ปัญหาสมบัติสภาพแข็งเกร็งที่ขอบของฟังก์ชันวิเคราะห์

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A BOUNDARY RIGIDITY PROBLEM OF ANALYTIC FUNCTION

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ในปี ค.ศ. 1994 เบิร์นส์และครานส์ได้พิสูจน์ว่าฟังก์ชันวิเคราะห์บนอาณาบริเวณวงกลมขนาด 1 หน่วยที่ลู่เข้าสู่จุดตรึงที่ขอบของวงกลมด้วยอัตราเร็วในระดับหนึ่งจะต้องเป็นฟังก์ชันเอกลักษณ์ ในวิทยานิพนธ์ฉบับนี้เราได้ขยายสมบัติข้อนี้ไปยังฟังก์ชันวิเคราะห์บนโดเมนเชื่อมต่อเชิงเดียว รวมไปถึงกรณีลิมิตที่ไม่สัมผัสขอบ นอกจากนี้เรายังได้พิจารณาในกรณีของฟังก์ชันผลคูณของบลาสกี

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In 1994, Burns and Krantz proved that an analytic function on the unit disc approaching a fixed point at the boundary with certain rate must be identity. This is essentially a boundary rigidity condition. In this thesis, we present the phenomena on simply connected domain with nice boundary and in the case of non-tangential limit. We also investigate the boundary rigidity problem on the Blascke's product.

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CHAPTER I
INTRODUCTION

Functions of one complex variable have many nice and interesting properties that are not valid for function of real variables. One of such properties is the rigidity phenomena; a condition that could be held by only a small group (maybe unique) of functions. For example, The Identity Principle states that zeros of a nonzero holomorphic function must be discrete on its (connected)domain. Nevalinna's Five Value Theorem states that two meromorphic functions that agree at five points (ignoring multiplicity) must be identically equal. Indeed, these properties are not held by real valued functions. Another example of rigidity phenomena is the equality part of the Schwarz's lemma. In this thesis , we denote $\mathbb{D} \subseteq \mathbb{C}$ the unit disc in the complex plane. $\mathbb{T}$ denotes $\partial\mathbb{D}$ and $\mathbb{C}_+ = \{z \in \mathbb{C} | Re(z) > 0\}$.
Classical Schwarz's Lemma: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$ then

$$|f(z)| \leq |z|, \forall z \in \mathbb{D} \text{ and } |f'(0)| \leq 1.$$

Furthermore, if $|f(z)| = |z|$, for some $z \in \mathbb{D} \setminus \{0\}$ or $|f'(0)| = 1$ then $f(z) = \lambda z$ where λ is a unimodular constant.

Schwarz's lemma, a simple consequence of the maximum modulus principle, has many interesting generalizations and plays an important role in studying geometric function theory. In 1938, Lars Ahlfors gave a geometric interpretation of this theorem : A holomorphic function on the unit disc are distance decreasing in Poincaré metric. One interesting consequence of this interpretation is a simple proof of the great Picard's theorem(see [7]). Schwarz's lemma can also be used to classify the automorphisms on the unit disc (Aut($\mathbb{D}$)) which is used to define Blaschke's product. Another easy consequence of Schwarz's lemma is that a non-identity holomorphic function on the unit disc has exactly one fixed point inside the disc. Farkas and Ritt also gave a condition that the iteration of function will converge to the fixed point (see [7]).

Our main interest is the conditions on the boundary. Löwner (see [10]) conducted a study in this direction with a motivation from distortion theorems. In 1994 Burns and Krantz [3] significantly improved the result of Löwner by removing many restrictive conditions in the theorem. They proved

Theorem 1.1 (Burns-Krantz's Theorem)**.** *Let* $f : \mathbb{D} \to \mathbb{D}$ *be holomorphic and*

$$f(z) = 1 + (z-1) + O(z-1)^4$$

as $z \to 1$. *Then* $f(z) \equiv z$ *on the unit disc.*

Roughly, this theorem says that if a holomorphic function on the unit disc has a fixed point at the boundary and f approaches to that point at a certain rate then f must be the identity map. They also provide an example to show that 4 in this theorem is sharp. Also "1" in the theorem can be replaced by any points on $\mathbb{T}$. Note that it could be seen from the proof that $O(z-1)^4$ may be strengthened to $o(z-1)^3$.

Burns-Krantz's theorem has been generalized to the finite Blaschke's product (see [4]); D.Chelest gave a condition on the boundary that forces the function to be a finite Blaschke's product. In [2] R.Tausaro and F.Vlacci gave the same result under weaker condition that $\lim_{r\to 1^-} \frac{F(r)-r}{(1-r)^3}$ which is equivalent to $\lim_{z\to 1} Re\frac{F(z)-z}{(1-z)^3} = 0$ where the limit is taken in Stolz angle (or nontangential limit region, see Section 3.3.) with the vertex at point $z = 1$.

In this thesis, we present some new results on generalization of Burns-Krantz's theorem. In chapter III, we give the boundary rigidity for analytic functions on more general simply connected domains with nice boundary and a result on nontangential limit case and the case of the finite Blaschke's product. We conclude with some suggestions on future work.

