เสถียรภาพของสมการเชิงฟังก์ชันไซน์ 3 มิติ

นายโอม สถิตยนาค

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2550

STABILITIES OF A 3-DIMENSIONAL SINE FUNCTIONAL EQUATION

Mr. Oam Sthityanak

A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic Year 2007

Thesis Title	STABILITIES OF A 3-DIMENSIONAL SINE FUNCTIONAL EQUATION
By	Mr. Oam Sthityanak
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Paisan Nakmahachalasint, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University, in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Science

(Professor Supot Hannongbua, Ph.D.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Patanee Udomkavanich, Ph.D.)

.... Thesis Advisor

(Assistant Professor Paisan Nakmahachalasint, Ph.D.)

.... Member

(Associate Professor Ajchara Harnchoowong, Ph.D.)

.... Member

(Assistant Professor Sajee Pianskool, Ph.D.)

.... Member

(Khamron Mekchay, Ph.D.)

นายโอม สถิตยนาค : เสถียรภาพของสมการเชิงฟังก์ชันไซน์ 3 มิติ.
(STABILITIES OF A 3-DIMENSIONAL SINE FUNCTIONAL
EQUATION) อ. ที่ปรึกษา : ผศ. ดร. ไพศาล นาคมหาชลาสินธุ์, 29 หน้า.

เสถียรภาพของสมการเชิงฟังก์ชันเป็นหัวข้อวิจัยทางคณิตศาสตร์ซึ่งกำลังเป็นที่สนใจ ผลงานวิจัยจำนวนมากได้รับการตีพิมพ์ภายหลัง ทีเอช เอ็ม ราสเซียสเสนอผลงานเป็นฉบับแรกในปี ค.ศ. 1987

วิทยานิพนธ์ฉบับนี้ได้ศึกษาผลเฉลยของสมการเชิงฟังก์ชันไซน์ 3 มิติ

$$cf(x)f(y)f(z) = f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z)$$

พร้อมทั้งหาเสถียรภาพของสมการ เราจะแบ่งปัญหาเป็น 2 กรณี คือ กรณี $c = 0$ และ $c \neq 0$ โดยในกรณี $c \neq 0$ จะเรียกว่า non-degenerate form ซึ่งเราสามารถพิสูจน์ได้ว่าผลเฉลยของสมการจะอยู่ในผลเฉลยของสมการเชิงฟังก์ชันไซน์ และได้ศึกษาเสถียรภาพยิ่งยวดของสมการ ส่วนในกรณี $c = 0$ เราจะเรียกว่า degenerate form ซึ่งผลเฉลยของสมการนี้จะเหมือนกับผลเฉลยของสมการเชิงฟังก์ชันของเจนเซน และสามารถหาเสถียรภาพแบบ ไฮเออร์ส-อูแลม-ราสเซียส ทั่วไปได้ โดยทำให้การพิสูจน์เสถียรภาพแบบ ไฮเออร์ส-อูแลม-ราสเซียส และเสถียรภาพแบบ ไฮเออร์ส-อูแลม เป็นกรณีเฉพาะ

ภาควิชา...........**คณิตศาสตร์**...........	ลายมือชื่อนิสิต.............โอม สถิตยนาค.................
สาขาวิชา..........**คณิตศาสตร์**..........	ลายมือชื่ออาจารย์ที่ปรึกษา..
ปีการศึกษา...........**2550**...............	

4872565023 : MAJOR MATHEMATICS
KEY WORD : STABILITY/ SINE FUNCTIONAL EQUATION

OAM STHITYANAK : STABILITIES OF A 3-DIMENSIONAL SINE FUNCTIONAL EQUATION. THESIS ADVISOR : ASST. PROF. PAISAN NAKMAHACHALASINT, Ph.D., 29 pp.

The stability of functional equations is currently active mathematical research problems. A number of research papers after the pioneer paper of Th. M. Rassias in 1978 were published.

In this thesis, we study the solution of a 3-dimensional sine functional equation

$$cf(x)f(y)f(z) = f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z)$$

and investigate its stabilities. We consider it into two cases when $c = 0$ and $c \neq 0$. For the case $c \neq 0$, it will be called the non-degenerate form. We can prove that its general solution is contained in the class of the general solution of the sine functional equation and we give its superstability. For the case $c = 0$, we will call the degenerate form. We get that its general solution is the same as that of the Jensens functional equation and we prove the Hyers-Ulam-Rassias stability of the degenerate case then treat the Hyers-Ulam-Rassias stability and Hyers-Ulam stability as special cases.

Department : .....**Mathematics**.... Student's signature : ..Oam Sthityana

Field of Study : ..**Mathematics**.... Advisor's signature : ....

Academic Year : ....**2007**..........

ACKNOWLEDGEMENTS

It is a great pleasure to thank my thesis advisor, Dr. Paisan Nakmahachalasint, for his excellent supervisions, helpful advices and very useful helps. Without his constructive suggestions and knowledgeable guidance in this study, this work would never have successfully been completed. I feel very thankful to the committee of my thesis and all the lecturers during my study.

Finally, my sincere gratitude and appreciation go to my beloved parents for their support and consistent encouragement throughout my study.

CONTENTS

page

ABSTRACT (THAI) iv

ABSTRACT (ENGLISH) v

ACKNOWLEDGEMENTS vi

CONTENTS vii

CHAPTER

I INTRODUCTION 1

II PRELIMINARIES 4

2.1 Functional Equations 4

2.2 General Solutions of a Functional Equation 5

2.3 Stabilities of a Functional Equation 6

III A 3-DIMENSIONAL SINE FUNCTIONAL EQUATION 8

3.1 Background 8

3.2 The General Solution of the Non-degenerate Form 9

3.3 The General Solution of the Degenerate Form 16

IV THE STABILITIES OF THE 3-DIMENSIONAL SINE FUNCTIONAL EQUATION 18

4.1 The Superstability of the Non-degenerate Form 18

4.2 The Stability of the Degenerate Form 21

REFERENCES 28

VITA 29

CHAPTER I

INTRODUCTION

"Functional equations", originally, are equations in which the unknowns are functions. So it includes all differential equations, integral equations and recurrence relations, for examples, $f''(x) + 2f'(x) + f(x) = 0$, $\int f(x)dx = f(x)$ and $f(n) = f(n-1)+1$. According to the book of J. Aczél and J. G. Dhombres[1], M. Kuczma[10], we know that the history of the study of functional equations may be back more than 2200 years when Archimedes made use of recurrences.

