FUNCTIONAL EQUATIONS WITH TRIGONOMETRIC FUNCTION SOLUTIONS

Miss Charinthip Hengkrawit

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สมการเชิงฟังก์ชันที่มีผลเฉลยเป็นฟังก์ชันตรีโกณมิติ

นางสาวจรินทร์ทิพย์ เฮงคราวิทย์

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Thesis Title	Functional Equations with Trigonometric Function Solutions
By	Miss Charinthip Hengkrawit
Field of Study	Mathematics
Thesis Advisor	Associate Professor Patanee Udomkavanich, Ph.D.
Thesis Co-Advisor	Professor Vichian Laohakosol, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree.

.... Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

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.... Chairman
(Associate Professor Ajchara Harnchoowong, Ph.D.)

.... Thesis Advisor
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.... Thesis Co-Advisor
(Professor Vichian Laohakosol, Ph.D.)

.... Examiner
(Associate Professor Paisan Nakmahachalasint, Ph.D.)

.... Examiner
(Associate Professor Phichet Chaoha, Ph.D.)

.... External Examiner
(Associate Professor Virool Boonyasombat, Ph.D.)

จรินทร์ทิพย์ เฮงคราวิทย์ : สมการเชิงฟังก์ชันที่มีผลเฉลยเป็นฟังก์ชันตรีโกณมิติ (FUNCTIONAL EQUATIONS WITH TRIGONOMETRIC FUNCTION SOLUTIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: รศ.ดร. พัฒนี อุดมกะวานิช, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม: ศ. ดร. วิเชียร เลาหโกศล, 44 หน้า.

ส่วนแรกของวิทยานิพนธ์ศึกษาปัญหาการจำแนกฟังก์ชันตรีโกณมิติและไฮเพอร์โบลิกไซน์-โคไซน์ ระเบียบวิธีของเรามาจากงานวิจัยของแคนแนพพันในปี ค.ศ. 2003 ซึ่งหาผลเฉลยของสมการเชิงฟังก์ชัน $f(x-y)=f(x)f(y)+g(x)g(y)$ สำหรับฟังก์ชันที่มีโดเมนเป็นกรุปและเรนจ์เป็นเซตย่อยของจำนวนเชิงซ้อนโดยไม่มีเงื่อนไขเพิ่มเติมใดๆ เราใช้เทคนิคของแคนแนพพันเพื่อหาผลเฉลยทั่วไปของสมการเชิงฟังก์ชัน $f(x+y)=f(x)f(y)-g(x)g(y)$ ซึ่งเมื่อรวมกับผลจากงานของแคนแนพพันจะได้การจำแนกที่สมบูรณ์ของฟังก์ชันตรีโกณมิติไซน์-โคไซน์ ถัดมาเรานำสมการเชิงฟังก์ชันในรูป $f(x-y)=f(x)f(y)-g(x)g(y)$ มาจำแนกฟังก์ชันไฮเพอร์โบลิกไซน์-โคไซน์และหาความสัมพันธ์ระหว่างฟังก์ชันผลเฉลยซึ่งเป็นเอกลักษณ์ของฟังก์ชันไฮเพอร์โบลิกไซน์-โคไซน์ที่รู้จักกันเป็นอย่างดีและเป็นนัยทั่วไปของสมการเชิงฟังก์ชันเดอ เลมเบิร์ต

ส่วนที่สองของวิทยานิพนธ์กล่าวถึงการจำแนกฟังก์ชันตรีโกณมิติและไฮเพอร์โบลิก แทนเจนต์-โคแทนเจนต์ ในส่วนนี้มี 2 วิธีการ วิธีการแรกอาศัยแนวทางของดอบส์ในปี ค.ศ. 1989 สำหรับฟังก์ชันตรีโกณมิติแทนเจนต์ วิธีการนี้เป็นการวิเคราะห์และใช้ความต่อเนื่องและการหาอนุพันธ์ได้ ณ จุดบ่งเฉพาะจุดหนึ่ง ดอบส์ได้นิยามชั้นของฟังก์ชันค่าจริง T ที่เรียกว่าฟังก์ชันแทนเจนต์เชียล คือฟังก์ชันที่สอดคล้องสมการเชิงฟังก์ชัน $T(u+v) = (T(u)+T(v))/(1-T(u)T(v))$ เราประยุกต์ผลที่ได้จากดอบส์เพื่อจำแนกฟังก์ชันตรีโกณมิติโคแทนเจนต์และใช้วิธีการของดอบส์เพื่อจำแนกฟังก์ชันไฮเพอร์โบลิกแทนเจนต์-โคแทนเจนต์ผ่านสมการเชิงฟังก์ชันที่สอดคล้องฟังก์ชันดังกล่าวตามลำดับ ฟังก์ชันที่พิจารณามีเซตของจำนวนจริงและ/หรือเซตย่อยของเซตของจำนวนจริงเป็นโดเมนและเรนจ์ วิธีการที่สองเป็นการศึกษางานวิจัยของรูมาในปี ค.ศ. 2005 ซึ่งให้ผลเฉลยรูปปิดของสมการเชิงฟังก์ชันในรูปของลำดับเวียนเกิดเชิงตรรกยะ $y_{n+2}=(y_n y_{n+1}-1)/(y_n+ y_{n+1})$ ซึ่งเป็นสมการเชิงฟังก์ชันที่น่าสนใจมากในขณะนี้ เราขยายเทคนิคของรูมาเพื่อหาผลเฉลยรูปปิดของสมการเชิงฟังก์ชันในรูปของลำดับเวียนเกิดเชิงตรรกยะและใช้ผลลัพธ์ดังกล่าวจำแนกฟังก์ชันตรีโกณมิติแทนเจนต์-โคแทนเจนต์และไฮเพอร์โบลิกแทนเจนต์ -โคแทนเจนต์

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ปีการศึกษา 2552	ลายมือชื่ออ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

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The first part of the thesis treats the problem of characterizing the trigonometric and hyperbolic sine-cosine functions. Our method arises from Kannappan's work of 2003 which solved the functional equation $f(x-y) = f(x)f(y) + g(x)g(y)$ for functions whose domain is a group, whose range is a subset of the complex field and without any additional conditions. We use Kannapan's technique to determine the general solutions of the functional equation $f(x+y) = f(x)f(y) - g(x)g(y)$ which together with Kannappan's result give a complete characterization of the trigonometric sine-cosine functions. Next, the functional equation $f(x-y) = f(x)f(y) - g(x)g(y)$ is used to characterize the hyperbolic sine-cosine functions, and inter-relations among the solution functions, resemble certain well-known hyperbolic sine-cosine identities and generalizing the classical d'Alembert functional equation, are obtained.

The second part of the thesis gives characterizations of the trigonometric and hyperbolic tangent-cotangent functions. There are two approaches in this part. The first approach is along the line treated by Dobbs in 1989 for the trigonometric tangent function. It is analytic in character and makes use of continuity and differentiabilty at one specific point. Dobbs defined the class of real-valued functions T of real variable, called tangential functions, as those satisfying the functional equation $T(u+v) = \frac{T(u)+T(v)}{1-T(u)T(v)}$. We apply the result of Dobbs to characterize the trigonometric cotangent function and then proceed to use Dobbs' approach to characterize the hyperbolic tangent-cotangent functions through their respective functional equations. The functions considered are to have the real numbers and/or its subset as their domain and range. The second approach is discrete in character and stems from the work of Rhouma in 2005 which gave a closed form solution to the recursive difference equation $y_{n+2} = \frac{y_n y_{n+1}-1}{y_n+y_{n+1}}$. This is a discrete functional equation of much recent interests in itself. We generalize the technique of Rhouma to find the closed form solutions of certain rational recursive equations and use the results to characterize the cotangent-tangent and the hyperbolic cotangent-tangent functions.

Department: Mathematics	Student's Signature Charinthip Hengkrawit
Field of Study: Mathematics	Advisor's Signature P. Udi...
Academic Year: 2009	Co-Advisor's Signature V. La...

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CONTENTS

CHAPTER I

INTRODUCTION

The principal investigation of this thesis deals with the problem of characterizing trigonometric and hyperbolic functions through the use of functional equations. Apart from the introduction, there are two main parts. The first part treats the problem of characterizing the trigonometric and hyperbolic sine-cosine functions and the second part treats a similar characterization for the trigonometric and hyperbolic tangent-cotangent functions.

Functional equations are equations in which the unknown (or unknowns) are functions. The origin of functional equations came about the same time as the modern definition of function. The best known, and most thoroughly studied, functional equations are those connected to the famous Cauchy functional equation, whose additive form, referred to as the *additive Cauchy functional equation*, is

$$A(x + y) = A(x) + A(y), \tag{1.1}$$

and whose solution is collectively referred to as an *additive function.* Through suitable change of variables, the additive Cauchy functional equation can be transformed to an *exponential Cauchy functional equation* of the form

$$E(x + y) = E(x)E(y), \tag{1.2}$$

whose nontrivial solution is referred to as an *exponential function.*

After the Cauchy functional equation, the next best known functional equations are those related to identities satisfied by the trigonometric sine and cosine functions and commonly called functional equations of d'Alembert type. During 1747 to 1750, J. d'Alembert published three papers. These three papers could be considered as first principal works on functional equations. The early significant growth of the discipline of functional equations was stimulated by the problem of the parallelogram law of forces, see e.g. Aczél (1966). In 1769, d'Alembert reduced this problem to finding solutions of the functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{1.3}$$

which is known as the *d'Alembert functional equation.* In 1821, Cauchy, [1], proved that the continuous nontrivial solution $f:\mathbb{R}\to\mathbb{R}$ of (1.3) is either

$$f(x)=\cosh(\alpha x) \text{ or } f(x)=\cos(\beta x),$$

where α and β are arbitrary real constants. There have been numerous related works since then. Let us mention some which are of interest to us. In 1924, Kaczmarz, [1], extended this result by showing that the same conclusion still holds if the continuity condition is replaced by measurability; his argument covers the case in which f takes complex values. Flett, [6], in 1963 proved that if $f:\mathbb{C}\to\mathbb{C}$ satisfies the d'Alembert functional equation (1.3) for all $x, y \in \mathbb{C}$ and f is continuous at a point, then f has one of the following forms

$$f\equiv 0, \quad f\equiv 1, \quad f(x+iy)=\cosh(\alpha x+\beta y) \qquad (z=x+iy\in\mathbb{C}),$$

where α, β are complex constants, not both zero. Over the domain of an arbitrary group, $(G, *)$, the d'Alembert functional equation (1.3) takes the form

$$f(x * y) + f(x * y^{-1}) = 2f(x)f(y). \tag{1.4}$$

This equation with $f : G \to \mathbb{C}$ was first studied by Pl. Kannappan, [7] in 1968. Kannappan proved, with a good deal of work and ingenuity, that any non-zero solution of (1.4) which also satisfies the condition

$$f(x * y * z) = f(x * z * y) \qquad (x, y, z \in G),$$

has the form

$$f(x) = \frac{E(x) + E^*(x)}{2}, \tag{1.5}$$

where E is an exponential function on G into $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $E^*(x) = 1/E(x)$.

