# $C_2$-COFINITENESS OF THE VERTEX ALGEBRA $V_L^+$ AND CLASSIFICATION OF IRREDUCIBLE MODULES OF THE VERTEX OPERATOR ALGEBRA $V_{\mathcal{L}}^{+^{\langle\tau\rangle}}$

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# $C_2$-โคไฟไนต์เนสของพีชคณิตเวอร์เท็กซ์ $V_L^+$ และ การจำแนกมอดูลที่ลดทอนไม่ได้ของพีชคณิตตัวดำเนินการเวอร์เท็กซ์ $V_{\mathcal{L}}^{+^{<\tau>}}$

นายพิเชษฐ์ จิตต์เจนการ



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ส่วนแรกของวิทยานิพนธ์เป็นการศึกษาทฤษฎีตัวแทนของพีชคณิตเวอร์เท็กซ์ $V_L^+$ เมื่อ $L$ คือแลตทิซคู่ที่ไม่เสื่อมลงซึ่งไม่เป็นบวกแน่นอน ในกรณีเฉพาะนั้น เราแสดงให้เห็นว่า พีชคณิตเวอร์เท็กซ์และมอดูลชนิดอ่อนที่ลดทอนไม่ได้ของมัน สอดคล้องกับเงื่อนไขที่เรียกว่า $C_2$- โคไฟไนต์เนส

สำหรับส่วนที่สองของวิทยานิพนธ์ เราศึกษาทฤษฎีตัวแทนของพีชคณิตตัวดำเนินการเวอร์เท็กซ์ $V_{\mathcal{L}}^{+<\tau>}$ ในที่นี้ $\mathcal{L}$ คือแลตทิซคู่บวกจำกัดเขตอันดับ 2 และ $\tau$ คืออัตสัณฐานของ $V_L^+$ เราสามารถจำแนกบางส่วนของ $V_{\mathcal{L}}^{+<\tau>}$ - มอดูลที่ลดทอนไม่ได้ ซึ่งอยู่ใน $V_L^+$ - มอดูลที่ลดทอนไม่ได้ ยิ่งกว่านั้น เราสร้างและจำแนก $\tau$ - ทวิสต์มอดูลของพีชคณิตเวอร์เท็กซ์ $M(1)^+$ ซึ่งเป็นพีชคณิตเวอร์เท็กซ์ย่อยของ $V_L^+$



ภาควิชา.....คณิตศาสตร์..... ลายมือชื่อนิสิต........พิเชษฐ์ จิตต์เจนการ........

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The first part of this thesis is the study of the representation theory of vertex algebras $V_L^+$ when $L$ are nondegenerate even lattices that are not positive definite. In particular, we show that such vertex algebras and their irreducible weak modules satisfy the so-called $C_2$-cofiniteness condition.

For the second part of this thesis, we study the representation theory of a vertex operator algebra $V_{\mathcal{L}}^{+\langle\tau\rangle}$. Here $\mathcal{L}$ is a positive definite even lattice of rank 2 and $\tau$ is an automorphism of $V_{\mathcal{L}}^+$. We are able to classify some irreducible $V_{\mathcal{L}}^{+\langle\tau\rangle}$-modules that are contained in irreducible $V_{\mathcal{L}}^+$-modules. Furthermore, we construct and classify some $\tau$-twisted modules of a vertex algebra $M(1)^+$. This vertex algebra is a vertex subalgebra of $V_{\mathcal{L}}^+$.



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# CHAPTER I

# INTRODUCTION

Throught this work, all vector spaces and algebras will be over $\mathbb{C}$ and $\mathbb{Z}_{\geq 0}$ is the set of positive integers.

## 1.1 Basic notions

We shall give some definitions of the notion of vertex algebra, as presented in [21] and [26].

### Formal calculus

For a vector space $V$, we will establish the following notations:

$$\begin{aligned}
V[z] &= \left\{ \sum_{n=0}^{\infty} v_n z^n \mid v_n \in V,\ v_n = 0 \text{ for all but finitely many } n \right\}, \\
V[z, z^{-1}] &= \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V,\ v_n = 0 \text{ for all but finitely many } n \right\}, \\
V[[z]] &= \left\{ \sum_{n=0}^{\infty} v_n z^n \mid v_n \in V \right\}, \\
V[[z, z^{-1}]] &= \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V \right\}.
\end{aligned}$$

Expanding formal series, we will use the following convention:

$$(x+y)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^{n-i} y^i, \quad (x-y)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^{n-i} (-y)^i$$

where $\binom{n}{0} = 1,\ \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},\ n \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$.

Let us define the formal $\delta$-function at $z = 1$ to be the series

$$\delta(z) = \textstyle\sum_{n\in\mathbb{Z}} z^n.$$

In the theory of vertex operator algebras we often use three-variable generating functions of the following sort:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) = \sum_{n\in\mathbb{Z}} \frac{(z_1 - z_2)^n}{z_0^{n+1}} = \sum_{i\in\mathbb{N}_0, n\in\mathbb{Z}} (-1)^i \binom{n}{i} z_0^{-n-1} z_1^{n-i} z_2^i$$

There are two basic properties of the $\delta$-function involving such expressions.

**Proposition 1.1.1.** ([26], p. 37).

$$z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right), \tag{1.1}$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right). \tag{1.2}$$

We also define a formal residue notation: for a positive integer $r$,

$$\mathrm{Res}_z \left(\textstyle\sum_{n\in\frac{1}{r}\mathbb{Z}} v_n z^n\right) = v_{-1}.$$

**Lattices**

A *lattice* $L$ of rank $n \in \mathbb{Z}_{>0}$ is a finite-rank $n$ free abelian group equipped with a symmetric nondegenerate $\mathbb{Q}$-valued $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle : L\times L \to \mathbb{Q}$. The nondegenerate property is the condition $\langle\alpha, L\rangle = 0$ implies $\alpha = 0$. There are some definitions of a lattice $L$ as follows:

1. $L$ is *even* if $\langle\alpha, \alpha\rangle \in 2\mathbb{Z}$ for $\alpha \in L$.

2. $L$ is *integral* if $\langle\alpha, \beta\rangle \in \mathbb{Z}$ for $\alpha, \beta \in L$.

3. $L$ is *positive definite* if $\langle \alpha, \alpha \rangle > 0$ for $\alpha \in L - \{0\}$.

4. $L$ is *negative definite* if $\langle \alpha, \alpha \rangle < 0$ for $\alpha \in L - \{0\}$.

A lattice isomorphism $\varphi$ from $L_1$ to $L_2$ is an *isometry*, i.e.

$$\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle \text{ for } \alpha, \beta \in L_1.$$

Notice that the even lattice $L$ is also integral, since

$$\langle \alpha, \beta \rangle = \tfrac{1}{2} \left( \langle \alpha + \beta, \alpha + \beta \rangle - \langle \alpha, \alpha \rangle - \langle \beta, \beta \rangle \right) \in \mathbb{Z} \quad \text{for } \alpha, \beta \in L.$$

The *dual lattice* $L^\circ$ of $L$ is defined to be

$$L^\circ = \{ \alpha \in L \otimes_{\mathbb{Z}} \mathbb{C} \mid \langle \alpha, L \rangle \subset \mathbb{Z} \}.$$

Then $L^\circ$ is a lattice whose rank is equal to the rank of $L$ and has as a base the dual base $\{\alpha_1^*, \ldots, \alpha_n^*\}$ of a given base $\{\alpha_1, \ldots, \alpha_n\}$ of $L$, defined by

$$\langle \alpha_i^*, \alpha_j \rangle = \delta_{i,j} \quad \text{for } i, j = 1, \ldots, n.$$

**Algebras**

An *algebra* is a vector space $V$ equipped with a bilinear map from $V \times V$ to $V$. The algebra $V$ is said to be *associative* if it contains an identity element 1 for multiplication and

$$(ab)c = a(bc) \quad \text{for } a, b, c \in V.$$

The algebra $V$ is said to be *commutative* if the commutative law holds:

$$ab = ba \quad \text{for } a, b \in V.$$

Given a group $G$, we define its *group algebra* to be the associative algebra $\mathbb{C}[G]$ which is formally the set of finite linear combinations of finitely many elements of $G$

with coefficients in $\mathbb{C}$. That is the set $G$ is a linear basis of $\mathbb{C}[G]$, and multiplication in $\mathbb{C}[G]$ is defined by linear extension of multiplication in $G$. The identity element of $\mathbb{C}[G]$ is the identity element of $G$. Notice that if the group $G$ is abelian, we will write $e^g$, $g \in G$, for the element of $\mathbb{C}[G]$ corresponding to $g \in G$, i.e. $\{e^g \mid g \in G\}$ is a linear basis. In particular,

$$\begin{aligned} e^0 &= 1, \\ e^{g_1}e^{g_2} &= e^{g_1+g_2} \quad \text{for } g_1, g_2 \in G. \end{aligned}$$

A *Lie algebra* is an algebra $\mathfrak{g}$ whose multiplication (called *bracket* and denoted by $[\cdot,\cdot]$) satisfes the following axioms:

1. $[x,x] = 0 \quad \text{for } x \in \mathfrak{g},$ (Skew-symmetry)

   (it is equivalent to $[x,y] = -[y,x] \quad \text{for } x, y \in \mathfrak{g}$)
2. $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \quad \text{for } x, y, z \in \mathfrak{g}.$ (Jacobi identity)

The *Virasoro algebra* is defined as the Lie algebra $\mathcal{L}$ with basis $\{L_n \mid n \in \mathbb{Z}\} \cup \{c\}$ equipped with the bracket relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}c$$

together with the condition that $c$ is a central element of $\mathcal{L}$.

We call a Lie algebra $\mathfrak{g}$ a *Heisenberg* (*Lie*) *algebra* if

$$\operatorname{Cent} \mathfrak{g} = \mathfrak{g}' \quad \text{and} \quad \dim \operatorname{Cent} \mathfrak{g} = 1$$

where $\operatorname{Cent} \mathfrak{g} = \{x \in \mathfrak{g} \mid [x,y] = 0 \;\; \forall y \in \mathfrak{g}\}$ and $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$.

For a Lie algebra $\mathfrak{g}$, set

$$T(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} T^i$$

where $T^0 = \mathbb{C}$, $T^1 = \mathfrak{g}$, $T^2 = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}$, $T^n = \overbrace{\mathfrak{g} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathfrak{g}}^{n \text{ times}}$. Define a bilinear map $T^m \times T^n$ to $T^{m+n}$ satisfying

$$(x_1 \otimes \cdots \otimes x_m) \cdot (y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n \tag{1.3}$$

for $x_i, y_j \in \mathfrak{g}$, and then extend this map by bilinearity to give a multiplication map $T(\mathfrak{g}) \times T(\mathfrak{g})$ to $T(\mathfrak{g})$. In this way, (1.3) $T(\mathfrak{g})$ becomes an associative algebra, called the *tensor algebra* of $\mathfrak{g}$.

The *universal enveloping algebra* $U(\mathfrak{g})$ is an associative algebra defined by $T(\mathfrak{g})/I$ where $I$ is the 2-sided ideal of $T(\mathfrak{g})$ generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y] \quad \text{for } x, y \in \mathfrak{g}.$$

If $\mathfrak{g}$ is an abelian Lie algebra, i.e. $[x, y] = 0$ for $x, y \in \mathfrak{g}$, the universal enveloping algebra $U(\mathfrak{g})$ will be denoted by $S(\mathfrak{g})$, called the *symmetric algebra* of $\mathfrak{g}$.

**Induced modules**

Let $B$ be a subalgebra of an associative algebra $A$ and let $V$ be a $B$-module. Set $A \otimes_B V$ be the quotient of the vector space $A \otimes_{\mathbb{C}} V$ by the subspace spanned by the elements

$$ab \otimes v - a \otimes b \cdot v$$

for $a \in A$, $b \in B$ and $v \in V$. Write $a \otimes v$ to stand for the image of $a \otimes v \in A \otimes_{\mathbb{C}} V$ in $A \otimes_B V$. Then in $A \otimes_B V$,

$$\begin{aligned} ab \otimes v &= a \otimes b \cdot v \quad \text{for } a \in A, b \in B, c \in V, \\ c \cdot (a \otimes v) &= ca \otimes v \quad\ \ \text{for } a, c \in A, v \in V. \end{aligned}$$

$A \otimes_B V$ is called the *$A$-module induced by the $B$-module $V$*. It is sometimes denoted by $\mathrm{Ind}\,_B^A V$.

Now we consider induced Lie algebra modules. Given a subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ and a $\mathfrak{h}$-module $V$, the $\mathfrak{g}$-module induced by $V$ is, by definition the $\mathfrak{g}$-module corresponding to the $U(\mathfrak{g})$-module,

$$\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V. \tag{1.4}$$

Given a subgroup $H$ of a group $G$ and an $H$-module $V$, we define the *$G$-module induced by $V$* to be the $G$-module associated with the induced $\mathbb{C}[G]$-module $\mathbb{C}\otimes_{\mathbb{C}[H]} V$. We sometimes write

$$\operatorname{Ind}_{H}^{G} V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V. \tag{1.5}$$

## 1.2 Vertex algebras and some fundamental properties

**Definition 1.2.1.** ([26], p. 8) A *vertex algebra* $V$ is a vector space equipped with a linear map

$$\begin{aligned} Y(\cdot, z) \quad &: V \to (\operatorname{End} V)[[z, z^{-1}]], \\ & v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \text{ where } v_n \in \operatorname{End} V \end{aligned}$$

and a distinguished vector $\mathbf{1} \in V$ which satisfies the following properties: for $u, v \in V$

1. $u_n v = 0$ for $n$ sufficiently large.

2. $Y(\mathbf{1}, z) = \mathrm{id}_V$.

3. $Y(v, z)\mathbf{1} \in V[[z]]$ and $\lim_{z\to 0} Y(v, z)\mathbf{1} = v$.

4. (the Jacobi identity)

$$\begin{aligned} z_0^{-1}\delta\left(\tfrac{z_1-z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\tfrac{z_2-z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1) \\ = z_2^{-1}\delta\left(\tfrac{z_1-z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \end{aligned}$$

We denote the vertex algebra just defined by $(V, Y, \mathbf{1})$ or, briefly, by $V$.

**Definition 1.2.2.** A *$\mathbb{Z}$-graded vertex algebra* is a vertex algebra

$$V = \bigoplus_{n\in\mathbb{Z}} V_n; \text{ for } v \in V_n, \quad n = \mathrm{wt}\, v,$$

equipped with a conformal vector $\omega \in V_2$ which satisfies the following relations:

- $[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V$ for $m, n \in \mathbb{Z}$, where $c_V \in \mathbb{C}$ (the central charge) and

$$Y(\omega, z) = \textstyle\sum_{n\in\mathbb{Z}} L(n)z^{-n-2} \left(= \sum_{m\in\mathbb{Z}} \omega_m z^{-m-1}\right);$$

- $L(0)v = nv = (\mathrm{wt}\, v)v$ for $n \in \mathbb{Z}$, and $v \in V_n$;

- $Y(L(-1)v, z) = \frac{d}{dz}Y(v, z)$.

We shall refer to the $\mathbb{Z}$-vertex algebra $V$ as $(V, Y, \mathbf{1}, \omega)$ if necessary.

**Definition 1.2.3.** A *vertex operator algebra* is a $\mathbb{Z}$-graded vertex algebra, $(V, Y, \mathbf{1}, \omega)$,

$$V = \bigoplus_{n\in\mathbb{Z}} V_n; \text{ for } v \in V_n, \quad n = \mathrm{wt}\, v,$$

which satisfies $\dim V_n < \infty$ for $n \in \mathbb{Z}$ and $V_n = 0$ for $n$ sufficiently negative.

For $l, m, n \in \mathbb{Z}$ and $u, v \in V$, the component form of the Jacobi identity by equating the coefficient of $z_0^{-l-1}z_1^{-m-1}z_2^{-n-1}$ gives

$$\sum_{i\geq 0}\binom{m}{i}(u_{l+i}v)_{m+n-i} = \sum_{i\geq 0}(-1)^i\binom{l}{i}u_{m+l-i}v_{n+i} - (-1)^l\sum_{i\geq 0}(-1)^i\binom{l}{i}v_{n+l-i}u_{m+i} \tag{1.6}$$

which is called *Borcherds's Identity*, in [5].