In more restrict sense, "functional equations"may be defined by the use of terms. It disposes all differential equations and integral equations which have conditions such as differentiability or integrability of functions that we have to concern. In this sense, it appeared that the first work was written in 1747 by d'Alembert[2]. His work was about the famous functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$ which was called the cosine functional equation or many ones referred to as d'Alembert's functional equation to honor him. After that many outstanding mathematicians such as Abel, Cauchy, Gauss and Euler studied d'Alembert's functional equation or posed individual equations.

The beginning of *a theory of functional equations* related to the work of Hungarian mathematician J. Aczél who was an excellent specialist in this field(see [10]) but at the beginning period it was not systemized so the published works were separately solved by individual techniques until A. R. Schweitzer[12] treated the subject more uniformly.

Theory of functional equations then has been developed gradually. Especially in the last two decades, it has grown rapidly. A lot of mathematical papers investigating functional equations were published. Moreover, functional equations are contained in the mathematical olympiad contest so students would widely learn in this branch of mathematics. Now functional equations become an important research field with its special methods, a number of interesting results and several applications.

To make us more familiar with the functional equations, we give the following famous examples [7]:

$f(x+y) = f(x) + f(y)$ (Cauchy's additive functional equation)

$f(x+y) = f(x)f(y)$ (Cauchy's exponential functional equation)

$f(xy) = f(x) + f(y)$ (Cauchy's logarithmic functional equation)

$f(x+y) + f(x-y) = 2f(x) + 2f(y)$ (quadratic functional equation)

$f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ (Jensen's functional equation)

$f(x+y) + f(x-y) = 2f(x)f(y)$ (d'Alembert's functional equation)

Although solving the functional equations is always the art that many mathematicians are crazy, investigating the functional inequalities becomes more interesting problem in this moment. Therefore, in this thesis, we will study both the solutions of the functional equations and the functional inequalities which is called the stability problem.

In Chapter II, we will introduce a definition of a functional equation based on the concept of terms. It gives us a sense of functional equations which are different from differential equations and integral equations. Then we explain what the general solutions and the stability problems are.

In chapter III, we introduce our new functional equation

$$cf(x)f(y)f(z) = f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z)$$

which is called a "3-dimensional sine functional equation"and give its general solutions. Background of our study are given for readers to see what we thought, how we did with our functional equation and why we separate the equation into two cases $c = 0$ and $c \neq 0$, which are called the degenerate form and the non-degenerate form, respectively.

In chapter IV, we investigate stabilities of the 3-dimensional sine functional equation. We give the superstability for the non-degenerate case and the generalized Hyers-Ulam-Rassias stability for the degenerate case.

CHAPTER II

PRELIMINARIES

2.1 Functional Equations

At first we will give a definition of a functional equation along which was given in the book of Kuczma[10]. It is based on the concept of the terms.

Definition 2.1.1. *A term* is defined by the following conditions:

1. Independent variables are terms.
2. If $t_1, \ldots, t_p$ are terms and $f(x_1, \ldots, x_p)$ is a function of p variables, then $f(t_1, \ldots, t_p)$ also is a term.
3. There exist no other terms.

Then a functional equation may be defined as follows:

Definition 2.1.2. *A functional equation* is an equality $t_1 = t_2$ between two terms t_1 and t_2 which contain at least one unknown function and a finite number of independent variables. This equality is to be satisfied identically with respect to all the occurring variables in a certain set (of any sort).

The notion of a functional equation as defined above does not contain differential, integral, operator equation and generally equation in which infinitesimal operations are performed.

As we know, the solutions of a functional equation must be functions. The next section will tell about the behaviour of the solutions of the functional equation that we have to concern when we solve the equation. This information is referred to [10].

2.2 General Solutions of a Functional Equation

Solution of a functional equation is directly related to the domain and the range which the equation is defined on. So we should clearly state what class the solution is in. The number and behaviour of solutions depends very strongly on this. It is one of the important differences between differential equations and functional equations.

To show the importance of the domain of functional equations, we give an example as followings.

If we want to find the solution of the functional equation $f(xy) = f(x) + f(y)$ where $f : \mathbb{R} \to \mathbb{R}$, we first set $y = 0$ in the equation. Then we get $f(0) = f(x) + f(0)$. Now we have that $f(x) = 0$ for all $x \in \mathbb{R}$, indeed, it also satisfies the investigated functional equation which means that the trivial function is the only solution of this functional equation.

But if we investigate $f : \mathbb{R}^+ \to \mathbb{R}$, 0 is excluded from the domain. So we cannot do as above. However we may easily see that both $f \equiv 0$ and $f(x) = \log x$ are solutions of the functional equation. This shows that the domain is important.

Algebraically, we have to concern about what operation we can do between variables and what properties we can use in the domain and the range that the equation is postulated.

In the next section, we provide the history of the study of stability problems which are now popular research problems for many mathematicians. The reader can find it in [11] and [7].

2.3 Stabilities of a Functional Equation

In 1940, S. M. Ulam[13] proposed the stability problem of a linear functional equation, in his talk before the Mathematics Club of the University of Wisconsin.

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y))) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?(see [11].)

In 1941, D. H. Hyers[8] answered this question affirmatively in the case of the additive mapping in Banach spaces.

The Ulam's question and the Hyers' theorem become a source of **the Hyers-Ulam stability**. **The Hyers-Ulam-Rassias stability** was from the theorem that Th.M. Rassias[11] proved in 1978. It weakened the condition for the bound of the norm of the Cauchy difference $f(x+y) - f(x) - f(y)$.

In 1979, J. A. Baker, J. Lawrence and F. Zorzitto[4] introduced that if f satisfies the stability inequality $||E_1(f) - E_2(f)|| \leq \delta$ where $E_1(f)$ and $E_2(f)$ are functional expressions of a function f and $\delta > 0$, then either f is bounded or $E_1(f) = E_2(f)$, then it is called a **superstability**.

Many mathematicians investigated this kind of stability problems. Baker[3] studied the superstability of the cosine functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$. P. W. Cholewa[5] showed the superstability of the sine functional equation $f(x+y)f(x-y) = f(x)^2 - f(y)^2$ on an abelian group. G. H. Kim[9] showed the superstability of the generalized sine functional equation of the form $g(x)h(x) = f(\frac{x+y}{2})^2 - f(\frac{x-y}{2})^2$.

Now we can briefly say that a study of stability of a funtional equation is to consider functional inequality and ask: does there exist a solution of the functional equation which approximates the solutions of the inequality within a given distance?

Nowadays, a trend of work in the branch of functional equation is to solve the functional equation and to investigate stability of the functional equation. So, in the next two chapters, we would like to present our new functional equation and investigate its general solution and, certainly, its stability. The history of study is provided in the background section. It will tell readers how we get the equation and what we discuss it.