Since the cosine function satisfies (1.3), the d'Alembert functional equation (1.3) is sometimes known as the *cosine equation*. Since the two trigonometric functions $f(x) = \cos x$ and $g(x) = \sin x$ satisfy

$$f(x - y) = f(x)f(y) + g(x)g(y), \tag{1.6}$$

this functional equation is sometimes called the *trigonometric functional equation*. The two trigonometric functions $f(x) = \cos x,\ g(x) = \sin x$ also satisfy the following three functional equations:

$$f(x+y) = f(x)f(y) - g(x)g(y) \tag{1.7}$$

$$g(x+y) = g(x)f(y) + f(x)g(y) \tag{1.8}$$

$$g(x-y) = g(x)f(y) - f(x)g(y). \tag{1.9}$$

The functional equation (1.6) was treated by Gerretsen (1939) and Vaughan (1955). The functional equation (1.7) was considered by Vietoris (1944) and also by van der Corput (1941). In the direction of finding inter-relations among the four functional equations (1.6)-(1.9), V.L. Klee, [11], in 1953 posed the following problem.

Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (1.6) *with* $f(t) = 1$ *and* $g(t) = 0$ *for some* $t \neq 0$. *Prove that* f *and* g *satisfy the functional equations* (1.7), (1.8) *and* (1.9).

A solution to this problem by T.S. Chihara appeared in [3], but it unfortunately had a gap. In 2003, Kannappan, [8], gave the following general solution of (1.6) for functions with more general domain and without any additional conditions.

Theorem 1.0.1. *Let* $(G, +)$ *be a two-divisible abelian group (i.e., a group for which to each* $x \in G$*, there exists a unique* $y \in G$ *such that* $x = 2y$*). If the functions* $f,\ g : G \to \mathbb{C}$ *satisfy the functional equation*

$$f(x-y) = f(x)f(y) + g(x)g(y), \tag{1.6}$$

then they also satisfy the equations (1.7), (1.8) *and* (1.9).

Moreover, the solution functions are given by

$$g(x) = \frac{1}{2}\left(E(x) + E^*(x)\right), \quad f(x) = b_0\left(E(x) - E^*(x)\right),$$

where $b_0^2 = -\frac{1}{4}$, $E : G \to \mathbb{C}^*$ *is an exponential function and* $E^*(x) = \frac{1}{E(x)}$.

Theorem 1.0.1 leads to ([1], [7]):

Corollary 1.0.2. *If* $f, g : \mathbb{R} \to \mathbb{C}$ *are nonconstant solutions of* (1.6) *and* g *is continuous, then* f *is also continuous and the two solution functions are of the form*

$$f(x) = \cos(k_0 x), \quad g(x) = b_0 \sin(k_0 x),$$

where $b_0^2 = -\frac{1}{4}$, $k_0 \in \mathbb{C}$.

Regarding the functional equation (1.8), the following result is known and its proof can be found in Aczél, [2] pages 210-211.

Theorem 1.0.3. *Let* $(G, +)$ *be a group and* $f, g : G \to \mathbb{C}$. *If* f *and* g *satisfy*

$$g(x + y) = g(x)f(y) + f(x)g(y), \tag{1.8}$$

then

$$\begin{cases} g(x) & \equiv 0, \\ f & \text{arbitrary}, \end{cases}$$

or

$$\begin{cases} g(x) & = \frac{E_1(x) - E_2(x)}{2\beta}, \\ f(x) & = \frac{E_1(x) + E_2(x)}{2}, \end{cases}$$

or

$$\begin{cases} g(x) & = E(x)A(x), \\ f(x) & = E(x), \end{cases}$$

where E, E_1, E_2 *are exponential functions,* A *is an additive function and* β *is a nonzero complex constant.*

Regarding the equation (1.9), the following result, whose proof can be found in Aczél, [2] pages 213-217, is known.

Theorem 1.0.4. *Let* $(G, +)$ *be a two-divisible abelian group and* $f, g : G \to \mathbb{C}$. *If* f *and* g *satisfy*

$$g(x - y) = g(x)f(y) - f(x)g(y), \tag{1.9}$$

then

$$\begin{cases} g(x) & \equiv 0, \\ f & \text{arbitrary}, \end{cases}$$

or

$$\begin{cases} g(x) & = \frac{\gamma}{2}(E(x) - E^*(x)), \\ f(x) & = \frac{1}{2}(E(x) + E^*(x)) + \frac{\beta}{2}(E(x) - E^*(x)), \end{cases}$$

or

$$\begin{cases} g(x) & = A(x), \\ f(x) & = 1 + \gamma A(x), \end{cases}$$

where E, E^* *are exponential functions,* A *is an additive function and* $\gamma, \beta \in \mathbb{C}$.

As we are unable to locate an explicit determination of the general solution of (1.7), we give an independent proof in Chapter 2. In Chapter 2, we investigate some functional equations which characterize the trigonometric sine-cosine functions. We consider only *functions whose domain is a group and whose codomain is* $\mathbb{C}$ *and without any regularity condition.* We start by solving (1.7) which together with Theorems 1.0.1–1.0.4 give a complete characterization of the trigonometric sine-cosine functions.

We then turn to hyperbolic sine-cosine functions. Since the hyperbolic cosine and hyperbolic sine function $\mathcal{F}(x) = \cosh x,\ \mathcal{G}(x) = \sinh x$ satisfy the functional equations

$$\mathcal{F}(x - y) = \mathcal{F}(x)\mathcal{F}(y) - \mathcal{G}(x)\mathcal{G}(y), \tag{1.10}$$

using a modification of Kannappan's technique in the trigonometric case, we prove a hyperbolic counterpart of Theorem 1.0.1, which renders a desired characterization of the hyperboloic sine-cosine function.

In Chapter 3, we take up the problem of charactering the trigonometric and hyperbolic tangent-cotangent functions. We attack this problem through two different approaches; one is analytical and the other is discrete in nature.

The analytical approach is based on a technique of Dobbs, [5], which makes use of the functional equation

$$T(u+v) = \frac{T(u)+T(v)}{1-T(u)T(v)},$$

satisfied by the trigonometric tangent function. Dobbs showed that by imposing certain analytical condition, the only solution to this equation is the trigonometric tangent function. We complement Dobbs' result by characterizing the trigonometric cotangent and the hyperbolic tangent-cotangent functions using the functional equations satisfied by them with some differntiabilty condition. *All functions considered in this approach are to have the real field* $\mathbb{R}$ *or its subset as both their domain and range.*

The discrete approach involves the concept of recursive equations, which is a functional equation with discrete arguments. In 2005, Rhouma, [14], gave a closed form solution to the recursive difference equation

$$y_{n+2} = \frac{y_n y_{n+1} - 1}{y_n + y_{n+1}}, \tag{1.11}$$

which was originated from an open problem in the book [12], see also [13], as follows:

Theorem 1.0.5. *Let* y_0 *and* y_1 *be arbitrary real numbers such that those* y_n *satisfying*

(1.11) *exists for all* $n \in \mathbb{N} \cup \{0\}$. *Let* F_n *be the Fibonacci sequence defined by*

$$F_0 = F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \qquad (n \geq 0).$$

The solution to equation (1.11) *exists for all* $n \in \mathbb{N} \cup \{0\}$ *if and only if*

$$F_{n-2}\theta_0 + F_{n-1}\theta_1 \not\equiv 0 \ (\mathrm{mod}\ 2\pi) \qquad (n \geq 2),$$

where $\theta_0 = -2\mathrm{arccot}\ y_0$ *and* $\theta_1 = -2\mathrm{arccot}\ y_1$.

When it exists, the solution to (1.11) *is given by*

$$y_n = -\cot\left(\frac{F_{n-2}\theta_0 + F_{n-1}\theta_1}{2}\right) = \cot(F_{n-2}\ \mathrm{arccot}\ y_0 + F_{n-1}\ \mathrm{arccot}\ y_1). \qquad (1.12)$$

Moreover,

1. *if* θ_0 *and* θ_1 *are both rational multiples of* π, *then either* $\{y_n\}$ *diverges in finitely many steps or* $\{y_n\}$ *is periodic;*

2. *if* θ_0 *is a rational multiple of* π *and* θ_1 *is not (or vice versa), then* $\{y_n\}$ *is aperiodic and does exist for all* n.