CHAPTER II
PRELIMINARIES

In this section we summarized some results used in the next chapter. Their proofs can be found in most standard analysis textbooks except Definition 2.10, Definition 2.11 and Theorem 2.12 which are taken from [5]. We include the proof of Theorem 2.7 which is taken from [6]. Firstly, we remark that a domain is a connected open subset of $\mathbb{C}$.

Definition 2.1 (Big-Oh Notation.)**.** *We write* $f(z) = O(g(z)), z \to a$ *to mean as* $z \to a$*, there is an constant* C *such that* $|f(z)| \leq C|g(z)|$.

Definition 2.2 (Little-Oh Notation.)**.** *We write* $f(z) = o(g(z)), z \to a$ *to mean as* $z \to a$, $\frac{f(z)}{g(z)} \to 0$.

Theorem 2.3 (Logarithm Function)**.** *If* U *is an open simply connected set in* $\mathbb{C}$ *that does not contain* 0*. We can find an analytic function* $\log : U \to \mathbb{C}$ *such that* $exp(\log z) = z$ *for all* $z \in U$*. The function* $\log$ *is unique up to addition of integer multiple of* $2\pi i$.

Theorem 2.4 (Riemann Mapping Theorem)**.** *If* $\Omega \subsetneq \mathbb{C}$ *is a simply connected domain then there exists a bijective function* $F : \Omega \to \mathbb{D}$ *such that* F, F^{-1} *are holomorphic. We will call* F *a Riemann map of* Ω.
Furthermore, for any $z_0 \in \mathbb{D}$*, we can choose* F *so that* $F(z_0) = 0$.

Theorem 2.5 (Maximum Modulus Principle)**.** *Let* $\varnothing \neq \Omega \subseteq \mathbb{C}$.

$$MVP(\Omega) = \{f \in C(\Omega) : \forall \omega \in \Omega, \exists \delta > 0 s.t. \forall 0 \leqslant r \leqslant \delta, f(\omega) = \frac{1}{2\pi}\int_0^{2\pi} f(\omega + re^{i\theta})d\theta\}$$

Note that $f \in C(\Omega)$ *is harmonic if and only if* $f \in MVP(\Omega)$ *and that every holomorphic function on* Ω *is harmonic on* Ω.

Maximum modulus principle states that if $f \in MVP(\Omega)$ *then if* $\exists \eta \in \Omega, |f(\eta)| \geq |f(z)| \forall z \in \Omega$, *then* f *is a constant function.*

Theorem 2.6 (Schwarz Reflection Principle). *If* $\Omega \subseteq \mathbb{C}$ *is a domain which is symmetric with respect to the real axis and let* $\Omega^+ = \Omega \cap \{z | Im(z) > 0\}$ *be the part of* Ω *in the upper half plane. Let u(z) be a real-valued harmonic function on* Ω^+ *such that* $u(z) \to 0$ *as* $z \in \Omega^+$ *tends to a point of* $\Omega \cap \mathbb{R}$. *Then* $u(z)$ *extends to a harmonic function on* Ω.
Consequently, an analytic function f *defined on a domain* Ω *with free analytic boundary arc (see Definitions 3.6-3.7) such that* $|f(z)| \to 1$, *as* $z \to \partial\Omega$ *could be holomorphically extended to the boundary.*

Theorem 2.7 (Pick's Lemma). *If* $f(z)$ *is analytic and satisfies* $|f(z)| < 1$ *for* $|z| < 1$, *then*

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, |z| < 1.$$

Furthermore, the equality holds if and only if f *is a conformal self map of* $\mathbb{D}$.

Proof. This proof is taken from [6]. Fix $z_0 \in \mathbb{D}$ and $w_0 = f(z_0)$. Let $g(z)$ and $h(z)$ be conformal self-maps of $\mathbb{D}$ mapping 0 to z_0 and w_0 to 0 respectively says,

$$g(z) = \frac{z + z_0}{1 + \bar{z}_0 z}, h(w) = \frac{w - w_0}{1 - \bar{w_0} w}$$

$\therefore h \circ f \circ g$ maps 0 to 0. $\therefore |(h \circ f \circ g)'(0)| = |h'(w_0) f'(z_0) g'(0)| \leq 1$. $\therefore |f'(z_0)| \leq \frac{1}{|g'(0)||h'(w_0)|}$. Substituting $g'(0) = 1 - |z_0|^2, h'(w_0) = \frac{1}{1 - |w_0|^2}$, we have $|f'(z_0)| \leq \frac{1 - |f(z_0)|^2}{1 - |z_0|^2}$. For the equality case, if f is a conformal self-map of $\mathbb{D}$, then so is $h \circ f \circ g$. Hence

$$|(h \circ f \circ g)'(0)| = |h'(w_0) f'(z_0) g'(0)| = 1.$$

Conversely, if the equality holds at one point z_0 then by calculation above $|(h \circ f \circ g)'(0)| = 1$. Then $h \circ f \circ g \equiv \lambda z, |\lambda| = 1$, is a conformal self-map so f is a conformal self map. $\square$