CHAPTER III

A 3-DIMENSIONAL SINE FUNCTIONAL EQUATION

3.1 Background

We aim to extend the sine functional equation from 2 variables to 3 variables so we observe many identities of sine function with 3 variables so that we obtain some interesting properties listed belows:

$$\begin{aligned}-4\sin(x)\sin(y)\sin(z) &= \sin(x+y+z) - \sin(x+y-z)\\ &\quad - \sin(x-y+z) + \sin(x-y-z),\\ 4\sinh(x)\sinh(y)\sinh(z) &= \sinh(x+y+z) - \sinh(x+y-z)\\ &\quad - \sinh(x-y+z) + \sinh(x-y-z).\end{aligned}$$

With these properties, we then study a functional equation

$$cf(x)f(y)f(z) = f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z), \quad (3.1.1)$$

where c is a constant, and we will call it "**a 3-dimensional sine functional equation**".

We can easily see from the above identities that when $c = -4$ or $c = 4$ the sine and hyperbolic sine are solutions of the equation, respectively. For another constant c, we ask if its solutions are the same as the solutions of the sine functional equation.

Because of the complexity of c, we will separate this problem into two cases, $c = 0$ and $c \neq 0$. We note that when $c = 0$ the 3-dimensional sine functional equation degenerates to the following form:

$$f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z) = 0 \tag{3.1.2}$$

which will be referred to as "**the degenerate form of the 3-dimensional sine functional equation**"or simply "**the degenerate form**". In the same sense, when $c \neq 0$ we will refer to

$$cf(x)f(y)f(z) = f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z) \tag{3.1.3}$$

as "**the non-degenerate form of the 3-dimensional sine functional equation**"or simply "**the non-degenerate form**".

We can solve this problem, in the case of the function $f : G \to \mathbb{C}$ where G is an abelian 2-divisible group, that the class of the general solution of the non-degenerate form is contained in the class of the general solution of the sine functional equation and the general solution of the degenerate form is the same as that of Jensen's functional equation. It appropriately supports our aim.

In the next two sections, we will give the general solutions of these two cases.

3.2 The General Solution of the Non-degenerate Form

To solve the non-degenerate form, we would like to give the preview of proof first. We begin with proving that the class of the general solution of the non-degenerate form is contained in the class of the general solution of the sine functional equation

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2. \tag{3.2.1}$$

It means that we can look for the solution only in the solution class of the sine functional equation. Then we dispose the class which does not satisfy the non-degenerate form and, finally, we prove that the remain class is our solution. Now let us move the first step of proof with the following proposition.

Proposition 3.2.1. *Let G be an abelian group. If $f : G \to \mathbb{C}$ satisfies the non-degenerate form (3.1.3) for all $x, y, z \in G$, then f satisfies the sine functional equation (3.2.1) for all $x, y, z \in G$.*

Proof. Let c be a nonzero complex number. It is easy to see that if f is the zero function then it satisfies both the non-degenerate form and the sine functional equation. Thus let a non-trivial function $f : G \to \mathbb{C}$ satisfy the non-degenerate form.

Putting $(x, y, z) = (x, y, y)$ in (3.1.3), we obtain

$$cf(x)f(y)^2 = f(x+2y) - f(x) - f(x) + f(x-2y) \tag{3.2.2}$$

Again putting $(x, y, z) = (x, z, z)$ in (3.1.3), we obtain

$$cf(x)f(z)^2 = f(x+2z) - f(x) - f(x) + f(x-2z) \tag{3.2.3}$$

Then subtracting (3.2.3) from (3.2.2), we get

$$\begin{aligned} &cf(x)\left[f(y)^2 - f(z)^2\right] \\ &\quad = f(x+2y) - f(x+2z) - f(x-2z) + f(x-2y) \\ &\quad = cf(x)f(y+z)f(y-z) \end{aligned}$$

So

$$cf(x)\left[\left(f(y)^2 - f(z)^2\right) - f(y+z)f(y-z)\right] = 0. \tag{3.2.4}$$

Since this equation is true for any $x, y, z \in G$ and $c \neq 0$ and $f \not\equiv 0$, there exists $a \in \mathbb{C}$ such that $f(a) \neq 0$. So we can set $x = a$ and divide both sides of (3.2.4) by $cf(a)$. Then we derive $f(y+z)f(y-z) = f(y)^2 - f(z)^2$ for all $x, y, z \in G$. This completes the proof. □

This proposition allows us to look for our solution only in the class of the general solution of the sine functional equation. So let us look at the theorem providing the general solution of the sine functional equation which is referred to [1]. We should note here that a *quadratically closed field* $\mathbb{F}$ is a field with the property that for all $c \in \mathbb{F}$, there exists $k \in \mathbb{F}$ such that $k^2 = c$ (see [1],p.209).

Theorem 3.2.2. *([1],p.219) Let* $\mathbb{F}$ *be a quadratically closed field divisible by* 2*,* G *be an abelian group divisible by* 2*. The general solutions* $f : G \to \mathbb{F}$ *of the sine functional equation (3.2.1) are given by*

$$f(x) = a\frac{h(x) - h(-x)}{2} \quad \text{and} \quad f(x) = l(x) \tag{3.2.5}$$

where h *and* l *are arbitrary solutions of* $h(x+y) = h(x)h(y)$ *and* $l(x+y) = l(x) + l(y)$ *and* a *is an arbitrary element of* $\mathbb{F}$*.*

It is still true when $\mathbb{F} = \mathbb{C}$ so we can see that there are only 2 possible classes which can be our solutions. The next process is to observe which class the solution of the non-degenerate form is. The lemma below will cut the class of additive function off.

Lemma 3.2.3. *Let* $f : G \to \mathbb{C}$ *be a complex-valued function defined on an abelian group* G*. A non-trivial additive function* f *does not satisfy the non-degenerate form (3.1.3).*

Proof. Assume on a contrary that f is an additive function satisfying (3.1.3). Using the properties of additive function that $f(x+y) = f(x)+f(y)$ and $f(-x) = -f(x)$, we have

$$\begin{aligned} cf(x)f(y)f(z) &= f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z) \\ &= f(x) + f(y) + f(z) - f(x) - f(y) + f(z) \\ &\quad - f(x) + f(y) - f(z) + f(x) - f(y) - f(z) \\ &= 0. \end{aligned}$$

Then we set $(x, y, z) = (x, x, x)$ in above equation, we get $cf(x)^3 = 0$. Since $c \neq 0$, so $f(x) = 0$ for all $x \in G$. It means that f is trivial so the proof is completed. $\square$

Now the only possibility is that the general solution of the non-degenerate form is given by $f(x) = a\frac{h(x)-h(-x)}{2}$ where h is an arbitrary solution of $h(x+y) = h(x)h(y)$ for all $x, y \in G$. But we can prove that a has to be in some specific form. The proof is given below.