It is easily checked that the cotangent function in (1.12) satisfies (1.11) showing that the rational recursive equation (1.11) does indeed characterize the cotangent function. Rhouma's technique is first to transform (1.11) to an equivalent form of

$$y_{n+2} = i\frac{(y_{n+1}+i)(y_n+i) + (y_{n+1}-i)(y_n-i)}{(y_{n+1}+i)(y_n+i) - (y_{n+1}-i)(y_n-i)} \qquad (i = \sqrt{-1}), \qquad (1.13)$$

or

$$\frac{y_{n+2}-i}{y_{n+2}+i} = \frac{y_{n+1}-i}{y_{n+1}+i} \cdot \frac{y_n - i}{y_n + i},$$

which is a difference equation of the shape

$$x_{n+2} = \alpha x_{n+1} x_n.$$

A closed form solution of this last equation is derived without difficulty. The difference equation (1.11) is interesting at least in two respects. First, it resembles the well-known identity of the cotangent function. Second, putting $y_n = \cot z_n$, with the values of z_n restricted to the open interval $(0, \pi)$, the difference equation leads to the Fibonacci sequence modulo π of the form $z_{n+2} \equiv z_n + z_{n+1} \mod \pi$. In the final part of this thesis, we carry out a far reaching extension of Rhouma's technique and results, both in the direction of recursive equations and in the direction of the Fibonacci sequence involved. As fruitful by-products, we establish our desired characterizations of both the trigonometric and hyperbolic tangent-cotangent functions. *The functions or sequences considered in this approach are to have discrete domain and the real field* $\mathbb{R}$ *or its subset as their range.*

CHAPTER II

TRIGONOMETRIC AND HYPERBOLIC SINE-COSINE FUNCTIONS

In this chapter, all functions considered are to have a group G as their domain and the complex field $\mathbb{C}$ as their codomain.

2.1 Trigonometric sine-cosine functions

Since the general solutions of the four functional equations (1.6)–(1.9) are already known, but we have not been able to find a proof for that of (1.7), we give here only a complete proof for this equation.

Theorem 2.1.1. *Let $(G, +)$ be an abelian group and $f, g : G \to \mathbb{C}$. Then the general solutions of*

$$f(x+y) = f(x)f(y) - g(x)g(y) \tag{1.7}$$

are

$$g(x) = E(x)A(x), \;\; f(x) = E(x)\left(1 \pm A(x)\right),$$

where E is an exponential function and A is an additive function.

Proof. If $g(x) \equiv 0$, then the functional equation gives

$$f(x+y) = f(x)f(y),$$

i.e., f is an exponential function.

Assuming now that $g(x) \not\equiv 0$, and let $\alpha \in G$ be such that $g(\alpha) \neq 0$. Using the functional equation (1.7) twice, we get

$$\begin{aligned} f((x+y)+z) &= f(x+y)f(z) - g(x+y)g(z) \\ &= f(x)f(y)f(z) - g(x)g(y)f(z) - g(x+y)g(z), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} f(x+(y+z)) &= f(x)f(y+z) - g(x)g(y+z) \\ &= f(x)f(y)f(z) - f(x)g(y)g(z) - g(x)g(y+z). \end{aligned} \tag{2.2}$$

Equating (2.1) and (2.2), we have

$$g(x)g(y)f(z) + g(x+y)g(z) = f(x)g(y)g(z) + g(x)g(y+z),$$

which simplifies to

$$g(x)\left[g(y+z) - g(y)f(z)\right] = g(z)\left[g(x+y) - f(x)g(y)\right].$$

Putting $z = \alpha$ and noting $g(\alpha) \neq 0$, we have

$$g(x+y) = g(x)h(y) + g(y)f(x), \tag{2.3}$$

where $h(y) := \dfrac{g(y+\alpha) - g(y)f(\alpha)}{g(\alpha)}$. Interchanging x and y in (2.3), we get

$$g(x+y) = g(y)h(x) + g(x)f(y). \tag{2.4}$$

Equating (2.3) and (2.4), we get

$$g(x)\left[h(y)-f(y)\right]=g(y)\left[h(x)-f(x)\right]. \tag{2.5}$$

Putting $y=\alpha$ in (2.5), we have

$$h(x)=\gamma g(x)+f(x),$$

where $\gamma:=\dfrac{h(\alpha)-f(\alpha)}{g(\alpha)}$. Adding (2.3) to (2.4), we have

$$2g(x+y)=g(x)\left[h(y)+f(y)\right]+g(y)\left[h(x)+f(x)\right]. \tag{2.6}$$

Define the function $\mathcal{F}:G\to\mathbb{C}$ by

$$\mathcal{F}(x)=\frac{h(x)+f(x)}{2}.$$

The equation (2.6) becomes

$$g(x+y)=g(x)\mathcal{F}(y)+g(y)\mathcal{F}(x). \tag{2.7}$$

The general solutions of (2.7) are, by Theorem 1.0.3, of the form

$$g(x)=E(x)A(x),\quad \mathcal{F}(x)=E(x),$$

or

$$g(x)=\frac{E_1(x)-E_2(x)}{2\beta},\quad \mathcal{F}(x)=\frac{E_1(x)+E_2(x)}{2}$$

where E, E_1, E_2 are exponential functions, A is an additive function and β is a nonzero complex constant.

Since $\mathcal{F}(x) = f(x) + \frac{\gamma}{2} g(x)$, the solutions of the equation (1.7) are given by

$$g(x) = E(x)A(x), \quad f(x) = E(x)\left(1 - \frac{\gamma A(x)}{2}\right). \tag{2.8}$$

or

$$g(x) = \frac{E_1(x) - E_2(x)}{2\beta}, \quad f(x) = E_1(x)\left(\frac{1}{2} - \frac{\gamma}{4\beta}\right) + E_2(x)\left(\frac{1}{2} + \frac{\gamma}{4\beta}\right) \tag{2.9}$$

Substituting the equation (2.8) into the equation (1.7), we have

$$\left(\frac{\gamma^2}{4} - 1\right) E(x)E(y)A(x)A(y) = 0. \tag{2.10}$$

Replacing $y = -x$, noting $E(-x) = \frac{1}{E(x)}$ and $A(-x) = -A(x)$ in the equation (2.10), we get

$$\begin{aligned} 0 &= \left(\frac{\gamma^2}{4} - 1\right) E(x)E(-x)A(x)A(-x) \\ &= \left(\frac{\gamma^2}{4} - 1\right) \left(-A(x)^2\right). \end{aligned}$$

Then

$$\left(\frac{\gamma^2}{4} - 1\right) = 0 \text{ or } A(x) = 0.$$

If $A(x) = 0$, then $g(x) \equiv 0$ and $f(x) = E(x)$.
If $\frac{\gamma^2}{4} - 1 = 0$, then $\gamma = \pm 2$.
Hence, the equation (2.8) becomes

$$g(x) = E(x)A(x), \; f(x) = E(x)\left(1 \pm A(x)\right).$$

Substituting the equation (2.9) into the equation (1.7) and using $\gamma = \pm 2$, we have

$$E_1(x)E_1(y) - E_1(x)E_2(y) - E_2(x)E_1(y) + E_2(x)E_2(y) = 0. \tag{2.11}$$

Replacing $y = -x$ and noting $E(-x) = \dfrac{1}{E(x)}$ in the equation (2.11), we get

$$E_1(x) = E_2(x).$$

Hence, the equation (2.9) becomes $g(x) \equiv 0$ and $f(x) = E_1(x)$. $\square$

Remark. Let us mention in passing that generalizing this result, Chung, Kannappan and Ng, [4], treated the functional equation

$$g(x+y) = g(x)f(y) + f(x)g(y) + h(x)h(y), \tag{2.12}$$

where $f, g, h : G \to \mathbb{C}$ and G is a group, an equation which contains (1.8). They determined its general solution consisting of a large number of possibilities too complicated to put down here.

2.2 Hyperbolic sine-cosine functions

Since the hyperbolic sine-cosine functions $\mathcal{G}(x) = \sinh x,\ \mathcal{F}(x) = \cosh x$ satisfy the functional equation

$$\mathcal{F}(x-y) = \mathcal{F}(x)\mathcal{F}(y) - \mathcal{G}(x)\mathcal{G}(y), \tag{1.10}$$

we ask whether results analogous to Theorem 1.0.1 and Corollary 1.0.2 hold for the hyperbolic sine-cosine functions. We affirmatively answer this by proving:

Theorem 2.2.1. *Let $(G,+)$ be a two-divisible abelian group and let $\mathcal{F},\mathcal{G} : G \to \mathbb{C}$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy the functional equation* (1.10).

I. If one of the functions $\mathcal{G},\mathcal{F}$ is a constant function, then so is the other, and the two constant functions are $\mathcal{G}(x) \equiv d,\ \mathcal{F}(x) \equiv c$ with $d^2 = c^2 - c$.

II. If both $\mathcal{G},\mathcal{F}$ are nonconstant functions, then they also satisfy

$$\mathcal{F}(x+y) = \mathcal{F}(x)\mathcal{F}(y) + \mathcal{G}(x)\mathcal{G}(y), \tag{2.13}$$

$$\mathcal{G}(x \pm y) = \mathcal{G}(x)\mathcal{F}(y) \pm \mathcal{F}(x)\mathcal{G}(y). \tag{2.14}$$

and the solution functions are given by

$$\mathcal{F}(x) = \frac{1}{2}\left(E(x) + E^*(x)\right), \quad \mathcal{G}(x) = b_1\left(E(x) - E^*(x)\right), \tag{2.15}$$

where $b_1^2 = \frac{1}{4}$, E is an exponential function and $E^*(x) = \frac{1}{E(x)}$.

Proof. By symmetry, the functional equation (1.10) implies

$$\mathcal{F}(y-x) = \mathcal{F}(y)\mathcal{F}(x) - \mathcal{G}(y)\mathcal{G}(x) = \mathcal{F}(x)\mathcal{F}(y) - \mathcal{G}(x)\mathcal{G}(y) = \mathcal{F}(x-y),$$

implying that $\mathcal{F}$ is an even function.

I. Assume first that $\mathcal{F}(x) \equiv c$, a constant function. The assertion trivially holds if $\mathcal{G}(x) \equiv 0$. Assume that $\mathcal{G}(x) \not\equiv 0$, there is $\alpha \in G$ such that $\mathcal{G}(\alpha) \neq 0$. Substituting into (1.10) yields

$$\mathcal{G}(x) = \frac{c^2 - c}{\mathcal{G}(\alpha)} =: d$$

and so $d^2 = c^2 - c$.

Next assume $\mathcal{G}(x) \equiv d$, a constant function. Replacing y by $-y$ in (1.10) and using the evenness of $\mathcal{F}$, we obtain

$$\mathcal{F}(x+y) = \mathcal{F}(x-y). \tag{2.16}$$

Putting $x = \dfrac{u+v}{2}, y = \dfrac{u-v}{2}$ $(u, v \in G)$ in (2.16), we get

$$\mathcal{F}(u) = \mathcal{F}(x+y) = \mathcal{F}(x-y) = \mathcal{F}(v),$$

i.e., $\mathcal{F}$ is a constant function and the two constants are related as shown before.