Definition 2.8. *Let X be a normed linear space. Let f_n be a sequence in X^*. The weak* convergence of f_n to $f \in X^*$ means that $f_n(x) \to f(x)$ pointwise as $n \to \infty$ for every $x \in X$.*

Theorem 2.9 (Banach-Alaoglu's Theorem)**.** *If V is a neighborhood of 0 in a topological vector space X and if $K = \{\Lambda \in X^* : |\Lambda x| \leq 1 \forall x \in V\}$ then K is weak* compact.*

Definition 2.10. *Let $B = C(\mathbb{T})$ then $B^* = M(\mathbb{T})$(Borel measure on $\mathbb{T}$.) The Fourier coefficients of measures are called Fourier-Stieltjes coefficients. Fourier series of measure is called Fourier-Stieltjes series.We say that a measure μ is positive if $\mu(E) > 0$ for every measurable set E or equivalently, if $\int f d\mu \geq 0$ whenever $f \in C(\mathbb{T})$ is nonnegative. If μ is absolutely continuous, that is, $\mu = \frac{1}{2\pi} g(t)dt, g \in L^1(\mathbb{T})$ (Radon-Nakodym's Theorem) then μ is positive if and only if $g \geq 0 a.e.$*

Definition 2.11. *A numerical sequence $\{a_{n,n\in\mathbb{Z}}\}$ is positive definit if for any choice of complex numbers $\{z_n\}$, we have*

$$\sum_{n,m} a_{n-m} z_n \bar{z}_m \geqslant 0.$$

Theorem 2.12 (Herglotz)**.** *A numerical sequence $\{a_{n,n\in\mathbb{Z}}\}$ is positive definit if and only if there is a positive measure $\mu \in M(\mathbb{T})$ such that $a_n = \hat{\mu}(n)$ for all n.*

CHAPTER III
MAIN RESULTS

In section 3.1, we present the idea that Burns and Krantz used in their paper [3]. We present our main results in Section 3.2-3.4. In Section 3.2, we generalize Burns-Krantz's theorem to some kind of simply connected domain. In Sections 3.3 and 3.4, the condition with non-tangential limit is investigated. We conclude with Section 3.5 for some suggestions on future work.

3.1 Burns-Krantz's Theorem

The main tools in [3] is the following two theorems. Theorem 3.1 is a standard result in partial differential equation. We give the proof of Theorem 3.2 following the idea given in [7]. For a thorough treatment of Theorem 3.2, see [1].

Theorem 3.1 (Hopf's Lemma)**.** *If u is a nonnegative nonconstant real-valued harmonic function on $\mathbb{D}$ and $\gamma \in \partial\mathbb{D}$ be such that u is continuous at γ and $u(z) \geq u(\gamma)$ for all $z \in \mathbb{D}$. Then the outer normal derivative of u at γ is negative.*

Theorem 3.2 (Herglotz's Representation Theorem)**.** *If g is a holomorphic function from $\mathbb{D}$ to $\mathbb{C}_+ = \{z \in \mathbb{C} | Re(z) > 0\}$. Then there is a positive measure μ on $[0, 2\pi)$ and a purely imaginary constant C such that*

$$g(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + \zeta}{e^{-i\theta} - \zeta} d\mu(\theta) + C.$$

Proof. For $0 < r < 1$ we define $G_r(e^{i\theta}) = Re\ g(re^{i\theta})$ which is positive and has mean value g(0). Thus they form a bounded set in $L^1[0, 2\pi) \subseteq M[0, 2\pi) = C[0, 2\pi)^*$. Here $C[0, 2\pi)$ is the space of all continuous functions from $[0, 2\pi)$ to $\mathbb{R}$ and $M[0, 2\pi)$ is the space of real finite borel measure on $[0, 2\pi)$. By Banach-Alaoglu's theorem, there is a subsequence G_{r_j} converging to a measure μ in $M[0, 2\pi)$ in the weak* topology. Now using Poisson's kernel we get

$$Re\ g(\zeta) = \lim_{j\to\infty} Re\ g(r_j\zeta) = \lim_{j\to\infty} \int_0^{2\pi} Re\ g(r_j e^{i\theta}) \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} d\theta$$

$$= \int_0^{2\pi} Re\ \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} d\mu(\theta).$$

Now $\int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} d\mu(\theta)$ defines an analytic function having the same real part as g. It follows that $g = \int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} d\mu(\theta) + iK, K \in \mathbb{R}$.

□

The idea of proving Burns-Krantz's theorem is to use the assumption on approaching rate to the fixed point at the boundary to construct a nonnegative harmonic function whose normal derivative at the point is zero. Now we present the idea of proving this theorem given in [3].