In Borcherds's Identity, let $l = 0$ and $m = 0$, we have two formulas as follows:

**Proposition 1.2.4.** *Let $u, v \in V$ and $m, n \in \mathbb{Z}$.*

*1.* $[u_m, v_n] = \sum_{i\geq 0} \binom{m}{i} (u_i v)_{m+n-i}$.

*2.* $(u_m v)_n = \sum_{i\geq 0} (-1)^i \binom{m}{i} (u_{m-i} v_{n+i} - (-1)^m v_{m+n-i} u_i)$.

From [26], taking $\mathrm{Res}_{z_0}$ of the Jacobi identity we obtain the commutator formula:

$$[Y(u, z_1), Y(v, z_2)] = \mathrm{Res}_{z_0} \left( z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2) \right) \tag{1.7}$$

Applying $u = \omega$ and taking $\mathrm{Res}_{z_1}$ and $\mathrm{Res}_{z_1} z_1$ to the commutator formula (1.7), and using (1.1) we have

$$[L(-1), Y(v, z)] = Y(L(-1)v, z) = \frac{d}{dz} Y(v, z), \tag{1.8}$$

$$[L(0), Y(v, z)] = zY(L(-1)v, z) + Y(L(0)v, z). \tag{1.9}$$

For homogeneous vectors $u, v$ of $V$, using (1.9), we have

$$\begin{aligned} [L(0), Y(v, z)]u &= (L(0)Y(v, z) - Y(v, z)L(0))u \\ &= L(0) \sum_{n\in\mathbb{Z}} v_n u z^{-n-1} - \sum_{n\in\mathbb{Z}} v_n (\mathrm{wt}\, u) u z^{-n-1} \\ &= z \sum_{n\in\mathbb{Z}} (-n-1) v_n u z^{-n-2} + (\mathrm{wt}\, v) \sum_{n\in\mathbb{Z}} v_n u z^{-n-1}. \end{aligned}$$

Taking $\mathrm{Res}_z z^n$ of both sides, we obtain

$$L(0)(v_n u) = (\mathrm{wt}\, u + \mathrm{wt}\, v - n - 1) v_n u,$$

$$\mathrm{wt}\, v_n = \mathrm{wt}\, v - n - 1,$$

that is, $v_n$ is a homogeneous operator that maps $V_{\mathrm{wt}\, u}$ to $V_{\mathrm{wt}\, u + \mathrm{wt}\, v - n - 1}$ for all $n \in \mathbb{Z}$.

Notice that for $n \in \mathbb{Z}_{\geq 0}$, from the property

$$L(-1)\mathbf{1} = L(n)\mathbf{1} = 0$$

so that the vacuum vector is homogeneous of weight 0.

**Definition 1.2.5.** ([26], p. 99). A *vertex* (*operator*) *subalgebra* of a vertex (operator) algebra $V$ is a vector subspace $U$ of $V$ such that $\mathbf{1} \in U$, ($\omega \in U$) and such that $U$ is itself a vertex (operator) algebra.

The two notions of the direct sum and the tensor product of finitely many vertex (operator) algebras can be found in [26], pages 111-116, as follows:

Let $(V_1, Y_1, \mathbf{1}), \ldots, (V_r, Y_r, \mathbf{1})$ be vertex algebras. Let $V = V_1 \oplus \cdots \oplus V_r$. Define the linear map $Y(\cdot, z)$ from $V$ to End $V[[z, z^{-1}]]$ by

$$Y(v^{(1)} \oplus \cdots \oplus v^{(r)}, z) = Y_1(v^{(1)}, z) \oplus \cdots \oplus Y_r(v^{(r)}, z)$$

for $v^{(i)} \in V_i$, $1 \leq i \leq r$ and the vacuum vector is $\mathbf{1} = \mathbf{1} \oplus \cdots \oplus \mathbf{1}$.

Suppose that for each $i = 1, \ldots, r$, $V_i$ is a vertex operator algebra with a conformal vector $\omega^{(i)}$ of the same central charge. Endow $V$ with the natural $\mathbb{Z}$-grading $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$, where

$$V_{(n)} = (V_1)_n \oplus \cdots \oplus (V_r)_n.$$

Then the vertex algebra $V_1 \oplus \cdots \oplus V_r$ is a vertex operator algebra of the same central charge, with the conformal vector

$$\omega = \omega^{(1)} \oplus \cdots \oplus \omega^{(r)}$$

where $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, $L(n) = L^{(1)}(n) \oplus \cdots \oplus L^{(r)}(n)$, $\omega^{(i)}_{n+1} = L^{(i)}(n)$ for $n \in \mathbb{Z}$.

The vertex algebra $V = V_1 \otimes \cdots \otimes V_r$ is constructed on the tensor product of vector spaces $V_1, \ldots, V_r$ where the linear map $Y$ is defined by

$$Y((v^{(1)} \otimes \cdots \otimes v^{(r)}), z) = Y(v^{(1)}, z) \otimes \cdots \otimes Y(v^{(r)}, z)$$

for $v^{(i)} \in V_i$, $1 \leq i \leq r$ and the vacuum vector is $\mathbf{1} = \mathbf{1} \otimes \cdots \otimes \mathbf{1}$.

Now suppose that each $V_i$ is a vertex operator algebra with a conformal vector $\omega^{(i)}$ of the central charge $c_i$ for $i = 1, \ldots, r$. Then $V = V_1 \otimes \cdots \otimes V_r$ is $\mathbb{Z}$-graded as $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$, where

$$V_{(n)} = \textstyle\sum_{m_1 + \cdots + m_r = n} (V_1)_{m_1} \otimes \cdots \otimes (V_r)_{m_r}.$$

Then the vertex algebra $V = V_1 \otimes \cdots \otimes V_r$ is a vertex operator algebra of the central charge $c_1 + \cdots + c_r$, with the conformal vector

$$\omega = \omega^{(1)} \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \omega^{(r)}$$

where $Y(\omega, z) = \displaystyle\sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, $L(n) = L^{(1)}(n) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes L^{(r)}(n)$, $\omega^{(i)}_{n+1} = L^{(i)}(n)$ for $n \in \mathbb{Z}$.

**Definition 1.2.6.** *Let $(V_1, Y_1, \mathbf{1})$ and $(V_2, Y_2, \mathbf{1})$ be vertex algebras. A* vertex algebra homomorphism *$f : V_1 \to V_2$ is defined to be a linear map such that*

$$f(Y_1(u, z)v) = Y_2(f(u), z) f(v) \quad \text{for } u, v \in V_1,$$

*or equivalently,* $f(u_n v) = f(u)_n f(v)$ *for* $u, v \in V_1$, $n \in \mathbb{Z}$, *and such that* $f(\mathbf{1}) = \mathbf{1}$.

If $V_1$ and $V_2$ are $\mathbb{Z}$-graded vertex algebras (vertex operator algebras), a homomorphism $f$ from $V_1$ to $V_2$ is required in addition to satisfy the condition

$$f(\omega^1) = \omega^2,$$

where $\omega^1$ and $\omega^2$ are the conformal vectors for $V_1$ and $V_2$, respectively.

**Remark 1.2.7.** From preserving of the conformal vector of a homomorphism $f$,

$$f L^1(n) = L^2(n) f \quad \text{for } n \in \mathbb{Z},$$

$(L^1(n) = \omega^1_{n+1},\ L^2(n) = \omega^2_{n+1})$ , $f$ is automatically grading-preserving. Furthermore, since

$$L^i(2)L^i(-2)\mathbf{1} = \tfrac{1}{2}c_{V_i}\mathbf{1} \ \text{ for } i = 1, 2,$$

$V_1$ and $V_2$ must have the same central charge.

The notions of isomorphism, endomorphism and automorphism are defined in the obvious ways.

We denote by $\mathrm{Aut}\,(V)$ the group of all automorphisms of $V$ . For a subgroup $G < \mathrm{Aut}\,(V)$ the fixed point set $V^G = \{a \in V \mid g(a) = a, g \in G\}$ has a canonical vertex operator algebra structure.

## 1.3 Modules and twisted modules

**Definition 1.3.1.** ([13]). Let $V$ be a vertex algebra, a *weak* $V$-module $M$ is a vector space equipped with a linear map

$$\begin{aligned} Y_M(\cdot, z):\ \ & V \to (\mathrm{End}\,M)\,[[z, z^{-1}]]\,, \\ & v \mapsto Y_M(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \ \text{ where } v_n \in \mathrm{End}\,M, \end{aligned}$$

which satisfies the following properties: for $v, u \in V$, and $w \in M$

1. $u_n w = 0$ for $n$ sufficiently large.

2. $Y(\mathbf{1}, z) = \mathrm{id}_M$.

3. (the Jacobi identity)

$$\begin{aligned} z_0^{-1}\delta\left(\tfrac{z_1 - z_2}{z_0}\right) Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta\left(\tfrac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2)Y_M(u, z_1) \\ = z_2^{-1}\delta\left(\tfrac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2). \end{aligned}$$

We denote the weak $V$-module $M$ by $(M, Y_M)$.

**Definition 1.3.2.** Let $V$ be a $\mathbb{Z}$-graded vertex algebra. An *admissible* $V$-module is a weak $V$-module, $(M, Y_M)$,

$$M = \bigoplus_{n=0}^{\infty} M(n), \text{ with top level } M(0) \neq 0,$$

satisfying $u_m M(n) \subset M(\operatorname{wt} u - m - 1 + n)$ for homogeneous $u \in V$, $n \in \mathbb{Z}$.

Set $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$. Then $\{L(n) \mid n \in \mathbb{Z}\}$ gives a representation of the Virasoro algebra on $M$ with central charge $c_V$ and the following relations hold on $M$: for $m, n \in \mathbb{Z}$ and $v \in V$ (see [13, 26])

$$[L(-1), Y_M(v, z)] = Y_M(L(-1)v, z) = \frac{d}{dz} Y_M(v, z), \tag{1.10}$$

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0} c_V, \tag{1.11}$$

$$[L(0), Y_M(v, z)] = zY_M(L(-1)v, z) + Y_M(L(0)v, z). \tag{1.12}$$

An *irreducible* weak (admissible) $V$-module is a weak $V$-module that has no weak (admissible) $V$-submodule except $\{0\}$ and itself. Here a weak (admissible) submodule is defined in the obvious way.

A weak (admissible) $V$-module is *completely reducible* if it can be rewritten as a direct sum of finitely many irreducible weak (admissible) $V$-modules.

**Proposition 1.3.3.** ([26], p. 130). *Let $W$ be a weak $V$-module, and let $\langle T \rangle$ be a weak $V$-submodule of $W$ generated by a subset $T$ of $W$. Then*

$$\langle T \rangle = \operatorname{Span}_{\mathbb{C}}\{ v_n w \mid v \in V, n \in \mathbb{Z}, w \in T \}.$$

**Corollary 1.3.4.** *If $W$ is an irreducible weak $V$-module then*

$$W = \operatorname{Span}_{\mathbb{C}}\{ v_n w \mid v \in V, n \in \mathbb{Z} \}.$$

*Here, $w$ is a non-zero element in $W$.*

**Definition 1.3.5.** Let $V$ be a vertex algebra. Let $M_1$ and $M_2$ be $V$-modules. A *$V$-homomorphism* from $M_1$ to $M_2$ is a linear map $\varphi$ such that

$$\varphi(Y_{M_1}(v,z)w) = Y_{M_2}(v,z)\varphi(w) \text{ for } v \in V,\, w \in M_1,$$

or equivalently, $\varphi(v_n w) = v_n \varphi(w)$ for $v \in V$, $w \in M_1$, $n \in \mathbb{Z}$.

Let $g$ be an automorphism of a vertex operator algebra $V = \coprod_{n\in\mathbb{Z}} V_n$. Then $g(V_n) = V_n$ for $n \in \mathbb{Z}$ and $V$ is a direct sum of the eigenspaces of $g$:

$$V = \bigoplus_{r=0}^{T-1} V^r$$

where $T$ is the order of $g$ and $V^r = \left\{ v \in V \mid g(v) = e^{\frac{-2\pi i r}{T}} v \right\}$. We adopt standard notation in using $W\{z\}$ to denote the space of $W$-valued formal series in arbitrary real powers of $z$ for a vector space $W$.

**Definition 1.3.6.** ([14]). A *weak $g$-twisted* $V$-module is a vector space equipped with a linear map

$$\begin{aligned} Y_M(\cdot,z):\ \ & V \to (\operatorname{End} M)\{z\}, \\ & v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Q}} v_n z^{-n-1}, \quad \text{where } v_n \in \operatorname{End} M, \end{aligned}$$

which satisfies the following conditions for all $0 \le r \le T-1$, $u \in V^r$, $v \in V$, $w \in M$

1. $Y_M(u,z) = \sum_{n\in\frac{r}{T}+\mathbb{Z}} u_n z^{-n-1}$.

2. There exists an integer $N$ such that $u_n w = 0$ for $n > \frac{r}{T} + N$.

3. $Y(\mathbf{1},z) = \mathrm{id}_M$.

4. (the twisted Jacobi identity)

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(u,z_1)Y_M(v,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(v,z_2)Y_M(u,z_1) \tag{1.13}$$

$$=z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{\frac{-r}{T}}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(u,z_0)v,z_2). \tag{1.14}$$

**Definition 1.3.7.** ([14]). An *admissible $g$-twisted* $V$-module is a weak $g$-twisted $V$-module $M$ which carries a $\frac{1}{T}\mathbb{Z}_{\geq 0}$-grading

$$M=\bigoplus_{n\in\frac{1}{T}\mathbb{Z}_{\geq 0}} M_n$$

such that $u_mM_n\subset M_{\mathrm{wt}\,u-m-1+n}$ for homogeneous $u\in V$ and $m,n\in\mathbb{Q}$.

If $g=\mathrm{id}_V$, these definitions reduce to the untwisted version used in [13].

**Remark 1.3.8.** ([20, 21]). One can prove that the twisted Jacobi identity as in (1.13) is equivalent to the following associativity formula:

$$(z_0+z_2)^{k+r/T}Y_M(u,z_0+z_2)Y_M(v,z_2)w=(z_2+z_0)^{k+r/T}Y_M(Y(u,z_0)v,z_2)w, \tag{1.15}$$

and commutator relation:

$$[Y_M(u,z_1),Y_M(v,z_2)]=\mathrm{Res}_{z_0}z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/T}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(u,z_0)v,z_2) \tag{1.16}$$

where $w\in M$ and $k\in\mathbb{Z}_{\geq 0}$ such that $z^{k+r/T}Y_M(u,z)w$ involves only positive power of $z$.

Equating the coefficients of $z_1^{-m-1}z_2^{-n-1}$ in (1.16) yields

$$[u_m,v_n]=\sum_{i=0}^{\infty}\binom{m}{i}(u_iv)_{m+n-i} \tag{1.17}$$

A $g$-twisted weak (admissible) $V$-*submodule* of a $g$-twisted weak (admissible) module $M$ is a subspace $N$ of $M$ such that $v_n N \subset N$ holds for all $v \in V$ and $n \in \mathbb{Q}$. If $M$ has no $g$-twisted weak (admissible) $V$-submodule except $\{0\}$ and $M$, $M$ is called an *irreducible* $g$-twisted weak (admissible) $V$-module.

A $g$-twisted weak(admissible) $V$-module is *completely reducible* if it can be rewritten as a direct sum of finitely many irreducible $g$-twisted weak(admissible) $V$-modules.

We say that $V$ is *$g$-rational* if every admissible $g$-twisted $V$-module is completely reducible.

# CHAPTER II

# THE VERTEX ALGEBRA $V_L^+$

The vertex algebras $V_L^+$ are one of the most important classes of vertex algebras along with those vertex algebras associated with lattices, affine Lie algebras and Virasoro algebras. They were originally introduced in the Frenkel-Lepowsky-Meurman construction of the moonshine module vertex algebra (see [21]). The representation theory of $V_L^+$ is well understood when $L$ is a positive definite even lattice. In fact, for such case, the classification of all irreducible weak $V_L^+$-modules, and the study of the complete reducibility property of weak $V_L^+$-modules was done by Abe, Dong, Jiang, and Nagatomo (see [1, 2, 4, 9, 18]).

When $L$ is a rank one negative definite even lattice, the classification of irreducible admissible $V_L^+$-modules was completed by Jordan in [23]. Later, in [30, 31], Yamskulna classified all irreducible admissible $V_L^+$-modules and showed the complete reducibility of admissible $V_L^+$-modules for the case when $L$ is a negative definite even lattice of arbitrary rank, and when $L$ is a non-degenerate even lattice that is neither positive definite nor negative definite.