II. Consider nonconstant solutions $\mathcal{G}$ and $\mathcal{F}$ of the equation (1.10). Using (1.10) and the evenness of $\mathcal{F}$, we obtain

$$\mathcal{F}(x+y) = \mathcal{F}(x-(-y)) = \mathcal{F}(x)\mathcal{F}(-y) - \mathcal{G}(x)\mathcal{G}(-y) = \mathcal{F}(x)\mathcal{F}(y) - \mathcal{G}(x)\mathcal{G}(-y). \tag{2.17}$$

Similarly,

$$\mathcal{F}(x+y) = \mathcal{F}(-x-y) = \mathcal{F}(-x)\mathcal{F}(y) - \mathcal{G}(-x)\mathcal{G}(y) = \mathcal{F}(x)\mathcal{F}(y) - \mathcal{G}(-x)\mathcal{G}(y). \tag{2.18}$$

The equations (2.17) and (2.18) together give

$$\mathcal{G}(x)\mathcal{G}(-y) = \mathcal{G}(-x)\mathcal{G}(y).$$

Since $\mathcal{G}(x) \not\equiv 0$, there is $\alpha \in G$ such that $\mathcal{G}(\alpha) \neq 0$. Thus,

$$\mathcal{G}(x) = \frac{\mathcal{G}(-\alpha)}{\mathcal{G}(\alpha)}\mathcal{G}(-x) = k\mathcal{G}(-x) = k^2\mathcal{G}(x),$$

where $k := \mathcal{G}(-\alpha)/\mathcal{G}(\alpha)$. Thus, $k = \pm\, 1$.

If $k = 1$, then $\mathcal{G}(x) = \mathcal{G}(-x)$, i.e., $\mathcal{G}$ is an even function. This together with (1.10) and (2.17) show that $\mathcal{F}(x - y) = \mathcal{F}(x + y)$. By the same argument as that leading to (2.2), we conclude that $\mathcal{F}$ is a constant function, which is a contradiction. Hence, $k = -1$, and so $\mathcal{G}(x) = -\mathcal{G}(-x)$, i.e., $\mathcal{G}$ is an odd function.

Using this and (2.17), we see that (2.13) holds.

Using (2.13) twice, we get

$$\mathcal{F}((x + y) + z) = \mathcal{F}(x)\mathcal{F}(y)\mathcal{F}(z) + \mathcal{G}(x)\mathcal{G}(y)\mathcal{F}(z) + \mathcal{G}(x + y)\mathcal{G}(z), \tag{2.19}$$

and

$$\mathcal{F}\left(x + (y + z)\right) = \mathcal{F}(x)\mathcal{F}(y)\mathcal{F}(z) + \mathcal{F}(x)\mathcal{G}(y)\mathcal{G}(z) + \mathcal{G}(x)\mathcal{G}(y + z). \tag{2.20}$$

Equating (2.19) and (2.20) and simplifying, we have

$$\mathcal{G}(x)\left(\mathcal{G}(y)\mathcal{F}(z) - \mathcal{G}(y + z)\right) = \left(\mathcal{F}(x)\mathcal{G}(y) - \mathcal{G}(x + y)\right)\mathcal{G}(z).$$

Putting $z = \alpha$ and noting $\mathcal{G}(\alpha) \neq 0$, we have

$$\mathcal{F}(x)\mathcal{G}(y) - \mathcal{G}(x + y) = h(y)\mathcal{G}(x), \tag{2.21}$$

where $h(y) := \frac{1}{\mathcal{G}(\alpha)}\left(\mathcal{G}(y)\mathcal{F}(\alpha) - \mathcal{G}(y + \alpha)\right)$. Replacing x by $-x$ in (2.21), using the oddness of $\mathcal{G}$ and the evenness of $\mathcal{F}$, we get

$$-\mathcal{G}(x - y) = \mathcal{G}(-x + y) = \mathcal{F}(-x)\mathcal{G}(y) - h(y)\mathcal{G}(-x) = \mathcal{F}(x)\mathcal{G}(y) + h(y)\mathcal{G}(x),$$

and so

$$\mathcal{G}(x-y) = -\mathcal{F}(x)\mathcal{G}(y) - h(y)\mathcal{G}(x). \tag{2.22}$$

Incorporating (2.21) and (2.22), we get

$$\mathcal{G}(x+y) + \mathcal{G}(x-y) = -2h(y)\mathcal{G}(x). \tag{2.23}$$

Interchanging x and y in (2.23) and using the oddness of $\mathcal{G}$, we have

$$\mathcal{G}(x+y) - \mathcal{G}(x-y) = -2h(x)\mathcal{G}(y). \tag{2.24}$$

Adding (2.23) to (2.24), we have

$$\mathcal{G}(x+y) = -h(y)\mathcal{G}(x) - h(x)\mathcal{G}(y). \tag{2.25}$$

Combining (2.21) and (2.25), we get

$$\mathcal{F}(x)\mathcal{G}(y) = -h(x)\mathcal{G}(y)$$

and so

$$\mathcal{F}(x) = -\frac{\mathcal{G}(\alpha)}{\mathcal{G}(\alpha)}h(x) = -h(x).$$

Putting this last relation back into the equation (2.21), we get one of the two relations in (2.14), namely,

$$\mathcal{G}(x+y) = \mathcal{F}(x)\mathcal{G}(y) + \mathcal{F}(y)\mathcal{G}(x) \tag{2.26}$$

Replacing y by $-y$ in (2.26), using the oddness of $\mathcal{G}$ and the evenness of $\mathcal{F}$, we have

$$\mathcal{G}(x-y) = \mathcal{F}(x)\mathcal{G}(-y) + \mathcal{F}(-y)\mathcal{G}(x) = \mathcal{G}(x)\mathcal{F}(y) - \mathcal{F}(x)\mathcal{G}(y) \tag{2.27}$$

which is the other equation in (2.14).

There remains to find general shapes of the two solution functions. From (1.10) and (2.13), we have

$$\mathcal{F}(x+y) + \mathcal{F}(x-y) = 2\mathcal{F}(x)\mathcal{F}(y) \tag{2.28}$$

which is the d'Alembert functional equation and by Kannappan's result, [7] see also the equation (1.5), we have

$$\mathcal{F}(x) = \frac{E(x) + E^*(x)}{2}, \tag{2.29}$$

where E is an exponential function and and $E^*(x) = 1/E(x)$.

To find $\mathcal{G}$, using (1.10), we have

$$\begin{aligned} &\frac{E(x)E(-y) + E^*(x)E^*(-y)}{2} = \frac{E(x-y) + E^*(x-y)}{2} = \mathcal{F}(x-y) \\ &= \mathcal{F}(x)\mathcal{F}(y) - \mathcal{G}(x)\mathcal{G}(y) = \left(\frac{E(x)+E^*(x)}{2}\right)\left(\frac{E(y)+E^*(y)}{2}\right) - \mathcal{G}(x)\mathcal{G}(y), \end{aligned}$$

i.e.,

$$\begin{aligned} \mathcal{G}(x)\mathcal{G}(y) &= \left(\frac{E(x)+E^*(x)}{2}\right)\left(\frac{E(y)+E^*(y)}{2}\right) - \frac{E(x)E(-y) + E^*(x)E^*(-y)}{2} \\ &= \frac{1}{4}\left(E(x) - E^*(x)\right)\left(E(y) - E^*(y)\right). \end{aligned}$$

Consequently,

$$\mathcal{G}(x) = \frac{1}{4}\frac{(E(\alpha) - E^*(\alpha))}{\mathcal{G}(\alpha)}(E(x) - E^*(x)) = b_1(E(x) - E^*(x)), \tag{2.30}$$

with $b_1^2 = \frac{1}{4}$. □

Immediate from Theorem 2.2.1 is:

Corollary 2.2.2. *If $\mathcal{G}, \mathcal{F} : \mathbb{R} \to \mathbb{C}$ are nonconstant solutions of (1.10) and $\mathcal{F}$ is continuous, then $\mathcal{G}$ is also continuous, and the two solution functions are given by*

$$\mathcal{F}(x) = \cosh(c_1 x), \quad \mathcal{G}(x) = b_1 \sinh(c_1 x),$$

where $b_1^2 = \frac{1}{4}$, $c_1 \in \mathbb{C}$.

CHAPTER III

TRIGONOMETRIC AND HYPERBOLIC TANGENT-COTANGENT FUNCTIONS

In this chapter, characterizations of trigonometric and hyperbolic tangent-cotangent functions are investigated based on the well-known trigonometric and hyperbolic addition formulas (identities). In section 1, we apply the result of Dobbs to characterize the cotangent function and use the approach of Dobbs to characterize the hyperbolic tangent-cotangent functions. In section 2, we use a technique of Rhouma to find closed form solutions of certain recursive equations in order to characterize the trigonometric and the hyperbolic cotangent-tangent functions.

3.1 Dobbs's method

From the additive formulas

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \tag{3.1}$$

$$\cot(u+v) = \frac{\cot u \cot v - 1}{\cot u + \cot v} \tag{3.2}$$

$$\tanh(u+v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v} \tag{3.3}$$

$$\coth(u+v) = \frac{\coth u \coth v + 1}{\cot u + \cot v}, \tag{3.4}$$

have appeared several articles, see e.g. [5], [14], [13], on the functional equations having the trigonometric tangent-cotangent function as the only possible solutions. A particular functional equation considered by Dobbs in [5] is based on the addition formula (3.1) for the tangent function. Dobbs began by defining the class of tangential functions.

Definition 3.1.1. *Let $T : D \to \mathbb{R}$ when $D \subseteq \mathbb{R}$. The function T is called a tangential function if T satisfies the functional equation*

$$T(u+v) = \frac{T(u)+T(v)}{1-T(u)T(v)}, \tag{3.5}$$

for all $u, v \in D$ with $1 - T(u)T(v) \neq 0$ and $u + v \in D$.