Proof of Theorem 1.1. Let $g(\zeta) = \frac{1+f(\zeta)}{1-f(\zeta)}$. By Herglotz's representation theorem there is a positive measure μ on $[0, 2\pi)$ and a purely imaginary constant C such that

$$g(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} d\mu(\theta) + C.$$

Now using the hypothesis on g and geometric series expansion, we have

$$g(\zeta) = \frac{1+\zeta+O(\zeta-1)^4}{1-\zeta-O(\zeta-1)^4} = \frac{1}{1-\zeta}(1+\zeta+O(\zeta-1)^4)(1+O(\zeta-1)^3).$$

$$= \frac{1+\zeta}{1-\zeta} + O(\zeta-1)^2.$$

Now write $\mu = \delta_0 + \nu$(where δ_0 is 2π times Dirac mass at the origin), we can use the above two equations to derive Fourier-Stieltjes expansion of $\delta_0 + \nu$ and apply Herglotz's criterion of positive measure (see [5]) to conclude the positivity of ν. Taking real part , we have

$$O(\zeta-1)^2 = Re(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} d\nu(\theta)).$$

Call the expression on the right $h(\theta)$ which is a positive harmonic function taking minimum at 1 (since it is zero there) and is $O(z-1)^2$ there. Hopf's lemma forces $h \equiv 0$, that is $\nu \equiv 0$. So $f(z) \equiv z$. □

Example 3.3. *In [3] the authors claimed that* $\phi(z) = z - \frac{1}{10}(1-z)^3$ *maps* $\mathbb{D}$ *to* $\mathbb{D}$. *This means that 4 in the hypothesis of Theorem 1.1 couldn't be replaced by 3. The only thing that is nontrivial is that* ϕ *maps into* $\mathbb{D}$. *By Maximum modulus principle we consider only* $z = e^{it}$. *By calculation,* $|f(e^{it})|^2 = -\frac{8}{100}cos^3(t) - \frac{16}{100}cos^2(t) + \frac{56}{100}cos(t) + \frac{68}{100}$. *It can be verified by standard calculus that the above expression has modulus at most 1. Our technique is useful for verifying or creating these kinds of examples.*

3.2 Burns-Krantz's Theorem on simply connected domain.

In this section we study Burns-Krantz's theorem on a simply connected domain rather than the unit disc. It turns out that our main tool is that the Riemann map (the map in Riemann Mapping Theorem) locally maps boundary to boundary. Therefore we focus on the domain that its boundary is locally mapped to the boundary of unit disc by the Riemann map.

Theorem 3.4. *Let* $\Omega \subsetneq \mathbb{C}$ *be a simply connected domain such that if* $F : \Omega \to \mathbb{D}$ *is a Riemann map then it can be extended locally holomorphically to its boundary. This means that for each* $a \in \partial\Omega$ *there is neighborhood* B *of* a *such that* F *can be extended holomorphically to* $B \cap \partial\Omega$ *and* F *maps* $B \cap \partial\Omega$ *into the boundary of disc. Now if* $f : \Omega \to \Omega$ *is holomorphic such that*

$$f(z) = z + O(z-a)^4$$

as $z \to a$. *Then* $f(z) \equiv z$.

Proof. Suppose $f : \Omega \to \Omega$ is analytic and $f(z) = z + O(z-a)^4, z \to a, a \in \partial\Omega$. Let $\phi : \Omega \to \mathbb{D}$ be the Riemann map such that $\phi(0) = 0$. Hence ϕ can be extended to a neighborhood of a. Now we have $\phi \circ f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic. Now by differentiability of ϕ, we have $\phi(z+h) = \phi(z) + O(h), h \to 0$. Also ϕ is a conformal map, $\phi(0) = 0, \phi^{-1}(0) = 0$ We have $\phi(h) = O(h)$ and $\phi^{-1}(x) - \phi^{-1}(y) = O(x-y), |x-y| \to 0$. Suppose that $z \to \phi(a) \in \partial\mathbb{D}$ then

$\phi^{-1}(z) \to a \in \partial\Omega$. Therefore, as $z \to \phi(a)$,

$$\begin{aligned}\phi(f(\phi^{-1}(z))) &= \phi(\phi^{-1}(z)) + O((\phi^{-1}(z) - a)^4) \\ &= \phi(\phi^{-1}(z)) + O((\phi^{-1}(z) - a))^4. \\ &= z + O(z - \phi(a))^4.\end{aligned}$$

Invoking Burns-Krantz's theorem on the unit disc, we have that $\phi \circ f \circ \phi^{-1}(z) \equiv z$ i.e. $f(z) \equiv z$.

□

Example 3.5. *In case that* Ω *is the strip* $\{z \in \mathbb{C}, -1 < Re(z) < 1\}$. *We can explicitly construct a Riemann map* $F : \mathbb{D} \to \Omega$ *by* $F(z) = \frac{2}{\pi}\log(i\frac{1-z}{1+z}) - 1$. *Here* log *is the logarithm with branch cut at negative imaginary axis and* $\log(1) = 0$. *(Note that* $\frac{1-z}{1+z}$ *maps* $\mathbb{D}$ *to* $\mathbb{C}_+$ *and* log *maps the upper half plane to* $\{z \in \mathbb{C}|0 < Im(z) < 1\}$. F *can be extended holomorphically to* $\overline{F} : \overline{\mathbb{D}} - \{-1, 1\} \to \overline{\Omega}$ *and* F *maps the upper half of* $\partial\mathbb{D}$ *onto the left half of* $\partial\Omega$ *and maps the lower half of* $\partial\mathbb{D}$ *onto the right half of* $\partial\Omega$. *We indeed have Burns-Krantz's theorem for analytic functions on the strip.*

Definitions 3.6, 3.7 are taken from [6].