In this chapter, we give the construction of $V_L^+$. Let $L$ be an even lattice of rank $d$. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend $\langle \cdot, \cdot \rangle$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Viewing $\mathfrak{h}$ as an abelian Lie algebra. Let

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

be the corresponding affine Lie algebra with the following commutator relations:

$$[\beta \otimes t^m, \gamma \otimes t^n] = \langle \beta, \gamma \rangle \, m\delta_{m+n,0} c \quad \text{and} \quad [c, \widehat{\mathfrak{h}}] = 0 \tag{2.1}$$

where $\beta, \gamma \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$.

Set

$$\widehat{\mathfrak{h}}^{+} = \mathfrak{h} \otimes t\mathbb{C}[t], \qquad \widehat{\mathfrak{h}}^{-} = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}] \ .$$

From [21], the subalgebra $\widehat{\mathfrak{h}}_{\mathbb{Z}} = \widehat{\mathfrak{h}}^{+} \oplus \widehat{\mathfrak{h}}^{-} \oplus \mathbb{C}c$ of $\widehat{\mathfrak{h}}$ has a structure of Heisenberg (Lie) algebra in the sense that its commutator subalgebra coincides with its center, which is one-dimensional.

Here for convenience, we write $\beta(m) := \beta \otimes t^m$ for $\beta \in \mathfrak{h}$ and $m \in \mathbb{Z}$.

Consider $\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$ as a subalgebra of $\widehat{\mathfrak{h}}$. Let $\mathbb{C}$ be a $\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$-module which $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially on $\mathbb{C}$, and $c$ acts as a multiplication by 1. From (1.4), we consider the induced $\widehat{\mathfrak{h}}$-module as follows:

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C})} \mathbb{C} \simeq S(\widehat{\mathfrak{h}}^{-}). \qquad \text{(linearly)}$$

Notice that $M(1)$ is spanned by

$$\{\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1} \mid \alpha_1, \ldots, \alpha_k \in \mathfrak{h}, \ n_1, \ldots, n_k \in \mathbb{Z}_{>0}, k \in \mathbb{Z}_{\geq 0}\}$$

where the action of $\widehat{\mathfrak{h}}$ on $M(1)$ is defined in the natural way and using the commutator relations (2.1) to arrange the position of $\beta(m)$, $\beta \in \mathfrak{h}, m \in \mathbb{Z}$, e.g.

$$\begin{aligned}
\beta(5) \cdot \alpha_1(-5)\alpha_2(-1)\mathbf{1} &= \beta(5)\alpha_1(-5)\alpha_2(-1)\mathbf{1} \\
&= [\alpha_1(-5)\beta(5) + 5\langle \beta, \alpha_1 \rangle c]\alpha_2(-1)\mathbf{1} \\
&= \alpha_1(-5)\alpha_2(-1)\beta(5)\mathbf{1} + 5\langle \beta, \alpha_1 \rangle \alpha_2(-1)c\mathbf{1} \\
&= 5\langle \beta, \alpha_1 \rangle \alpha_2(-1)\mathbf{1}.
\end{aligned}$$

Then there exists a linear map $Y : M(1) \to \operatorname{End}(M(1))[[z, z^{-1}]]$ such that $M(1)$ becomes a simple $\mathbb{Z}$-graded vertex algebra with the vacuum vector $\mathbf{1}$ and the Virasoro element $\omega = \frac{1}{2}\sum_{i=1}^{d}\beta_i(-1)^2\mathbf{1}$ (see [21]). Here, $\{\beta_1, \ldots, \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$.

Next, we let $\hat{L}$ be a canonical central extension of $L$ by the cyclic group of order 2, we have an exact sequence:

$$1 \to \langle \pm 1 \rangle \to \hat{L} \xrightarrow{-} L \to 0$$

with the commutator map $c_0 : L \times L \to \langle \pm 1 \rangle$, $c_0(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$.

**Remark 2.0.9.** The commutator map $c_0$ is an alternating $\mathbb{Z}$-bilinear form (*alternating* means that $c_0(\alpha, \alpha) = 1$ for $\alpha \in L$) and it characterizes the central extension uniquely up to equivalence (see Proposition 5.2.3 in [21]).

Let

$$e : L \to \hat{L},\ \alpha \mapsto e_\alpha, \tag{2.2}$$

be a section of $\hat{L}$, that is, $- \circ e = 1$, such that $e_0 = 1$ as identity element of $\hat{L}$.

Define the $\mathbb{Z}$-bilinear map $\epsilon : L \times L \to \langle \pm 1 \rangle$ determined by

$$\epsilon(\alpha_i, \alpha_j) = \begin{cases} c_0(\alpha_i, \alpha_j) = (-1)^{\langle \alpha_i, \alpha_j \rangle} & \text{if } i \leq j, \\ 1 & \text{if } i > j \end{cases} \tag{2.3}$$

where $\{\alpha_1, \ldots, \alpha_d\}$ is a $\mathbb{Z}$-base of $L$. Then $\epsilon$ is the corresponding bimultiplicative 2-cocycle, satisfying

$$\begin{aligned} e_\alpha e_\beta &= \epsilon(\alpha, \beta) e_{\alpha+\beta}, \\ \epsilon(\alpha, \beta)\epsilon(\alpha+\beta, \gamma) &= \epsilon(\beta, \gamma)\epsilon(\alpha, \beta+\gamma), \\ \epsilon(\alpha, \beta)\epsilon(\beta, \alpha) &= (-1)^{\langle \alpha, \beta \rangle}, \\ \epsilon(\alpha, 0) &= \epsilon(0, \alpha) = 1 \end{aligned}$$

for $\alpha, \beta, \gamma \in L$ (see [21], pp. 104-107).

**Remark 2.0.10.** Viewing $\hat{L}$ as $L \times \langle \pm 1 \rangle$, $\hat{L}$ under operator $(\alpha, (-1)^s)(\beta, (-1)^t) = (\alpha + \beta, (-1)^{s+t}\epsilon(\alpha, \beta))$ for $\alpha, \beta \in L$ and $s, t \in \mathbb{Z}$, is a group which $\langle \pm 1 \rangle \subset Z(\hat{L})$. In particular, $e_\alpha e_\beta = (\alpha, 1)(\beta, 1) = (\alpha + \beta, \epsilon(\alpha, \beta)) = \epsilon(\alpha, \beta) e_{\alpha+\beta}$ for $\alpha, \beta \in L$.

Consider $\mathbb{C}$ as a $\langle \pm 1 \rangle$-module which $(-1) \cdot 1 = -1$. Denote by $\mathbb{C}\{L\}$ the induced $\hat{L}$-module

$$\mathbb{C}\{L\} := \mathrm{Ind}\,_{\langle \pm 1 \rangle}^{\hat{L}} \mathbb{C} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\langle -1 \rangle]} \mathbb{C}.$$

For $a, b \in \hat{L}$, we write $\iota(a) := a \otimes 1 \in \mathbb{C}\{L\}$ and the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is

$$a \cdot \iota(b) = \iota(ab), \quad -1 \cdot \iota(b) = -\iota(b). \tag{2.4}$$

Recall (2.2), $\mathbb{C}\{L\}$ is viewed as a vector space, by the linear isomorphism

$$\mathbb{C}[L] \to \mathbb{C}\{L\}, \ e^\alpha \mapsto \iota(e_\alpha) \ \text{ for } \ \alpha \in L.$$

The action of $\hat{L}$ on $\mathbb{C}[L]$ is given by

$$e_\alpha \cdot e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}, \quad -1 \cdot e^\beta = -e^\beta \text{ for } \ \alpha, \beta \in L. \tag{2.5}$$

In particular, $\quad e_\alpha \cdot 1 = e_\alpha \cdot e^0 = \epsilon(\alpha, 0) e^0 = e^0.$

Consider the dual lattice $L^\circ$ of $L$. There is an $\hat{L^\circ}$-module structure on $\mathbb{C}\{L^\circ\} := \mathbb{C}[\hat{L^\circ}] \otimes_{\mathbb{C}[\langle -1 \rangle]} \mathbb{C} \simeq \mathbb{C}[L^\circ] = \oplus_{\lambda \in L^\circ} \mathbb{C} e^\lambda$ such that the action of $\hat{L}^\circ$ is defined as (2.4). We set

$$L^\circ = \cup_{i \in L^\circ / L} (L + \lambda_i)$$

the coset decomposition such that $\lambda_0 = 0$, and we define

$$\mathbb{C}[L+\lambda_i]=\bigoplus_{\alpha\in L}\mathbb{C}e^{\alpha+\lambda_i}.$$

Then $\mathbb{C}[L+\lambda_i]$ is an $\hat{L}$-module under the following action:

$$e_\alpha\cdot e^{\beta+\lambda_i}=\epsilon(\alpha,\beta)e^{\alpha+\beta+\lambda_i}\text{ for }\alpha,\beta\in L.$$

Now, we set

$$V_{L+\lambda}=M(1)\otimes\mathbb{C}[L+\lambda]. \tag{2.6}$$

Let $z$ be a formal variable. The action of $L$, $\hat{\mathfrak{h}}$, $z^h$ $(h\in\mathfrak{h})$ will act naturally on $V_{L+\lambda}$ by:

$$\begin{aligned}
\beta(-m)\cdot u\otimes e^{n\alpha+\lambda} &= \beta(-m)u\otimes e^{n\alpha+\lambda}\text{ for }m>0,\\
\beta(0)\cdot u\otimes e^{n\alpha+\lambda} &= \langle\beta,n\alpha+\lambda\rangle u\otimes e^{n\alpha+\lambda},\\
e_\beta\cdot u\otimes e^{n\alpha+\lambda} &= u\otimes e^{\beta+n\alpha+\lambda}\text{ for }\beta\in L,\\
z^\beta\cdot u\otimes e^{n\alpha+\lambda} &= u\otimes z^{\langle\beta,n\alpha+\lambda\rangle}e^{\beta+n\alpha+\lambda}\text{ for }\beta\in L.
\end{aligned}$$

For homogeneous $v=\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes e^{m\alpha}\in V_L$ $(=V_{L+\lambda_0})$, let

$$Y(e^{m\alpha},z):=\exp\left(\sum_{n=1}^{\infty}\frac{m\alpha(-n)}{n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{m\alpha(n)}{n}z^{-n}\right)e_{m\alpha}z^{m\alpha},$$

define

$$Y(v,z)=:\partial^{(n_1-1)}\alpha_1(z)\cdots\partial^{(n_r-1)}\alpha_k(z)Y(e^{m\alpha},z):, \tag{2.7}$$

where $\partial^{(n)}=\frac{1}{n!}\left(\frac{d}{dz}\right)^n$, $\alpha(z):=\sum_{n\in\mathbb{Z}}\alpha(n)z^{-n-1}$ and the normal ordering $:\cdot:$ is an operation which reorders the operators so that $\alpha(n)$ $(\alpha\in\mathfrak{h},n<0)$ and $e_{m\alpha}$ to be placed to the left of $\alpha(n)$ $(\alpha\in\mathfrak{h},n\geq 0)$ and $z^{m\alpha}$.

**Remark 2.0.11.** Notice that for $v=\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes e^{m\alpha}$, we have that $\operatorname{wt}v=n_1+\cdots+n_k+\frac{1}{2}\langle m\alpha,m\alpha\rangle$.

**Proposition 2.0.12.** ([5, 7, 21]).

1. *The space* $V_L$ *is a simple* $\mathbb{Z}$*-graded vertex algebra with a Virasoro element* $\omega = \frac{1}{2}\sum_{i=1}^{d} h_i(-1)^2 \otimes e^0$*. Here* $\{h_i \mid 1 \leq i \leq d\}$ *is an orthonormal basis of* $\mathfrak{h}$*. Moreover,* $M(1)$ *is a* $\mathbb{Z}$*-graded vertex sub-algebra of* $V_L$*.*

2. $\{\ V_{L+\lambda_i} \mid i \in L^\circ/L\ \}$ *is the set of all inequivalent irreducible* $V_L$*-modules.*

Next, define a linear isomorphism $\theta : V_{L+\lambda_i} \to V_{L-\lambda_i}$ by

$$\theta(\alpha_1(-n_1)\alpha_2(-n_2)\cdots\alpha_k(-n_k)e^{\alpha+\lambda_i}) = (-1)^k\alpha_1(-n_1)\cdots\alpha_k(-n_k)e^{-\alpha-\lambda_i}$$

if $2\lambda_i \notin L$ and

$$\theta(\alpha_1(-n_1)\alpha_2(-n_2)\cdots\alpha_k(-n_k)e^{\alpha+\lambda_i}) = (-1)^k c_{2\lambda_i}\epsilon(\beta, 2\lambda_i)\alpha_1(-n_1)\cdots\alpha_k(-n_k)e^{-\alpha-\lambda_i}$$

if $2\lambda_i \in L$. Here $c_{2\lambda_i}$ is a square root of $\epsilon(2\lambda_i, 2\lambda_i)$. Then $\theta$ is a linear automorphism of $V_{L^\circ}$. Moreover, $\theta$ is an automorphism of $V_L$ and $M(1)$.

For any stable $\theta$-subspace $U$ of $V_{L^\circ}$, a $\pm 1$ eigenspace of $U$ for $\theta$ is denoted by $U^\pm$. We have

$$V_L^+ := V_L^{\langle\theta\rangle} = \{v \in V_L \mid \theta(v) = v\} \quad \text{and}$$
$$M(1)^+ := M(1)^{\langle\theta\rangle} = \{u \in M(1) \mid \theta(u) = u\}.$$

**Proposition 2.0.13.** ([11, 16]).

1. $M(1)^+$ *and* $V_L^+$ *are simple* $\mathbb{Z}$*-graded vertex algebras.*

2. $(V_{L+\lambda_i} + V_{L-\lambda_i})^\pm$ *for* $i \in L^\circ/L$ *are irreducible weak* $V_L^+$*-modules. Moreover, if* $2\lambda_i \notin L$ *then* $(V_{L+\lambda_i} + V_{L-\lambda_i})^\pm$*,* $V_{L+\lambda_i}$ *and* $V_{L-\lambda_i}$ *are isomorphic.*

Let $\mathfrak{h}[-1] = \mathfrak{h} \otimes t^{\frac{1}{2}}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the twisted affine Lie algebra with the commutator relation

$$[\beta \otimes t^m, \gamma \otimes t^n] = \langle \beta, \gamma \rangle\, m\delta_{m+n,0}c \quad \text{and} \quad [c, \mathfrak{h}[-1]] = 0$$

where $\beta, \gamma \in \mathfrak{h}$ and $m, n \in \mathbb{Z} + \frac{1}{2}$. Next we let $M(1)(\theta) = S(\mathfrak{h} \otimes t^{-\frac{1}{2}}\mathbb{C}[t^{-1}])$ be the unique irreducible $\mathfrak{h}[-1]$-module such that $c$ acts as 1, and when $n > 0$, $\beta \otimes t^n$ acts on 1 as zero.

By abusing the notation, we use $\theta$ to denote an automorphism of $\hat{L}$ defined by $\theta(e_\alpha) = e_{-\alpha}$ and $\theta(\kappa) = \kappa$. We set $K = \left\{a^{-1}\theta(a) \mid a \in \hat{L}\right\}$. Also, we let $\chi$ be a central character of $\hat{L}/K$ such that $\chi(\kappa) = -1$ and we let $T_\chi$ be the irreducible $\hat{L}/K$-module with central character $\chi$. We define

$$V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi.$$

We define an action of $\theta$ on $V_L^{T_\chi}$ in the following way:

$$\theta(\beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)t) \;\;=\;\; (-1)^k\beta_1(-n_1)\cdots\beta_k(-n_k)t.$$

Here, $\beta_i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ and $t \in T_\chi$. We denote by $V_L^{T_\chi,\pm}$ the $\pm 1$-eigenspace of $V_L^{T_\chi}$ for $\theta$.