Lemma 3.2.4. *For a given nonzero complex number c, let $f : G \to \mathbb{C}$ be a complex-valued function defined on an abelian group G. If $f(x) = a\frac{h(x)-h(-x)}{2}$ where $a^2 = \frac{4}{c}$ and $h(x + y) = h(x)h(y)$ for all $x, y \in G$, then f satisfies the non-degenerate form (3.1.3).*

Proof. Assume that $f(x) = a\frac{h(x)-h(-x)}{2}$ where $a^2 = \frac{4}{c}$ and $h(x + y) = h(x)h(y)$ for all $x, y \in G$. Begin with the right-hand side of (3.1.3) which may be defined as $g(x, y, z)$, for convenience, and then transform it into the form of h by using the assumption. So we get

$$\begin{aligned} g(x,y,z) &= f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z) \\ &= \frac{a}{2}[h(x+y+z) - h(-x-y-z) - h(x+y-z) + h(-x-y+z) \\ &\quad - h(x-y+z) + h(-x+y-z) + h(x-y-z) - h(-x+y+z)]. \end{aligned} \tag{3.2.6}$$

Then rearrange the derived equation (3.2.6) and use the property of h, we get

$$
\begin{aligned}
g(x,y,z) = \frac{a}{2}[&h(x+y+z) - h(x+y-z) - h(x-y+z) \\
&+ h(x-y-z) - h(-x+y+z) + h(-x+y-z) \\
&+ h(-x-y+z) - h(-x-y-z)] \\
= \frac{a}{2}[&h(x)h(y)h(z) - h(x)h(y)h(-z) - h(x)h(-y)h(z) \\
&+ h(x)h(-y)h(-z) - h(-x)h(y)h(z) + h(-x)h(y)h(-z) \\
&+ h(-x)h(-y)h(z) - h(-x)h(-y)h(-z)].
\end{aligned}
$$

Now, we can factorize the above equation to

$$
\begin{aligned}
g(x,y,z) &= \frac{a}{2}[(h(x)h(y) - h(x)h(-y) - h(-x)h(y) + h(-x)h(-y))\,(h(z) - h(-z))] \\
&= \frac{a}{2}\left[(h(x) - h(-x))\,(h(y) - h(-y))\,(h(z) - h(-z))\right].
\end{aligned}
$$

Then, by simple calculations, we can change the right-hand side into the form of function f.

$$
\begin{aligned}
g(x,y,z) &= \frac{4}{a^2} f(x)f(y)f(z) \\
&= cf(x)f(y)f(z).
\end{aligned}
$$

It means that f satisfies the non-degenerate form. This completes the proof. □

The Lemma 3.2.4 shows that the constant a of the solution depends on the constant c of the non-degenerate form and must be only in that form. Hence we get the theorem which gives the general solution of the non-degenerate form.

Theorem 3.2.5. *The general solution $f : G \to \mathbb{C}$, where G is an abelian 2-divisible group, of the non-degenerate form (3.1.3) is given by*

$$
f(x) = a\frac{h(x) - h(-x)}{2}
$$

where $a^2 = \frac{4}{c}$ and h is an arbitrary solution of $h(x+y) = h(x)h(y)$ for all $x, y \in G$.

When $f : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{R} \to \mathbb{R}$, the solutions, which are not necessary continuous, of the sine functional equation are given in following corollary.

Corollary 3.2.6. *([1],p.219) The general solution $f : \mathbb{R} \to \mathbb{C}$ or $f : \mathbb{R} \to \mathbb{R}$ of the sine functional equation in the class of measurable functions on a proper interval are given by*

$$f(x) = a \sinh bx, \quad f(x) = ax,$$

and

$$f(x) = a \sinh bx, \quad f(x) = a \sin bx, \quad f(x) = ax,$$

where a and b are complex or real constants, respectively.

Remark 3.2.1. To solve the equation for the case $f : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{R} \to \mathbb{R}$, we can conclude from Lemma 3.2.3 that non-trivial additive functions $f(x) = ax$ are not the solution of the non-degenerate form and it is straightforward from Theorem 3.2.5 that the solution $f : \mathbb{R} \to \mathbb{C}$ of the non-degenerate form has to be given by $f(x) = a \sinh bx$ where $a^2 = \frac{4}{c}$ and $b \in \mathbb{C}$.

Therefore we obtain the next corollary.

Corollary 3.2.7. *The general solution $f : \mathbb{R} \to \mathbb{C}$ of the non-degenerate form (3.1.3) in the class of measurable functions on a proper interval are given by*

$$f(x) = a \sinh bx$$

where $b \in \mathbb{C}$ and $a^2 = \frac{4}{c}$.

For the case $f : \mathbb{R} \to \mathbb{R}$, the constant a of the solution has to be classified. We give it in the next theorem.

Theorem 3.2.8. *The general solution* $f:\mathbb{R}\to\mathbb{R}$ *of the non-degenerate form (3.1.3) in the class of measurable functions on a proper interval are given by*

$$f(x)=a\sinh bx \quad \text{where} \quad a^2=\frac{4}{c},\ c>0 \quad \text{or}$$
$$f(x)=a\sin bx \quad \text{where} \quad a^2=\frac{4}{-c},\ c<0,$$

where $b\in\mathbb{R}$.

Proof. It is obvious for the trivial function that it is a solution of the non-degenerate form so we will discuss for the non-trivial function. From Corollary 3.2.6 and Remark 3.2.1, we can conclude that the solution must be in the class that are given in Corollary 3.2.6. Thus we remain to classify the constant a. Let us consider the followings.

If $f(x)=a\sinh bx$ satisfies (3.1.3), we must have

$$\begin{aligned} ca^2\sinh(bx)\sinh(by)\sinh(bz) &= \sinh(bx+by+bz)-\sinh(bx+by-bz)\\ &\quad -\sinh(bx-by+bz)+\sinh(bx-by-bz). \end{aligned} \tag{3.2.7}$$

If $f(x)=a\sin bx$ satisfies (3.1.3), we must have

$$\begin{aligned} ca^2\sin(bx)\sin(by)\sin(bz) &= \sin(bx+by+bz)-\sin(bx+by-bz)\\ &\quad -\sin(bx-by+bz)+\sin(bx-by-bz). \end{aligned} \tag{3.2.8}$$

As the identities of hyperbolic sine and sine that we gave at the Background section, we can see that the equation (3.2.7) and (3.2.8) will be valid only when $ca^2=4$ and $ca^2=-4$, respectively. In the other words, we can get the conclusions as stated in theorem. This completes the proof. □

From Lemma 3.2.3, we can conclude that if $c=0$ then the additive function is a solution of the degenerate form. This is an interesting result to study and gives us a trail to solve the degenerate form.