Definition 3.6. *Let* $\Omega \subseteq \mathbb{C}$ *be a domain. An analytic curve(analytic arc)* γ *is a curve* γ *such that for every point of* γ, *one has an open neighborhood* U *for which there is a conformal map* $\zeta \mapsto z(\zeta)$ *of a disc* D *centered on the real line such that image of* $D \cap \mathbb{R}$ *coincides with* $U \cap \gamma$.

Definition 3.7. *If* Ω *is a domain. An analytic arc* $\gamma \subseteq \partial\Omega$ *is a* ***free analytic boundary arc*** *of* Ω *if every point of* γ *is contained in a disc* U *such that* $U\backslash\gamma$ *has two components , one contained in* Ω *and the other disjoint from* Ω.

Theorem 3.8. *Let* $\Omega \neq \mathbb{C}$ *be a simply connected domain in* $\mathbb{C}$ *whose the boundary is free analytic boundary arc. Then the Riemann map* ϕ *of* Ω *onto* $\mathbb{D}$ *extends locally analytically across any free analytic boundary arc of* Ω.

Proof. For each $\epsilon > 0$, the set $\{z \in \Omega ||\phi(z)| \leq 1-\epsilon\}$ is a compact subset of Ω , which is has positive distance from $\partial\Omega$. Hence $|\phi(z)| \to 1$ as $z \to \partial\Omega$. Let $p \in \partial\Omega$. Note that there is exactly one point $w \in \Omega$ such that $\phi(w) = 0$. Now since ϕ is analytic on $\Omega, \log|\phi|$ is harmonic on $\Omega\backslash l$ where l is the branch cut of $\log|\phi|$ that does not intersect p. To see this, let $\phi(z_0) \neq 0, z_0 \in \Omega\backslash l$ then there is a $\delta > 0$ such that f is nonzero on $B_\delta(z_0)$.This means that there is an analytic function g on $B_\delta(z_0)$ such that $\phi \equiv e^g$ so $\log|\phi| \equiv Re(g)$ is harmonic on $B_\delta(z_0)$. Now apply Schwarz's Reflection Principle for harmonic function to $\log|\phi|$, we have that $\log|\phi|$ extends harmonically across any free analytic boundary arc of $\Omega\backslash l$ i.e. ϕ extends analytically across any free analytic boundary arc of $\Omega\backslash l$. This means that ϕ extends analytically locally at p. □

Corollary 3.9. *If Ω is a domain with free analytic boundary arc. Let $f : \Omega \to \Omega$ is analytic and $p \in \partial\Omega$. If $f(z) = z + O(z-p)^4, z \to p$ then $f(z) \equiv z$.*

3.3 Non-Tangential Limit Case.

We begin with the definition of non-tangential region (or Stolz region) taken from [7]. Roughly, we say that a sequence z_k converges non-tangentially if z_k is convergent and z_k lies in the non-tangential region.

Definition 3.10 (non-tangential region). *Let $e^{i\theta} \in \partial\mathbb{D}, 1 < \alpha < \infty$, then define the Stolz region (or nontangential approach region) with vertex $e^{i\theta}$ and aperture α to be*

$$\Gamma_\alpha(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}| < \alpha(1-|z|)\}.$$

Now we present the result of theorem 1.1 in the case of non-tangential limit. In general domains, it is difficult to define non-tangential convergence. However, in the case of domains with free analytic boundary arc, we have a Riemann map that maps locally from boundary to boundary. We define non-tangential convergence as follows.

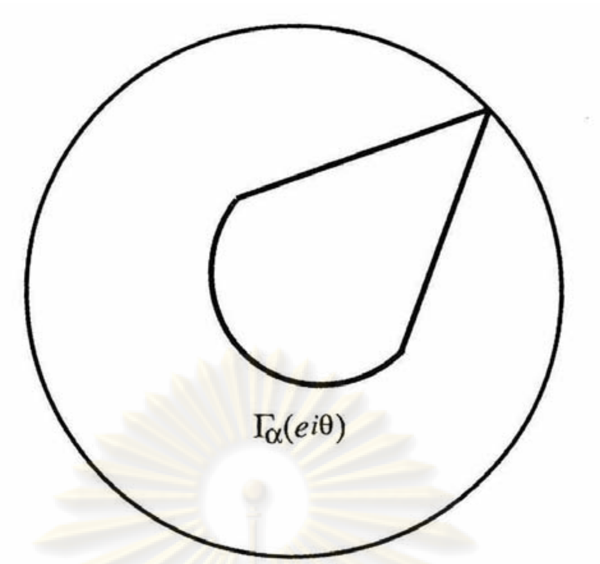

Figure 3.1: A Stolz region

Definition 3.11. *Let* Ω *be a simply connected domain with free analytic boundary arc and* $p \in \partial\Omega$*. Suppose that* $z_k \to p \in \partial\Omega$*. We say that* $z_k \to p$ *non-tangentially if there is a Riemann map* $\phi : \Omega \to \mathbb{D}$ *that maps boundary to boundary locally at* p *and* $\phi(z_k) \to \phi(p) \in \partial\mathbb{D}$ *in* $\mathbb{D}$ *non-tangentially.*