**Proposition 2.0.14.** ([15]). *Let $\chi$ be a central character of $\hat{L}/K$ such that $\chi(\iota(\kappa)) = -1$ and let $T_\chi$ be the irreducible $\hat{L}/K$-module with central character $\chi$. Then $V_L^{T_\chi,\pm}$ are irreducible weak $V_L^+$-modules.*

**Remark 2.0.15.** From Propositon 7.4.8 in [21] page 167, there are exactly $|R/2L|$ central characters, $\chi : \hat{R}/K \to \mathbb{C}^\times$, of $\hat{L}/K$ such that $\chi(\kappa) = -1$, where $R = \{\alpha \in L \mid \langle \alpha, L \rangle \subset 2\mathbb{Z}\}$.

# CHAPTER III

# THE $C_2$-COFINITENESS OF THE VERTEX ALGEBRA $V_L^+$ AND ITS WEAK MODULES

For the rest of this section, we assume that $V$ is a $\mathbb{Z}$-graded vertex algebra. We will work on the vertex algebra $V_L^+$ when $L$ is a non-degenerate even lattice.

**Definition 3.0.16.** ([33]). For a vertex algebra $V$, we define

$$C_2(V) = \mathrm{Span}_{\mathbb{C}}\{\ u_{-2}v \mid u, v \in V\ \}.$$

$V$ is said to satisfy the *cofiniteness $C_2$-condition* if $V/C_2(V)$ is finite dimensional.

The $C_2$-condition was first appeared in [33] when Zhu used this condition, as well as other assumptions, to show the modular invariance of certain trace functions. Since its introduction, the $C_2$-condition has proven to be a power tool in the study of theory of vertex algebras. In particular, it has played an important role in the study of structure of modules of vertex algebras which satisfy it (cf. [6, 12, 22, 24, 27, 28]).

**Proposition 3.0.17.** ([33]).

1. *$L(-1)u \in C_2(V)$ for $u \in V$.*

2. *$v_{-n}u \in C_2(V)$ for $u, v \in V$ and $n \geq 2$.*

3. *$C_2(V)$ is an ideal of $V$ with respect to $(-1)^{st}$-product.*

4. *$V/C_2(V)$ is a commutative associative algebra under $(-1)^{st}$-product.*

**Definition 3.0.18.** Let $M$ be a weak $V$-module. We define

$$\widetilde{C}_2(M) = Span_{\mathbb{C}}\{\ u_{-2}w \mid u \in V,\ w \in M\ \}.$$

The space $M$ is called *$C_2$-cofinite* if $M/\widetilde{C}_2(M)$ has a finite dimension.

The following is the key proposition.

**Proposition 3.0.19.** *Let $W$ be an irreducible weak $V$-module. If $V = C_2(V)$, then $W = \widetilde{C}_2(W)$.*

*Proof.* We first show that $a_n w \in \widetilde{C}_2(W)$ for all $a \in V, w \in W$ and $n \in \mathbb{Z}$. Clearly, this statement is true when $n \leq -2$. Since $V = C_2(V)$, one may write $a_n w$ as

$$a_n w = \sum_{j=1}^{l} (u^j_{-2} v^j)_n w = \sum_{j=1}^{l} \sum_{i \geq 0} (-1)^i \binom{-2}{i} \left( u^j_{-2-i} v^j_{n+i} - v^j_{-2+n-i} u^j_i \right) w.$$

Here $u^j, v^j \in V$. By using an induction on $n$, we can show that $a_n w \in \widetilde{C}_2(W)$ for all $a \in V, w \in W$ and $n \geq -1$.

Next, by using the facts that $W$ is irreducible and $a_n w \in \widetilde{C}_2(W)$ for all $a \in V, w \in W$, and $n \in \mathbb{Z}$, we can conclude immediately that $W = \widetilde{C}_2(W)$. □

**Corollary 3.0.20.** *Assume that $V = C_2(V)$. If a weak $V$-module $M$ is completely reducible, then $M = \widetilde{C}_2(M)$.*

**Proposition 3.0.21.** *Let $V'$ be a $\mathbb{Z}$-graded vertex algebra and let $V$ be a $\mathbb{Z}$-graded vertex subalgebra of $V'$ such that $V = C_2(V)$. If $V'$ is completely reducible as a weak $V$-module, then $V'$ is $C_2$-cofinite. In particular, $V' = C_2(V')$.*

*Proof.* By Corollary 3.0.20, we can conclude that $V' = \widetilde{C}_2(V')$. Since $\widetilde{C}_2(V')$ is a subset of $C_2(V')$, we then have that $V' = C_2(V')$. □

**Proposition 3.0.22.** *Let $V^1, \ldots, V^n$ be $\mathbb{Z}$-graded vertex algebras. Assume that $V^j = C_2(V^j)$ for some $j \in \{1, \ldots, n\}$. Then $V^1 \otimes \cdots \otimes V^n$ satisfies the $C_2$-condition. In particular, $V^1 \otimes \cdots \otimes V^n = C_2(V^1 \otimes \cdots \otimes V^n)$.*

*Proof.* The statements follow from the fact that

$$\begin{aligned} V^1 \otimes \cdots \otimes V^n &= V^1 \otimes \cdots \otimes C_2(V^j) \otimes \cdots \otimes V^n \\ &\subset C_2(V^1 \otimes \cdots \otimes V^n). \end{aligned}$$

□

When $L$ is a positive definite even lattice, it was shown by Abe, Buhl and Dong, and Yamskulna that the vertex algebras $V_L^+$ and their irreducible weak modules satisfy the $C_2$ condition (cf. [3, 29]). In this section, we will extend their results to the case when $L$ is a negative definite even lattice and when $L$ is a non-degenerate even lattice that is neither positive definite nor negative definite.

## 3.1 Case I: $L$ is a rank one negative definite even lattice

For the rest of this subsection, we assume that $L = \mathbb{Z}\alpha$ is a rank one negative definite even lattice such that $\langle \alpha, \alpha \rangle = -2k$. Here, $k$ is a positive integer.

Let $m \in \mathbb{Z}_{>0}$. For convenience, we set

$$E^m = e^{m\alpha} + e^{-m\alpha}, \quad F^m = e^{m\alpha} - e^{-m\alpha}, \quad \text{and}$$
$$V_L^+(m) = M(1)^+ \otimes E^m \oplus M(1)^- \otimes F^m.$$

Clearly, $V_L^+ = M(1)^+ \oplus \bigoplus_{m=1}^{\infty} V_L^+(m)$.

**Proposition 3.1.1.** ([8, 18, 23]).

1. *As a vertex algebra, $M(1)^+$ is generated by the Virasoro element $\omega$ and any singular vector of weight greater than zero. In particular, $M(1)^+$ is generated by $\omega$ and $J$ where $J = \frac{1}{4k^2}\alpha(-1)^4\mathbf{1} + \frac{1}{k}\alpha(-3)\alpha(-1)\mathbf{1} - \frac{3}{4k}\alpha(-2)^2\mathbf{1}$.*

*2. The vertex algebra* $M(1)^+$ *is spanned by*

$$L(-m_1)\cdots L(-m_s)J_{-n_1}\cdots J_{-n_t}\mathbf{1}$$

*where* $m_1 \geq \cdots \geq m_s \geq 2$, $n_1 \geq \cdots \geq n_t \geq 1$.

*3.* $V_L^+(m)$ *is spanned by* $L(-n_1)L(-n_2)\cdots L(-n_r)\otimes E^m$ *where* $n_r \geq \cdots \geq n_2 \geq n_1 \geq 1$.

Next, for $m > 0$, we set

$$V_L^+(m) = \bigoplus_{n\in\mathbb{Z}} V_L^+(m,n)$$

where $V_L^+(m,n)$ is the weight $n$ subspace of $V_L^+(m)$. Clearly, $V_L^+(m,-km^2+6)$ has the following basis elements:

<u>Basis of $V_L^+(m,-km^2+6)$</u>

$g_1 = \alpha(-6)F^m$, $g_2 = \alpha(-5)\alpha(-1)E^m$, $g_3 = \alpha(-4)\alpha(-2)E^m$,
$g_4 = \alpha(-4)\alpha(-1)^2F^m$, $g_5 = \alpha(-3)^2E^m$, $g_6 = \alpha(-3)\alpha(-2)\alpha(-1)F^m$,
$g_7 = \alpha(-3)\alpha(-1)^3E^m$, $g_8 = \alpha(-2)^3F^m$, $g_9 = \alpha(-2)^2\alpha(-1)^2E^m$,
$g_{10} = \alpha(-2)\alpha(-1)^4F^m$, $g_{11} = \alpha(-1)^6E^m$.

Furthermore, the following elements form bases of $V_L^+(m,-km^2+5)$ and $V_L^+(m,-km^2+3)$, respectively:

<u>Basis of $V_L^+(m,-km^2+5)$</u>

$f_1 = \alpha(-5)F^m$, $f_2 = \alpha(-4)\alpha(-1)E^m$, $f_3 = \alpha(-3)\alpha(-2)E^m$,
$f_4 = \alpha(-3)\alpha(-1)^2F^m$, $f_5 = \alpha(-2)^2\alpha(-1)F^m$, $f_6 = \alpha(-2)\alpha(-1)^3E^m$,
$f_7 = \alpha(-1)^5F^m$

<u>Basis of $V_L^+(m,-km^2+3)$</u>

$$h_1 = \alpha(-3)F^m, \quad h_2 = \alpha(-2)\alpha(-1)E^m, \quad h_3 = \alpha(-1)^3F^m.$$

**Lemma 3.1.2.** *The set* $\{\ L(-1)f_i,\ L(-3)h_j,\ (\alpha(-1)^4\mathbf{1})_{-3}E^m \mid 1 \le i \le 7,\ 1 \le j \le 3\}$ *is a basis of* $V_L^+(m, -km^2 + 6)$.

*Proof.* The table 1 describes expressions of $L(-1)f_i\ \ (i = 1, \ldots, 7)$, $L(-3)h_j\ \ (j = 1, 2, 3)$, and $(\alpha(-1)^4\mathbf{1})_{-3}E^m$ in terms of $g_l$ $(l = 1, \ldots, 11)$. We will denote this table by a $11 \times 11$ matrix $A$. Since $\det A = -6144m^3k^2\,(m^2k + 1)$, and $m, k$ are positive integers, we can conclude that $A$ is invertible. Moreover, $L(-1)f_i\ \ (i = 1, \ldots, 7)$, $L(-3)h_j\ \ (j = 1, 2, 3)$, and $(\alpha(-1)^4\mathbf{1})_{-3}E^m$ form a basis of $V_L^+(m, -km^2 + 6)$. $\square$

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $L(-1)f_1$ | $5$ | $m$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $L(-1)f_2$ | $0$ | $4$ | $1$ | $m$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $L(-1)f_3$ | $0$ | $0$ | $3$ | $0$ | $2$ | $m$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $L(-1)f_4$ | $0$ | $0$ | $0$ | $3$ | $0$ | $2$ | $m$ | $0$ | $0$ | $0$ | $0$ |
| $L(-1)f_5$ | $0$ | $0$ | $0$ | $0$ | $0$ | $4$ | $0$ | $1$ | $m$ | $0$ | $0$ |
| $L(-1)f_6$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $2$ | $0$ | $3$ | $m$ | $0$ |
| $L(-1)f_7$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $5$ | $m$ |
| $2kL(-3)h_1$ | $6k$ | $0$ | $0$ | $0$ | $2km$ | $-1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $2kL(-3)h_2$ | $0$ | $4k$ | $2k$ | $0$ | $0$ | $2km$ | $0$ | $0$ | $-1$ | $0$ | $0$ |
| $2kL(-3)h_3$ | $0$ | $0$ | $0$ | $6k$ | $0$ | $0$ | $2km$ | $0$ | $0$ | $-1$ | $0$ |
| $(\alpha(-1)^4\mathbf{1})_{-3}E^m$ | $-32k^3m^3$ | $48k^2m^2$ | $48k^2m^2$ | $-24km$ | $24k^2m^2$ | $-48km$ | $4$ | $-8km$ | $6$ | $0$ | $0$ |

Table 1.

**Proposition 3.1.3.** *For* $m \in \mathbb{Z}_{>0}$, *we have* $V_L^+(m, -km^2 + 2n)$ *is a subset of* $C_2(V_L^+)$ *when* $n \ge 3$.

*Proof.* By Proposition 3.0.17 and Lemma 3.1.2, we can conclude immediately that $V_L^+(m, -km^2+6)$ is contained in $C_2(V_L^+)$.

Next, we will show that $V_L^+(m, -km^2+2n)$ is a subset of $C_2(V_L^+)$ for $n \geq 4$. Since $(L(-2))^3E^m \in C_2(V_L^+)$, it implies that for $j \geq 4$,

$$(L(-2))^jE^m = (L(-2))^{j-3}(L(-2))^3E^m \in C_2(V_L^+).$$

By Proposition 3.1.1, we can conclude further that $V_L^+(m, -km^2+2n)$ is a subset of $C_2(V_L^+)$ when $n \geq 4$. $\square$

**Theorem 3.1.4.** *For $m \in Z_{>0}$, $V_L^+(m)$ is a subset of $C_2(V_L^+)$.*

*Proof.* First, we set $\exp(\sum_{n=1}^{\infty} \frac{x_n}{n} z^n) = \sum_{j=0}^{\infty} p_j(x_1, x_2, \ldots)z^j$. Since

$$\begin{aligned} E_{-2kn}\alpha(-1)F^n &= p_{4nk-1}(\alpha)\alpha(-1) \otimes e^{\alpha(n+1)} - p_{4nk-1}(-\alpha)\alpha(-1) \otimes e^{-\alpha(n+1)} \\ &\quad +2k\left(p_{4nk}(\alpha) \otimes e^{\alpha(n+1)} + p_{4nk}(-\alpha) \otimes e^{-\alpha(n+1)}\right) - 2kE^{n-1}, \end{aligned}$$

and for $n \geq 2$, $V_L^+(n+1, -k(n+1)^2+4nk)$ is contained in $C_2(V_L^+)$ (cf. Proposition 3.1.3), it follows that $E^{n-1} \in C_2(V_L^+)$ for $n \geq 2$. By Proposition 3.1.1 we can conclude further that $V_L^+(m)$ is a subset of $C_2(V_L^+)$ for all $m \in \mathbb{Z}_{>0}$. $\square$

**Theorem 3.1.5.** *$V_L^+$ satisfies the $C_2$-cofiniteness condition. In particular,*

$$V_L^+/C_2(V_L^+) = \{0 + C_2(V_L^+)\}.$$

*Proof.* We will show that $V_L^+ = C_2(V_L^+)$. It is enough to show that $M(1)^+$ is contained in $C_2(V_L^+)$. Since $E_{-2k-1}E = p_{4k}(\alpha) \otimes e^{2\alpha} + p_{4k}(-\alpha) \otimes e^{-2\alpha} + 2$ and $V_L^+(2)$ is a subset of $C_2(V_L^+)$, these imply that $\mathbf{1} \in C_2(V_L^+)$. By Proposition 3.1.1, we can conclude that $M(1)^+$ is a subset of $C_2(V_L^+)$. Consequently, $V_L^+ = C_2(V_L^+)$. $\square$

**Corollary 3.1.6.** *Every irreducible weak $V_L^+$-module satisfies the $C_2$-condition.*

*Proof.* This follows immediately from Proposition 3.0.19 and Theorem 3.1.5. □

## 3.2 Case II: $L$ is a negative definite even lattice

For the rest of this subsection, we assume that $L$ is a rank $d$ negative definite even lattice.

**Theorem 3.2.1.** *The vertex algebra $V_L^+$ is $C_2$-cofinite.*

*Proof.* We will show that $V_L^+ = C_2(V_L^+)$. We will follow the proof in [3] very closely. Let $K$ be a direct sum of $d$ orthogonal rank one negative definite even lattice $L_1, \ldots, L_d$. We write $L = \cup_{i \in L/K}(K + \lambda_i)$ as a direct sum of its coset decompositon with respect to $K$. Then $V_L = \bigoplus_{i \in L/K} V_{K+\lambda_i}$.

