เศษส่วนต่อเนื่องที่มีรูปแบบชัดแจ้งบางแบบ

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SOME TYPES OF EXPLICIT CONTINUED FRACTIONS

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การที่สามารถทำนายแบบรูปของเศษส่วนต่อเนื่องปกติมิได้เพียงน่าสนใจเท่านั้น หากแต่ในบางครั้งยังได้มาซึ่งข้อมูลเพิ่มเติมจากเศษส่วนต่อเนื่องปกตินั้นอีกด้วย เป็นที่ทราบกันดีว่า ในฟีลด์ของจำนวนจริงและฟีลด์ของอนุกรมตามแบบแผนเหนือฟีลด์ใด ๆ การพิจารณาว่าเศษส่วนต่อเนื่องปกติเป็นแบบจำกัดหรืออนันต์ถูกนำมาใช้เพื่อแสดงลักษณะเฉพาะของความเป็นตรรกยะหรืออตรรกยะ และยังทราบอีกด้วยว่าเศษส่วนต่อเนื่องปกติแบบคาบใด ๆ สอดคล้องกับสมาชิกที่เป็นอตรรกยะกำลังสองเท่านั้น นอกจากนี้มีงานวิจัยจำนวนมากที่เกี่ยวข้องกับการจัดหาเกณฑ์เพื่อตรวจสอบความเป็นอดิศัยผ่านเศษส่วนต่อเนื่องปกติ

ส่วนสำคัญของวิทยานิพนธ์ฉบับนี้ คือ การสร้างสูตรชัดแจ้งของเศษส่วนต่อเนื่อง โดยเริ่มต้นจากการจัดตั้งเอกลักษณ์สำหรับเศษส่วนต่อเนื่องที่มีแบบรูปพิเศษต่าง ๆ รวมทั้งแบบรูปพาลินโดรมซึ่งคือแบบรูปที่การอ่านจากซ้ายไปขวาเหมือนกันกับการอ่านจากขวามาซ้าย โดยการใช้ประโยชน์จากเอกลักษณ์เหล่านี้เราได้มาซึ่งเศษส่วนต่อเนื่องที่มีรูปแบบชัดแจ้งที่เขียนแทนจำนวนที่อยู่ในรูปอนุกรมที่แน่นอน เศษส่วนต่อเนื่องที่มีรูปแบบชัดแจ้งเหล่านี้ครอบครองคุณสมบัติที่สวยงาม คือ ในทุกช่วงความยาวที่เหมาะสม การอ่านลำดับของเศษส่วนย่อยจากซ้ายไปขวาจะเหมือนกันกับการอ่านจากขวามาซ้าย และดังนั้น จากการใช้เกณฑ์เพื่อตรวจสอบความเป็นอดิศัยของ Adamczewski และ Bugeaud ที่ทำไว้ในปี 2007 สามารถสรุปได้ว่าจำนวนจริงที่ถูกเขียนแทนด้วยเศษส่วนต่อเนื่องที่มีรูปแบบชัดแจ้งเหล่านี้เป็นจำนวนอดิศัย นอกจากสูตรชัดแจ้งสำหรับเศษส่วนต่อเนื่องแล้ว การมีขอบเขตของเศษส่วนย่อยของเศษส่วนต่อเนื่องเป็นเรื่องที่น่าสนใจและถูกพิจารณาในฐานะที่เป็นส่วนสำคัญอันดับสองในวิทยานิพนธ์นี้ ในส่วนนี้ เราได้ให้เกณฑ์ในการตรวจสอบการมีขอบเขตของเศษส่วนย่อยของเศษส่วนต่อเนื่องปกติที่เขียนแทนการแปลงเชิงเส้นของอนุกรมตามแบบแผน และยังให้ตัวอย่างที่น่าสนใจอย่างยิ่งของจำนวนตรรกยะที่เศษส่วนย่อยของเศษส่วนต่อเนื่องปกติมีค่าไม่เกิน 5 กล่าวคือ เราได้พิสูจน์ข้อความคาดการณ์ที่มีชื่อเสียงข้อความหนึ่งของ Zaremba สำหรับจำนวนเต็มที่อยู่ในรูป $2^s \cdot 3^t$ เมื่อ s และ t เป็นจำนวนเต็มที่ไม่เป็นลบ

ภาควิชา คณิตศาสตร์ ลายมือชื่อนิสิต อรนิตย์ พันธ์ประสิทธิ์เวช

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ORANIT PANPRASITWECH : SOME TYPES OF EXPLICIT CONTINUED FRACTIONS. THESIS ADVISOR : TUANGRAT CHAICHANA, Ph.D., 98 pp.

Being able to predict a pattern of a regular continued fraction is not only interesting in its own right but it sometimes yields more informations about that regular continued fraction. In the real number field and in the field of formal series over any base field, it is well-known that the termination of a regular continued fraction can be used to characterize rationality and is also known that any periodic regular continued fraction corresponds exactly to a quadratic irrational element. There are a number of researches about transcendental criteria via regular continued fractions.

The major part of this thesis is devoted to the establishing of explicit formulae for continued fractions. First, identities for continued fractions with specific patterns, including palindromic patterns, are realized over an arbitrary field. Then by making use of these identities, explicit continued fractions representing the numbers expressible explicitly by certain series are obtained. These explicit continued fractions possess a beautiful property, that is, sequences of their partial quotients begin in arbitrarily long palindromes. By using a transcendental criterion of Adamczewski and Bugeaud in 2007, it can be concluded that the real numbers represented by these explicit continued fractions are transcendental. Besides explicit formulae for continued fractions, boundedness of the partial quotients of a continued fraction is of interest and is considered as the second main part in this thesis. In this part, a criterion of boundedness of the partial quotients of the regular continued fraction representing a linear fractional transformation of a formal series is given. Also, a fascinating example of rational numbers represented by regular continued fractions which their partial quotients are bounded by 5 is provided by proving a famous conjecture attributed to Zaremba for integers being of the form $2^s \cdot 3^t$ where s, t are non-negative integers.

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CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

It is generally difficult to explicitly obtain a regular continued fraction representing a quantity expressed in other form, see e.g., [28], [18] and [9]. However, being able to predict a pattern in a regular continued fraction of a quatity is not only interesting in its own right but it sometimes enable us to derive more informations about that quantity from its regular continued fraction. In the real number field and in the field $\mathbb{F}((x^{-1}))$ of formal series, over a base field $\mathbb{F}$, which is the completion of the field of rational functions with respect to the degree valuation, it is well-known that the termination of a regular continued fraction can be used to characterize rationality and is also known that any periodic regular continued fraction corresponds exactly to a quadratic irrational element. Various researches, for examples [5], [1], [2], [3], [4], [6] and [12], gave transcendence criteria depending on special patterns in regular continued fractions.

The main objectives of this thesis are to establish explicit formulae of continued fractions. It is well-known that the theory of continued fractions for formal series goes parallel with that for real numbers; for detail see [24]. The work in this thesis centers around continued fractions both in the real number field and in the field $\mathbb{F}((x^{-1}))$. However, it is useful to define continued fractions over a general field K and some results in this work are widely provided for any field K.

In chapter II, boundedness of partial quotients of regular continued fractions representing some certain quantities is mentioned as the second objectives in this work. We say that any irrational number has bounded partial quotients if the supremum of all its partial quotients is finite. Lagarias and Shallit [17] proved, using the so-called Lagrange constant through a result of Cusick and Mendès France [11], that if a irrational number has bounded partial quotients, so does its linear fractional transformation. We show here that this is also the case in $\mathbb{F}((x^{-1}))$. Also, a bound of the partial quotients of a regular continued fraction representing a linear fractional tranformation of a rational elements in $\mathbb{F}((x^{-1}))$ is obtained. Next, a fascinating example of rational numbers represented by regular continued fractions whose their partial quotients are bounded by a small integer is provided by proving a famous conjecture attributed to Zaremba. This conjecture of Zaremba, see e.g., [20], states that for a positive integer $m \geq 2$, there exists an integer $1 \leq a < m$, coprime to m such that all of the partial quotients in the regular continued fraction of $\frac{a}{m}$ are less than or equal to 5. This conjecture has been verified for m being a power of 2, 3 and 5 by Niederreiter [20], and for m being a power of 6 by Yodphotong and Laohakosol [30]. In 2005, this conjecture was verified by Komatsu, [16], for m being the $c \cdot 2^n$-th power of 7 where $n \geq 0$ and c is an odd number less than or equal to 11. In this thesis, evidence of Zaremba's conjecture for m being the form $2^s \cdot 3^t$ where s, t are non-negative integers is presented.