It is shown in [5] that (3.5) may possess several weird solutions such as

1. $T(x) = 1 \quad (x \in \mathbb{R})$;
2. $T(x) = -1 \quad (x \in \mathbb{R})$;
3. Let p be a fixed prime number. If $x \in \mathbb{R}$, put

$$T(x) = \begin{cases} 0 & \text{if } x = \frac{m}{p^n} \text{ for some } m, n \in \mathbb{Z}; \\ 1 & \text{otherwise.} \end{cases}$$

Dobbs went on to prove the following results which lead to the fact that under certain regularity conditions, the only solution of (3.5) is the tangent function.

Theorem 3.1.2. *Let T be a tangential function such that $T'(0) = 1$ and each real number $x \neq \frac{\pi}{2} + m\pi \quad (m \in \mathbb{Z})$ is in the domain of T. Then T is the trigonometric tangent function.*

We now apply this result of Dobbs to characterize the trigonometric cotangent function.

Definition 3.1.3. *Let $C : D \to \mathbb{R}$ when $D \subseteq \mathbb{R}$. The function C is called a cotangential function if C satisfies the functional equation*

$$[C(u) + C(v)]\, C(u + v) = C(u)C(v) - 1, \tag{3.6}$$

for all $u, v \in D$ with $u + v \in D$.

Our characterization of the trigonometric cotangent function by Dobbs' method is:

Theorem 3.1.4. *Let C be a cotangential function whose domain is $\mathbb{R} \setminus \{m\pi : m \in \mathbb{Z}\}$. Assume that*

(i) C is differentiable at $\frac{\pi}{2}$ with $C'(\frac{\pi}{2}) = -1$;

(ii) $C(x) = 0$ if and only if $x = \frac{\pi}{2} + m\pi$ for all $m \in \mathbb{Z}$.

Then C is the trigonometric cotangent function.

Proof. Define

$$S(x) = -C\left(x + \frac{\pi}{2}\right) \qquad \left(x \in \mathbb{R} \setminus \left\{\frac{m\pi}{2} : m \in \mathbb{Z}\right\}\right).$$

We claim that S is a tangential function.
Let $x, y \in \mathbb{R} \setminus \left\{\frac{m\pi}{2} : m \in \mathbb{Z}\right\}$ with $x + y \in \mathbb{R} \setminus \left\{\frac{m\pi}{2} : m \in \mathbb{Z}\right\}$.

To see this, by the functional equation (3.6), we have

$$\begin{aligned}(1-S(x)S(y))S(x+y) &= \left[1-C\left(x+\frac{\pi}{2}\right)C\left(y+\frac{\pi}{2}\right)\right]\left[-C\left(x+y+\frac{\pi}{2}\right)\right]\\ &= \left[C\left(x+\frac{\pi}{2}\right)C\left(y+\frac{\pi}{2}\right)-1\right]C\left(x+y+\frac{\pi}{2}\right)\\ &= \left[\left(C\left(x+\frac{\pi}{2}\right)+C\left(y+\frac{\pi}{2}\right)\right)C\left(x+y+\frac{\pi}{2}+\frac{\pi}{2}\right)\right]C\left(x+y+\frac{\pi}{2}\right).\end{aligned}$$

Replacing $u=x+y+\frac{\pi}{2}$ and $v=\frac{\pi}{2}$ in equation (3.6) and noting $C(\frac{\pi}{2})=0$, we get

$$C\left(x+y+\frac{\pi}{2}+\frac{\pi}{2}\right)C\left(x+y+\frac{\pi}{2}\right)=-1.$$

Hence,

$$[1-S(x)S(y)]\,S(x+y)=-C\left(x+\frac{\pi}{2}\right)-C\left(y+\frac{\pi}{2}\right)=S(x)+S(y).$$

Thus S is a tangential function. Note that $S(0)=-C\left(\frac{\pi}{2}\right)=0$. Since C is differentiable at $\frac{\pi}{2}$ and $C'(\frac{\pi}{2})=-1$, we have $-1=C'\left(\frac{\pi}{2}\right)=-S'(0)$, i.e., S is a differentiable at 0 with $S'(0)=1$.

By Theorem 3.1.2, we deduce that

$$S(x)=\tan x \quad \left(x\in\mathbb{R}\setminus\{\frac{m\pi}{2}:m\in\mathbb{Z}\}\right).$$

Thus,

$$C(x)=-\tan\left(x-\frac{\pi}{2}\right)=\cot x \qquad \left(x\in\mathbb{R}\setminus\{\frac{m\pi}{2}:m\in\mathbb{Z}\}\right).$$

For $m\in\mathbb{Z}$, since $C(\frac{\pi}{2}+m\pi)=0=\cot(\frac{\pi}{2}+m\pi)$, we conclude that

$$C(x)=\cot x \qquad (x\in\mathbb{R}\setminus\{m\pi:m\in\mathbb{Z}\}).$$

□

As the hyperbolic tangent-cotangent functions satisfy, respectively, the functional equations, similar to those satisfied by the trigonometric tangent-cotangent functions, namely,

$$\tanh(u+v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

and

$$\coth(u+v) = \frac{\coth u \coth v + 1}{\coth u + \coth v},$$

this prompts us to introduce the classes of hyperbolic tangential and hyperbolic cotangential functions as follows:

Definition 3.1.5. *A. Let $h_T : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. The function h_T is called a hyperbolic-tangential function if h_T satisfies the functional equation*

$$[1 + h_T(u)h_T(v)]\, h_T(u+v) = h_T(u) + h_T(v), \tag{3.7}$$

for all $u, v \in D$ with $u + v \in D$.

B. Let $h_C : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. The function h_C is called a hyperbolic-cotangential function if h_C satisfies the functional equation

$$[h_C(u) + h_C(v)]\, h_C(u+v) = h_C(u)h_C(v) + 1, \tag{3.8}$$

for all $u, v \in D$ with $u + v \in D$.

As in Dobbs's work, there are solutions to the functional equation (3.7) other than the hyperbolic tangent functions as seen in the following examples.

Example 3.1.6. *Each of the following functions is the hyperbolic-tangential from $\mathbb{R}$ into $\mathbb{R}$:*

1. $h_T(x) = 1 \quad (x \in \mathbb{R})$;

2. $h_T(x) = -1 \quad (x \in \mathbb{R})$;

3. *Let* $x \in \mathbb{R}$, *put*

$$h_T(x) = \begin{cases} -1 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x < 0. \end{cases}$$

Similar hyperbolic cotangential functions which are not the hyperbolic cotangent functions can be constructed in the same manner.

In order to characterize the hyperbolic tangent-cotangent functions by Dobbs' method, we first prove:

Proposition 3.1.7. *Let* h_T *be a hyperbolic-tangential function. Assume that* $h_T(0) = 0$.

(*a*) *If* h_T *is continuous at* 0, *then* h_T *is continuous at each* x.

(*b*) *If* h_T *is differentiable at* 0 *and* $h_T'(0) = 1$, *then* h_T *is differentiable at each* x *with* $h_T'(x) = 1 - h_T(x)^2$.

Proof. Since h_T is continuous at 0, we have $\lim_{\xi \to 0} h_T(\xi) = h_T(0) = 0$. Take any x in the domain of h_T, and for any ξ in the domain of h_T sufficiently small so that $x + \xi$ is in the domain of h_T. Thus,

$$\begin{aligned}&[1 + h_T(x)h_T(\xi)]\,[h_T(x+\xi) - h_T(x)] \\ &\qquad = [1 + h_T(x)h_T(\xi)]\,h_T(x+\xi) - [1 + h_T(x)h_T(\xi)]\,h_T(x). \qquad (3.9)\end{aligned}$$

Noting h_T is the hyperbolic-tangential function, we get

$$\begin{aligned}[1+h_T(x)h_T(\xi)]\,h_T(x+\xi)-[1+h_T(x)h_T(\xi)]\,h_T(x)&\\ &=[h_T(x)+h_T(\xi)]-h_T(x)-h_T(x)^2h_T(\xi)\\ &=h_T(\xi)\left[1-h_T(x)^2\right].\end{aligned}$$

Since $\lim_{\xi\to 0}(1+h_T(x)h_T(\xi))=1$ and $\lim_{\xi\to 0}\left(h_T(\xi)[1-h_T(x)^2]\right)=0$, we have

$$\lim_{\xi\to 0}(h_T(x+\xi)-h_T(x))=0,$$

i.e., h_T is continuous at x and this proves part (a).

To prove (b), first observe that

$$\lim_{\xi\to 0}\frac{h_T(\xi)}{\xi}=h_T'(0)=1.$$

Take any x in the domain of h_T, and for any ξ in the domain of h_T sufficiently small so that $x+\xi$ is in the domain of h_T. Using (3.9), we get

$$[1+h_T(x)h_T(\xi)]\left[\frac{h_T(x+\xi)-h_T(x)}{\xi}\right]=\frac{h_T(\xi)}{\xi}\left[1-h_T(x)^2\right].$$

Since $\lim_{\xi\to 0}(1+h_T(x)h_T(\xi))=1$ and $\lim_{\xi\to 0}\left(\frac{h_T(\xi)}{\xi}[1-h_T(x)^2]\right)=1-h_T(x)^2$, we get

$$h_T'(x)=\lim_{\xi\to 0}\frac{h_T(x+\xi)-h_T(x)}{\xi}=1-h_T(x)^2.$$

□

Our characterization of the hyperbolic tangent function by Dobbs' method is:

Theorem 3.1.8. *Let $h_T : \mathbb{R} \to \mathbb{R}$ be a hyperbolic-tangential function. Assume that*

(i) $h_T(0) = 0$;

(ii) h_T *is differentiable at* 0 *with* $h'_T(0) = 1$;

(iii) $|h_T(x)| < 1$.

Then h_T is the trigonometric hyperbolic tangent function.

Proof. By Proposition 3.1.7(b), we have

$$h'_T(x) = 1 - h_T(x)^2.$$

Since $|h_T(x)| < 1$, solving this first order differential equation yields, see e.g. p. 276 of [16],

$$x = \operatorname{arctanh} h_T(x) + k$$

for some constant $k \in \mathbb{R}$. From $h_T(0) = 0$, we get $k = 0$ and so $h_T(x) = \tanh x$. □

To characterize the hyperbolic cotangent function, we also need an auxliary result.