3.3 The General Solution of the Degenerate Form

For the degenerate form we can show that the degenerate form (3.1.2) has the same general solution as the solution of Jensen's functional equation,

$$f(x) + f(y) = 2f(\frac{x+y}{2}). \tag{3.3.1}$$

This means that the degenerate form extends Jensen's functional equation to 3 dimensions.

Theorem 3.3.1. *Let G be an abelian 2-divisible group. A complex-valued function $f : G \to \mathbb{C}$ satisfies the degenerate form (3.1.2) for all $x, y, z \in G$ if and only if f satisfies Jensen's functional equation (3.3.1) for all $x, y, z \in G$*

Proof. To prove the necessity, let $f : G \to \mathbb{C}$ satisfy (3.1.2).
Substituting $(x, y, z) = (x, \frac{y+z}{2}, \frac{y-z}{2})$ in (3.1.2). Thus we have

$$f(x+y) - f(x+z) - f(x-z) + f(x-y) = 0. \tag{3.3.2}$$

Putting $(x, y, z) = (\frac{x+y}{2}, \frac{x-y}{2}, 0)$ in (3.3.2), we obtain

$$f(x) + f(y) = 2f(\frac{x+y}{2}) \;\; \forall x, y \in G.$$

Conversely, let $f : G \to \mathbb{C}$ satisfy the functional equation (3.3.1).
Putting $(x, y) = (x+y, x-y)$ and $(x, y) = (x+z, x-z)$ in (3.3.1), we obtain

$$f(x+y) + f(x-y) = 2f(x)$$

and

$$f(x+z) + f(x-z) = 2f(x),$$

respectively. With these two equations, we obtain

$$f(x+y) + f(x-y) = f(x+z) + f(x-z) \; \forall x, y, z \in G. \tag{3.3.3}$$

Then setting $(x,y,z)=(x,y+z,y-z)$ in (3.3.3), we have

$$f(x+y+z)-f(x+y-z)-f(x-y+z)+f(x-y-z)=0$$

for all $x,y,z\in G$ as desired. □

We can easily see that if f satisfies Cauchy's additive functional equation so $f(0)=0$ then it satisfies Jensen's functional equation. It implies that additive function is a solution of the degenerate form. With this fact, it is reasonable to add a condition $f(0)=0$ and prove that the general solution of the degenerate form is contained in the solution class of the sine functional equation.

Proposition 3.3.2. *If $f:G\to\mathbb{C}$ satisfies the degenerate form (3.1.2) for all $x,y,z\in G$ and $f(0)=0$, then f satisfies the sine functional equation.*

Proof. Let f satisfy (3.1.2). Then setting $(x,y,z)=(\frac{x+y}{2},\frac{x}{2},\frac{y}{2})$, we get

$$f(x+y)-f(x)-f(y)+f(0)=0.$$

Since $f(0)=0$, we have

$$f(x+y)=f(x)+f(y).$$

Hence f satisfies Cauchy's functional equation, by Theorem 3.2.2, we can conclude that f satisfies the sine functional equation. □

This concludes that if $f(0)$ is fixed to be 0, then the solutions of our 3-dimensional sine functional equation are contained in the solution class of the sine functional equation for all $c\in\mathbb{C}$.

We have provided all solutions of the 3-dimensional sine functional equation so we will pass to the next chapter to study about the stabilities of the 3-dimensional sine functional equation.

CHAPTER IV

THE STABILITIES OF THE 3-DIMENSIONAL SINE FUNCTIONAL EQUATION

Similar to the study about solutions, the stability of our 3-dimensional sine functional equation will be separated into two cases, the non-degenerate form and the degenerate form. For the non-degenerate form, it appears that its general solution is of the form of sine function which is one of trigonometric functions. According to many papers investigating cosine functional equation and sine functional equation (see [3],[5],[6],[9]), the superstability is the most popular stability for this type. So we will study the superstability of the non-degenerate form as they do with the classical sine functional equation.

4.1 The Superstability of the Non-degenerate Form

Now we consider the superstability of the non-degenerate form.

Theorem 4.1.1. *Let $\delta > 0, c \in \mathbb{C} - \{0\}$ and G be an abelian group. If a function $f : G \to \mathbb{C}$ satisfies the inequality*

$$|cf(x)f(y)f(z) - f(x+y+z) + f(x+y-z) + f(x-y+z) - f(x-y-z)| \leq \delta, \quad (4.1.1)$$

for all $x, y, z \in G$, then either f is bounded or satisfies the non-degenerate form

$$cf(x)f(y)f(z) = f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z). \quad (4.1.2)$$

Proof. Assume that f is an unbounded complex-valued function satisfying the inequality (4.1.1) for all $x, y, z \in G$. So we can choose a sequence x_n in G such that $|cf(x_n)^2| > 2$ and $|f(x_n)| \to \infty$ as $n \to \infty$.

Therefore, we have

$$|cf(x_n)^2 + 2| \geq |cf(x_n)^2| - 2 > 0.$$

Next, putting $(x, y, z) = (x, x_n, x_n)$ in (4.1.1), then we obtain

$$\begin{aligned}
\left|cf(x)f(x_n)^2 - f(x+2x_n) + f(x) + f(x) - f(x-2x_n)\right| &\leq \delta \\
\left|f(x)(cf(x_n)^2+2) - f(x+2x_n) - f(x-2x_n)\right| &\leq \delta \\
\left|f(x) - \frac{f(x+2x_n)+f(x-2x_n)}{cf(x_n)^2+2}\right| &\leq \frac{\delta}{|cf(x_n)^2+2|}
\end{aligned}$$

Taking limit as $n \to \infty$, then the right-hand side tends to 0. Thus we have

$$f(x) = \lim_{n\to\infty} \frac{f(x+2x_n)+f(x-2x_n)}{cf(x_n)^2+2}. \tag{4.1.3}$$

Putting $(x, y, z) = (x, y, z+2x_n)$ and $(x, y, z) = (x, y, z-2x_n)$ in (4.1.1), then we get

$$\begin{aligned}
|cf(x)f(y)f(z+2x_n) &- f(x+y+z+2x_n) + f(x+y-z-2x_n) \\
&+ f(x-y+z+2x_n) - f(x-y-z-2x_n)| \leq \delta
\end{aligned} \tag{4.1.4}$$

and

$$\begin{aligned}
|cf(x)f(y)f(z-2x_n) &- f(x+y+z-2x_n) + f(x+y-z+2x_n) \\
&+ f(x-y+z-2x_n) - f(x-y-z+2x_n)| \leq \delta,
\end{aligned} \tag{4.1.5}$$

respectively. From the inequality (4.1.4), the inequality (4.1.5) and the triangle inequality, we derive

$$\begin{aligned}
&|(cf(x)f(y)f(z+2x_n) + cf(x)f(y)f(z-2x_n)) \\
&- (f(x+y+z+2x_n) + f(x+y+z-2x_n)) \\
&+ (f(x+y-z-2x_n) + f(x+y-z+2x_n)) \\
&+ (f(x-y+z+2x_n) + f(x-y+z-2x_n)) \\
&- (f(x-y-z-2x_n) + f(x-y-z+2x_n))| \leq 2\delta,
\end{aligned}$$