A group of researchers [25], [26], [28], [29], [15], [22] and [23] have found continued fractions for numbers or formal series expressed by certain types of series. An important tool used in these results is the so-called Folding lemma, an identity, first appears in [19], for continued fractions which has folding symmetry in their partial quotients. In Chapter III, we attach significance to identities for continued fractions with some interesting patterns. Many identities for continued fractions with some patterns in their partial quotients were tied together as a single phenomenon in the work of Clemens, Merrill and Roeder in 1995, see [9]. They worked in the real number field. This phenonmenon is generalized to continued fractions over a general field K and then many identities for continued fractions with some interesting patterns are realized. One of these results is an identity for continued fractions whose their partial quotients have palindromic property. This is of particular interest since this palindromic property leads to a special property which is useful for finding explicit continued fractions. Next, similar to the work of Cohn in 1996, [10], which generalized the Folding lemma, a generalization of the identity for continued fractions whose their partial quotients have palindromic property is investigated using a modification of a technique due to [9].

In the final chapter, Chapter IV, along the line which Tamura [27] did for two classes of real numbers $\tilde{\theta}(T;\tilde{f}^{(1)})$ and $\tilde{\theta}(T;\tilde{f}^{(2)})/T$ defined by

$$\tilde{\theta}(T;f)=\sum_{m=0}^{\infty}\frac{1}{f_0(T)f_1(T)\dots f_m(T)},$$

where $f(T)\in\mathbb{Z}[T]$, $f_0(T)=T$ and for all $i\geq 1$, $f_i(T)=f(f_{i-1}(T))$, and

$$\tilde{f}^{(1)}(T)=T(T+2)(T-2)\tilde{g}^{(1)}(T)+T^2-2,$$
$$\tilde{f}^{(2)}(T)=T^2(T+2)(T-2)\tilde{g}^{(2)}(T)+T^2-2,$$

with suitable $\tilde{g}^{(1)}(T)$, $\tilde{g}^{(2)}(T)\in\mathbb{Z}[T]$ and $T\in\mathbb{N}$, explicit formulae for regular continued fractions for two classes of real numbers $\theta(T;f^{(1)})$ and $\theta(T;f^{(2)})/T$ expressed by the following series

$$\theta(T;f)=\sum_{m=0}^{\infty}\frac{(-1)^m}{f_0(T)f_1(T)\dots f_m(T)},$$

where $f(T)\in\mathbb{Z}[T]$, $f_0(T)=T$ and for all $i\geq 1$, $f_i(T)=f(f_{i-1}(T))$, and

$$f^{(1)}(T) = T(T+2)(T-2)g^{(1)}(T) - T^2 + 2,$$
$$f^{(2)}(T) = T^2(T+2)(T-2)g^{(2)}(T) - T^2 + 2,$$

with suitable $g^{(1)}(T),\ g^{(2)}(T) \in \mathbb{Z}[T]$ and $T \in \mathbb{N}$ are given. An identity for continued fractions with palindromic property is used as a guideline to produce these formulae. We found that partial quotients of these explicit regular continued fractions begin in arbitrarily long palidromes and using a transcendental criterion given by Adamczewski and Bugeaud in [3] it can be concluded that the numbers in these classes are transcendental. Analogues of these explicit formulae are also established for formal series. In the formal series case, our explicit continued fractions also have a beautiful pattern but it is different from the real number case because we cannot assure that these continued fractions are regular. Using the same technique, we give analogues of the works of Tamura as well.

In the rest of this chapter, we collect definitions and elementary properties, mainly without proofs, to be used throughout the entire thesis.

1.2 Preliminaries

Definition 1.2.1. A *valuation* on a field K is a real-valued function $a \mapsto |a|$ defined on K which satisfies the following conditions:

(i) $\forall a \in K, \quad |a| \geq 0$ and $|a| = 0 \Leftrightarrow a = 0$

(ii) $\forall a, b \in K, \quad |ab| = |a|\,|b|$

(iii) $\forall a, b \in K, \quad |a+b| \leq |a| + |b|$.

A valuation on K is called *non-Archimedean* if the condition (iii), called the *triangle inequality*, is replaced by a stronger condition: $\forall a, b \in K, \quad |a+b| \leq \max\{|a|, |b|\}$, called the *strong triangle inequality*. Any other valuation on K is called *Archimedean*.

An important consequence of the strong triangle inequality is if $|\cdot|$ is a non-Archimedean valuation on a field K, then

$$\forall x, y \in K, \quad |x| \neq |y| \text{ implies } |x+y| = \max\{|x|, |y|\}. \tag{1.1}$$

Examples 1) For $K = \mathbb{Q}$, the ordinary absolute value $|\cdot|$ is an Archimedean valuation on K.

2) Consider the field $\mathbb{F}(x)$ of rational functions over a field $\mathbb{F}$. Let $\frac{f(x)}{g(x)} \in \mathbb{F}(x) \setminus \{0\}$. Define the degree valuation $|\cdot|_\infty$ by

$$\left|\frac{f(x)}{g(x)}\right|_\infty = 2^{\deg f - \deg g} \quad \text{and} \quad |0|_\infty = 0.$$

Then $|\cdot|_\infty$ is a non-Archimedean valuation on $\mathbb{F}(x)$.

Let K be an arbitrary field equipped with a valuation $|\cdot|$. We adjoin to K an element, called *infinity*, and denoted by ∞. The set $K \cup \{\infty\}$ will be denoted by $\widehat{K}$ and will be called the *extended field*. Arithmetic operations involving ∞ are defined for all $a, b \in K$ with $a \neq 0$ as follows:

$$a \cdot \infty = \infty, \quad \frac{a}{\infty} = 0, \quad \frac{a}{0} = \infty, \quad b + \infty = \infty, \text{ and } \infty + \infty = \infty.$$

A sequence $\{x_n\}$ in $\widehat{K}$ is said to *converge* to an element $x \in K$ if for all sufficiently large n,

$$x_n \in K \quad \text{and} \quad \lim_{n\to\infty} |x_n - x| = 0.$$

A *continued fraction* over K is defined formally to be an ordered pair

$$\langle\langle\{a_n\}, \{b_n\}\rangle, \{\gamma_n\}\rangle,$$

where $a_1, a_2, \ldots \in K \setminus \{0\}$, $b_0, b_1, \ldots \in K$ and $\{\gamma_n\}$ is a sequence in $\widehat{K}$ given by

$$\gamma_n = S_n(0), \qquad n = 0, 1, 2, 3, \ldots,$$

where $S_n : \widehat{K} \to \widehat{K}$ is defined depending on $s_n : \widehat{K} \to \widehat{K}$ as follows

$$\begin{aligned} &S_0(w) = s_0(w), && S_n(w) = S_{n-1}(s_n(w)), && n = 1, 2, 3, \ldots, \\ &s_0(w) = b_0 + w, && s_n(w) = \tfrac{a_n}{b_n + w}, && n = 1, 2, 3, \ldots. \end{aligned}$$

We call a_n and b_n the n^{th} *partial numerator* and *denominator* of the continued fraction, respectively, and call γ_n the n^{th} *approximant.* If $\{a_n\}$ and $\{b_n\}$ are infinite sequences, then $\langle\langle\{a_n\}, \{b_n\}\rangle, \{\gamma_n\}\rangle$ is called an *infinite* or *non-terminating* continued fraction. It is called a *finite* or *terminating* continued fraction if $\{a_n\}$ and $\{b_n\}$ have only a finite number of terms $a_1, a_2, \ldots, a_k$ and $b_0, b_1, b_2, \ldots, b_k$.

It can be seen that the n^{th} approximant is given by

$$\gamma_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\ddots + \cfrac{a_n}{b_n}}}.$$

It is more convenient to use the notation

$$[b_0; a_1, b_1; a_2, b_2; \ldots; a_n, b_n; \ldots]$$

denote a continued fraction $\langle\langle\{a_n\}, \{b_n\}\rangle, \{\gamma_n\}\rangle$ and if $a_n = 1$ for all $n \geq 1$, we denote

$$[b_0; b_1, b_2, \ldots, b_n, \ldots] := [b_0; 1, b_1; 1, b_2; \ldots; 1, b_n; \ldots],$$

in this case $b_0, b_1, b_2, \ldots$ are called *partial quotients* of $[b_0; b_1, b_2, \ldots, b_n, \ldots]$.