Proposition 3.1.9. *Let h_C be a hyperbolic-cotangential function. Assume that h_C is an odd function.*

(a) *If h_C is continuous at a point, then h_C is continuous at each x.*

(b) *If h_C is differentiable at a point, say x_0, and $h'_C(x_0) = 1 - h_C(x_0)^2 \,(\neq 0)$, then h_C is differentiable at each x in the domain of h_C and $h'_C(x) = 1 - h_C(x)^2$.*

Proof. To prove (a), assume that h_C is continuous at x_0. Then $\lim_{\xi \to x_0} h_C(\xi) = h_C(x_0)$. Take any x in the domain of h_C and for any ξ in the domain of h_C sufficiently small so that $x+\xi$ is in the domain of h_C. Thus,

$$[h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)]\,[h_C(x+\xi-x_0)-h_C(x)]$$
$$=[h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)]\,h_C(x+\xi-x_0)$$
$$-[h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)]\,h_C(x). \tag{3.10}$$

Now,

$$[h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)]\,h_C(x+\xi-x_0)$$
$$=h_C(x)h_C(\xi)h_C(x+\xi-x_0)+h_C(x+\xi-x_0)$$
$$+h_C(-x_0)h_C(x)h_C(x+\xi-x_0)+h_C(\xi)h_C(-x_0)h_C(x+\xi-x_0)$$
$$=[h_C(x)h_C(\xi)+1]\,h_C(x+\xi-x_0)$$
$$+h_C(-x_0)h_C(x)h_C(x+\xi-x_0)+h_C(\xi)h_C(-x_0)h_C(x+\xi-x_0)$$
$$=([h_C(x)+h_C(\xi)]\,h_C(x+\xi))\,h_C(x+\xi-x_0)$$
$$+h_C(-x_0)h_C(x)h_C(x+\xi-x_0)+h_C(\xi)h_C(-x_0)h_C(x+\xi-x_0)$$
$$=h_C(x)h_C(x+\xi)h_C(x+\xi-x_0)+h_C(\xi)h_C(x+\xi)h_C(x+\xi-x_0)$$
$$+h_C(-x_0)h_C(x)h_C(x+\xi-x_0)+h_C(\xi)h_C(-x_0)h_C(x+\xi-x_0)$$
$$=[h_C(x+\xi)+h_C(-x_0)]\,h_C(x)h_C(x+\xi-x_0)$$
$$+[h_C(x+\xi)+h_C(-x_0)]\,h_C(\xi)h_C(x+\xi-x_0)$$

$$= [h_C(x+\xi)h_C(-x_0)+1]\, h_C(x) + [h_C(x+\xi)h_C(-x_0)+1]\, h_C(\xi)$$

$$= h_C(x+\xi)h_C(-x_0)h_C(x) + h_C(x) + h_C(x+\xi)h_C(-x_0)h_C(\xi) + h_C(\xi)$$

$$= [h_C(x)+h_C(\xi)]\, h_C(x+\xi)h_C(-x_0) + h_C(x) + h_C(\xi)$$

$$= [h_C(x)h_C(\xi)+1]\, h_C(-x_0) + h_C(x) + h_C(\xi)$$

$$= h_C(x)h_C(\xi)h_C(-x_0) + h_C(-x_0) + h_C(x) + h_C(\xi).$$

Consequently,

$$\begin{aligned}
&[h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)]\,[h_C(x+\xi-x_0)-h_C(x)] \\
&= h_C(x)h_C(\xi)h_C(-x_0) + h_C(-x_0) + h_C(x) + h_C(\xi) \\
&\quad - h_C(x)^2h_C(\xi) - h_C(x) - h_C(x)^2h_C(-x_0) - h_C(\xi)h_C(-x_0)h_C(x) \\
&= h_C(-x_0) + h_C(\xi) - h_C(x)^2h_C(\xi) - h_C(x)^2h_C(-x_0). \qquad (3.11)
\end{aligned}$$

Since h_C is an odd function, we have

$$\lim_{\xi\to x_0}\big(h_C(-x_0)+h_C(\xi)-h_C(x)^2h_C(\xi)-h_C(x)^2h_C(-x_0)\big)=0$$

and

$$\lim_{\xi\to x_0}\,(h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)) = 1-h_C(x_0)^2\ (\neq 0),$$

yielding $\lim\limits_{\xi\to x_0}(h_C(x+\xi-x_0)-h_C(x))=0$. i.e., h_C is continuous at x.

To prove (b), take any x in the domain of h_C and for any ξ in the domain of h_C

sufficiently small so that $x+\xi$ is in the domain of h_C. Using (3.11) in (a), we have

$$\begin{aligned}&[h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)]\,\frac{h_C(x+\xi-x_0)-h_C(x)}{\xi-x_0}\\ &=\frac{h_C(-x_0)+h_C(\xi)-h_C(x)^2h_C(\xi)-h_C(x)^2h_C(-x_0)}{\xi-x_0}.\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{\xi\to x_0}[h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)]\,\frac{h_C(x+\xi-x_0)-h_C(x)}{\xi-x_0}\\ &=\lim_{\xi\to x_0}\frac{h_C(-x_0)+h_C(\xi)-h_C(x)^2h_C(\xi)-h_C(x)^2h_C(-x_0)}{\xi-x_0}\\ &=\left[1-h_C(x_0)^2\right]\left\{1-h_C(x)^2\right\}.\end{aligned}$$

But

$$\lim_{\xi\to x_0}\{h_C(x)h_C(\xi)+1+h_C(-x_0)h_C(x)+h_C(\xi)h_C(-x_0)\}=1-h_C(x_0)^2\ (\neq 0),$$

i.e.,

$$h_C'(x)=\lim_{\xi\to x_0}\frac{h_C(x+\xi-x_0)-h_C(x)}{\xi-x_0}=1-h_C(x)^2.$$

□

We come now to our characterization of the hyperbolic cotangent function by Dobbs' method.

Theorem 3.1.10. *Let $h_C:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a hyperbolic-cotangential function. Assume that*

(i) h_C is an odd function;

(ii) there exists x_0 such that h_C is differentiable at x_0 with $h_C'(x_0)=1-h_C(x_0)^2$;

(iii) $|h_C(x)| > 1$.

Then $h_C(x)$ is the trigonometric hyperbolic cotangent function.

Proof. By Proposition 3.1.9(b), we have

$$h'_C(x) = 1 - h_C(x)^2.$$

Since $|h_C(x)| > 1$, solving this first order differential equation yields, see e.g. p. 276 of [16],

$$x = \operatorname{arccoth} h_C(x) + k$$

for some constant $k \in \mathbb{R}$, i.e., $h_C(x) = \coth(x - k)$. Since h_C is odd, we deduce that $k = 0$. Therefore, $h_C(x) = \coth x$. □

3.2 Rhouma's method

We start by finding a closed form solution for any rational recursive equation extending

$$y_{n+2} = i\frac{(y_{n+1}+i)(y_n+i) + (y_{n+1}-i)(y_n-i)}{(y_{n+1}+i)(y_n+i) - (y_{n+1}-i)(y_n-i)} \qquad (i = \sqrt{-1}), \tag{1.13}$$

of the form

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\dots(y_n+i)^{A_\ell} + (y_{n+\ell-1}-i)^{A_1}\dots(y_n-i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\dots(y_n+i)^{A_\ell} - (y_{n+\ell-1}-i)^{A_1}\dots(y_n-i)^{A_\ell}}, \tag{3.12}$$

and determine its asymptotic behavior. Our first lemma follows easily from a simple calculation whose trivial proof is omitted.

Lemma 3.2.1. *Let $\ell \in \mathbb{N},\ \ell \geq 2;\ b, x_1, \ldots, x_\ell, z$ complex numbers and let $A_1, \ldots, A_\ell$ be nonzero integers such that*

$$b(x_1+b)^{A_1}\ldots(x_\ell+b)^{A_\ell}\left\{(x_1+b)^{A_1}\ldots(x_\ell+b)^{A_\ell}-(x_1-b)^{A_1}\ldots(x_\ell-b)^{A_\ell}\right\}\neq 0.$$

Then

$$\frac{z-b}{z+b}=\left(\frac{x_1-b}{x_1+b}\right)^{A_1}\cdots\left(\frac{x_\ell-b}{x_\ell+b}\right)^{A_\ell}$$

if and only if

$$z=b\,\frac{(x_1+b)^{A_1}\cdots(x_\ell+b)^{A_\ell}+(x_1-b)^{A_1}\cdots(x_\ell-b)^{A_\ell}}{(x_1+b)^{A_1}\cdots(x_\ell+b)^{A_\ell}-(x_1-b)^{A_1}\cdots(x_\ell-b)^{A_\ell}}.$$

The next lemma relates generalized Fibonacci numbers with elements in rational recursive sequences.

Lemma 3.2.2. *Let $\ell \in \mathbb{N},\ \ell \geq 2$ and let $\{F_n\}$ be the (generalized Fibonacci) sequence satisfying a linear recurrence relation of the form*

$$F_{n+\ell}=A_1F_{n+\ell-1}+A_2F_{n+\ell-2}+\cdots+A_\ell F_n, \tag{3.13}$$

where $A_1, \ldots, A_\ell$ are nonzero integers such that $A_1+\ldots+A_\ell \neq 0$, with initial values

$$F_0=A_\ell,\ F_1=A_{\ell-1},\ldots,\ F_{\ell-1}=A_1.$$

If

$$x_{n+\ell}=\alpha^{A_1^2+\cdots+A_\ell^2-\frac{(A_1^2+\cdots+A_\ell^2)}{A_1+\cdots+A_\ell}}x_{n+\ell-1}^{A_1}x_{n+\ell-2}^{A_2}\cdots x_n^{A_\ell}\quad(n\geq 0), \tag{3.14}$$

then

$$x_n=\alpha^{F_n-\frac{(A_1^2+\cdots+A_\ell^2)}{A_1+\cdots+A_\ell}}x_0^{F_{n-\ell}}x_1^{F_{n-\ell+1}}\cdots x_{\ell-1}^{F_{n-1}} \tag{3.15}$$

for all $n \geq \ell$.