Then dividing both sides by $|cf(x_n)^2+2|$, we derive

$$\begin{aligned}\Big|&\left(\frac{cf(x)f(y)(f(z+2x_n)+f(z-2x_n))}{cf(x_n)^2+2}\right)\\&-\left(\frac{f(x+y+z+2x_n)+f(x+y+z-2x_n)}{cf(x_n)^2+2}\right)\\&+\left(\frac{f(x+y-z-2x_n)+f(x+y-z+2x_n)}{cf(x_n)^2+2}\right)\\&+\left(\frac{f(x-y+z+2x_n)+f(x-y+z-2x_n)}{cf(x_n)^2+2}\right)\\&-\left(\frac{f(x-y-z-2x_n)+f(x-y-z+2x_n)}{cf(x_n)^2+2}\right)\Big|\le\frac{2\delta}{|cf(x_n)^2+2|}.\end{aligned}$$

By the definition of $f(x)$ in (4.1.3) and taking limit as $n\to\infty$ in the above inequality, we have

$$cf(x)f(y)f(z)=f(x+y+z)-f(x+y-z)-f(x-y+z)+f(x-y-z).$$

So f satisfies the non-degenerate form. This completes the proof. $\square$

We would like to give some examples of bounded solutions of the inequality (4.1.1).

Example 4.1.1. For $c\neq 0$ and $\delta>0$, we set $f(x)=\sqrt[3]{\frac{\delta}{2|c|}}$ for all $x\in G$. So f is a constant function and it is bounded. Next consider that:

$$\begin{aligned}&|cf(x)f(y)f(z)-f(x+y+z)+f(x+y-z)+f(x-y+z)-f(x-y-z)|\\&\quad=|c\cdot\frac{\delta}{2|c|}-0|\\&\quad=\frac{\delta}{2}\le\delta.\end{aligned}$$

Example 4.1.2. For a non-linear function, we consider in the case of $G=\mathbb{R}$, $c=4$ and $\delta=2$. First we recall that when $c=4$ the solution of the degenerate form is $f(x)=a\sinh(bx)$ where $a^2=1$. Then we set $f(x)=\frac{1}{4}\sin(x)$ where $x\in\mathbb{R}$. It is obvious that f is bounded by $\frac{1}{4}$ and f is not a solution of the degenerate form with $c = 4$. Next substituting f in the inequality and using identities of sine, we get that:

$$\begin{aligned}
&|4f(x)f(y)f(z) - f(x+y+z) + f(x+y-z) + f(x-y+z) - f(x-y-z)| \\
&\quad = |\frac{1}{16}\sin(x)\sin(y)\sin(z) - \frac{1}{4}(\sin(x+y+z) \\
&\quad\quad - \sin(x+y-z) - \sin(x-y+z) + \sin(x-y-z))| \\
&\quad = |(\frac{1}{16}+1)\sin(x)\sin(y)\sin(z)| \\
&\quad \leq 2 = \delta.
\end{aligned}$$

4.2 The Stability of the Degenerate Form

We will give the stability of the degenerate form in the sense of generalized Hyers-Ulam-Rassias and treat Hyers-Ulam-Rassias stability and Hyers-Ulam stability as special cases.

Here we use a Banach space as the range of the solution so we recall that a *Banach space* is a normed linear space which is complete in the metric defined by its norm.

Theorem 4.2.1. *Let E be a Banach space, G be an abelian 2-divisible group and $\phi : G^3 \to [0, \infty)$. If $f : G \to E$ satisfies the inequality*

$$||f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z)|| \leq \phi(x,y,z), \quad (4.2.1)$$

where
$$\begin{cases} \Phi(x,y,z) := \sum_{k=1}^{\infty} 2^{-k}\phi(2^{k-1}x, 2^{k-2}y, 2^{k-2}z) < \infty \quad \forall x,y,z \in G. & \text{(I)} \\ \Phi(x,y,z) := \sum_{k=0}^{\infty} 2^{k}\phi(2^{-k-1}x, 2^{-k-2}y, 2^{-k-2}z) < \infty \quad \forall x,y,z \in G. & \text{(II)} \end{cases}$$

Then there exists a unique function $A : G \to E$ as a solution of the degenerate form (3.1.2) such that $A(0) = f(0)$ and

$$||A(x) - f(x)|| \leq \Phi(x,x,x), \ \forall x \in G. \quad (4.2.2)$$

Proof. First we will prove for the case (I). Suppose that $f : G \to E$ satisfies the inequality (4.2.1) where

$$\Phi(x, y, z) := \sum_{k=1}^{\infty} \frac{1}{2^k} \phi(2^{k-1}x, 2^{k-2}y, 2^{k-2}z) < \infty$$

for $x, y, z \in G$.

Then putting $(x, y, z) = (x, \frac{x}{2}, \frac{x}{2})$ in (4.2.1), we get

$$||f(2x) + f(0) - 2f(x)|| \le \phi(x, \frac{x}{2}, \frac{x}{2}), \quad x \in G. \tag{4.2.3}$$

Let $F(x) := f(x) - f(0)$. Then $F(0) = 0$ and, from (4.2.3), we obtain

$$||F(2x) - 2F(x)|| = ||f(2x) + f(0) - 2f(x)|| \le \phi(x, \frac{x}{2}, \frac{x}{2}). \tag{4.2.4}$$

Replacing x by $2^n x$ in the inequality (4.2.4) and dividing its result by 2^{n+1}, we get

$$\left\| \frac{F(2^{n+1}x)}{2^{n+1}} - \frac{F(2^n x)}{2^n} \right\| \le \frac{1}{2^{n+1}} \phi(2^n x, 2^{n-1}x, 2^{n-1}x), \tag{4.2.5}$$

for all $x \in G$ and all nonnegative integers n. Using (4.2.5) and the triangle inequality we have

$$\left\| \frac{F(2^m x)}{2^m} - \frac{F(2^n x)}{2^n} \right\| \le \sum_{k=m+1}^{n} \frac{1}{2^k} \phi(2^{k-1}x, 2^{k-2}x, 2^{k-2}x) \tag{4.2.6}$$

for all $x \in G$ and all nonnegative integers m and n with $m < n$. This show that $\left\{ \frac{F(2^n x)}{2^n} \right\}$ is a Cauchy sequence for all $x \in G$ since the right side of (4.2.6) converges to zero by the assumption of ϕ when $m \to \infty$.