Corresponding to each continued fraction $[b_0; a_1, b_1; a_2, b_2; \ldots]$, two sequences $\{p_n\}$

and $\{q_n\}$ are defined by the system of second order linear difference equations

$$p_{-1} = 1, \qquad p_0 = b_0, \qquad q_{-1} = 0, \qquad q_0 = 1;$$

$$p_n = b_n p_{n-1} + a_n p_{n-2} \quad \text{and} \quad q_n = b_n q_{n-1} + a_n q_{n-2} \qquad (n \geq 1), \tag{1.2}$$

these $p_n,\ q_n$ are called the n^{th} *numerator* and *denominator*, respectively, and the fraction $\frac{p_n}{q_n}$ is called the n^{th} *convergent*.

Some important properties of these numerators and denominators of continued fractions are presented in the following lemma whose proof is straightforward by induction.

Lemma 1.2.2. *For an arbitrary field* K, *let* $[b_0; a_1, b_1; a_2, b_2; \ldots]$ *be a continued fraction over* K *and* $\alpha \in K$. *Then for* $n \geq 0$,

$$S_n(w) = \frac{p_n + p_{n-1}w}{q_n + q_{n-1}w} \qquad (n \geq 1,\ w \in \hat{K}), \tag{1.3}$$

$$\frac{p_n}{q_n} = [b_0; a_1, b_1; a_2, b_2; \ldots; a_n, b_n], \tag{1.4}$$

$$\frac{q_{n-1}}{q_n} = [0; 1, b_n; a_n, b_{n-1}; \ldots; a_2, b_1] \qquad (n \geq 1), \tag{1.5}$$

$$\frac{\alpha p_n + p_{n-1}}{\alpha q_n + q_{n-1}} = [b_0; a_1, b_1; a_2, b_2; \ldots; a_n, b_n; 1, \alpha], \tag{1.6}$$

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1} \prod_{k=1}^{n} a_k, \quad \textit{where} \quad \prod_{k=1}^{0} a_k = 1. \tag{1.7}$$

The following theorem is a classical result about convergence of continued fractions of Pringsheim in 1899, for detail see [14]:

Theorem 1.2.3. *Let* K *be arbitrary field together with a prescribed valuation. The continued fraction* $[b_0; a_1, b_1; a_2, b_2; \ldots; a_n, b_n; \ldots]$ *converges to an element in* K *if*

$$|b_n| \geq |a_n| + 1, \qquad \textit{for all } n \geq 1.$$

Definition 1.2.4. An infinite continued fraction $[b_0; a_1, b_1; a_2, b_2; \ldots]$ is said to be *periodic* if there exist positive integers k, N such that

$$a_n = a_{n+k} \quad \text{and} \quad b_n = b_{n+k} \quad \text{for all } n \geq N,$$

and is denoted by

$$[b_0; a_1, b_1; \ldots; a_{N-1}, b_{N-1}; \overline{a_N, b_N; \ldots; a_{N+k-1}, b_{N+k-1}}\ ].$$

Definition 1.2.5. A continued fraction $[b_0; b_1, b_2, \ldots, b_n]$ $(n \geq 1)$ is said to be *palindromic* if the word $b_1 b_2 \ldots b_n$ is equal to its reversal.

Remark 1.2.6. If a continued fraction $[b_0; b_1, b_2, \ldots, b_n]$ $(n \geq 1)$ is palindromic, then by (1.4) and (1.5) we have the following specific property

$$p_n = b_0 q_n + q_{n-1}.$$

Definition 1.2.7. For any terminating continued fraction $[b_0; b_1, \ldots, b_n]$ $(n \geq 0)$, n is said to be the *length* of $[b_0; b_1, \ldots, b_n]$.

In $\mathbb{R}$, it is known that any real number can be represented as a continued fraction of the form

$$[b_0; b_1, b_2, \ldots, b_n, \ldots],$$

where $b_0 \in \mathbb{Z}$, $b_i \in \mathbb{N}$ $(i \geq 1)$. This continued fraction is called a *regular* or *simple* continued fraction.

The construction of a regular continued fraction for $\xi \in \mathbb{R} \setminus \{0\}$ runs as follows: Define $\xi = [\xi] + (\xi)$, where $[\xi]$ denote the greatest integer less than or equals ξ and $(\xi) := \xi - [\xi]$. We call $[\xi]$ and (ξ) the *head* and *tail* parts of ξ, respectively. Clearly, the head and tail parts of ξ are uniquely determined. Let $b_0 = [\xi] \in \mathbb{Z}$.

If $(\xi) = 0$, then the process stops.

If $(\xi) \neq 0$, then write $\xi = b_0 + \xi_1^{-1}$, where $\xi_1^{-1} = (\xi)$ with $\xi_1 > 1$. Next we write $\xi_1 = [\xi_1] + (\xi_1)$. Let $b_1 = [\xi_1] \in \mathbb{N}$.

If $(\xi_1) = 0$, then the process stops.

If $(\xi_1) \neq 0$, then write $\xi_1 = b_1 + \xi_2^{-1}$, where $\xi_2^{-1} = (\xi_1)$ with $\xi_2 > 1$. Next we write $\xi_2 = [\xi_2] + (\xi_2)$. Let $b_2 = [\xi_2] \in \mathbb{N}$.

Again, if $(\xi_2) = 0$, then the process stops. If $(\xi_2) \neq 0$, then we continue in the same manner. By doing so, we obtain the unique representation

$$\xi = [b_0; b_1, b_2, \ldots, b_{n-1}, \xi_n],$$

where $b_0 \in \mathbb{Z}$, $b_i \in \mathbb{N}$ $(i \geq 1)$ and ξ_n is referred to as the n^{th} *complete quotient* of ξ.

If $(\xi_n) = 0$ for some n, then $\xi = [b_0; b_1, b_2, \ldots, b_n]$, i.e., its regular continued fraction to ξ is *terminating* or *finite*. Otherwise, $(\xi_n) \neq 0$ for all n and the regular continued fraction is infinite and this is the case of interest from now on. In order to establish convergence, we make use of the following properties which are easily verified by using Lemma 1.2.2 and (1.2).

Lemma 1.2.8. *For $n \geq 1$, let $\frac{p_n}{q_n}$ be the n^{th} convergent corresponding to the above $b_0, b_1, \ldots, b_n$. Then*

(i) p_n *and* q_n *are relatively prime,*

(ii) $q_n > q_{n-1} > 0$,

(iii) $\xi - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(\xi_{n+1} q_n + q_{n-1})}$.

Using Lemma 1.2.8 *(iii)*, we get the approximation

$$\left| \xi - \frac{p_n}{q_n} \right| = \frac{1}{q_n(\xi_{n+1} q_n + q_{n-1})} \quad \to 0 \quad \text{as } n \to \infty,$$

since the integer q_n are increasing with n, by Lemma 1.2.8 (ii), and ξ_{n+1} is positive. This immediately implies that $\frac{p_n}{q_n} \to \xi$, and enable us to write $\xi = [b_0; b_1, b_2, \ldots]$.

The regular continued fraction is unique for an irrational number, but for rational numbers, we have the following characterization; for details, see e.g., [21, Chapter 7].

Theorem 1.2.9. *Any finite regular continued fraction represents a rational number. Conversely, any rational number can be expressed as a finite regular continued fraction. and in exactly two ways,*

$$[b_0; b_1, b_2, \ldots, b_n] = [b_0; b_1, b_2, \ldots, b_{n-1}, b_n - 1, 1],$$

where $b_n \geq 2$.

A well-known theorem of Lagrange characterizing periodic regular continued fractions, whose proof can be found in [21, Chapter 7], states that:

Theorem 1.2.10. *A periodic regular continued fraction is a quadratic irrational number, and conversely.*

Next, continued fractions in the field $\mathbb{F}((x^{-1}))$ of formal series over a field $\mathbb{F}$ are mentioned. It is well-known, see e.g., [7, Chapter 1], that every element $\xi \in \mathbb{F}((x^{-1})) \setminus \{0\}$ can be uniquely written as

$$\xi := \sum_{i=r}^{\infty} w_{-i} x^{-i},$$

where $r \in \mathbb{Z}$, and the coefficients $w_{-i} \in \mathbb{F}$ $(i \geq r)$ with $w_{-r} \neq 0$. The degree valuation $|\xi|_\infty$ in $\mathbb{F}((x^{-1}))$ is defined by putting

$$|0|_\infty = 0, \quad |\xi|_\infty = 2^{-r} \quad \text{if} \;\; \xi = \sum_{i=r}^{\infty} w_{-i} x^{-i} \;\; \text{with } w_{-r} \neq 0.$$

Definition 1.2.11. Let $\xi = \sum_{i=r}^{\infty} w_{-i}x^{-i} \in \mathbb{F}((x^{-1}))$. The *head* part, $[\xi]$, and the *tail* part, (ξ), of ξ are defined by

$$[\xi] := \begin{cases} \sum_{i=r}^{0} w_{-i}x^{-i} & \text{if } r \leq 0, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad (\xi) := \xi - [\xi]\,.$$

In $\mathbb{F}((x^{-1}))$, there is a continued fraction algorithm similar to the case of real numbers. Each element can be uniquely represented as the regular continued fraction of the form

$$[b_0; b_1, b_2, \ldots],$$

where $b_0 \in \mathbb{F}[x]$ and $b_i \in \mathbb{F}[x] \setminus \mathbb{F}$ $(i \geq 1)$.