First recall that $O(z-1)^4$ in the hypothesis of Theorem 1.1 could be replaced by $o(z-1)^3$

Theorem 3.12. *Let* Ω *be a simply connected domain with free analytic boundary arc. If* $f : \Omega \to \Omega$ *is holomorphic and there is a sequence* $z_k \in \Omega, z_k \to p \in \partial\Omega$ *nontangentially in the sense that if* $\phi : \Omega \to \mathbb{D}$ *is a Riemann map then* $\phi(z_k) \to \phi(p) \in \partial\mathbb{D}$ *nontangentially. Assume that* $f(z_k) = z_k + O(z_k - p)^4, k \to \infty$*, then* $f(z) \equiv z$*.*

Proof. Let $g = \phi \circ f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$ and $\phi(z_k) \to \phi(p) \in \partial\mathbb{D}$ nontangentially. Without loss of generality, let $\phi(p) = 1$. Firstly we follow idea in proving the classical Julia's theorem in [11]. For $a \in \mathbb{D}$, let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \in Aut(\mathbb{D})$(which interchanges a and 0). Define $h(z) = \varphi_{g(a)} \circ g \circ \varphi_a$, Hence $h(0) = 0$ and by the classical Schwarz's lemma, $|\varphi_{g(a)}(g(\varphi_a(z)))| \leq |z|$. Since $\varphi_a \circ \varphi_a$ is the identity, we have $|\varphi_{g(a)}(g(z))| \le |\varphi_a(z)|$. Next, a straightforward calculation shows that

$$1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

Hence

$$\left|\frac{1-\overline{g(a)}g(z)}{1-|g(z)|^2}\right| \le \frac{1-|g(a)|^2}{1-|a|^2}\frac{|1-\bar{a}z|^2}{1-|z|^2}. \tag{3.1}$$

Take $a = \phi(z_k)$. By the same method in the proof of theorem 3.4, we have $g(\phi(z_k)) = \phi(z_k)+O(\phi(z_k)-1)^4$. Hence $1\pm|g(\phi(z_k))| = 1\pm|\phi(z_k)|+O(\phi(z_k)-1)^4$. Therefore we have $1-|g(\phi(z_k))|^2 = 1-|\phi(z_k)|^2+O(\phi(z_k)-1)^4$. So

$$\frac{1-|g(\phi(z_k))|^2}{1-|\phi(z_k)|^2} = 1+O(\frac{(\phi(z_k)-1)^4}{1-|\phi(z_k)|}) = 1+O(\phi(z_k)-1)^3.$$

Note that we use the nontangential convergence assumption in the last equality. Hence in equation (3.3.1), letting $k\to\infty$,we get

$$\frac{|1-g(z)|^2}{1-|g(z)|^2} \le \frac{|1-z|^2}{1-|z|^2}. \tag{3.2}$$

Let $\xi(z) = Re(\frac{1+g(z)}{1-g(z)} - \frac{1+z}{1-z})$ which is a nonnegative harmonic function (by (3.3.2) since $\frac{1}{2}(\frac{1+g(z)}{1-g(z)} + \frac{1+\overline{g(z)}}{1-\overline{g(z)}} - \frac{1+z}{1-z} - \frac{1+\bar{z}}{1-\bar{z}}) = \frac{1-|g(z)|^2}{|1-g(z)|^2} - \frac{1-|z|^2}{|1-z|^2}$. We also have from the assumption that $|g(\phi(z_k))-\phi(z_k)|/|1-\phi(z_k)| < 1$ for sufficiently large k. Therefore by geometric series expansion,

$$\begin{aligned}
\frac{1+g(\phi(z_k))}{1-g(\phi(z_k))} &= \frac{(1+\phi(z_k)+g(\phi(z_k)))-\phi(z_k)/(1-\phi(z_k))}{1-(g(\phi(z_k))-\phi(z_k))/(1-\phi(z_k))} \\
&= \frac{(1+\phi(z_k))/(1-\phi(z_k))+(g(\phi(z_k))-\phi(z_k))/(1-\phi(z_k))}{1-(g(\phi(z_k))-\phi(z_k))/(1-\phi(z_k))}. \\
&= \frac{1+\phi(z_k)}{1-\phi(z_k)}\sum_{n=0}^{\infty}\left(\frac{g(\phi(z_k))-\phi(z_k)}{1-\phi(z_k)}\right)^n + \sum_{n=1}^{\infty}\left(\frac{g(\phi(z_k))-\phi(z_k)}{1-\phi(z_k)}\right)^n \\
&= \frac{1+\phi(z_k)}{1-\phi(z_k)} + \frac{2}{1-\phi(z_k)}\frac{g(\phi(z_k))-\phi(z_k)}{1-g(\phi(z_k))} \\
&= \frac{1+\phi(z_k)}{1-\phi(z_k)} + \frac{2}{1-\phi(z_k)}\frac{O(\phi(z_k)-1)^4}{1-\phi(z_k)+o(\phi(z_k)-1)^3} \\
&= \frac{1+\phi(z_k)}{1-\phi(z_k)} + o(\phi(z_k)-1).
\end{aligned}$$