Since E is complete, the Cauchy sequence $\left\{ \frac{F(2^n x)}{2^n} \right\}$ is converge. So we can define a mapping $A : G \to E$ by

$$A(x) := \lim_{n \to \infty} \frac{F(2^n x)}{2^n} + f(0), \quad \forall x \in G. \tag{4.2.7}$$

Setting $x = 0$ in (4.2.7), we derive $A(0) = f(0)$. To show that A satisfies the inequality (4.2.2), putting $m = 0$ in (4.2.6), we have

$$\left\| F(x) - \frac{F(2^n x)}{2^n} \right\| \le \sum_{k=1}^{n} \frac{1}{2^k} \phi(2^{k-1}x, 2^{k-2}x, 2^{k-2}x) \tag{4.2.8}$$

and then taking limit as $n \to \infty$ in (4.2.8), we obtain

$$\|f(x) - A(x)\| \leq \Phi(x, x, x),$$

as desired.

Next we will show that A satisfies the degenerate form, let us consider:

$$\begin{aligned}
&||A(x+y+z) - A(x+y-z) - A(x-y+z) + A(x-y-z)|| \\
&= \lim_{n\to\infty} ||\frac{F(2^n x + 2^n y + 2^n z)}{2^n} - \frac{F(2^n x + 2^n y - 2^n z)}{2^n} \\
&\quad - \frac{F(2^n x - 2^n y + 2^n z)}{2^n} + \frac{F(2^n x - 2^n y - 2^n z)}{2^n}|| \\
&= \lim_{n\to\infty} \frac{1}{2^n} ||F(2^n x + 2^n y + 2^n z) - F(2^n x + 2^n y - 2^n z) \\
&\quad - F(2^n x - 2^n y + 2^n z) + F(2^n x - 2^n y - 2^n z)|| \\
&= \lim_{n\to\infty} \frac{1}{2^n} ||f(2^n x + 2^n y + 2^n z) - f(2^n x + 2^n y - 2^n z) \\
&\quad - f(2^n x - 2^n y + 2^n z) + f(2^n x - 2^n y - 2^n z)|| \\
&\leq \lim_{n\to\infty} \frac{1}{2^n} \phi(2^{n-1} \cdot 2x, 2^{n-2} \cdot 2^2 y, 2^{n-2} \cdot 2^2 z)
\end{aligned}$$

By the property of Φ, the right-hand side of above inequality converges to 0 as $n \to \infty$. So

$$A(x+y+z) - A(x+y-z) - A(x-y+z) + A(x-y-z) = 0.$$

Finally, we will show the uniqueness of A. Assume that there is another function $\tilde{A} : G \to \mathbb{C}$ satisfying the degenerate form (3.1.3), $\tilde{A}(0) = f(0)$ and satisfies the inequality $||\tilde{A}(x) - f(x)|| \leq \Phi(x, x, x)\ \forall x \in G$.

Since A and $\tilde{A}$ satisfy the degenerate form, they also satisfy

$$f(2x) + f(0) - 2f(x) = 0 \quad \forall x \in G$$

or the simpler form

$$f(x) = \frac{1}{2}[f(2x) + f(0)] \quad \forall x \in G. \tag{4.2.9}$$

Substituting x by $2x, 2^2x, \ldots, 2^{n-1}x$ in (4.2.9) where n is a positive integer, we get

$$\begin{aligned} f(2x) &= \frac{1}{2}[f(2^2x) + f(0)] \\ f(2^2x) &= \frac{1}{2}[f(2^3x) + f(0)] \\ &\vdots \\ f(2^{n-1}x) &= \frac{1}{2}[f(2^nx) + f(0)]. \end{aligned}$$

By reverse substitutions of the above system, we derive

$$f(x) = \frac{1}{2^n} f(2^nx) + \sum_{k=1}^{n} \frac{1}{2^k} f(0)$$

which is satisfied by A and $\tilde{A}$.

Hence we obtain

$$||A(x) - \tilde{A}(x)|| = \frac{1}{2^n}||A(2^nx) - \tilde{A}(2^nx)|| + \sum_{k=1}^{n} \frac{1}{2^k}||A(0) - \tilde{A}(0)||. \qquad (4.2.10)$$

Since $A(0) = f(0) = \tilde{A}(0)$, $||A(0) - \tilde{A}(0)|| = 0$.

Then we take limit as $n \to \infty$, we derive

$$\begin{aligned} ||A(x) - \tilde{A}(x)|| &= \lim_{n\to\infty} \frac{1}{2^n}||A(2^nx) - \tilde{A}(2^nx)|| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\Big(||A(2^nx) - f(2^nx)|| + ||f(2^nx) - \tilde{A}(2^nx)||\Big) \\ &\le \lim_{n\to\infty} \frac{1}{2^n} 2\Phi(2^nx, 2^nx, 2^nx) \\ &\le \lim_{n\to\infty} \frac{1}{2^n} 2\sum_{k=1}^{\infty} \frac{1}{2^k}\phi(2^{k+n-1}x, 2^{k+n-2}x, 2^{k+n-2}x) \\ &\le \lim_{n\to\infty} 2\sum_{k=n+1}^{\infty} \frac{1}{2^k}\phi(2^{k-1}x, 2^{k-2}x, 2^{k-2}x) \\ &= 0. \end{aligned}$$

We can conclude that $A(x) = \tilde{A}(0)$. This proves the uniqueness of A.

Next we will consider for the case (II). The procedure of proof in this case runs similarly to the proof of the case (I). The differences are when we have the equation (4.2.4), we will replace x by $2^{-n}x$ and divide its result by 2^{-n+1}, instead. Then we will get that $\left\{\frac{F(2^{-n}x)}{2^{-n}}\right\}$ is a Cauchy sequence. And we can define

$$A(x) := \lim_{n\to\infty} \frac{F(2^{-n}x)}{2^{-n}} + f(0), \quad \forall x \in G. \tag{4.2.11}$$

Then the proof runs along the same as that of the case (I). □

We will give the stability of the degenerate form in the sense of Ulam-Hyers-Rassias. For this type of stability, we will use a normed space E_1 instead of a group G.