The construction of the regular continued fraction for $\xi \in \mathbb{F}((x^{-1})) \setminus \{0\}$ runs as follows:

Consider $\xi = [\xi] + (\xi)$. Let $b_0 = [\xi] \in \mathbb{F}[x]$.

If $(\xi) = 0$, then the process stops.

If $(\xi) \neq 0$, then write $\xi = b_0 + \xi_1^{-1}$, where $\xi_1^{-1} = (\xi)$ with $|\xi_1|_\infty > 1$. Next we write $\xi_1 = [\xi_1] + (\xi_1)$. Let $b_1 = [\xi_1] \in \mathbb{F}[x] \setminus \mathbb{F}$.

If $(\xi_1) = 0$, then the process stops.

If $(\xi_1) \neq 0$, then write $\xi_1 = b_1 + \xi_2^{-1}$, where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2|_\infty > 1$. Next we write $\xi_2 = [\xi_2] + (\xi_2)$. Let $b_2 = [\xi_2] \in \mathbb{F}[x] \setminus \mathbb{F}$.

Again, if $(\xi_2) = 0$, then the process stops. If $(\xi_2) \neq 0$, then we continue in the same manner. By doing so, we obtain the unique representation

$$\xi = [b_0; b_1, b_2, \ldots, b_{n-1}, \xi_n],$$

where $b_0 \in \mathbb{F}[x]$ and $b_i \in \mathbb{F}[x] \setminus \mathbb{F}$ $(i \geq 1)$ and ξ_n is referred to as the n^{th} *complete quotient* of ξ.

If $(\xi_n) = 0$ for some n, then $\xi = [b_0; b_1, b_2, \ldots, b_n]$, i.e., its regular continued fraction to ξ is *terminating* or *finite*. Otherwise, $(\xi_n) \neq 0$ for all n and the regular continued fraction is infinite and this is the case of interest from now on. The following lemma collects basic properties of regular continued fractions whose proof is easily verified by using Lemma 1.2.2 and (1.2).

Lemma 1.2.12. *For $n \geq 1$, let $\frac{p_n}{q_n}$ be the n^{th} convergent corresponding to the above $b_0, b_1, \ldots, b_n$. Then*

(i) p_n *and* q_n *are relatively prime;*

(ii) $|q_{n-1}|_\infty > |q_{n-2}|_\infty$, $|\xi_n|_\infty = |b_n|_\infty$;

(iii) $|q_n|_\infty = |b_1 b_2 \ldots b_n|_\infty$;

(iv) $\xi - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(\xi_{n+1}q_n + q_{n-1})}$;

(v) p_n *is the head part of* $q_n\xi$.

Since $|\xi_{n+1}|_\infty = |b_{n+1}|_\infty \geq 2$, Lemma 1.2.12 *(ii)* and *(iii)* give

$$|q_n(\xi_{n+1}q_n + q_{n-1})|_\infty = |q_n|_\infty^2 \, |b_{n+1}|_\infty \geq 2^{2n+1}.$$

Using Lemma 1.2.12 *(iv)*, we get the approximation

$$\left|\xi - \frac{p_n}{q_n}\right|_\infty \leq 2^{-(2n+1)} \quad \to 0 \quad \text{as } n \to \infty,$$

which immediately implies that $\frac{p_n}{q_n} \to \xi$, and enable us to write $\xi = [b_0; b_1, b_2, \ldots]$.

As in the classical case, the following characterization of rational elements in $\mathbb{F}((x^{-1}))$ via their regular continued fractions is well-known, see e.g., [24].

Theorem 1.2.13. *Let* $\xi \in \mathbb{F}((x^{-1}))$. *Then* ξ *is rational if and only if its regular continued fraction is finite.*

Specific properties of periodic regular continued fractions for formal series are stated as in Theorem 1.2.14 and 1.2.15, whose proofs can be found in [8].

Theorem 1.2.14. *Let* $\xi \in \mathbb{F}((x^{-1}))$. *If the regular continued fraction of* ξ *is periodic, then* ξ *is an irrational root of a quadratic equation of the form* $at^2 + bt + c = 0$ *where* $a, b, c \in \mathbb{F}[x],\ a \neq 0$.

Theorem 1.2.15. *Let* $\xi \in \mathbb{F}((x^{-1}))$. *If* ξ *is an irrational root of a quadratic equation of the form* $at^2 + bt + c = 0$ *where* $a, b, c \in \mathbb{F}[x],\ a \neq 0$, *then the regular continued fraction of* ξ *is periodic.*

CHAPTER II

CONTINUED FRACTIONS WITH BOUNDED PARTIAL QUOTIENTS

In this chapter, the boundedness of partial quotients of regular continued fractions representing linear fractional transformations of elements in the field $\mathbb{F}((x^{-1}))$ of formal series over a based field $\mathbb{F}$ is considered. In the last section, we verify a famous conjecture involving a bound of partial quotients of regular continued fractions for some rational numbers.

2.1 Linear fractional transformations of bounded continued fractions

Definition 2.1.1. Let θ be an irrational element in $\mathbb{F}((x^{-1}))$ whose infinite regular continued fraction expansion is $[b_0; b_1, b_2, \ldots]$. Define

$$K(\theta) := \sup_{i \geq 1} |b_i|_\infty \quad \text{and} \quad K_\infty(\theta) := \limsup_{i \geq 1} |b_i|_\infty.$$

We say that θ *has bounded partial quotients* if $K(\theta)$ is finite.

Clearly, $K_\infty(\theta) \leq K(\theta)$ and $K(\theta)$ is finite if and only if $K_\infty(\theta)$ is finite.

The main result reads:

Theorem 2.1.2. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *where* $a, b, c, d \in \mathbb{F}[x]$ *be such that* $\det M \neq 0$. *If the regular continued fraction of an irrational element* $\theta \in \mathbb{F}((x^{-1}))$ *has bounded* *partial quotients, then*

$$\frac{1}{|\det\ M|_\infty} K_\infty(\theta) \le K_\infty\left(\frac{a\theta+b}{c\theta+d}\right) \le |\det M|_\infty\, K_\infty(\theta), \tag{2.1}$$

and

$$K\left(\frac{a\theta+b}{c\theta+d}\right) \le \max\left\{|\det M|_\infty\, K(\theta),\ |c(c\theta+d)|_\infty\right\}. \tag{2.2}$$

Theorem 2.1.2 is proved by making use of the following results.

The next lemma is known as the best approximation property, cf. Theorem 7.13 in [21] for the real case.

Lemma 2.1.3. *Let θ be an irrational element in $\mathbb{F}((x^{-1}))$ whose regular continued fraction expansion is $[b_0; b_1, b_2, \ldots]$. If $u, v \in \mathbb{F}[x]$ with $v \neq 0$ satisfy, for some $n \ge 0$,*

$$|v\theta - u|_\infty < |q_n\theta - p_n|_\infty, \tag{2.3}$$

then $|v|_\infty \ge |q_{n+1}|_\infty$.

Proof. Suppose that

$$|v|_\infty < |q_{n+1}|_\infty. \tag{2.4}$$

Consider the system of linear equations (in y, z)

$$yq_n + zq_{n+1} = v \tag{2.5}$$

$$yp_n + zp_{n+1} = u. \tag{2.6}$$

By (1.7), $\det\begin{pmatrix} q_n & q_{n+1} \\ p_n & p_{n+1} \end{pmatrix} = (-1)^n$, and so

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} (-1)^n p_{n+1} & (-1)^{n+1} q_{n+1} \\ (-1)^{n+1} p_n & (-1)^n q_n \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix},$$

implying that y and z are in $\mathbb{F}[x]$.

We claim that neither y nor z is zero. If $y = 0$, then $0 \neq v = zq_{n+1}$, and so $|v|_\infty \geq |q_{n+1}|_\infty$, which contradicts (2.4). Then $y \neq 0$. If $z = 0$, then $u = yp_n$ and $v = yq_n$. Since $|y|_\infty \geq 1$, we have $|v\theta - u|_\infty = |y(q_n\theta - p_n)|_\infty \geq |q_n\theta - p_n|_\infty$, contradicting (2.3).