Since

$$V_K = V_{L_1} \otimes \cdots \otimes V_{L_d} = \sum_{\epsilon_i = \pm} V_{L_1}^{\epsilon_1} \otimes \cdots \otimes V_{L_d}^{\epsilon_d},$$

it follows that $V_K^+ = \sum_{\epsilon_i=\pm,\ \prod_i |\epsilon_i|=1} V_{L_1}^{\epsilon_1} \otimes \cdots \otimes V_{L_d}^{\epsilon_d}$. Here $|\pm| = \pm 1$. By using the fact that $V_{L_j}^+ = C_2(V_{L_j}^+)$ for all $1 \leq j \leq d$, we then have

$$V_{L_1}^+ \otimes \cdots \otimes V_{L_d}^+ = C_2\left(V_{L_1}^+ \otimes \cdots \otimes V_{L_d}^+\right).$$

Since $V_{L_1}^{\epsilon_1} \otimes \cdots \otimes V_{L_d}^{\epsilon_d}$ is an irreducible weak $V_{L_1}^+ \otimes \cdots \otimes V_{L_d}^+$-module, it follows from Proposition 3.0.21 that $V_K^+ = C_2(V_K^+)$. Since $V_L^+$ is completely reducible as a weak $V_K^+$-module, we can conclude that $V_L^+ = C_2(V_L^+)$ (cf. Proposition 3.0.21). □

**Corollary 3.2.2.** *Every irreducible weak $V_L^+$-module satisfies the $C_2$-condition.*

## 3.3 Case III: $L$ is a rank $d$ non-degenerate even lattice that is neither positive definite nor negative definite

**Theorem 3.3.1.** *Assume that $L$ is a non-degenerate even lattice of a finite rank that is neither positive definite nor negative definite. Then the vertex algebra $V_L^+$ and its irreducible weak modules satisfy the $C_2$-condition. In particular, we have $V_L^+ = C_2(V_L^+)$ and $M = \widetilde{C}_2(M)$ for all irreducible weak $V_L^+$-module.*

*Proof.* The proof is very similar to the proof of Theorem 3.2.1. In fact, we can obtain the above results by using the fact that $L$ has a rank one negative definite even sublattice, and follow the proof in Theorem 3.2.1 step by step. $\square$

# CHAPTER IV

# IRREDUCIBLE MODULES OF THE VERTEX OPERATOR ALGEBRA $V_{\mathcal{L}}^{+\langle\tau\rangle}$

Let $V$ be a simple vertex operator algebra and let $G$ be a finite automorphism group of $V$ . In [19], it was shown that any irreducible (admissible) $g$-twisted $V$-module is a completely reducible $V^G$-module for the case $g \in Z(G)$ where $V^G$ is the $G$-invariant vertex operator subalgebra of $V$.

**Remark 4.0.2.** From [15], for $g$ an automorphism of finite order $T$ of $V$ and an admissible $g$-twisted $V$-module $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_{\geq 0}} M_n$, we write

$$M = \bigoplus_{r=0}^{T-1}(\bigoplus_{n\in\mathbb{Z}} M_{n+\frac{r}{T}}).$$

We define a $\langle g\rangle$-action on $M$ such that $g$ acts on $\bigoplus_{n\in\mathbb{Z}} M_{n-\frac{r}{T}}$ by the scalar $e^{\frac{2\pi i}{T}r}$. With this action, let $u \in V^r$ and $w \in \bigoplus_{n\in\mathbb{Z}} M_{n+\frac{j}{T}}$ be homogeneous elements. For $n \in \frac{r}{T}+\mathbb{Z}$, $u_n(w)$ has weight $\mathrm{wt}\, u+\mathrm{wt}\, w-n-1 \in \mathrm{wt}\, w-\frac{r}{T}+\mathbb{Z}$. Then $u_n(w) \in \bigoplus_{n\in\mathbb{Z}} M_{n-\frac{r}{T}+\frac{j}{T}}$ and $g(u_n(g^{-1}w)) = g(e^{\frac{2\pi i}{T}j}u_n(w)) = e^{\frac{2\pi i}{T}j}e^{\frac{2\pi i}{T}(r-j)}u_n(w) = e^{\frac{2\pi i}{T}r}u_n w = (gu)_n w$. Then

$$gY_M(u,z)g^{-1} = Y_M(gu,z) \text{ for all } u \in V.$$

**Proposition 4.0.3.** ([15]). *Let $V$ be a simple vertex operator algebra and $M$ an irreducible $g$-twisted module where $g$ is an automorphism of finite order $T$ of $V$. Then, in the decomposition $M = M^0 + M^1 + \cdots + M^{T-1}$, $M^0, \ldots, M^{T-1}$ under $g$-action on $M$ are nonzero and non-isomorphic irreducible $V^{\langle g\rangle}$-modules.*

For $g \in G$, let $(M, Y_M)$ be an irreducible $g$-twisted $V$-module. For $h \in G$, we set

$$(M, Y_M) \circ h = (M \circ h, Y_{M\circ h}).$$

Here, $M \circ h = M$ as vector spaces and

$$Y_{M\circ h}(v, z) = Y_M(h(v), z) \text{ for } v \in V.$$

Note that $M \circ h$ is an irreducible $h^{-1}gh$-twisted $V$-module. It was shown in [19] that $M \circ h$ is an irreducible $g$-twisted $V$-module if and only if $h \in C_G(g)$. Here $C_G(g)$ denotes the centralizer of $g$ in $G$. If $g = 1$ then $M \circ h$ is an irreducible $V$-module.

Set

$$G_M = \{h \in G \mid M \circ h \cong M \ \text{ as } g\text{-twisted } V\text{-modules}\}.$$

It was proved in [17] that $g \in G_M$ and $G_M$ is a subgroup of $C_G(g)$.

**Proposition 4.0.4.** ([19, 32]).

1. *Assume that $g \in Z(G)$. Let $M$ be an irreducible $g$-twisted $V$-module. If $G_M$ is a cyclic group, say $\langle h \rangle$, then the eigenspaces of $M$ with respect to the action of $h$ are irreducible $V^G$-modules.*

2. *Suppose $G$ is a cyclic group of prime order $p$. Let $M$ be an irreducible $g^i$-twisted $V$-module where $1 \le i \le p-1$. Hence the eigenspaces of $M$ are irreducible $V^G$-modules.*

*In particular, if $G_M = \{1\}$, then $M$ is an irreducible $V^G$-module.*

In this chapter we work on a simple vertex operator algebra $V_L^+$, $L$ is positive definite. Here, let $\mathcal{L} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ be a rank 2 positive definite even lattice equipped with a symmetric nondegenerate $\mathbb{Q}$-valued $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ and having $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 2k$.

Let $\tau : \mathcal{L} \to \mathcal{L}$ be a map from $\mathcal{L}$ into itself that sends

$$\alpha_1 \mapsto -\alpha_2 \text{ and } \alpha_2 \mapsto \alpha_1.$$

**Remark 4.0.5.** Notice that $V_{\mathcal{L}}^{+}$ is spanned by the vectors of the form

$$\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{i\alpha_1+j\alpha_2}+e^{-i\alpha_1-j\alpha_2}) \quad \text{and}$$

$$\alpha_1(-m_1)\cdots\alpha_1(-m_r)\alpha_2(-n_1)\cdots\alpha_2(-n_s)\otimes(e^{i\alpha_1+j\alpha_2}-e^{-i\alpha_1-j\alpha_2})$$

where $k \geq 0, l \geq 0, r \geq 0, s \geq 0, i, j \in \mathbb{Z}$, $k+l$ is even and $r+s$ is odd.
Here, the conformal vector $\omega = \frac{1}{2}(h_1(-1)^2 \otimes 1 + h_2(-1)^2 \otimes 1)$ where $h_i = \frac{1}{\sqrt{2k}}\alpha_i$ $(= \frac{1}{\sqrt{2k}} \otimes \alpha_i \in \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L})$ for $i = 1, 2$.

Define $\tau : V_{\mathcal{L}}^{+} \to V_{\mathcal{L}}^{+}$ by the following actions: $\alpha(-n) \mapsto \tau(\alpha)(-n)$ and $e^{\alpha} \mapsto e^{\tau(\alpha)}$.

**Proposition 4.0.6.** *$\tau$ is an automorphism of $V_{\mathcal{L}}^{+}$ of order 2.*

*Proof.* Clearly $\tau$ is a linear isomorphism which $\tau(\mathbf{1}) = \mathbf{1}$, $\tau(\omega) = \omega$ and ,(2.7), $\tau(Y(u,z)v) = Y(\tau(u),z)\tau(v)$ for $u, v \in V_{\mathcal{L}}^{+}$. Then $\tau$ is an automorphism of $V_{\mathcal{L}}^{+}$. Since, for $k \geq 0, l \geq 0, i \geq 0, j \geq 0$, we have

$$\begin{aligned}
&\tau(\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{i\alpha_1+j\alpha_2}\pm e^{-i\alpha_1-j\alpha_2}))\\
=\;&\tau(\alpha_1)(-m_1)\cdots\tau(\alpha_1)(-m_k)\tau(\alpha_2)(-n_1)\cdots\tau(\alpha_2)(-n_l)\otimes(e^{i\tau(\alpha_1)+j\tau(a_2)}\pm e^{-i\tau(\alpha_1)-j\tau(\alpha_2)})\\
=\;&(-1)^k\alpha_2(-m_1)\cdots\alpha_2(-m_k)\alpha_1(-n_1)\cdots\alpha_1(-n_l)\otimes(e^{-i\alpha_2+j\alpha_1}\pm e^{i\alpha_2-j\alpha_1}),
\end{aligned}$$

it implies that

$$\begin{aligned}
&\tau^2(\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{i\alpha_1+j\alpha_2}\pm e^{-i\alpha_1-j\alpha_2}))\\
=\;&\tau((-1)^k\alpha_2(-m_1)\cdots\alpha_2(-m_k)\alpha_1(-n_1)\cdots\alpha_1(-n_l)\otimes(e^{-i\alpha_2+j\alpha_1}\pm e^{i\alpha_2-j\alpha_1}))\\
=\;&(-1)^{k+l}\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{-i\alpha_1-j\alpha_2}\pm e^{i\alpha_1+j\alpha_2}).
\end{aligned}$$

If $k+l$ is even, we then have that

$$\begin{aligned}
&\tau^2(\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{i\alpha_1+j\alpha_2}+e^{-i\alpha_1-j\alpha_2}))\\
=\;&\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{-i\alpha_1-j\alpha_2}+e^{i\alpha_1+j\alpha_2}). \qquad (4.1)
\end{aligned}$$

If $k+l$ is odd, it implies that

$$\begin{aligned}
& \tau^2(\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{i\alpha_1+j\alpha_2}-e^{-i\alpha_1-j\alpha_2})) \\
= \; & -\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{-i\alpha_1-j\alpha_2}-e^{i\alpha_1+j\alpha_2}) \\
= \; & \alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\otimes(e^{i\alpha_1+j\alpha_2}-e^{-i\alpha_1-j\alpha_2}). \qquad (4.2)
\end{aligned}$$

From Remark 4.0.5, (4.1) and (4.2), we can conclude that the order of $\tau$ is 2. $\square$

Next, we will find the vertex operator subalgebra $V_{\mathcal{L}}^{+\langle\tau\rangle}$. For convenience, we set

$$\begin{aligned}
v_{m_1,\ldots,m_k;n_1,\ldots,n_l} &:= \alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l)\mathbf{1}, \\
E^{i,j} &:= e^{i\alpha_1+j\alpha_2}+e^{-i\alpha_1-j\alpha_2}, \text{ and} \\
F^{i,j} &:= e^{i\alpha_1+j\alpha_2}-e^{-i\alpha_1-j\alpha_2}.
\end{aligned}$$

Since $\tau(E^{i,j})=E^{j,-i}$ and $\tau(F^{i,j})=F^{j,-i}$, we have

$$\begin{aligned}
\tau(E^{m,n}+E^{n,-m}) &= E^{n,-m}+E^{-m,-n}=E^{n,-m}+E^{m,n}, \\
\tau(E^{m,n}-E^{n,-m}) &= E^{n,-m}-E^{m,n}=-(E^{m,n}-E^{n,-m}), \\
\tau(F^{m,n}+F^{n,-m}) &= F^{n,-m}-F^{m,n}=-(F^{m,n}-F^{n,-m}), \\
\tau(F^{m,n}-F^{n,-m}) &= F^{n,-m}+F^{m,n}.
\end{aligned}$$

If $k$ and $l$ are even, then

$\tau(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}+v_{n_1,\ldots,n_l;m_1,\ldots,m_k})=v_{m_1,\ldots,m_k;n_1,\ldots,n_l}+v_{n_1,\ldots,n_l;m_1,\ldots,m_k}$,

$\tau(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}-v_{n_1,\ldots,n_l;m_1,\ldots,m_k})=-(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}-v_{n_1,\ldots,n_l;m_1,\ldots,m_k})$.

If $k$ and $l$ are odd, then

$\tau(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}+v_{n_1,\ldots,n_l;m_1,\ldots,m_k})=-(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}+v_{n_1,\ldots,n_l;m_1,\ldots,m_k})$,

$\tau(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}-v_{n_1,\ldots,n_l;m_1,\ldots,m_k})=v_{m_1,\ldots,m_k;n_1,\ldots,n_l}-v_{n_1,\ldots,n_l;m_1,\ldots,m_k}$.

If $k$ is even and $l$ is odd, then

$\tau(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}+v_{n_1,\ldots,n_l;m_1,\ldots,m_k})=-(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}-v_{n_1,\ldots,n_l;m_1,\ldots,m_k})$,

$\tau(v_{m_1,\ldots,m_k;n_1,\ldots,n_l}-v_{n_1,\ldots,n_l;m_1,\ldots,m_k})=v_{m_1,\ldots,m_k;n_1,\ldots,n_l}+v_{n_1,\ldots,n_l;m_1,\ldots,m_k}$.

We note that $E^{-i,-j} = E^{i,j}$ and $F^{-i,-j} = -F^{i,j}$.

Set $S^+$ to be

$$
\begin{aligned}
Span_{\mathbb{C}}\{ \quad & v_{m_1,\ldots,m_k;n_1,\ldots,n_l} E^{m,n} + v_{n_1,\ldots,n_l;m_1,\ldots,m_k} E^{n,-m}, \\
& v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_{l+1}} E^{m,n} - v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_{k+1}} E^{n,-m}, \\
& v_{m_1,\ldots,m_k;n_1,\ldots,n_{l+1}} F^{m,n} + v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_k} F^{n,-m}, \\
& v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_l} F^{m,n} - v_{n_1,\ldots,n_l;m_1,\ldots,m_{k+1}} F^{n,-m} \qquad : \ k,l \in 2\mathbb{N}_0 \ \}.
\end{aligned}
$$

Set $S^-$ to be

$$
\begin{aligned}
Span_{\mathbb{C}}\{ \quad & v_{m_1,\ldots,m_k;n_1,\ldots,n_l} E^{m,n} - v_{n_1,\ldots,n_l;m_1,\ldots,m_k} E^{n,-m}, \\
& v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_{l+1}} E^{m,n} + v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_{k+1}} E^{n,-m}, \\
& v_{m_1,\ldots,m_k;n_1,\ldots,n_{l+1}} F^{m,n} - v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_k} F^{n,-m}, \\
& v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_l} F^{m,n} + v_{n_1,\ldots,n_l;m_1,\ldots,m_{k+1}} F^{n,-m} \qquad : \ k,l \in 2\mathbb{N}_0 \ \}.
\end{aligned}
$$

It is clear that $\forall v \in S^+$, $\tau(v) = v$ and $\forall v \in S^-$, $\tau(v) = -v$. We have the decomposition $V_{\mathcal{L}}^+ = S^+ \oplus S^-$. Therefore we have the $\langle\tau\rangle$-invariant vertex operator subalgebra

$$V_{\mathcal{L}}^{+\langle\tau\rangle} = \left\{ a \in V_{\mathcal{L}}^+ \mid \tau(a) = a \right\} = S^+.$$

Since $V_{\mathcal{L}}^+$ is a simple vertex operator algebra, $V_{\mathcal{L}}^{+\langle\tau\rangle}$ is also simple (see [16]). Now we have $\langle\tau\rangle = \{1, \tau\}$ is a cyclic group, from Proposition 4.0.4(1), we have $V_{\mathcal{L}}^{+\langle\tau\rangle}$ and $S^-$ are two eigenspaces with respect to $\tau$. Then both of them become two irreducible $V_{\mathcal{L}}^{+\langle\tau\rangle}$-modules.

