3 = \omega(G_4)$. Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be as in Figure 5.2.1 and let $H_3 = P_4(a_1, a_2, a_3, a_4) \subseteq G_3$ and $H_4 = P_4(b_1, b_2, b_3, b_4) \subseteq G_3$ Define isomorphism $f : H_1 \rightarrow H_2$ by $f(u_i) = v_i$ for all $i = 2, 3, 4, 5, 6, 7$ and isomorphism $g : H_3 \rightarrow H_4$ by $g(a_i) = b_i$ for all $i = 1, 2, 3, 4$. We get $\underset{H_3 \cong_g H_4}{G_3 \Diamond G_4} = G_1$ and $\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2}$ as in Figure 5.2.1. Labels of vertices in $\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2}$ are colors. We can see that $\chi(\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2}) \leq 4$. But $\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2}$ contains K_4, so $\chi(\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2}) = 4 = \omega(\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2})$. We observe that $\omega(G_1) < \chi(G_1)$ and $\omega(G_2) < \chi(G_2)$ but $\omega(\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2}) = \chi(\underset{H_1 \cong_f H_2}{G_1 \Diamond G_2})$ while $\omega(G_3) = \chi(G_3)$ and $\omega(G_4) = \chi(G_4)$ but $\omega(\underset{H_3 \cong_g H_4}{G_3 \Diamond G_4}) < \chi(\underset{H_3 \cong_g H_4}{G_3 \Diamond G_4})$.

   Hence an open problem is to find a condition to make the two statements hold.

3. The total chromatic number of any graph is introduced in [5]. Let G be any

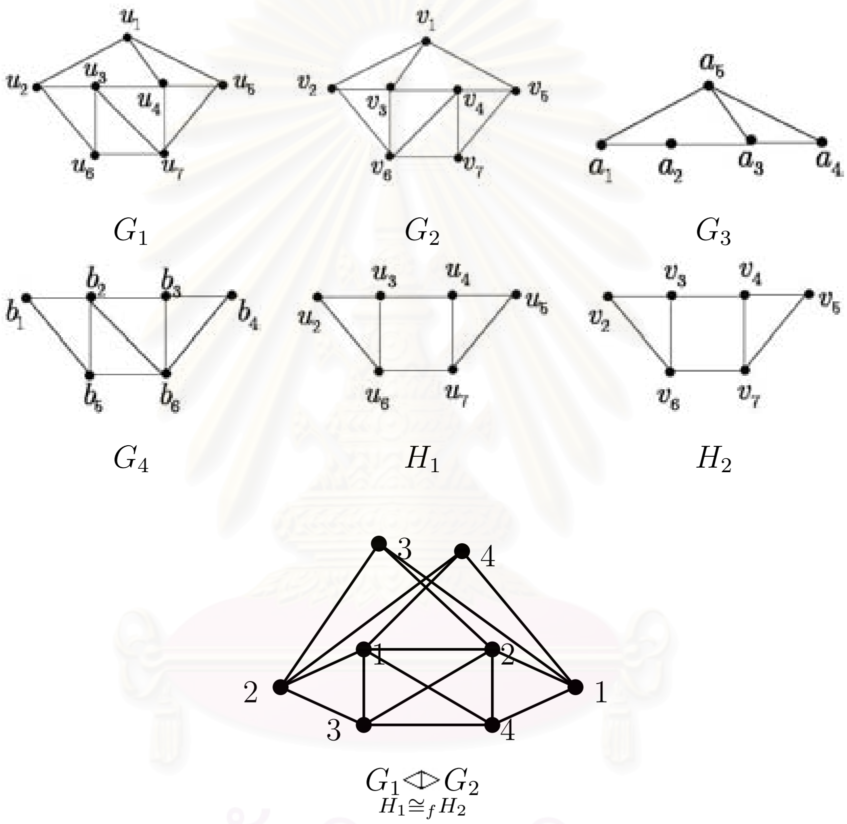


Figure 5.2.1: An open problem.

graph. The total chromatic number $\chi''(G)$ is the smallest number of colors needed to color all the elements of $V(G) \cup E(G)$ in such a way that no two adjacent or incident elements receive the same color. Bounds of the total chromatic number of any graph is showed in [5]. This motivates a future work to investigate bounds of the total chromatic numbers of glued graphs.

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APPENDICES

Definitions and Notations

A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and relations that associates with each edge two vertices(not necessarily distinct) called its **endpoints**. A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints. A **simple graph** is a graph having no loops or multiple edges. A **non-simple graph** is a graph which is not simple. In a simple graph, when u and v are the end points of an edge e, denoted by $e = uv$(or $e = vu$), they are **adjacent** and are **neighbors**. We write $u \leftrightarrow v$ for "u is adjacent to v". Also we denote $u \not\leftrightarrow v$ for "u is not adjacent to v". Let e_1 and e_2 be edges of a graph G. We say e_1 and e_2 are **incident** if e_1 and e_2 share a common vertex.

Graph G having at least one edge is called **non-trivial**. For each vertex v in a loopless graph G, the **degree** of vertex v in G, denoted by $\deg_G(v)$, is the number of incident edges. The maximum degree of a graph G is denoted by $\Delta(G)$ while the minimum degree of graphs G is denoted by $\delta(G)$. For a graph G, if $\Delta(G) = \delta(G)$, then we call that G is **regular**. The **order** of a graph G is the number of vertices in G. An n**-vertex graph** is a graph of order n. The **size** of graph G is the number of edges in G.

A **subgraph** H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that "G **contains** H". Given a subset $V' \subseteq V(G)$. We call V' as an **induced subgraph** of G, denoted by $G[V']$, if V' is a subgraph in which vertices of V' are adjacent in $G[V']$ whenever they are adjacent in G.

For $S \subseteq V(G)$ and $M \subseteq E(G)$, we write $G \backslash S$ for the subgraph of G obtained by deleting the set of vertices S. We write $G \backslash M$ for the subgraph of G obtained by deleting the set of edges M. Let H be a subgraph of a graph G. We write $G \backslash H$ for the subgraph of G obtained by deleting the set of vertices $V(H)$ and the set of edges $E(H)$.

A **path** is a simple graph P of the form $V(P) = \{x_0, x_1, \ldots, x_l\}$, $E(P) = \{x_0x_1, x_1x_2, \ldots, x_{l-1}x_l\}$ where l is a positive integer. A u, v**-path** is a path whose vertices of degree 1 are u and v. We called u and v as its **endpoints**. A **cycle** is a graph such that two vertices are adjacent if and only if they appear consecutively along the circle. The **length** of a cycle or path is its number of edges. An **odd cycle** is a cycle of an odd length while an **even cycle** is a cycle of an even length. A **complete graph** or a **clique** is a graph that every pair of vertices are adjacent. The unlabeled path, cycle and complete graph with n vertices are denoted as P_n, C_n and K_n, respectively. The labeled path, cycle and complete graph on the vertex set $\{u_1, u_2, \dots, u_n\}$ are denoted as $P_n(u_1, u_2, \dots, u_n)$, $C_n(u_1, u_2, \dots, u_n)$ and $K_n(u_1, u_2, \dots, u_n)$.

A graph G is **connected** if it has a u, v-path whenever $u, v \in V(G)$. Otherwise, G is disconnected. The **components** of a graph G are its maximum connected subgraphs. A **vertex-cut** of a graph G is a set $S \subseteq V(G)$ such that removing vertices in S from $V(G)$ increases the number of components.

VITA

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