Next, we show that

$$|y(q_n\theta - p_n)|_\infty \neq |z(q_{n+1}\theta - p_{n+1})|_\infty. \tag{2.7}$$

Suppose $|y(q_n\theta - p_n)|_\infty = |z(q_{n+1}\theta - p_{n+1})|_\infty$. By Lemma 1.2.12 (*ii*) and (*iv*), we have

$$|q_i\theta - p_i|_\infty = \frac{1}{|\theta_{i+1}q_i + q_{i-1}|_\infty} = \frac{1}{|q_{i+1}|_\infty} \quad (i \geq 0),$$

and so $|yq_{n+2}|_\infty = |zq_{n+1}|_\infty$. Since $|yq_n|_\infty < |yq_{n+2}|_\infty$, (1.1) and (2.5) yield $|zq_{n+1}|_\infty = |v|_\infty$ implying that $|q_{n+1}|_\infty \leq |v|_\infty$. This contradicts (2.4). Thus, (2.7) holds.

Finally, consider $|v\theta - u|_\infty = |y(q_n\theta - p_n) + z(q_{n+1}\theta - p_{n+1})|_\infty$. Using (2.7), (1.1) and $y \in \mathbb{F}[x] \setminus \{0\}$, we have

$$\begin{aligned}|v\theta - u|_\infty &= \max\{|y(q_n\theta - p_n)|_\infty,\ |z(q_{n+1}\theta - p_{n+1})|_\infty\} \\ &\geq |y(q_n\theta - p_n)|_\infty \geq |q_n\theta - p_n|_\infty,\end{aligned}$$

which contradicts (2.3), and the lemma follows. □

Remark 2.1.4. The best approximation property presented in the above lemma also holds the convergents of finite regular continued fractions. Namely, for a rational element $\theta \in \mathbb{F}((x^{-1}))$ whose finite regular continued fraction is $[b_0; b_1, \ldots, b_n]$, if $u, v \in \mathbb{F}[x]$ with $v \neq 0$ satisfy, for some $0 \leq i < n$, $|v\theta - u|_\infty < |q_i\theta - p_i|_\infty$, then $|v|_\infty \geq |q_{i+1}|_\infty$.

Definition 2.1.5. For $\xi \in \mathbb{F}((x^{-1}))$, the *distance to the head part* $\|\xi\|$ of ξ is defined as $\|\xi\| = |\xi - [\xi]|_\infty$.

Hence for an irrational element $\theta \in \mathbb{F}((x^{-1}))$ whose regular continued fraction expansion is $\theta = [b_0; b_1, b_2, \ldots]$, by Lemma 1.2.12 (v), we have $\|q_n\theta\| = |q_n\theta - p_n|$, and so Lemma 1.2.12 (ii) and (iv) together yield

$$|q_n|_\infty \ \|q_n\theta\| = \frac{1}{|\theta_{n+1} + q_{n-1}/q_n|_\infty} = \frac{1}{|b_{n+1}|_\infty}, \tag{2.8}$$

where $\theta_{n+1} = [b_{n+1}; b_{n+2}, \ldots]$ is the $(n+1)^{th}$ complete quotient of $[b_0; b_1, b_2, \ldots]$.

Definition 2.1.6. For an irrational element $\theta \in \mathbb{F}((x^{-1}))$, define its *type* and its *Lagrange constant*, respectively, by

$$L(\theta) = \sup_{|q|_\infty \geq 1} \left(|q|_\infty \ \|q\theta\|\right)^{-1} \quad \text{and} \quad L_\infty(\theta) = \limsup_{|q|_\infty \geq 1} \left(|q|_\infty \|q\theta\|\right)^{-1}.$$

To determine the type and the Lagrange constant, it suffices to use the partial denominators as we show now.

Lemma 2.1.7. *Let θ be an irrational element in $\mathbb{F}((x^{-1}))$ whose regular continued fraction is $[b_0; b_1, b_2, \ldots]$. Then*

$$L(\theta) = \sup_{i \geq 0} \left(|q_i|_\infty \|q_i\theta\|\right)^{-1} \ \textit{and} \ L_\infty(\theta) = \limsup_{i \geq 0} \left(|q_i|_\infty \|q_i\theta\|\right)^{-1}. \tag{2.9}$$

Proof. Let $q \in \mathbb{F}[x] \setminus \{0\}$. Since the regular continued fraction of any irrational is infinite, there exists $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $|q_m|_\infty \leq |q|_\infty < |q_{m+1}|_\infty$. By Lemma 2.1.3,

$$\frac{1}{|q|_\infty \ \|q\theta\|} \leq \frac{1}{|q|_\infty \ \|q_m\theta\|} \leq \frac{1}{|q_m|_\infty \ \|q_m\theta\|},$$

and the result follows. $\square$

Corollary 2.1.8. *A) For an irrational element $\theta \in \mathbb{F}((x^{-1}))$, we have*

$$K(\theta) = L(\theta) \quad \text{and} \quad K_\infty(\theta) = L_\infty(\theta). \tag{2.10}$$

B) Let $\phi = [d_0; d_1, d_2, \ldots]$, $\gamma = [e_0; e_1, e_2, \ldots]$ be two irrational elements in $\mathbb{F}((x^{-1}))$. If there exist $s_1, s_2 \in \mathbb{N}_0$ such that $|d_{s_1+i}|_\infty = |e_{s_2+i}|_\infty$ $(i \geq 0)$, then

$$K_\infty(\phi) = K_\infty(\gamma) \quad \text{and} \quad L_\infty(\phi) = L_\infty(\gamma).$$

Proof. Part A) follows immediately from the definitions of $K(\theta)$ and $K_\infty(\theta)$, (2.8) and Lemma 2.1.7. Part B) follows from at once the definition of K_∞, Lemma 2.1.7 and (2.10). □

The next lemma is proved by modifying the proofs of Theorems 172 and 175 of [13] in the real to the formal series case.

Lemma 2.1.9. *Let $[b_0; b_1, b_2, \ldots]$ be the regular continued fraction for an irrational element $\theta \in \mathbb{F}((x^{-1}))$ with $|\theta|_\infty > 1$, and let $\psi = \frac{a\theta+b}{c\theta+d}$, where $a, b, c, d \in \mathbb{F}[x]$ be such that $|ad - bc|_\infty = 1$.*

A) If $|c|_\infty > |d|_\infty > 0$, then b/d and a/c equal two consecutive convergents of the regular continued fraction for ψ and if b/d and a/c equal the $(n-1)^{th}$ and n^{th} convergents of the regular continued fraction for ψ, respectively, we have that the $(n+1)^{th}$ complete quotient is of the form $\delta\theta$ for some $\delta \in \mathbb{F} \setminus \{0\}$.

B) If the regular continued fraction for ψ is $[c_0; c_1, c_2, \ldots]$, then there exist $k, n \in \mathbb{N}_0$ such that

$$|b_{k+i}|_\infty = |c_{n+i}|_\infty \quad \text{for all } i \geq 0.$$

Proof. Denote the regular continued fraction expansion of a/c by $[c_0; c_1, \ldots, c_n]$ and let p_n/q_n be its n^{th} (last) convergent. Since $|ad-bc|_\infty = 1$, we have, by Lemma 1.2.12 (i), $\gcd(a,c) = 1 = \gcd(p_n, q_n)$ and hence $a = \gamma p_n,\ c = \gamma q_n$ for some $\gamma \in \mathbb{F} \setminus \{0\}$. Thus,

$$|p_n d - q_n b|_\infty = |ad - bc|_\infty = 1 = |p_n q_{n-1} - p_{n-1} q_n|_\infty,$$

yielding $p_n d - q_n b = \delta'(p_n q_{n-1} - p_{n-1} q_n)$ for some $\delta' \in \mathbb{F} \setminus \{0\}$, and so

$$p_n(d - \delta' q_{n-1}) = q_n(b - \delta' p_{n-1}). \tag{2.11}$$

Since $\gcd(p_n, q_n) = 1$, the relation (2.11) gives

$$q_n | (d - \delta' q_{n-1}). \tag{2.12}$$

From $|q_n|_\infty = |c|_\infty > |d|_\infty > 0$ and $|q_n|_\infty > |q_{n-1}|_\infty \geq 0$, we get $|d - \delta' q_{n-1}|_\infty < |q_n|_\infty$, which is consistent with (2.12) only when $d - \delta' q_{n-1} = 0$, i.e., when $d = \delta' q_{n-1},\ b = \delta' p_{n-1}$. Consequently, $\psi = \frac{p_n \delta\theta + p_{n-1}}{q_n \delta\theta + q_{n-1}}$ for some $\delta \in \mathbb{F} \setminus \{0\}$, and so by (1.6),

$$\psi = [c_0; c_1, \ldots, c_n, \delta\theta].$$

If we develop $\delta\theta$ as a continued fraction, we obtain $\delta\theta = [c_{n+1}; c_{n+2}, \ldots]$ with $|c_{n+1}|_\infty > 1$. Hence $\psi = [c_0; c_1, \ldots, c_n, c_{n+1}, c_{n+2}, \ldots]$.