Proof. For the starting case, the condition (3.14) and the recurrence (3.13) yield

$$x_\ell = \alpha^{A_1^2+\cdots+A_\ell^2-\frac{(A_1^2+\ldots+A_\ell^2)}{A_1+\cdots+A_\ell}} x_{\ell-1}^{A_1} x_{\ell-2}^{A_2} \cdots x_0^{A_\ell} = \alpha^{F_\ell-\frac{(A_1^2+\cdots+A_\ell^2)}{A_1+\cdots+A_\ell}} x_{\ell-1}^{A_1} x_{\ell-2}^{A_2} \cdots x_0^{A_\ell},$$

which agrees with (3.15) when $n = \ell$.

Next, suppose that (3.15) is true for all $\ell \leq n \leq k$. From (3.14), using the induction hypothesis and the recurrence (3.13), we get

$$\begin{aligned}
x_{k+1} &= \alpha^{A_1^2+\cdots+A_\ell^2-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}} x_k^{A_1} \cdots x_{k-\ell+1}^{A_\ell} \\
&= \alpha^{A_1^2+\cdots+A_\ell^2-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}} \left(\alpha^{-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}+F_k} x_0^{F_{k-\ell}} x_1^{F_{k-\ell+1}} \cdots x_{\ell-1}^{F_{k-1}} \right)^{A_1} \times \cdots \\
&\times \left(\alpha^{-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}+F_{k-\ell+1}} x_0^{F_{k-\ell+1-\ell}} x_1^{F_{k-\ell+1-\ell+1}} \cdots x_{\ell-1}^{F_{k-\ell+1-1}} \right)^{A_\ell} \\
&= \alpha^{-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}(-(A_1+\cdots+A_\ell)+1+A_1+\cdots+A_\ell)+A_1F_k+\cdots+A_\ell F_{k-\ell+1}} \times \\
&\quad \times x_0^{A_1F_{k-\ell}+\cdots+A_\ell F_{k-\ell-\ell+1}} \cdots x_{\ell-1}^{A_1F_{k-1}+\cdots+A_\ell F_{k-1-\ell+1}} \\
&= \alpha^{-\frac{(A_1^2+\cdots+A_\ell^2)}{A_1+\cdots+A_\ell}+F_{k+1}} x_0^{F_{(k+1)-\ell}} \cdots x_{\ell-1}^{F_{(k+1)-1}}.
\end{aligned}$$

□

Our characterization of the trigonometric cotangent function by Rhouma's method is:

Theorem 3.2.3. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $\{F_n\}$ be the sequence satisfying a linear recurrence relation of the form*

$$F_{n+\ell} = A_1F_{n+\ell-1} + A_2F_{n+\ell-2} + \ldots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots,$ $F_{\ell-1} = A_1$. Let $y_0, \ldots, y_{\ell-1}$ be real *numbers such that those y_n which satisfy*

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\dots(y_n+i)^{A_\ell}+(y_{n+\ell-1}-i)^{A_1}\dots(y_n-i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\dots(y_n+i)^{A_\ell}-(y_{n+\ell-1}-i)^{A_1}\dots(y_n-i)^{A_\ell}}, \tag{3.12}$$

exist for all $n \in \mathbb{N}\cup\{0\}$.

Then the solution to the equation (3.12) exists if and only if

$$A_1F_{n-\ell}\theta_0 + \ldots + A_\ell F_{n-1}\theta_{\ell-1} \neq 0 \pmod{2\pi} \quad (n \in \mathbb{N}\cup\{0\}),$$

where $\theta_j = \dfrac{-2}{A_{j+1}}$ arccot y_j for all $j \in \{0,\ldots,\ell-1\}$.

When it exists, the solution is given by

$$y_n = i\frac{(y_0+i)^{F_{n-\ell}}\cdots(y_{\ell-1}+i)^{F_{n-1}}+(y_0-i)^{F_{n-\ell}}\cdots(y_{\ell-1}-i)^{F_{n-1}}}{(y_0+i)^{F_{n-\ell}}\cdots(y_{\ell-1}+i)^{F_{n-1}}-(y_0-i)^{F_{n-\ell}}\cdots(y_{\ell-1}-i)^{F_{n-1}}}, \tag{3.16}$$

or

$$\begin{aligned} y_n &= \cot\left(\frac{-A_1F_{n-\ell}\theta_0 - \cdots - A_\ell F_{n-1}\theta_{\ell-1}}{2}\right) \\ &= \cot(F_{n-\ell}\,\text{arccot}\ y_0 + \cdots + F_{n-1}\,\text{arccot}\ y_{\ell-1}). \end{aligned}$$

Moreover,

1. *if all the θ_j's are rational multiples of π, then either $\{y_n\}$ diverges in finitely many steps or y_n is periodic;*

2. *if $\theta_0,\theta_1,\ldots,\theta_{\ell-1},\pi$ are linearly independent over $\mathbb{Q}$, and $A_1,\ldots,A_\ell$ are nonzero integers, then y_n exists for all n and the sequence $\{y_n\}$ is never periodic.*

Proof. Taking $z = y_{n+\ell}$, $x_1 = y_{n+\ell-1},\ldots,x_\ell = y_n$, $b = i$ in Lemma 3.2.1, the rational recursive equation (3.12) is equivalent to

$$\frac{y_{n+\ell}-i}{y_{n+\ell}+i}=\left(\frac{y_{n+\ell-1}-i}{y_{n+\ell-1}+i}\right)^{A_1}\cdots\left(\frac{y_n-i}{y_n+i}\right)^{A_\ell}. \tag{3.17}$$

Putting $U_n=\dfrac{y_n-i}{y_n+i}$, the relation (3.17) becomes

$$U_{n+\ell}=U_{n+\ell-1}^{A_1}U_{n+\ell-2}^{A_2}\cdots U_n^{A_\ell},$$

whose solution is, by virtue of Lemma 3.2.2,

$$U_n=U_0^{F_{n-\ell}}U_1^{F_{n-\ell+1}}\cdots U_{\ell-1}^{F_{n-1}}\quad(n\geq\ell),$$

and so

$$\frac{y_n-i}{y_n+i}=\left(\frac{y_0-i}{y_0+i}\right)^{F_{n-\ell}}\left(\frac{y_1-i}{y_1+i}\right)^{F_{n-\ell+1}}\cdots\left(\frac{y_{\ell-1}-i}{y_{\ell-1}+i}\right)^{F_{n-1}}, \tag{3.18}$$

which, by Lemma 3.2.1, becomes

$$y_n=i\frac{(y_0+i)^{F_{n-\ell}}\cdots(y_{\ell-1}+i)^{F_{n-1}}+(y_0-i)^{F_{n-\ell}}\cdots(y_{\ell-1}-i)^{F_{n-1}}}{(y_0+i)^{F_{n-\ell}}\cdots(y_{\ell-1}+i)^{F_{n-1}}-(y_0-i)^{F_{n-\ell}}\cdots(y_{\ell-1}-i)^{F_{n-1}}}. \tag{3.19}$$

Next, setting $e^{i\theta_0A_1}=\dfrac{y_0-i}{y_0+i},\ \ldots\ ,e^{i\theta_{\ell-1}A_\ell}=\dfrac{y_{\ell-1}-i}{y_{\ell-1}+i}$, we have

$$\frac{y_n-i}{y_n+i}=e^{i(A_1F_{n-\ell}\theta_0+\cdots+A_\ell F_{n-1}\theta_{\ell-1})},$$

i.e.,

$$\begin{aligned} y_n &= i\left(\frac{1+e^{i(A_1F_{n-\ell}\theta_0+\cdots+A_\ell F_{n-1}\theta_{\ell-1})}}{1-e^{i(A_1F_{n-\ell}\theta_0+\cdots+A_\ell F_{n-1}\theta_{\ell-1})}}\right) \\ &= \cot\left(\frac{-A_1F_{n-\ell}\theta_0-\cdots-A_\ell F_{n-1}\theta_{\ell-1})}{2}\right) \\ &= \cot(F_{n-\ell}\ \text{arccot}\ y_0+\cdots+F_{n-1}\ \text{arccot}\ y_{\ell-1}), \end{aligned}$$

provided $A_1F_{n-\ell}\theta_0+\cdots+A_\ell F_{n-1}\theta_{\ell-1}\not\equiv 0 \pmod{2\pi}$.

If all the θ_j $(j=0,1,\ldots,\ell-1)$ are rational multiples of π, say,

$$\theta_j=\frac{m_j\pi}{t_j} \text{ with } m_j,t_j\ (>0)\in\mathbb{Z},\ \gcd(m_j,t_j)=1,$$

then it is easily checked that $\sum_{k=1}^{\ell}A_kF_{n-\ell+k-1}\theta_{k-1} \pmod{2\pi}$ is equivalent to

$$G_n=\sum_{k=1}^{\ell}A_kF_{n-\ell+k-1}m_{k-1}\prod_{\substack{j=0\\ j\neq k}}^{\ell-1}t_j \quad \left(\text{mod } \prod_{j=0}^{\ell-1}2t_j,\right)$$

Since each G_n takes at most $\prod_{j=0}^{\ell-1}(2t_j)$ distinct values, each ℓ-tuple $(G_t,\ldots,G_{t+\ell-1})$ takes at most $\prod_{j=0}^{\ell-1}(2t_j)^\ell$ distinct values. Since the sequence $\{(G_t,\ldots,G_{t+\ell-1})\}_{t\geq 0}$ is infinite, there are integers $N_1\neq N_2$ such that

$$(G_{N_1},\ldots,G_{N_1+\ell-1})=(G_{N_2},\ldots,G_{N_2+\ell-1}).$$

Since

$$G_{j+\ell}=G_j+\cdots+G_{j+\ell-1} \quad (j\in\mathbb{N}),$$

we deduce that $G_{N_1+k}\equiv G_{N_2+k}$ for all $k\in\mathbb{N}$, i.e., the sequence $\{G_n\}$ is periodic. If some G_n is zero, then clearly the sequence $\{y_n\}$ diverges.