An application of Hopf's lemma shows that $\xi\equiv 0$. Hence $g(z)\equiv z$ and hence $f(z)\equiv z$.

$\square$

3.4 Finite Blaschke's Product and Non-Tangential Limit.

In this section, we discuss the study of boundary rigidity condition for Blascke's product (see theorem 4.2). First recall that $Aut(\mathbb{D}) = \{\omega\frac{z-\lambda}{1-\bar{\lambda}z}, |\omega| = 1, \lambda \in \mathbb{D}\}$.

Definition 3.13 (finite Blaschke's Product)**.** *The product of the form*

$$\prod_{j=1}^{n} \omega_j(\frac{z-\lambda_j}{1-\overline{\lambda_j}z}), |\omega_j| = 1, \lambda_j \in \mathbb{D}$$

is called the finite Blaschke's product of degree n. It is an n to 1 function from $\mathbb{D}$ onto $\mathbb{D}$. Also the product has modulus 1 on $\mathbb{T}$. In fact , a holomorphic on $\mathbb{D}$ which can be extended holomorphically to $\overline{\mathbb{D}}$ must be finite Blaschke's product.

The following theorem is due to D.Chelest [4].

Theorem 3.14. *If $\phi : \mathbb{D} \to \mathbb{D}$ be analytic and f is a finite Blaschke's product i.e.*

$$f : \mathbb{D} \to \mathbb{D}, f(z) = \prod_{k=1}^{n}(\omega_k \frac{z-\lambda_k}{1-\lambda_k z}), |\omega_k| = 1, |\lambda_k| < 1$$

Suppose that f equals $\tau \in \partial\mathbb{D}$ on a finite set $A_f \subseteq \mathbb{D}$ and if:

1) For a given $\gamma_0 \in A_f$, $\phi(z) = f(z) + O((z-\gamma_0)^4)$, as $z \to \gamma_0$ and,

2) For all $\gamma \in A_f - \{\gamma_0\}$, $\phi(z) = f(z) + O((z-\gamma)^{k_\gamma})$, $k_\gamma \geq 2$, as $z \to \gamma$

then $\phi \equiv f$.

The idea of proving this theorem in [4] is to define a harmonic function similar to the proof of theorem 3.12. Our main task is to show the positivity of this function. The first step is to use the assumption to show that it is positive everywhere (the main property of finite Blaschke's product we used it that it is analytic through $\overline{\mathbb{D}}$ and has modulus 1 on $\mathbb{T}$.) then we can use the compactness property to show the positivity of the harmonic function. We investigate the above theorem in the case n=1 with non-tangential convergence condition.

Theorem 3.15. *If $\phi : \mathbb{D} \to \mathbb{D}$ is analytic and suppose $f(z) = \omega\frac{\lambda - z}{1-\bar{\lambda}z.}$ equals $\tau \in \partial\mathbb{D}$ at $\gamma \in \partial\mathbb{D}$. Let $\lambda \in \mathbb{D} \cap \{z \in \mathbb{C} : |z - \frac{\gamma}{2}| \geq \frac{1}{2}\}, \omega \in \partial\mathbb{D}$. Suppose there is a* *sequence* $z_k \in \mathbb{D}, z_k \to \gamma$, *non-tangentially such that*

$$\phi(z_k) = \omega \frac{\lambda - z_k}{1 - \bar{\lambda} z_k} + O(z_k - \gamma)^4, k \to \infty.$$

Then $\phi(z) \equiv \omega \frac{\lambda - z}{1 - \bar{\lambda} z}$.

Proof. Without loss of generality,we may assume $\tau = 1$. Define a harmonic function on the unit disc $g(z) = Re(\frac{1+\phi(z)}{1-\phi(z)}) - Re(\frac{1+f(z)}{1-f(z)})$. In the view of equation 3.3.1, we have

$$\frac{|1 - \overline{\phi(a)}\phi(f(z))|^2}{1 - |\phi(f(z))|^2} \leq \frac{1 - |\phi(a)|^2}{1 - |a|^2} \frac{|1 - \bar{a} f(z)|^2}{1 - |f(z)|^2}. \tag{3.3}$$