To prove part B), from (1.6), we have

$$\theta = [b_0; b_1, \ldots, b_{k-1}, \theta_k] = \frac{p_{k-1}\theta_k + p_{k-2}}{q_{k-1}\theta_k + q_{k-2}},$$

which implies

$$\psi = \frac{P\theta_k + R}{Q\theta_k + S},$$

where

$$P = aA_{k-1} + bq_{k-1},\ R = ap_{k-2} + bq_{k-2},\ Q = cA_{k-1} + dq_{k-1} \ \text{ and } \ S = cp_{k-2} + dq_{k-2}$$

are in $\mathbb{F}[x]$ with $|PS - QR|_\infty = |(ad - bc)(p_{k-1}q_{k-2} - p_{k-2}q_{k-1})|_\infty = 1$. From Lemma 1.2.12 (iv), we have $|\theta - \frac{p_i}{q_i}|_\infty = \frac{1}{|q_i(\theta_{i+1}q_i + q_{i-1})|_\infty} < \frac{1}{|q_i^2|_\infty}$ for all $i \geq 0$, and so

$$p_{k-1} = \theta q_{k-1} + \frac{\beta_1}{q_{k-1}}, \quad p_{k-2} = \theta q_{k-2} + \frac{\beta_2}{q_{k-2}},$$

where $|\beta_1|_\infty < 1,\ |\beta_2|_\infty < 1$. Thus,

$$Q = (c\theta + d)q_{k-1} + \frac{c\beta_1}{q_{k-1}}, \quad S = (c\theta + d)q_{k-2} + \frac{c\beta_2}{q_{k-2}}.$$

Since $c\theta + d \neq 0,\ |q_{k-1}|_\infty > |q_{k-2}|_\infty \to \infty$ $(k \to \infty)$, we have $|Q|_\infty > |S|_\infty > 0$ for all large k. For such k, part A) ensures that there exists $\delta \in \mathbb{F} \setminus \{0\}$ such that $\delta\theta_k = \psi_n$ for some n, i.e., $|b_{k+i}|_\infty = |c_{n+i}|_\infty$ for all $i \geq 0$. □

Lemma 2.1.9 and Corollary 2.1.8 immediately yield:

Lemma 2.1.10. *Let θ be an irrational element in $\mathbb{F}((x^{-1}))$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{F}[x]$ and denote $M(\theta) := \frac{a\theta+b}{c\theta+d}$. If $|\det M|_\infty = 1$, then*

$$L_\infty(M(\theta)) = L_\infty(\theta).$$

For a transformation with non-unit determinant, we have weaker results.

Lemma 2.1.11. *Let θ be an irrational element in $\mathbb{F}((x^{-1}))$; $h,\ d_1,\ d_3 \in \mathbb{F}[x] \setminus \{0\}$ and $d_2 \in \mathbb{F}[x]$. Then*

$$L_\infty(h\theta) \leq |h|_\infty L_\infty(\theta) \tag{2.13}$$

and

$$L_\infty\left(\frac{d_1\theta + d_2}{d_3}\right) \leq |d_1 d_3|_\infty\, L_\infty(\theta). \tag{2.14}$$

Proof. If θ has unbounded partial quotients, i.e., $L_\infty(\theta) = \infty$, both inequalities are trivial. Now assume θ has bounded partial quotients. For $h \in \mathbb{F}[x] \setminus \{0\},\ k \in \mathbb{N}_0$, clearly,

$$\sup_{\deg q \geq k} \left(|qh|_\infty \|qh\theta\|\right)^{-1} \leq \sup_{\deg q \geq k} \left(|q|_\infty \|q\theta\|\right)^{-1}$$

and

$$\limsup_{|q|_\infty \geq 1} \left(|qh|_\infty \|qh\theta\|\right)^{-1} \leq \limsup_{|q|_\infty \geq 1} \left(|q|_\infty \|q\theta\|\right)^{-1}.$$

Consequently,

$$\begin{aligned} L_\infty(h\theta) &= \limsup_{|q|_\infty \geq 1} \left(|q|_\infty \|qh\theta\|\right)^{-1} = |h|_\infty \limsup_{|q|_\infty \geq 1} \left(|qh|_\infty \|qh\theta\|\right)^{-1} \\ &\leq |h|_\infty \limsup_{|q|_\infty \geq 1} \left(|q|_\infty \|q\theta\|\right)^{-1} = |h|_\infty L_\infty(\theta), \end{aligned}$$

which proves (2.13).

To verify (2.14), from Corollary 2.1.8 *B*) and (2.13), we have

$$\begin{aligned} L_\infty\left(\frac{d_1\theta + d_2}{d_3}\right) &= L_\infty\left(\frac{d_3}{d_1\theta + d_2}\right) \\ &\leq |d_3|_\infty L_\infty\left(\frac{1}{d_1\theta + d_2}\right) \\ &= |d_3|_\infty L_\infty(d_1\theta + d_2) \leq |d_1|_\infty |d_3|_\infty L_\infty(\theta). \end{aligned}$$

□

Now we are ready to prove our main theorem.

Proof of Theorem 2.1.2. By Corollary 2.1.8, it suffices to prove the two results for $L_\infty,\ L$ in place of $K_\infty,\ K$, respectively. Let $\psi := \frac{a\theta+b}{c\theta+d} = M(\theta)$.

We start by showing that there exists $M_2 \in GL_2(\mathbb{F}[x])$ such that

$$|\det M_2|_\infty = 1,\ M_2M = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \in GL_2(\mathbb{F}[x]),\ |\alpha\gamma|_\infty = |\det M|_\infty.$$

Write $M_2 = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$. To fulfil the matrix equality, it is required that $Ga + Hc = 0$.

If $a = 0$, then $c \neq 0$ and so we must take $H = 0$. Choose $F \in \mathbb{F} \setminus \{0\},\ G = 1/F$ and arbitrary $E \in \mathbb{F}[x]$ to fulfil all requirements.

If $c = 0$, then $a \neq 0$ and we must take $G = 0$. Choose $E \in \mathbb{F} \setminus \{0\},\ H = 1/E$ and arbitrary $F \in \mathbb{F}[x]$ to fulfil all requirements.

If both $a \neq 0$ and $c \neq 0$, then take $G = \frac{\text{l.c.m.}(a,c)}{a}$ and $H = -\frac{\text{l.c.m.}(a,c)}{c}$. Since $\gcd(G, H) = 1$, there are $\mu, \nu \in \mathbb{F}[x]$ such that $\mu G + \nu H = 1$. Setting $E = \nu$ and $F = -\mu$, all the requirements are fulfilled.

After we obtain such M_2, we apply Lemma 2.1.10 to get

$$L_\infty(\psi) = L_\infty(M_2(\psi)) = L_\infty(M_2M(\theta)) = L_\infty\left(\frac{\alpha\theta + \beta}{\gamma}\right),$$

and the second inequality of (2.1) now follows from the inequality (2.14) of Lemma 2.1.11.

To prove the first inequality of (2.1), we consider the adjoint matrix

$$M' := \text{adj }(M) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

which has $M'M = (\det\ M)I_2$, and so

$$M'(\psi) = M'(M(\theta)) = M'M(\theta) = \theta.$$

Applying the second inequality of (2.1) to ψ, we have

$$L_\infty(\theta) = L_\infty(M'(\psi)) \le |\det\ M'|_\infty L_\infty(\psi) = |\det\ M|_\infty L_\infty(\psi),$$

and the first inequality of (2.1) follows.