Corollary 4.2.2. *Let E be a Banach space and E_1 be a normed space. If $f : E_1 \to E$ satisfies the inequality*

$$||f(x+y+z)-f(x+y-z)-f(x-y+z)+f(x-y-z)|| \leq \theta(\|x\|^p+\|y\|^p+\|z\|^p), \tag{4.2.12}$$

for all $x, y, z \in E_1$, for some $\theta \geq 0$ and some $0 < p < 1$ or $p > 1$. Then there exists a unique function $A : E_1 \to E$ as a solution of the degenerate form (3.1.2) such that $A(0) = f(0)$ and

$$||A(x) - f(x)|| \leq \theta \frac{2^{1-p}+1}{|2-2^p|}\|x\|^p. \tag{4.2.13}$$

Proof. For the case $0 < p < 1$, we have that $-1 < p-1 < 0$ so $2^{-1} < 2^{p-1} < 1$. Then setting $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ in the inequality (4.2.1) and consider that:

$$\begin{aligned}
\sum_{k=1}^{\infty} 2^{-k}\phi(2^{k-1}x, 2^{k-2}y, 2^{k-2}z) &= \sum_{k=1}^{\infty} \frac{1}{2^k}\theta(\|2^{k-1}x\|^p + \|2^{k-2}y\|^p + \|2^{k-2}z\|^p) \\
&= \sum_{k=1}^{\infty} \frac{1}{2^k}\theta 2^{(k-1)p}(\|x\|^p + \|\frac{y}{2}\|^p + \|\frac{z}{2}\|^p) \\
&= \sum_{k=1}^{\infty} \frac{\theta}{2^p} 2^{(p-1)k}(\|x\|^p + \|\frac{y}{2}\|^p + \|\frac{z}{2}\|^p) \\
&= \frac{\theta}{2^p}\frac{2^{p-1}}{1-2^{p-1}}(\|x\|^p + \|\frac{y}{2}\|^p + \|\frac{z}{2}\|^p) \\
&= \theta\frac{2^{-1}}{1-2^{p-1}}(\|x\|^p + \|\frac{y}{2}\|^p + \|\frac{z}{2}\|^p) \\
&= \theta\frac{1}{2-2^p}(\|x\|^p + \|\frac{y}{2}\|^p + \|\frac{z}{2}\|^p) < \infty. \qquad (4.2.14)
\end{aligned}$$

So we can set $\Phi(x, y, z) := \sum_{k=1}^{\infty} 2^{-k}\phi(2^{k-1}x, 2^{k-2}y, 2^{k-2}z)$ which satisfies the condition (I) in Theorem 4.2.1. Next consider that

$$\Phi(x, x, x) = \theta\frac{1}{2-2^p}(\|x\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p) = \theta\frac{2^{1-p}+1}{2-2^p}\|x\|^p \geq 0.$$

Then the proof runs along the same procedure as Theorem 4.2.1 and it implies the conclusion (4.2.13).

For the case $p > 1$, we can set ϕ in the same way as above but we can prove that ϕ will satisfy the condition (II) of Theorem 4.2.1 so we remain to observe $\Phi(x, x, x)$.

$$\begin{aligned}
\Phi(x, x, x) :&= \sum_{k=0}^{\infty} 2^k\phi(2^{-k-1}x, 2^{-k-2}x, 2^{-k-2}x) \\
&= \sum_{k=0}^{\infty} 2^k\theta(\|2^{-k-1}x\|^p + \|2^{-k-2}y\|^p + \|2^{-k-2}z\|^p) \\
&= \sum_{k=0}^{\infty} 2^k\theta 2^{(-k-1)p}(\|x\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p) \\
&= \sum_{k=0}^{\infty} \frac{\theta}{2^p} 2^{(1-p)k}(\|x\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p).
\end{aligned}$$

Since $1 - p < 0$, $2^{1-p} < 1$. Thus the sum in the above equation is convergent.

Then we get

$$\Phi(x,x,x) = \frac{\theta}{2^p}\frac{1}{1-2^{1-p}}(\|x\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p)$$

which can arrange into the form

$$\begin{aligned}\Phi(x,x,x) &= \theta\frac{1+2^{1-p}}{2^p-2}\|x\|^p \\ &= \theta\frac{2^{1-p}+1}{2^p-2}\|x\|^p \geq 0.\end{aligned}$$

Thus we proved the case $p > 1$. □

In the case $p = 0$, we observe that it can be treated to the case of Hyers-Ulam stability as follow.

Corollary 4.2.3. *Let $\delta > 0$, E be a Banach space and G be an abelian 2-divisible group. If $f : G \to E$ satisfies the inequality*

$$||f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z)|| \leq \delta, \qquad (4.2.15)$$

for all $x, y, z \in G$. Then there exists a unique function $A : G \to E$ as a solution of the degenerate form (3.1.2) such that $A(0) = f(0)$ and

$$||A(x) - f(x)|| \leq \delta, \quad x \in G. \qquad (4.2.16)$$

Proof. Putting $\phi(x,y,z) = \delta$ in inequality (4.2.1), then it implies (4.2.15). The proof runs along the same procedure as Theorem 4.2.1. □

REFERENCES

[1] Aczél, J., and Dhombres, J. *Functional equations in several variables.* Cambridge University Press, 1989.

[2] d'Alembert, J. Recheches sur la courbe que forme une corde tendue mise en vibration. *Histoire Académie Berlin* (1747): 214-249.

[3] Baker, J. A. The stability of the cosine equation. *Proc. Amer. Math. Soc.* 80 (1980): 411-416.

[4] Baker, J., Lawrence, J., and Zorzitto, F. The stability of the equation $f(x+y) = f(x)f(y)$. *Proc. Amer. Math. Soc.* **74** (1979): 242-246.

[5] Cholewa, P. W. The stability of the sine equation. *Proc. Amer. Math. Soc.* **88** (1983): 631-634.

[6] Chung, J., and Kim, D. Sine functional equation in several variables. *Arch. Math.* **86** (2006): 425-429.

[7] Czerwik, S. *Funtional equations and inequalities in several variables.* World Scientific Publishing, 2002.

[8] Hyers, D. H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. U.S.A.* **27** (1941): 222-224.

[9] Kim, G. H. On the stability of the generalized sine functional equations. *J. Math. Anal. Appl.* **331** (2004): 886-894.

[10] Kuczma, M. A survey of the theory of functional equations. *L'université Belgrade* **130** (1964): 1-64.

[11] Rassias, Th. M. On the stability of functional equations and a problem of ulam. *Acta. Appl. Math.* **62** (2000): 23-130.

[12] Schweitzer, A. R. On the history of functional equations. *Bull. Amer. Math. Soc.* **25** (1918/19): 439.

[13] Ulam, S. M. *A collection of mathematical problems.* Interscience, New York, 1960.

VITA

Mr.Oam Sthityanak was born on August 17, 1983 in Bangkok, Thailand. He received a financial support from the NSTDA in 2001-2004. He graduated with a Bachelor Degree with the first class honors of Science in Mathematics from Thammasat University, Patumthani Thailand in 2004. He also graduated in Minor of Insurance and Computer. For the master degree program, he had studied Mathematics at the Faculty of Science, Chulalongkorn University.