## 4.1 Irreducible (1-twisted) $V_{\mathcal{L}}^+$-modules

**Theorem 4.1.1.** ([4]). *Let $L$ be a positive definite even lattice. Then any irreducible admissible $V_L^+$-module is isomorphic to one of irreducible modules*

$$V_L^+, V_L^-, V_{L+\lambda} \textit{ for } \lambda \in L^\circ \textit{ with } 2\lambda \notin L,$$

$$V_{L+\lambda}^+, V_{L+\lambda}^- \textit{ for } \lambda \in L^\circ \textit{ with } 2\lambda \in L \textit{ and}$$

$$V_L^{T_\chi,+}, V_L^{T_\chi,-} \textit{ for any irreducible } \hat{L}/K\textit{-module } T_\chi \textit{ with central character } \chi.$$

For the case $V_{\mathcal{L}}^+$, it was considered by above. Next, we will find all $\lambda$ satisfied in Theorem 4.1.1. Recall that $\mathcal{L}^\circ = \{\beta \in \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L} \mid \langle \beta, \mathcal{L} \rangle \subset \mathbb{Z}\}$.

<u>Case</u> $\langle \alpha_1, \alpha_2 \rangle = 0$. We have

$$\begin{aligned}
\mathcal{L}^\circ &= \{\beta \in \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L} \mid \langle \beta, \mathcal{L} \rangle \subset \mathbb{Z}\} && (4.3)\\
&= \{\frac{m}{2k}\alpha_1 + \frac{n}{2k}\alpha_2 \mid m, n \in \mathbb{Z}\} && (4.4)\\
&= \bigcup_{i,j=0}^{2k-1} (\mathcal{L} + \lambda_{i,j}) \quad \text{where } \lambda_{i,j} = \frac{i}{2k}\alpha_1 + \frac{j}{2k}\alpha_2. && (4.5)
\end{aligned}$$

<u>Case</u> $\langle \alpha_1, \alpha_2 \rangle = \pm 2k$. We have

$$\begin{aligned}
\mathcal{L}^\circ &= \{\beta \in \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L} \mid \langle \beta, \mathcal{L} \rangle \subset \mathbb{Z}\} && (4.6)\\
&= \{\frac{i}{2k}\alpha_1 + \frac{j-i}{2k}\alpha_2 \mid i, j \in \mathbb{Z}\} && (4.7)\\
&= \bigcup_{i,j=0}^{2k-1} (\mathcal{L} + \nu_{i,j-i}) \quad \text{where } \nu_{i,j-i} = \frac{i}{2k}\alpha_1 + \frac{j-i}{2k}\alpha_2. && (4.8)
\end{aligned}$$

<u>Case</u> $\langle \alpha_1, \alpha_2 \rangle \in \mathbb{Z} - \{0, \pm 2k\}$. We have

$$\begin{aligned}
\mathcal{L}^\circ &= \{\beta \in \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L} \mid \langle \beta, \mathcal{L} \rangle \subset \mathbb{Z}\} && (4.9)\\
&= \{\left[\frac{\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_2 \rangle^2 - 4k^2} j - \frac{2k}{\langle \alpha_1, \alpha_2 \rangle^2 - 4k^2} i\right] \alpha_1 && (4.10)\\
&\quad + \left[\frac{\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_2 \rangle^2 - 4k^2} i - \frac{2k}{\langle \alpha_1, \alpha_2 \rangle^2 - 4k^2} j\right] \alpha_2 \mid i, j \in \mathbb{Z}\} && (4.11)\\
&= \bigcup_{i,j=0}^{|\langle \alpha_1, \alpha_2 \rangle^2 - 4k^2| - 1} (\mathcal{L} + \epsilon_{\alpha_1(j,i) + \alpha_2(i,j)}) && (4.12)
\end{aligned}$$

where $\epsilon_{\alpha_1(j,i)+\alpha_2(i,j)} = \left[\frac{\langle\alpha_1,\alpha_2\rangle}{\langle\alpha_1,\alpha_2\rangle^2-4k^2}j - \frac{2k}{\langle\alpha_1,\alpha_2\rangle^2-4k^2}i\right]\alpha_1 + \left[\frac{\langle\alpha_1,\alpha_2\rangle}{\langle\alpha_1,\alpha_2\rangle^2-4k^2}i - \frac{2k}{\langle\alpha_1,\alpha_2\rangle^2-4k^2}j\right]\alpha_2.$

**Proposition 4.1.2.** *$V_{\mathcal{L}}^{+} = S^{+} \oplus S^{-}$ and $V_{\mathcal{L}}^{-} = U^{+i} \oplus U^{-i}$ where $S^{\pm} = \{v \in V_{\mathcal{L}}^{+} \mid \tau(v) = \pm v\}$, and $U^{\pm i} = \{v \in V_{\mathcal{L}}^{-} \mid \tau(v) = \pm iv\}$.*

*Proof.* Recall that $V_{\mathcal{L}}^{-}$ is spanned by the vectors of the form

$$v_{m_1,\ldots,m_s;n_1,\ldots,n_r} \otimes F^{m,n} \quad \text{and} \quad v_{m_1,\ldots,m_k;n_1,\ldots,n_l} \otimes E^{m,n}$$

where $s \geq 0, r \geq 0, k \geq 0, l \geq 0, m, n \in \mathbb{Z}$, $s + r$ is even and $k + l$ is odd.

Set $U^{+i}$ to be

$$\begin{aligned} Span_{\mathbb{C}}\{ \quad & (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} + iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (E^{m,n} + E^{n,-m}), \\ & (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} - iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (E^{m,n} - E^{n,-m}), \\ & (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} + v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (F^{m,n} - iF^{n,-m}), \\ & (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} - v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (F^{m,n} - iF^{n,-m}), \\ & (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} + v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (F^{m,n} + iF^{n,-m}), \\ & (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} - v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (F^{m,n} + iF^{n,-m}) \\ & \qquad \mid k, p, q \in 2\mathbb{N}_0 + 1, \; l, s, r \in 2\mathbb{N}_0 \qquad \}. \end{aligned}$$

Set $U^{-i}$ to be

$$\begin{aligned} Span_{\mathbb{C}}\{ \quad & (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} - iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (E^{m,n} + E^{n,-m}), \\ & (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} + iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (E^{m,n} - E^{n,-m}), \\ & (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} + v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (F^{m,n} + iF^{n,-m}), \\ & (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} - v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (F^{m,n} + iF^{n,-m}), \\ & (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} + v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (F^{m,n} - iF^{n,-m}), \\ & (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} - v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (F^{m,n} - iF^{n,-m}) \\ & \qquad \mid k, p, q \in 2\mathbb{N}_0 + 1, \; l, s, r \in 2\mathbb{N}_0 \qquad \}. \end{aligned}$$

It is clear that $\forall v \in U^{+i}$, $\tau(v) = iv$ and $\forall v \in U^{-i}$, $\tau(v) = -iv$. Moreover, $V_{\mathcal{L}}^{-} = U^{+i} \oplus U^{-i}$. $\square$

**Remark 4.1.3.** If $2\epsilon_{\alpha_1(j,i)+\alpha_2(i,j)} \in \mathcal{L}$, we have $i = j = 0$, that is, $V^+_{\mathcal{L}+\epsilon_{\alpha_1(0,0)+\alpha_2(0,0)}} = V^+_{\mathcal{L}} = S^+ \oplus S^-$, it was done. For $2\nu_{i,j-i}, 2\lambda_{i,j} \in \mathcal{L}$, we have $i, j \in \{0, k\}$.

**Proposition 4.1.4.** *If $2\lambda_{i,j} \in \mathcal{L}$ for $i, j \in \{0, k\}$, we have*

$$V^+_{\mathcal{L}+\lambda_{i,j}} = S^+_{\lambda_{i,j}} \oplus S^-_{\lambda_{i,j}} \text{ and } V^-_{\mathcal{L}+\lambda_{i,j}} = U^{+i}_{\lambda_{i,j}} \oplus U^{-i}_{\lambda_{i,j}}$$

*where $S^{\pm}_{\lambda_{i,j}} = \{v \in V^+_{\mathcal{L}+\lambda_{i,j}} \mid \tau(v) = \pm v\}$, and $U^{\pm i}_{\lambda_{i,j}} = \{v \in V^-_{\mathcal{L}+\lambda_{i,j}} \mid \tau(v) = \pm iv\}$.*

*Proof.* Recall that $V^+_{\mathcal{L}+\lambda_{i,j}}$ is spanned by the vectors of the form

$$v_{m_1,\ldots,m_k;n_1,\ldots,n_l} \otimes (e^{\lambda_{i,j}+m\alpha_1+n\alpha_2} + e^{-\lambda_{i,j}-m\alpha_1-n\alpha_2}),$$
$$v_{m_1,\ldots,m_r;n_1,\ldots,n_s} \otimes (e^{\lambda_{i,j}+m\alpha_1+n\alpha_2} - e^{-\lambda_{i,j}-m\alpha_1-n\alpha_2}).$$

$V^-_{\mathcal{L}+\lambda_{i,j}}$ is spanned by the vectors of the form

$$v_{m_1,\ldots,m_k;n_1,\ldots,n_l} \otimes (e^{\lambda_{i,j}+m\alpha_1+n\alpha_2} - e^{-\lambda_{i,j}-m\alpha_1-n\alpha_2}),$$
$$v_{m_1,\ldots,m_r;n_1,\ldots,n_s} \otimes (e^{\lambda_{i,j}+m\alpha_1+n\alpha_2} + e^{-\lambda_{i,j}-m\alpha_1-n\alpha_2})$$

where $k \geq 0, l \geq 0, r \geq 0, s \geq 0, m, n \in \mathbb{Z}$, $k + l$ is even and $r + s$ is odd. Let

$$\begin{aligned}\widetilde{E}^{i,j,m,n} &= e^{\lambda_{i,j}+m\alpha_1+n\alpha_2} + e^{-\lambda_{i,j}-m\alpha_1-n\alpha_2} \text{ and}\\ \widetilde{F}^{i,j,m,n} &= e^{\lambda_{i,j}+m\alpha_1+n\alpha_2} - e^{-\lambda_{i,j}-m\alpha_1-n\alpha_2}.\end{aligned}$$

We note that $\tau(\widetilde{E}^{i,j,m,n}) = \widetilde{E}^{j,-i,n,-m}$, $\tau(\widetilde{F}^{i,j,m,n}) = \widetilde{F}^{j,-i,n,-m}$ and $\widetilde{E}^{-i,-j,-m,-n} = \widetilde{E}^{i,j,m,n}$ , $\widetilde{F}^{-i,-j,-m,-n} = -\widetilde{F}^{i,j,m,n}$. Set $S^+_{\lambda_{i,j}}$ to be

$$\begin{aligned}Span_{\mathbb{C}}\{ \quad & v_{m_1,\ldots,m_k;n_1,\ldots,n_l}\widetilde{E}^{i,j,m,n} + v_{n_1,\ldots,n_l;m_1,\ldots,m_k}\widetilde{E}^{j,-i,n,-m},\\ & v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_{l+1}}\widetilde{E}^{i,j,m,n} - v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_{k+1}}\widetilde{E}^{j,-i,n,-m},\\ & v_{m_1,\ldots,m_k;n_1,\ldots,n_{l+1}}\widetilde{F}^{i,j,m,n} + v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_k}\widetilde{F}^{j,-i,n,-m},\\ & v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_l}\widetilde{F}^{i,j,m,n} - v_{n_1,\ldots,n_l;m_1,\ldots,m_{k+1}}\widetilde{F}^{j,-i,n,-m} \quad : \ k, l \in 2\mathbb{N}_0 \ \}.\end{aligned}$$

Set $S^-_{\lambda_{i,j}}$ to be

$$
\begin{aligned}
Span_{\mathbb{C}}\{ \quad & v_{m_1,\ldots,m_k;n_1,\ldots,n_l}\widetilde{E}^{i,j,m,n} - v_{n_1,\ldots,n_l;m_1,\ldots,m_k}\widetilde{E}^{j,-i,n,-m},\\
& v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_{l+1}}\widetilde{E}^{i,j,m,n} + v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_{k+1}}\widetilde{E}^{j,-i,n,-m},\\
& v_{m_1,\ldots,m_k;n_1,\ldots,n_{l+1}}\widetilde{F}^{i,j,m,n} - v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_k}\widetilde{F}^{j,-i,n,-m},\\
& v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_l}\widetilde{F}^{i,j,m,n} + v_{n_1,\ldots,n_l;m_1,\ldots,m_{k+1}}\widetilde{F}^{j,-i,n,-m} \qquad : \ k,l \in 2\mathbb{N}_0 \ \}.
\end{aligned}
$$

Set $U^{+i}_{\lambda_{i,j}}$ to be

$$
\begin{aligned}
Span_{\mathbb{C}}\{ \quad & (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} + iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (\widetilde{E}^{i,j,m,n} + \widetilde{E}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} - iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (\widetilde{E}^{i,j,m,n} - \widetilde{E}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} + v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (\widetilde{F}^{i,j,m,n} - i\widetilde{F}^{j,-i,n,-mn}),\\
& (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} - v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (\widetilde{F}^{i,j,m,n} - i\widetilde{F}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} + v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (\widetilde{F}^{i,j,m,n} + i\widetilde{F}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} - v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (\widetilde{F}^{i,j,m,n} + i\widetilde{F}^{j,-i,n,-m})\\
& \qquad\qquad\qquad \mid k,p,q \in 2\mathbb{N}_0+1, \ l,s,r \in 2\mathbb{N}_0\}.
\end{aligned}
$$

Set $U^{-i}_{\lambda_{i,j}}$ to be

$$
\begin{aligned}
Span_{\mathbb{C}}\{ \quad & (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} - iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (\widetilde{E}^{i,j,m,n} + \widetilde{E}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_k;n_1,\ldots,n_l} + iv_{n_1,\ldots,n_l;m_1,\ldots,m_k}) \otimes (\widetilde{E}^{i,j,m,n} - \widetilde{E}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} + v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (\widetilde{F}^{i,j,m,n} + i\widetilde{F}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_s;n_1,\ldots,n_r} - v_{n_1,\ldots,n_r;m_1,\ldots,m_s}) \otimes (\widetilde{F}^{i,j,m,n} + i\widetilde{F}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} + v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (\widetilde{F}^{i,j,m,n} - i\widetilde{F}^{j,-i,n,-m}),\\
& (v_{m_1,\ldots,m_p;n_1,\ldots,n_q} - v_{n_1,\ldots,n_q;m_1,\ldots,m_p}) \otimes (\widetilde{F}^{i,j,m,n} - i\widetilde{F}^{j,-i,n,-m})\\
& \qquad\qquad\qquad \mid k,p,q \in 2\mathbb{N}_0+1, \ l,s,r \in 2\mathbb{N}_0\}.
\end{aligned}
$$

It is clear that $S^{\pm}_{\lambda_{i,j}} = \{v \in V^{+}_{\mathcal{L}+\lambda_{i,j}} \mid \tau(v) = \pm v\}$, and $U^{\pm i}_{\lambda_{i,j}} = \{v \in V^{-}_{\mathcal{L}+\lambda_{i,j}} \mid \tau(v) = \pm iv\}$. Moreover, $V^{+}_{\mathcal{L}+\lambda_{i,j}} = S^{+}_{\lambda_{i,j}} \oplus S^{-}_{\lambda_{i,j}}$ and $V^{-}_{\mathcal{L}+\lambda_{i,j}} = U^{+i}_{\lambda_{i,j}} \oplus U^{-i}_{\lambda_{i,j}}$. $\square$

**Proposition 4.1.5.** *If $2\lambda_{i,j}, 2\nu_{i,j-i}, 2\epsilon_{\alpha_1(j',i')+\alpha_2(i',j')} \notin \mathcal{L}$ for $i,j$ as in (4.5) and $i',j'$ as in (4.12), we have $V_{\mathcal{L}+\lambda_{i,j}}$, $V_{\mathcal{L}+\nu_{i,j-i}}$, $V_{\mathcal{L}+\epsilon_{\alpha_1(j',i')+\alpha_2(i',j')}}$ are simple $V_{\mathcal{L}}^{+^{\langle\tau\rangle}}$-modules.*

*Proof.* Recall that $V_{\mathcal{L}+\lambda_{i,j}}$ is spanned by the vectors of the form

$$v_{m_1,\dots,m_k;n_1,\dots,n_l} \otimes e^{\lambda_{i,j}+m\alpha_1+n\alpha_2}$$

where $k,l \in \mathbb{N}_0,\ m,n \in \mathbb{Z}$. We define the map $f$ from $V_{\mathcal{L}+\lambda_{j,-i}}$ onto $V_{\mathcal{L}+\lambda_{i,j}} \circ \tau$ by this action:

$$\alpha_1(-m_1)\cdots\alpha_1(-m_k)\alpha_2(-n_1)\cdots\alpha_2(-n_l) \otimes (e^{\lambda_{j,-i}+m\alpha_1+n\alpha_2})$$
$$\downarrow$$
$$(\tau\alpha_1)(-m_1)\cdots(\tau\alpha_1)(-m_k)(\tau\alpha_2)(-n_1)\cdots(\tau\alpha_2)(-n_l) \otimes (e^{\lambda_{i,j}-m\alpha_2+n\alpha_1})$$

and then extend by linearity. Clearly $f$ is well-defined and bijective. Moreover, $f$ satisfies the following condition:

$$fY_M(u,z) = Y_{M\circ\tau}(u,z)f = Y_M(\tau u,z)f \ \text{ for } u \in V_{\mathcal{L}}^+.$$

Then as $V_L^+$-modules, $V_{\mathcal{L}+\lambda_{i,j}} \circ \tau \cong V_{\mathcal{L}+\lambda_{j,-i}}$ but $V_{\mathcal{L}+\lambda_{i,j}} \ncong V_{\mathcal{L}+\lambda_{j,-i}}$. Therefore $G_M = \{1\}$, $V_{\mathcal{L}+\lambda_{i,j}}$ is an irreducible $V_{\mathcal{L}}^{+^{\langle\tau\rangle}}$-module. For the other, the proof is similarly. □

**Remark 4.1.6.** From Remark 2.0.15, we consider in two cases:

1. When $\langle\alpha_1,\alpha_2\rangle \in 2\mathbb{Z}$, we have

   $$\mathcal{L} = \{x \in \mathcal{L} \mid \langle x,\mathcal{L}\rangle \subset 2\mathbb{Z}\}.$$

   Since $\mathcal{L}/2\mathcal{L}$ is an abelian group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathcal{L}/2\mathcal{L}$ has two irreducible modules $T_1, T_2$ such that $x+2\mathcal{L}$ acts as multiplication by 1 and $-1$, respectively. Then there are two irreducible $\theta$-twisted $V_{\mathcal{L}}$-modules, $V_{\mathcal{L}}^{T_i} = M_{\mathbb{Z}+\frac{1}{2}}(1) \otimes T_i \simeq S(\hat{\mathfrak{h}}_{\mathbb{Z}+\frac{1}{2}}) \otimes T_i$, for $i = 1,2$.