We turn now to the second assertion of the theorem. For each $q \in \mathbb{F}[x] \setminus \{0\}$, let

$$x_q = |q|_\infty \, \|q\psi\| = |q|_\infty \left| q\left(\frac{a\theta+b}{c\theta+d}\right) - p \right|_\infty \qquad \left(p = \left[q\left(\frac{a\theta+b}{c\theta+d}\right)\right]\right).$$

If $c = 0$, then $|\det M|_\infty = |ad|_\infty \ne 0$ and so

$$|ad|_\infty \, x_q = |aq|_\infty \, |aq\theta - (dp - bq)|_\infty \ge |aq|_\infty \, \|aq\theta\| \ge 1/L(\theta),$$

yielding

$$L(\psi) = \sup_{|q|_\infty \ge 1} (|q|_\infty \, \|q\psi\|)^{-1} \le |ad|_\infty L(\theta),$$

which is the first term in the right hand expression of (2.2).

If $c \ne 0$, then

$$|c\theta + d|_\infty \, x_q = |q|_\infty \, |(qa - pc)\theta - (pd - qb)|_\infty. \tag{2.15}$$

Since θ has bounded partial quotients, both $K(\theta)$ and $K_\infty(\theta)$ are finite. The result of the first part shows then that $K_\infty(\psi)$ is finite and so is $K(\psi)$. Corollary 2.1.8 in turn shows that $L(\psi)$ is finite. Thus, there is an infinite sequence of non-zero approximations

$$x_{q^{(i)}} = |q^{(i)}|_\infty \left\|q^{(i)}\psi\right\|$$

such that

$$L(\psi) - \frac{1}{2^i} \le \frac{1}{x_{q^{(i)}}} \le L(\psi) \qquad \text{for all } i \ge 0. \tag{2.16}$$

By taking a suitable subsequence, we may reduce to the case where either all of the approximations have $q^{(i)}a - p^{(i)}c = 0$ or all of them have $q^{(i)}a - p^{(i)}c \neq 0$.

We first treat the subcase $q^{(i)}a - p^{(i)}c = 0$ for all $i \geq 0$. Since $ad - bc = \det M \neq 0$, we have $p^{(i)}d - q^{(i)}b \in \mathbb{F}[x] \setminus \{0\}$ and so (2.15) gives

$$|c\theta + d|_\infty \, x_{q^{(i)}} = |q^{(i)}|_\infty \, |p^{(i)}d - q^{(i)}b|_\infty \geq 1.$$

Consequently,

$$L(\psi) - \tfrac{1}{2^i} \leq \tfrac{1}{x_{q^{(i)}}} \leq |c\theta + d|_\infty \leq |c(c\theta + d)|_\infty \quad \text{for all } i \geq 0.$$

Letting $i \to \infty$, we get the second term in the right hand expression of (2.2).

Finally, we consider the subcase that $q^{(i)}a - p^{(i)}c \neq 0$ for all $i \geq 0$. From (2.15), we have

$$\begin{aligned} |c\theta + d|_\infty \left| \frac{q^{(i)}a - p^{(i)}c}{q^{(i)}} \right|_\infty x_{q^{(i)}} &= |q^{(i)}a - p^{(i)}c|_\infty |(q^{(i)}a - p^{(i)}c)\theta - (p^{(i)}d - q^{(i)}b)|_\infty \\ &\geq |q^{(i)}a - p^{(i)}c|_\infty \left\|(q^{(i)}a - p^{(i)}c)\theta\right\| \geq \frac{1}{L(\theta)}. \end{aligned} \tag{2.17}$$

Using the first inequality in (2.16) and the inequality (2.17), we get

$$\begin{aligned} L(\psi) - \frac{1}{2^i} \leq \frac{1}{x_{q^{(i)}}} &\leq |c\theta + d|_\infty \left| \frac{q^{(i)}a - p^{(i)}c}{q^{(i)}} \right|_\infty L(\theta) \\ &= |c\theta + d|_\infty \frac{|c|_\infty}{|q^{(i)}|_\infty} \left| \frac{q^{(i)}a - p^{(i)}c}{c} \right|_\infty L(\theta). \end{aligned} \tag{2.18}$$

Using the strong triangle inequality, we have

$$\begin{aligned} \left| q^{(i)} \left(\frac{a}{c}\right) - p^{(i)} \right|_\infty &\leq \max \left\{ \left| q^{(i)} \left(\frac{a\theta + b}{c\theta + d}\right) - q^{(i)} \left(\frac{a}{c}\right) \right|_\infty, \ \left| q^{(i)} \left(\frac{a\theta + b}{c\theta + d}\right) - p^{(i)} \right|_\infty \right\} \\ &= \max \left\{ \frac{|q^{(i)}|_\infty | \det \ (M)|_\infty}{|c(c\theta + d)|_\infty}, \ \frac{x_{q^{(i)}}}{|q^{(i)}|_\infty} \right\}. \end{aligned} \tag{2.19}$$

Combining (2.18) and (2.19) gives

$$L(\psi) - \frac{1}{2^i} \le L(\theta) \max\left\{ |\det M|_\infty,\ |c(c\theta+d)|_\infty \frac{x_{q^{(i)}}}{|(q^{(i)})|_\infty^2} \right\}.$$

Using the first inequality in (2.16), i.e., $x_{q^{(i)}} \le \frac{1}{L(\psi)-1/2^i}$, we deduce that

$$L(\psi) - \frac{1}{2^i} \le \max\left\{ |\det M|_\infty L(\theta),\ \frac{|c(c\theta+d)|_\infty}{|(q^{(i)})|_\infty^2} \cdot \frac{L(\theta)}{L(\psi)-1/2^i} \right\}. \tag{2.20}$$

If $L(\theta) \ge L(\psi)$, then the inequality (2.2) holds trivially, using the first term in the right hand expression. If $L(\theta) < L(\psi)$, then letting $i \to \infty$ in (2.20), the ratio $\frac{L(\theta)}{L(\psi)-1/2^i}$ becomes ≤ 1 in the limit, and (2.2) follows. □

Next, the boundedness of partial quotients of regular continued fractions representing linear fractional tranformations of rational elements in $\mathbb{F}((x^{-1}))$ is investigated.

Definition 2.1.12. Let ϕ be an element in $\mathbb{F}(x) \setminus \mathbb{F}[x]$ whose regular continued fraction expansion is $[b_0; b_1, \ldots, b_n]$. Define $R(\phi) := \max_{1\le i\le n} |b_i|_\infty$.

Lemma 2.1.13. *Let ϕ be an element in $\mathbb{F}(x) \setminus \mathbb{F}[x]$ whose regular continued fraction expansion is $[b_0; b_1, \ldots, b_n]$. Then*

$$R(\phi) = \max_{0\le i<n} (|q_i|_\infty \|q_i\phi\|)^{-1} = \max_{0<|q|_\infty<|q_n|_\infty} (|q|_\infty \|q\phi\|)^{-1}.$$

Proof. From Lemma 1.2.12 (v), we have for $0 \le i < n$, $\|q_i\phi\| = |q_i\phi - p_i|$, and so Lemma 1.2.12$(ii)$ and (iv) together yield

$$|q_i|\ \|q_i\phi\| = \frac{1}{|\phi_{i+1} + q_{i-1}/q_i|} = \frac{1}{|b_{i+1}|}.$$

This implies the first desired equality. The second desired equality follows immediately from the best approximation property of regular continued fraction convergents according to Remark 2.1.4. $\square$

Proposition 2.1.14. *Let ϕ be an element in $\mathbb{F}(x) \setminus \mathbb{F}[x]$ with $|\phi|_\infty > 1$ whose regular continued fraction expansion is $[b_0; b_1, \ldots, b_n]$ and let*

$$\psi = \frac{a\phi + b}{c\phi + d},$$

where $a, b, c, d \in \mathbb{F}[x]$ be such that $|ad - bc|_\infty = 1$, $|c|_\infty > |d|_\infty > 0$ and $c\phi + d \neq 0$. Assume that $\psi \in \mathbb{F}(x) \setminus \mathbb{F}[x]$. Then

A) b/d and a/c equal two consecutive convergents, say $(m-1)^{th}$ and m^{th} convergents, respectively, of the regular continued fraction for ψ and we have that the $(m+1)^{th}$ complete quotient is of the form $\delta\theta$ for some $\delta \in \mathbb{F} \setminus \{0\}$.

B) $R(\psi) = \max\{|b_0|_\infty, R(\phi), R(a/c)\}$.