Now we have $\phi(z_k) = f(z_k) + O(z_k - \gamma)^4, k \to \infty$. Then

$$1 \pm |\phi(z_k)| = 1 \pm |f(z_k)| + O(z_k - \gamma)^4.$$

So

$$1 - |\phi(z_k)|^2 = 1 - |f(z_k)|^2 + O(z_k - \gamma)^4.$$

That is

$$\frac{1 - |\phi(z_k)|^2}{1 - |z_k|^2} = \frac{1 - |f(z_k)|^2}{1 - |z_k|^2} + O(\frac{z_k - \gamma}{1 - |z_k|}(z_k - \gamma)^3) = \frac{1 - |f(z_k)|^2}{1 - |z_k|^2} + O(z_k - \gamma)^3.$$

Now since f is an automorphism on the unit disc. Invoking the equality part of Pick's lemma (theorem 2.7), we have

$$|f'(z)| = \frac{1 - |f(z)|^2}{1 - |z|^2}, z \in \mathbb{D}. \tag{3.4}$$

Now $f'(z) = \frac{|\lambda|^2 - 1}{(1 - \bar{\lambda} z)^2}$ which is bounded by 1 as $z \to \gamma$ since $\lambda \in \mathbb{D}$. Hence $\frac{1 - |f(z_k)|^2}{1 - |z_k|^2}$ is bounded by 1 as $z_k \to \gamma$. It follows that

$$\sup_{k \in \mathbb{N}} \frac{1 - |\phi(z_k)|^2}{1 - |z_k|^2} \leq 1.$$

Hence substituing $a = z_k$ in equation 3.4.1 and let $k \to \infty$, we get

$$\frac{|1 - \phi(z)|^2}{1 - |\phi(z)|^2} \leq \frac{|1 - f(z)|^2}{1 - |f(z)|^2}.$$

It follows that $g(z) = Re(\frac{1+\phi(z)}{1-\phi(z)}) - Re(\frac{1+f(z)}{1-f(z)})$ is a nonnegative harmonic function. Now we claim that $f(z) - 1$ is not $o(z_k - \gamma), k \to \infty$. To see this, Suppose $f(z_k) - 1 = o(z_k - \gamma)$, consider a positive harmonic function

$$Re(1 - f(z)).$$

Note that f has a continuous extension to the boundary. Since $f(z_k) - 1$ is $o(z_k - \gamma)$, it has a minimum at $z = \gamma$. This contradicts Hopf's lemma, and we have the claim. Next, write $h = \phi - f$, we have

$$g(z) = Re\frac{2h(z)}{(1 - f(z) - h(z))(1 - f(z))}$$

Each term in the denominator is not $o(z_k - \gamma)$. Hence g is $O(z_k - \gamma)^2, z \to \gamma$. By Hopf's lemma, we have $g \equiv 0$ i.e. $\phi \equiv f$. □

3.5 Conclusion.

In this section we will suggest some more work that could be done further on our results. In section 3.2, we proved Burns-Krantz's theorem on simply connected domain that its Riemann map could be extended holomorphically. In [11] it is proved a simply connected domain whose boundary is a Jordan's curve has a Riemann map that extended continuously that maps boundary to boundary. Hence if we can strengthen the hypothesis from holomorphically extended to continuously extended. We would have the result on a larger class of domains. In section 3.3 and we deals with non-tangential limit. It is interesting to ask if the non-tangential condition is necessary. If we arbitrarily pick a sequence $z_k, z_k \to 1$ and $f(z_k) = z_k + O(z_k - 1)^4$,can we conclude that $f(z) \equiv z$? Finally we would like to extend the result to the case of general finite Blaschke's product.This would follow if we can demonstrate that

$$\sup_{k\in\mathbb{N}} \frac{1 - |f(z_k)|^2}{1 - |z_k|^2} < \infty, z_k \to \gamma$$

when f is a finite Blaschke's product, $\gamma \in f^{-1}(1)$. Furthermore, since infinite Blaschke's product has many properties similar to finite Blaschke's product, we expect a similar rigidity condition for infinite product as well. We summarize the property of infinite product as follow.

Theorem 3.16 (Properties of Infinite Blaschke's Product.). *Let* $\{\alpha_n\}$ *be a sequence in* $\mathbb{D}$ *with* $\alpha_n \neq 0$ *and* $\sum_{n=1}^{\infty}(1\text{-}|\alpha_n|) < \infty$. *If* k *is a non-negative integer* *then*

$$f(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}$$

converges locally uniformly in $\mathbb{D}$ *and defines an analytic function in* $\mathbb{D}$ *having precisely the zeros* α_n *(and at 0 if* $k \geqslant 1$*)and no other zeros.We have* $|f(z)|=1$ *a.e. on* $\mathbb{T}$*.Conversely, if* f *is bounded analytic on* $\mathbb{D}$ *which is not identically zero. If zeros of* f *are* α_n *then* $\Sigma_{n=1}^{\infty}(1 - |\alpha_n|) < \infty$*.*

Note that if λ is a zero of f then $\frac{1}{\lambda}$ is a pole of f outside $\mathbb{D}$. In this case, we have infinitely many zeros and the zeros couldn't be accumulated. Hence zeros will approach the boundary and then so is the pole. Thus in this case, we expect a singularity at the boundary which ensures the difficulty of the problem.

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