2. When $\langle\alpha_1,\alpha_2\rangle \in 2\mathbb{Z}+1$, we have

$$2\mathcal{L} = \{x \in \mathcal{L} \mid \langle x, \mathcal{L}\rangle \subset 2\mathbb{Z}\}.$$

Then $2\mathcal{L}/2\mathcal{L}$ has only one irreducible module $T$, which $V_{\mathcal{L}}^T = M_{\mathbb{Z}+\frac{1}{2}}(1) \otimes T \simeq S(\hat{\mathfrak{h}}_{\mathbb{Z}+\frac{1}{2}}) \otimes T$.

**Proposition 4.1.7.** *For $i = 1, 2$,*

$$V_{\mathcal{L}}^{T_i,+} = S_{T_i}^{+} \oplus S_{T_i}^{-} \quad \text{and} \quad V_{\mathcal{L}}^{T_i,-} = U_{T_i}^{+i} \oplus U_{T_i}^{-i}$$

*where $S_{T_i}^{\pm} = \{v \in V_{\mathcal{L}}^{T_i,+} \mid \tau(v) = \pm v\}$ and $U_{T_i}^{\pm i} = \{v \in V_{\mathcal{L}}^{T_i,-} \mid \tau(v) = \pm iv\}$.*

*Proof.* Recall that $V_L^{T_i,+}$ is spanned by the vectors of the form: $v_{m_1,\dots,m_k;n_1,\dots,n_l} \otimes t$, and $V_{\mathcal{L}}^{T_i,-}$ is spanned by the vectors of the form: $v_{m_1,\dots,m_s;n_1,\dots,n_r} \otimes t$, where $k \geq 0, l \geq 0, s \geq 0, r \geq 0$, $k+l$ is even, $s+r$ is odd, $m_i, n_j \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, and $t \in T_i$. Set

$$\begin{aligned} S_{T_i}^{+} = Span_{\mathbb{C}}\{ \quad & v_{m_1,\dots,m_k;n_1,\dots,n_l} \otimes t + v_{n_1,\dots,n_l;m_1,\dots,m_k} \otimes t, \\ & v_{m_1,\dots,m_{k+1};n_1,\dots,n_{l+1}} \otimes t - v_{n_1,\dots,n_{l+1};m_1,\dots,m_{k+1}} \otimes t \quad \mid k, l \in 2\mathbb{N}_0,\ t \in T_i\}, \\ S_{T_i}^{-} = Span_{\mathbb{C}}\{ \quad & v_{m_1,\dots,m_k;n_1,\dots,n_l} \otimes t - v_{n_1,\dots,n_l;m_1,\dots,m_k} \otimes t, \\ & v_{m_1,\dots,m_{k+1};n_1,\dots,n_{l+1}} \otimes t + v_{n_1,\dots,n_{l+1};m_1,\dots,m_{k+1}} \otimes t \quad \mid k, l \in 2\mathbb{N}_0, t \in T_i\}, \end{aligned}$$

$$U_{T_i}^{+i} = Span_{\mathbb{C}}\{(v_{m_1,\dots,m_{k+1};n_1,\dots,n_l} + iv_{n_1,\dots,n_l;m_1,\dots,m_{k+1}}) \otimes t \mid k, l \in 2\mathbb{N}_0, t \in T_i\},$$

$$U_{T_i}^{-i} = Span_{\mathbb{C}}\{(v_{m_1,\dots,m_{k+1};n_1,\dots,n_l} - iv_{n_1,\dots,n_l;m_1,\dots,m_{k+1}}) \otimes t \mid k, l \in 2\mathbb{N}_0,\ t \in T_i\}.$$

It is clear that $S_{T_i}^{\pm} = \{v \in V_{\mathcal{L}}^{T_i,+} \mid \tau(v) = \pm v\}$ and $U_{T_i}^{\pm i} = \{v \in V_{\mathcal{L}}^{T_i,-} \mid \tau(v) = \pm iv\}$ which $V_{\mathcal{L}}^{T_i,+} = S_{T_i}^{+} \oplus S_{T_i}^{-}$ and $V_{\mathcal{L}}^{T_i,-} = U_{T_i}^{+i} \oplus U_{T_i}^{-i}$. □

Dong, Li and Mason have shown in [12] that if $V$ is a simple vertex operator algebra which satisfies condition $C_2$ and $g \in \operatorname{Aut} V$ has finite order. Then $V$ has at least one simple $g$-twisted $V$-module.

Here the vertex operator algebra $V_{\mathcal{L}}^{+}$ has at least one simple $\tau$-twisted $V_{\mathcal{L}}^{+}$-module.

## 4.2 An irreducible $\tau \circ \theta$-twisted $M(1)$-module

There is a construction of $\nu$-twisted $M(1)$-module (see [10]), under assumption that $\nu$ is an isometry. To get $\tau$ or $\tau\circ\theta$ having that property, we will assume that $\langle \alpha_1, \alpha_2 \rangle = 0$.

Consider $\tau \circ \theta$,

$$\alpha_1 \underset{\tau\circ\theta}{\mapsto} \alpha_2 \quad \text{and} \quad \alpha_2 \underset{\tau\circ\theta}{\mapsto} -\alpha_1$$

we have $\tau \circ \theta$ induce an automorphism on $M(1)$ of order 4. Decompose $M(1)$ with respect to $\tau \circ \theta$:

$$M(1) = M^+ \oplus M^- \oplus M^{+i} \oplus M^{-i} \tag{4.13}$$

where $M^{\pm} = \{\beta \in M(1) \mid (\tau \circ \theta)\beta = \pm\beta\}$, $M^{\pm i} = \{\beta \in M(1) \mid (\tau \circ \theta)\beta = \pm i\beta\}$ which

$$M^+ = Span_{\mathbb{C}}\{ \quad v_{m_1,\ldots,m_k;n_1,\ldots,n_l}\mathbf{1} + v_{n_1,\ldots,n_l;m_1,\ldots,m_k}\mathbf{1}, \tag{4.14}$$

$$v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_{l+1}}\mathbf{1} - v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_{k+1}}\mathbf{1} \tag{4.15}$$

$$\mid k, l \in 2\mathbb{N}_0,\ t \in T_i\}, \tag{4.16}$$

$$M^- = Span_{\mathbb{C}}\{ \quad v_{m_1,\ldots,m_k;n_1,\ldots,n_l}\mathbf{1} - v_{n_1,\ldots,n_l;m_1,\ldots,m_k}\mathbf{1}, \tag{4.17}$$

$$v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_{l+1}}\mathbf{1} + v_{n_1,\ldots,n_{l+1};m_1,\ldots,m_{k+1}}\mathbf{1} \tag{4.18}$$

$$\mid k, l \in 2\mathbb{N}_0, t \in T_i\}, \tag{4.19}$$

$$M^{+i} = Span_{\mathbb{C}}\{(v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_l} - iv_{n_1,\ldots,n_l;m_1,\ldots,m_{k+1}})\mathbf{1} \mid k, l \in 2\mathbb{N}_0,\ t \in T_i\}, \tag{4.20}$$

$$M^{-i} = Span_{\mathbb{C}}\{(v_{m_1,\ldots,m_{k+1};n_1,\ldots,n_l} + iv_{n_1,\ldots,n_l;m_1,\ldots,m_{k+1}})\mathbf{1} \mid k, l \in 2\mathbb{N}_0, t \in T_i\}. \tag{4.21}$$

Next, consider a primitive 4-th root of unity $\omega_4 = e^{\frac{2\pi i}{4}} = i$ in $\mathbb{C}$. For $n \in \mathbb{Z}$, set

$$\mathfrak{h}_{(n)} := \{\alpha \in \mathfrak{h} \mid (\tau \circ \theta)\alpha = i^n \alpha\} \subset \mathfrak{h}.$$

Then $\mathfrak{h} = \bigoplus_{n \in \mathbb{Z}/4\mathbb{Z}} \mathfrak{h}_{(n)} = \mathfrak{h}_{(0)} \oplus \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)} \oplus \mathfrak{h}_{(3)}$ where

$$\mathfrak{h}_{(0)} = \{0\}, \quad \mathfrak{h}_{(1)} = Span_{\mathbb{C}}\{\alpha_1 - i\alpha_2\},$$
$$\mathfrak{h}_{(2)} = \{0\}, \quad \mathfrak{h}_{(3)} = Span_{\mathbb{C}}\{\alpha_1 + i\alpha_2\}.$$

We defined the $\tau \circ \theta$-twisted affine Lie algebra $\hat{\mathfrak{h}}[\tau \circ \theta]$ associated with the abelian Lie algebra $\mathfrak{h}$ to be

$$\begin{aligned} \hat{\mathfrak{h}}[\tau \circ \theta] \; &:= \textstyle\coprod_{n \in \frac{1}{4}\mathbb{Z}} \mathfrak{h}_{(4n)} \otimes t^n \oplus \mathbb{C}c. \\ &= \sum_{n \in \mathbb{Z}} \mathfrak{h}_{(1)} \otimes t^{\frac{1}{4}+n} \oplus \mathbb{C}c + \sum_{n \in \mathbb{Z}} \mathfrak{h}_{(3)} \otimes t^{\frac{3}{4}+n} \oplus \mathbb{C}c \end{aligned}$$

with

$$[x \otimes t^m, y \otimes t^n] = \langle x, y \rangle m \delta_{m+n,0} c \;\text{ for } x \in \mathfrak{h}_{(4m)}, y \in \mathfrak{h}_{(4n)},\; m, n \in \tfrac{1}{4}\mathbb{Z},$$
$$\left[c, \hat{\mathfrak{h}}[\tau \circ \theta]\right] = 0.$$

Set

$$\hat{\mathfrak{h}}[\tau \circ \theta]^+ = \sum_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{h}_{(1)} \otimes t^{\frac{1}{4}+n} \oplus \mathbb{C}c + \sum_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{h}_{(3)} \otimes t^{\frac{3}{4}+n} \oplus \mathbb{C}c,$$

$$\hat{\mathfrak{h}}[\tau \circ \theta]^- = \sum_{n \in \mathbb{Z}_{<0}} \mathfrak{h}_{(1)} \otimes t^{\frac{1}{4}+n} \oplus \mathbb{C}c + \sum_{n \in \mathbb{Z}_{<0}} \mathfrak{h}_{(3)} \otimes t^{\frac{3}{4}+n} \oplus \mathbb{C}c,$$

Then the subalgebra

$$\hat{\mathfrak{h}}[\tau \circ \theta]^+ \oplus \hat{\mathfrak{h}}[\tau \circ \theta]^- \oplus \mathbb{C}c$$

of $\hat{\mathfrak{h}}[\tau\circ\theta]$ is a Heisenberg algebra. Form the induced $\hat{\mathfrak{h}}[\tau\circ\theta]$-module

$$M(1)[\tau\circ\theta] = U(\hat{\mathfrak{h}}[\tau\circ\theta]) \otimes_{U(\hat{\mathfrak{h}}[\tau\circ\theta]^{+}\oplus\mathfrak{h}_{(0)}\oplus\mathbb{C}c)} \mathbb{C} \simeq S(\hat{\mathfrak{h}}[\tau\circ\theta]^{-}) \text{ (linearly)} \tag{4.22}$$

where $\hat{\mathfrak{h}}[\tau\circ\theta]^{+}$ acts trivially on $\mathbb{C}$, and $c$ acts as 1. We will use the notation $\alpha^{\tau\circ\theta}(n)$ $(\alpha\in\mathfrak{h}_{(4n)}, n\in\frac{1}{4}\mathbb{Z})$ for the action of $\alpha\otimes t^{n}\in\hat{\mathfrak{h}}[\tau\circ\theta]$ on $M(1)[\tau\circ\theta]$.

Set

$$\alpha^{\tau\circ\theta}(z) := \sum_{n\in\frac{1}{4}\mathbb{Z}} \alpha(n)z^{-n-1}.$$

For $v = \alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes\mathbf{1}\in M(1)$, we let

$$W(v,z) =: \partial^{(n_1-1)}\alpha_1^{\tau\circ\theta}(z)\cdots\partial^{(n_k-1)}\alpha_k^{\tau\circ\theta}(z) :,$$

where $\partial^{(n)} = \frac{1}{n!}\left(\frac{d}{dz}\right)^{n}$ and define $Y^{\tau\circ\theta}(v,z) = W(e^{\Delta_z}v,z)$. Here, $\Delta_z$ is a certain formal operator involving the formal variable $z$, and defined as follows: For an orthonormal basis of $\mathfrak{h}$, namely $\{\beta_1 = \frac{1}{\sqrt{2k}}\alpha_1, \beta_2 = \frac{1}{\sqrt{2k}}\alpha_2\}$, we set

$$\begin{aligned}
\Delta_z &= \textstyle\sum_{m,n\geq 0}\sum_{l=0}^{3}\sum_{j=1}^{2} c_{mnl}((\tau\circ\theta)^{-l}\beta_j)(m)\beta_j(n)z^{-m-n}\\
&= \textstyle\sum_{m,n\geq 0}\sum_{l=0}^{3} c_{mnl}\frac{1}{2k}\left[((\tau\circ\theta)^{-l}\alpha_1)(m)\alpha_1(n) + ((\tau\circ\theta)^{-l}\alpha_2)(m)\alpha_2(n)\right] z^{-m-n}\\
&= \textstyle\sum_{m,n\geq 0} c_{mn0}\frac{1}{2k}\left[((\tau\circ\theta)^{-0}\alpha_1)(m)\alpha_1(n) + ((\tau\circ\theta)^{-0}\alpha_2)(m)\alpha_2(n)\right] z^{-m-n} + \cdots\\
&\quad + c_{mn3}\frac{1}{2k}\left[((\tau\circ\theta)^{-3}\alpha_1)(m)\alpha_1(n) + ((\tau\circ\theta)^{-3}\alpha_2)(m)\alpha_2(n)\right] z^{-m-n}\\
&= \textstyle\sum_{m,n\geq 0}\frac{1}{2k}[c_{mn0}[\alpha_1(m)\alpha_1(n)+\alpha_2(m)\alpha_2(n)]\\
&\qquad + c_{mn1}[-\alpha_2(m)\alpha_1(n)+\alpha_1(m)\alpha_2(n)]\\
&\qquad + c_{mn2}[-\alpha_1(m)\alpha_1(n)-\alpha_2(m)\alpha_2(n)]\\
&\qquad + c_{mn3}[\alpha_2(m)\alpha_1(n)-\alpha_1(m)\alpha_2(n)]]z^{-m-n}
\end{aligned}$$

where constants $c_{mnl} \in \mathbb{C}$ for $m, n \in \mathbb{Z}_{\geq 0}$, $l = 0, 1, 2, 3$ are defined by the formulas

$$\sum_{m,n\geq 0} c_{mn0} x^m y^n = -\frac{1}{2}\sum_{r=1}^{3} \log\left(\frac{(1+x)^{1/4} - i^{-r}(1+y)^{1/4}}{1-i^{-r}}\right), \qquad (4.23)$$

$$\sum_{m,n\geq 0} c_{mnl} x^m y^n = \frac{1}{2}\log\left(\frac{(1+x)^{1/4} - i^{-l}(1+y)^{1/4}}{1-i^{-l}}\right), \quad \text{for } l \neq 0. \qquad (4.24)$$

It has been established in [10, 25] that $(M(1)[\tau\circ\theta], Y^{\tau\circ\theta})$ is the irreducible $\tau\circ\theta$-twisted $M(1)$-module.