If $\theta_0, \theta_1, \ldots, \theta_{\ell-1}, \pi$ are linearly independent over $\mathbb{Q}$, then

$$A_1 F_{n-\ell}\theta_0 + \ldots + A_\ell F_{n-1}\theta_{\ell-1} \neq 2k\pi \quad (k \in \mathbb{Z}),$$

showing that y_n exists for each n and the sequence $\{y_n\}$ is never periodic. $\square$

As the equation (1.11) characterizes the cotangent function, it is natural to consider its counterpart

$$y_{n+2} = \frac{y_n + y_{n+1}}{1 - y_n y_{n+1}}, \tag{3.20}$$

or equivalently,

$$y_{n+2} = i\frac{(y_{n+1}+i)(y_n+i) - (-y_{n+1}+i)(-y_n+i)}{(y_{n+1}+i)(y_n+i) + (-y_{n+1}+i)(-y_n+i)}, \tag{3.21}$$

which clearly has the tangent function as a solution. Our next objective is to find a closed form solution for any rational recursive equation extending (3.21) of the form

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} - (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} + (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}}, \tag{3.22}$$

and determine its asymptotic behavior.

Corollary 3.2.4. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $\{F_n\}$ be a sequence satisfying a linear recurrence relation of the form*

$$F_{n+\ell} = A_1 F_{n+\ell-1} + A_2 F_{n+\ell-2} + \ldots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots,$ $F_{\ell-1} = A_1$. Let $y_0, \ldots, y_{\ell-1}$ be real *numbers such that those y_n which satisfy the rational recursive equation*

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} - (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} + (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}}, \tag{3.23}$$

exist for all $n \in \mathbb{N} \cup \{0\}$.

Then the solution to the equation (3.23) exists if and only if

$$A_1F_{n-\ell}\theta_0 + A_2F_{n-\ell+1}\theta_1 + \cdots + A_\ell F_{n-1}\theta_{\ell-1}$$

is not an odd multiple of π, where $\theta_j = \dfrac{-2}{A_{j+1}}\arctan\ y_j \quad (j = 0, 1, \ldots, \ell-1)$.

When the solution exists, it is given by

$$y_n = i\frac{(y_0+i)^{A_1}\cdots(y_{\ell-1}+i)^{A_\ell} - (-y_0+i)^{A_1}\cdots(-y_{\ell-1}+i)^{A_\ell}}{(y_0+i)^{A_1}\cdots(y_{\ell-1}+i)^{A_\ell} + (-y_0+i)^{A_1}\cdots(-y_{\ell-1}+i)^{A_\ell}}, \tag{3.24}$$

or

$$\begin{aligned} y_n &= -\tan\left(\frac{A_1F_{n-\ell}\theta_0 + \cdots + A_\ell F_{n-1}\theta_{\ell-1}}{2}\right) \\ &= \tan\left(F_{n-\ell}\arctan\ y_0 + \cdots + F_{n-1}\arctan\ y_{\ell-1}\right). \end{aligned}$$

Moreover,

1. *if all θ_j's are rational multiples of π, then the sequence either $\{y_n\}$ diverges in finitely many steps or is periodic;*

2. *if $\theta_0, \theta_1, \ldots, \theta_{\ell-1}, \pi$ are linearly independent over $\mathbb{Q}$, and $A_1, \ldots, A_\ell$ are nonzero integers, then the sequence y_n exists for all n and the sequence $\{y_n\}$ is never periodic.*

Proof. Substituting y_n by $\dfrac{1}{y_n}$ turns the equation (3.23) into a rational recursive equation of the form (3.12) and so the corollary follows at once from Theorem 3.2.3. □

Remarks. Although the substitution y_n by $\dfrac{1}{y_n}$ employed in Corollary 3.2.4 allows us to obtain a closed form solution of the equation (3.23), there remains a difficulty should there exist integer N such that $y_N = 0$. To overcome this short-coming, we may either interpret the infinite value of the two expressions on both sides of the solution as equal or repeat the technique used in the proof of Theorem 3.2.3 to solve the equation (3.23) using auxiliary results analogous to Lemmas 3.2.1 and 3.2.2.

Finally, we consider rational recursive equations which can be used to characterize the hyperbolic tangent-cotangent functions.

Corollary 3.2.5. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $\{F_n\}$ be a sequence satisfying a linear recurrence relation of the form*

$$F_{n+\ell} = A_1 F_{n+\ell-1} + A_2 F_{n+\ell-2} + \ldots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots,$ $F_{\ell-1} = A_1$. Let $y_0, \ldots, y_{\ell-1}$ be real numbers such that those y_n which satisfy

$$y_{n+\ell} = \frac{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} + (y_{n+\ell-1}-1)^{A_1} \cdots (y_n-1)^{A_\ell}}{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} - (y_{n+\ell-1}-1)^{A_1} \cdots (y_n-1)^{A_\ell}}, \tag{3.25}$$

exist for all $n \in \mathbb{N} \cup \{0\}$.

Then the solution to the equation (3.25) exists and is given by

$$y_n = \frac{(y_0+1)^{F_{n-\ell}} \cdots (y_{\ell-1}+1)^{F_{n-1}} + (y_0-1)^{F_{n-\ell}} \cdots (y_{\ell-1}-1)^{F_{n-1}}}{(y_0+1)^{F_{n-\ell}} \cdots (y_{\ell-1}+1)^{F_{n-1}} - (y_0-1)^{F_{n-\ell}} \cdots (y_{\ell-1}-1)^{F_{n-1}}}, \tag{3.26}$$

or

$$y_n = \coth\left(F_{n-\ell} \text{ arccoth } y_0 + \ldots + F_{n-1} \text{ arccoth } y_{\ell-1}\right).$$

Proof. Substituting y_n by iy_n in the equation (3.25) turns it into a rational recursive equation of the form (3.12) and so Theorem 3.2.3 yields the desired result. □

Corollary 3.2.6. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $\{F_n\}$ be a sequence satisfying a linear recurrence relation of the form*

$$F_{n+\ell} = A_1 F_{n+\ell-1} + A_2 F_{n+\ell-2} + \cdots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots,$ $F_{\ell-1} = A_1$. Let $y_0, \ldots, y_{\ell-1}$ be real numbers such that those y_n which satisfy

$$y_{n+\ell} = \frac{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} - (-y_{n+\ell-1}+1)^{A_1} \cdots (-y_n+1)^{A_\ell}}{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} + (-y_{n+\ell-1}+1)^{A_1} \cdots (-y_n+1)^{A_\ell}}, \tag{3.27}$$

exist for all $n \in \mathbb{N} \cup \{0\}$.
Then the solution to equation (3.27) exists and is given by

$$y_n = \frac{(y_0+1)^{F_{n-\ell}} \ldots (y_{\ell-1}+1)^{F_{n-1}} - (-y_0+1)^{F_{n-\ell}} \ldots (-y_{\ell-1}+1)^{F_{n-1}}}{(y_0+1)^{F_{n-\ell}} \ldots (y_{\ell-1}+1)^{F_{n-1}} + (-y_0+1)^{F_{n-\ell}} \ldots (-y_{\ell-1}+1)^{F_{n-1}}}, \tag{3.28}$$

or

$$y_n = \tanh\left(\text{arctanh } y_0 F_{n-\ell} + \cdots + \text{arctan}h \ y_{\ell-1} F_{n-1}\right).$$

Proof. Replacing y_n by iy_n in the equation (3.27), we get a rational recursive equation of the form (3.23) and Corollary 3.2.4 yields the required result. □

REFERENCES

[1] J. Aczél, Lectures on Functional Equation and their Applications, Academic Press, New York, 1966.

[2] J. Aczél and J. Dhombres, "Functional Equations in Several Variables", Cambridge University Press, Cambridge, 1988.

[3] T. S. Chihara, *Solution to E1079.* Amer. Math. Monthly, **61**(1954), 197.

[4] J.K. Chung, Pl. Kannappan, and C.T. Ng, *A generalization of the cosine-sine functional equation on groups*, Linear Algebra and Its Applications, **66**(1985), 259277.

[5] D.E. Dobbs, *A characterization of the tangent function*, Elemente der Mathematik. **44**(1989), 101-104.

[6] T. M. Flett, *Continuous solutions of the functional equation* $f(x+y)+f(x-y) = 2f(x)f(y)$, Amer. Math. Monthly, **70**(1963), 392397.

[7] Pl. Kannappan, *The functional equations* $f(xy) + f(xy^{-1}) = 2f(x)F(y)$ *for groups,* Proc. Amer. Math. Soc. **19**(1968), 69-74.

[8] Pl. Kannappan, *Klee's trigonometry problem.* Amer. Math. Monthly, **110** (2003), 940-944.

[9] Pl. Kannappan, P. Sahoo, Introduction to Functional Equations, 2008.

[10] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer, 2009.

[11] V. L. Klee, *Problem E1079*. Amer. Math. Monthly, **60** (1953), 479.

[12] V. L. Kocic and G. Ladas, Global Behaviour of Nonlinear Difference Equation of Higher Order with Applications, Kluwer Academic Press, Dordrecht, 1993.

[13] X. Li and D. Zhu, *Two rational recursive sequences*, Computers and Mathematics with Applications **47**(2004),1487-1494.

[14] M.B.H. Rhouma, *The Fibonacci sequence modulo* π*, chaos and some rational recursive equations*, J. Math. Anal. Appl. **310**(2005),506-517.

[15] W. Rudin, Principal of Mathematical Analysis, McGraw-Hill, New York, 1964.

[16] G.B. Thomas, Calculus and Analytic Geometry 4^{th} edition, Addison-Wesley, Reading, 1974.

VITA

Miss Charinthip Hengkrawit was born on January 30, 1979 in Phetchaburi, Thailand. She graduated with a Bachelor of Science Degree in Mathematics from Kasetsart University in 2001 and graduated with a Master's degree of Science Degree in Mathematics from Thammasat University in 2004. She has received a scholarship from the University Development Commission since 2003. For her Doctoral degree, she has studied Mathematics at the Department of Mathematics, Faculty of Science, Chulalongkorn University.