We define a linear map $\tau\circ\theta$ on $M(1)[\tau\circ\theta]$ by the following action:

$$\begin{aligned} & x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r) \\ & \downarrow \\ & ((\tau\circ\theta)x)(-n_1)\cdots((\tau\circ\theta)x)(-n_s)((\tau\circ\theta)y)(-m_1)\cdots((\tau\circ\theta)y)(-m_r) \\ & = (-1)^r i^{s+r} x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r) \end{aligned}$$

where $(\tau\circ\theta)x = ix$, $(\tau\circ\theta)y = -iy$ for $x \in \mathfrak{h}_{(1)}$ and $y \in \mathfrak{h}_{(3)}$. We now decompose $M(1)[\tau\circ\theta]$ into its eigenspaces with respect to $\tau\circ\theta$:

$$M(1)[\tau\circ\theta] = M(1)[\tau\circ\theta]^{+} \oplus M(1)[\tau\circ\theta]^{-} \oplus M(1)[\tau\circ\theta]^{+i} \oplus M(1)[\tau\circ\theta]^{-i}$$

where $M(1)[\tau\circ\theta]^{\pm} = \{v \in M(1)[\tau\circ\theta] \mid (\tau\circ\theta)v = \pm v\}$,

$M(1)[\tau\circ\theta]^{\pm i} = \{v \in M(1)[\tau\circ\theta] \mid (\tau\circ\theta)v = \pm iv\}$.

**Remark 4.2.1.** For $x \in \mathfrak{h}_{(1)}$, $y \in \mathfrak{h}_{(3)}$ and $[i] \in \mathbb{Z}_4$, we have

1. $M(1)[\tau\circ\theta]^{+}$ is spanned by vectors having forms:

$$x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r)$$

where $(s,r) \in \bigcup_{i=0}^{3} [i] \times [i]$.

2. $M(1)[\tau \circ \theta]^-$ is spanned by vectors having forms:

$$x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r)$$

where $(s,r) \in [0] \times [2] \ \cup \ [1] \times [3] \ \cup \ [2] \times [0] \ \cup \ [3] \times [1]$.

3. $M(1)[\tau \circ \theta]^{+i}$ is spanned by vectors having forms:

$$x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r)$$

where $(s,r) \in [0] \times [3] \ \cup \ [1] \times [0] \ \cup \ [2] \times [1] \ \cup \ [3] \times [2]$.

4. $M(1)[\tau \circ \theta]^{-i}$ is spanned by vectors having forms:

$$x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r)$$

where $(s,r) \in [0] \times [1] \ \cup \ [1] \times [2] \ \cup \ [2] \times [3] \ \cup \ [3] \times [0]$.

Notice that from Proposition 4.0.3, we have $M^{\pm}, M^{\pm i}, M(1)[\tau \circ \theta]^{\pm}, M(1)[\tau \circ \theta]^{\pm i}$ are eight simple $M^+$-modules , $M^+$ defined as (4.14).

## 4.3 Irreducible $\tau$-twisted $M(1)^+$-modules

A linear automorphism map $\theta$ on $M(1)[\tau \circ \theta]$ is defined as (4.22), by the following action:

$$\begin{gathered} x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r) \\ \downarrow \\ (\theta x)(-n_1)\cdots(\theta x)(-n_s)(\theta y)(-m_1)\cdots(\theta y)(-m_r) \\ = (-1)^{s+r}x(-n_1)\cdots x(-n_s)y(-m_1)\cdots y(-m_r). \end{gathered}$$

Clearly, $\theta Y^{\tau\circ\theta}(v,z)\theta^{-1} = Y^{\tau\circ\theta}(\theta v, z)$ for $v \in M(1)$, the action of $\theta$ on $M(1)[\tau\circ\theta]$ gives an automorphism on $M(1)[\tau\circ\theta]$. Then we now decompose $M(1)[\tau\circ\theta]$ into its eigenspaces with respect to $\theta$:

$$M(1)[\tau\circ\theta] = M(1)[\tau\circ\theta]^{+\theta} \oplus M(1)[\tau\circ\theta]^{-\theta}$$

where $M(1)[\tau\circ\theta]^{\pm\theta} = \{v \in M(1)[\tau\circ\theta] \mid \theta v = \pm v\}$. Note that from Remark 4.2.1, we have

$$\begin{aligned} M(1)[\tau\circ\theta]^{+\theta} &= M(1)[\tau\circ\theta]^{+} \oplus M(1)[\tau\circ\theta]^{-}, \text{ and} \\ M(1)[\tau\circ\theta]^{-\theta} &= M(1)[\tau\circ\theta]^{+i} \oplus M(1)[\tau\circ\theta]^{-i} \end{aligned}$$

Next, to show that $M(1)[\tau\circ\theta]^{+\theta}$ and $M(1)[\tau\circ\theta]^{-\theta}$ are $\tau$-twisted $M(1)^+$-modules, it suffices to prove that they satisfy twisted Jacobi identity.

**Remark 4.3.1.** From (4.13)-(4.21), we have

$$\begin{aligned} M(1) &= M^0_{\tau\circ\theta} + M^1_{\tau\circ\theta} + M^2_{\tau\circ\theta} + M^3_{\tau\circ\theta}, \quad M^n_{\tau\circ\theta} = \{u \in M(1) \mid (\tau\circ\theta)u = i^n u\} \\ &\quad \text{where } \ M^0_{\tau\circ\theta} = M^+, M^1_{\tau\circ\theta} = M^{+i}, M^2_{\tau\circ\theta} = M^-, M^3_{\tau\circ\theta} = M^{-i}, \ \text{ and} \\ M(1) &= M^0_{\tau} + M^1_{\tau} + M^2_{\tau} + M^3_{\tau}, \quad M^n_{\tau} = \{u \in M(1) \mid \tau u = i^n u\} \\ &\quad \text{where } \ M^0_{\tau} = M^+, M^1_{\tau} = M^{-i}, M^2_{\tau} = M^-, M^3_{\tau} = M^{+i}. \end{aligned}$$

We observe that $M(1)^+ = M^0_\tau + M^2_\tau$ which $M^0_\tau = M^0_{\tau\circ\theta}$ and $M^2_\tau = M^2_{\tau\circ\theta}$. Recall twisted Jacobi identity (1.13): For $u \in V^r \ \ (= M^r_{\tau\circ\theta})$, $r = 0,1,2,3$,

$$\begin{aligned} z_0^{-1}\delta\left(\tfrac{z_1-z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\left(\tfrac{z_2-z_1}{-z_0}\right) Y_M(v,z_2)Y_M(u,z_1) \\ = z_2^{-1}\left(\tfrac{z_1-z_0}{z_2}\right)^{\frac{-r}{T}} \delta\left(\tfrac{z_1-z_0}{z_2}\right) Y_M(Y(u,z_0)v,z_2). \end{aligned}$$

Since for $u \in M^0_{\tau\circ\theta} = M^0_\tau$ or $u \in M^2_{\tau\circ\theta} = M^2_\tau$, $\tau\circ\theta$ has order 4 on $M(1)$ and $\tau$ has order 2 on $M(1)^+$, these imply that $M(1)[\tau\circ\theta]^{+\theta}$ and $M(1)[\tau\circ\theta]^{-\theta}$ are $\tau$-twisted $M(1)^+$-modules. As $(M(1)[\tau\circ\theta], Y^{\tau\circ\theta})\circ\theta = (M(1)[\tau\circ\theta]\circ\theta, Y^{\tau\circ\theta}_\theta)$, where $M(1)[\tau\circ\theta]\circ\theta = M(1)[\tau\circ\theta]$ as vector spaces and $Y^{\tau\circ\theta}_\theta(v,z) = Y^{\tau\circ\theta}(\theta(v),z)$ for

$v \in M(1)$. We have $M(1)[\tau \circ \theta] \circ \theta \cong M(1)[\tau \circ \theta]$ as $\tau \circ \theta$-twisted $M(1)$-modules. Since $\langle \theta \rangle$ is an automorphism of $M(1)$ of order 2, by applying Proposition 4.0.3 or 4.0.4, we have $M(1)[\tau \circ \theta]^{+\theta}$ and $M(1)[\tau \circ \theta]^{-\theta}$ are two irreducible $\tau$-twisted $M(1)^{+}$-modules.

In the future, we will use these results to construct irreducible $\tau$-twisted $V_{\mathcal{L}}^{+}$-modules.

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# APPENDIX

The calculation of Table 1.

Recall that $\omega = -\frac{1}{4k}\alpha(-1)^2\mathbf{1}$, $\langle\alpha,\alpha\rangle = -2k$, $\alpha(0)F^m = -2kmE^m$, $\alpha(0)E^m = -2kmF^m$ and

$$\begin{aligned} Y(\omega,z) &= Y(-\tfrac{1}{4k}\alpha(-1)^2\mathbf{1},z) \\ &= -\tfrac{1}{4k}\sum_{m,n\in\mathbb{Z}} :\alpha(m)\alpha(n): z^{-m-n-2}. \end{aligned}$$

$$\begin{aligned} L(-1)f_1 &= \omega_0\alpha(-5)F^m \\ &= -\tfrac{1}{4k}\sum_{m+n=-1} :\alpha(m)\alpha(n):\alpha(-5)F^m \\ &= -\tfrac{1}{4k}\left[2\alpha(-6)\alpha(5)+2\alpha(-1)\alpha(0)\right]\alpha(-5)F^m \\ &= -\tfrac{1}{4k}\left[2\alpha(-6)\left(\alpha(-5)\alpha(5)+5\langle\alpha,\alpha\rangle\right)+2\alpha(-1)\alpha(-5)\alpha(0)\right]F^m \\ &= 5\alpha(-6)F^m+m\alpha(-5)\alpha(-1)E^m, \end{aligned}$$

$$\begin{aligned} L(-1)f_2 &= -\tfrac{1}{4k}\left[2\alpha(-5)\alpha(4)+2\alpha(-1)\alpha(0)+2\alpha(-2)\alpha(1)\right]\alpha(-4)\alpha(-1)E^m \\ &= 4\alpha(-5)\alpha(-1)E^m+m\alpha(-4)\alpha(-1)^2F^m+\alpha(-4)\alpha(-2)E^m, \end{aligned}$$

$$\begin{aligned} L(-1)f_3 &= -\tfrac{1}{4k}\left[2\alpha(-4)\alpha(3)+2\alpha(-3)\alpha(2)+2\alpha(-1)\alpha(0)\right]\alpha(-3)\alpha(-2)E^m \\ &= 3\alpha(-4)\alpha(-2)E^m+2\alpha(-3)^2E^m+m\alpha(-3)\alpha(-2)\alpha(-1)F^m, \end{aligned}$$

$$\begin{aligned} L(-1)f_4 &= -\tfrac{1}{4k}\left[2\alpha(-4)\alpha(3)+2\alpha(-2)\alpha(1)+2\alpha(-1)\alpha(0)\right]\alpha(-3)\alpha(-1)^2F^m \\ &= 3\alpha(-4)\alpha(-1)^2F^m+2\alpha(-3)\alpha(-2)\alpha(-1)F^m+m\alpha(-3)\alpha(-1)^3E^m, \end{aligned}$$

$$\begin{aligned} L(-1)f_5 &= -\tfrac{1}{4k}\left[2\alpha(-3)\alpha(2)+2\alpha(-2)\alpha(1)+2\alpha(-1)\alpha(0)\right]\alpha(-2)^2\alpha(-1)F^m \\ &= 4\alpha(-3)\alpha(-2)\alpha(-1)F^m+\alpha(-2)^3F^m+m\alpha(-2)^2\alpha(-1)^2E^m, \end{aligned}$$

$$\begin{aligned} L(-1)f_6 &= -\tfrac{1}{4k}\left[2\alpha(-3)\alpha(2)+2\alpha(-2)\alpha(1)+2\alpha(-1)\alpha(0)\right]\alpha(-2)\alpha(-1)^3E^m \\ &= 2\alpha(-3)\alpha(-1)^3E^m+3\alpha(-2)^2\alpha(-1)^2E^m+m\alpha(-2)\alpha(-1)^4F^m, \end{aligned}$$

$$\begin{aligned} L(-1)f_7 &= -\tfrac{1}{4k}\left[2\alpha(-2)\alpha(1)+2\alpha(-1)\alpha(0)\right]\alpha(-1)^5F^m \\ &= 5\alpha(-2)\alpha(-1)^4F^m+m\alpha(-1)^6E^m, \end{aligned}$$

$$\begin{aligned} 2kL(-3)h_1 &= -2k\tfrac{1}{4k}\left[2\alpha(-6)\alpha(3)+2\alpha(-3)\alpha(0)+2\alpha(-2)\alpha(-1)\right]\alpha(-3)F^m \\ &= 6k\alpha(-6)F^m+2km\alpha(-3)^2E^m-\alpha(-3)\alpha(-2)\alpha(-1)F^m, \end{aligned}$$

$$\begin{aligned}
2kL(-3)h_2 &= -2k\tfrac{1}{4k}[2\alpha(-5)\alpha(2)+2\alpha(-4)\alpha(1)+2\alpha(-3)\alpha(0) \\
&\quad +2\alpha(-2)\alpha(-1)]\alpha(-2)\alpha(-1)E^m \\
&= 4k\alpha(-5)\alpha(-1)E^m+2k\alpha(-4)\alpha(-2)E^m+2km\alpha(-3)\alpha(-2)\alpha(-1)F^m \\
&\quad -\alpha(-2)^2\alpha(-1)^2E^m, \\
2kL(-3)h_3 &= -2k\tfrac{1}{4k}\left[2\alpha(-4)\alpha(1)+2\alpha(-3)\alpha(0)+2\alpha(-2)\alpha(-1)\right]\alpha(-1)^3F^m \\
&= 6k\alpha(-4)\alpha(-1)^2F^m+2km\alpha(-3)\alpha(-1)^3E^m-\alpha(-2)\alpha(-1)^4F^m, \\
(\alpha(-1)^4\mathbf{1})_{-3}E^m &= \sum_{n_1+\cdots+n_4=-6} :\alpha(n_1)\alpha(n_2)\alpha(n_3)\alpha(n_4): E^m \\
&= [4\alpha(-6)\alpha(0)^3+12\alpha(-5)\alpha(-1)\alpha(0)^2+12\alpha(-4)\alpha(-2)\alpha(0)^2 \\
&\quad +12\alpha(-4)\alpha(-1)^2\alpha(0)+6\alpha(-3)^2\alpha(0)^2 \\
&\quad +24\alpha(-3)\alpha(-2)\alpha(-1)\alpha(0)+4\alpha(-3)\alpha(-1)^3 \\
&\quad +4\alpha(-2)^3\alpha(0)+6\alpha(-2)^2\alpha(-1)^2]E^m \\
&= -32k^3m^3g_1+48k^2m^2g_2+48k^2m^2g_3-24kmg_4+24k^2m^2g_5 \\
&\quad -48kmg_6+4g_7-8kmg_8+6g_9.
\end{aligned}$$

# VITA

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