Proof. Denote the regular continued fraction expansion of a/c by $[c_0; c_1, \ldots, c_m]$ and let p_m/q_m be its m^{th} (last) convergent. Since $|ad - bc|_\infty = 1$, we have, by Lemma 1.2.12 (i), $\gcd(a, c) = 1 = \gcd(p_m, q_m)$ and $a = \gamma p_m$, $c = \gamma q_m$ for some $\gamma \in \mathbb{F} \setminus \{0\}$. Thus,

$$|p_m d - q_m b|_\infty = |ad - bc|_\infty = 1 = |p_m q_{m-1} - p_{m-1} q_m|_\infty,$$

yielding $p_m d - q_m b = \delta'(p_m q_{m-1} - p_{m-1} q_m)$ for some $\delta' \in \mathbb{F} \setminus \{0\}$, and so

$$p_m(d - \delta' q_{m-1}) = q_m(b - \delta' p_{m-1}). \tag{2.21}$$

Since $\gcd(p_m, q_m) = 1$, the relation (2.21) gives

$$q_m | (d - \delta' q_{m-1}). \tag{2.22}$$

From $|q_m|_\infty = |c|_\infty > |d|_\infty > 0$, and $|q_m|_\infty > |q_{m-1}|_\infty \geq 0$, we get $|d - \delta' q_{m-1}|_\infty < |q_m|_\infty$, which is consistent with (2.22) only when $d - \delta' q_{m-1} = 0$, i.e., when $d = \delta' q_{m-1},\ b = \delta' p_{m-1}$. Consequently, $\psi = \frac{\delta p_m \phi + p_{m-1}}{\delta q_m \phi + q_{m-1}}$ for some $\delta \in \mathbb{F} \setminus \{0\}$, and so

$$\psi = [c_0; c_1, \ldots, c_m, \delta\phi].$$

This immediately yields

$$\psi = \begin{cases} [c_0; c_1, \ldots c_m, \delta b_0, \delta^{-1} b_1, \ldots, \delta^{-1} b_{n-1}, \delta b_n] & \text{if } n \text{ is odd,} \\ [c_0; c_1, \ldots c_m, \delta b_0, \delta^{-1} b_1, \ldots, \delta b_{n-1}, \delta^{-1} b_n] & \text{if } n \text{ is even.} \end{cases}$$

Since $|b_i|_\infty = |\delta b_i|_\infty = |\delta^{-1} b_i|_\infty, \quad 0 \leq i \leq n$, we have

$$R(\psi) = \max\{|b_0|_\infty, R(\phi), R(a/c)\}$$

as desired. $\square$

Theorem 2.1.15. *Let ϕ be an element in $\mathbb{F}(x) \setminus \mathbb{F}[x]$ whose regular continued fraction expansion is $[b_0; b_1, \ldots, b_n]$ and let p_n/q_n be its n^{th} (last) convergent and let*

$$\psi = \frac{a\phi + b}{c\phi + d},$$

where $a, b, c, d \in \mathbb{F}[x],\ ad - bc \neq 0$ and $c\theta + d \neq 0$. Assume that $\psi \in \mathbb{F}(x) \setminus \mathbb{F}[x]$. If $|a|_\infty \neq |\psi c|_\infty$, then

$$R(\psi) \leq \max\left\{\left|\frac{1}{\psi}\right|_\infty, |ad - bc|_\infty R(\phi)\right\}.$$

Proof. Let $[\tilde{b}_0; \tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_s]$ be the regular continued fraction of ψ and denote its i^{th} convergent by $\tilde{p}_i/\tilde{q}_i$. Choose a k^{th} denominator, $\tilde{q}_k$, of ψ such that

$$0 \leq k < s \qquad \text{and} \qquad R(\psi) = \frac{1}{|\tilde{q}_k|_\infty \, \|\tilde{q}_k \psi\|}.$$

From Lemma 1.2.12 (iv),

$$\tilde{q}_k\psi - \tilde{p}_k = \frac{(-1)^k}{\psi_{k+1}\tilde{q}_k + \tilde{q}_{k-1}},$$

so that we can write

$$\tilde{p}_k = \tilde{q}_k\psi - \beta,$$

for some $\beta \in \mathbb{F}(x)$ such that $|\beta|_\infty < 1$.

Case 1. $\tilde{p}_k \neq 0$.

Hence $|\tilde{p}_k|_\infty \geq 1 > |\beta|_\infty$, and so by (1.1) $|\tilde{q}_k\psi|_\infty = |\tilde{p}_k + \beta|_\infty = |\tilde{p}_k|_\infty > |\beta|_\infty$. We get $|\tilde{q}_k a - \tilde{p}_k c|_\infty = |\tilde{q}_k(a - \psi c) + \beta c|_\infty$. By the assumption $|a|_\infty \neq |\psi c|_\infty$, so we have

$$|\tilde{q}_k a - \psi\tilde{q}_k c|_\infty = \max\{|\tilde{q}_k a|_\infty,\ |\psi\tilde{q}_k c|_\infty\} \geq |\psi\tilde{q}_k c|_\infty > |\beta c|_\infty.$$

Then by (1.1)

$$|\tilde{q}_k a - \tilde{p}_k c|_\infty = |\tilde{q}_k a - \psi\tilde{q}_k c + \beta c|_\infty = |\tilde{q}_k a - \psi\tilde{q}_k c|_\infty \neq 0.$$

Thus

$$\begin{aligned}
|ad - bc|_\infty \frac{1}{R(\psi)} &= |c\phi + d|_\infty |a - \psi c|_\infty |\tilde{q}_k|_\infty \, \|\tilde{q}_k\psi\| \\
&= |c\phi + d|_\infty \left|\frac{\tilde{q}_k a - \psi\tilde{q}_k c}{\tilde{q}_k}\right|_\infty |\tilde{q}_k|_\infty \, \|\tilde{q}_k\psi\| \\
&= |c\phi + d|_\infty |\tilde{q}_k a - \tilde{p}_k c|_\infty \, \|\tilde{q}_k\psi\| \\
&= |c\phi + d|_\infty |\tilde{q}_k a - \tilde{p}_k c|_\infty \left|\tilde{q}_k\left(\frac{a\phi + b}{c\phi + d}\right) - \tilde{p}_k\right|_\infty \\
&= |\tilde{q}_k a - \tilde{p}_k c|_\infty |(\tilde{q}_k a - \tilde{p}_k c)\phi - (\tilde{p}_k d - \tilde{q}_k b)|_\infty. \qquad (2.23)
\end{aligned}$$

Subcase 1.1. $|\tilde{q}_k a - \tilde{p}_k c|_\infty < |q_n|_\infty$.

Since $|ad - bc|_\infty R(\psi) \neq 0$, $|(\tilde{q}_k a - \tilde{p}_k c)\phi - (\tilde{p}_k d - \tilde{q}_k b)|_\infty \neq 0$. Hence by the definition of the distance

$$|(\tilde{q}_k a - \tilde{p}_k c)\phi - (\tilde{p}_k d - \tilde{q}_k b)|_\infty \geq \|(\tilde{q}_k a - \tilde{p}_k c)\phi\|,$$

and then Lemma 2.1.13 together with (2.23) yield

$$|ad - bc|_\infty \frac{1}{R(\psi)} \geq |\tilde{q}_k a - \tilde{p}_k c|_\infty \, \|(\tilde{q}_k a - \tilde{p}_k c)\phi\| \geq \frac{1}{R(\phi)}.$$

Subcase 1.2. $|\tilde{q}_k a - \tilde{p}_k c|_\infty \geq |q_n|_\infty$.

Write $\phi = \dfrac{E}{F}$, where $E \in \mathbb{F}[x]$, $F \in \mathbb{F}[x] \smallsetminus \mathbb{F}$ and $\gcd(E, F) \in \mathbb{F} \smallsetminus \{0\}$. Then by (2.23)

$$\begin{aligned} |ad - bc|_\infty \frac{1}{R(\psi)} &= |\tilde{q}_k a - \tilde{p}_k c|_\infty \left| (\tilde{q}_k a - \tilde{p}_k c)\frac{E}{F} - (\tilde{p}_k d - \tilde{q}_k b) \right|_\infty \\ &= |\tilde{q}_k a - \tilde{p}_k c|_\infty \frac{|(\tilde{q}_k a - \tilde{p}_k c)E - (\tilde{p}_k d - \tilde{q}_k b)F|_\infty}{|F|_\infty}. \end{aligned}$$

We have by Lemma 1.2.12 (i) and the definition of F that $|q_n|_\infty = |F|_\infty$, and so combines with the facts that $|\tilde{q}_k a - \tilde{p}_k c|_\infty \geq |q_n|_\infty$ and $|(\tilde{q}_k a - \tilde{p}_k c)E - (\tilde{p}_k d - \tilde{q}_k b)F|_\infty \geq 1$ we get

$$|ad - bc|_\infty \frac{1}{R(\psi)} \geq |q_n|_\infty \frac{1}{|q_n|_\infty} = 1 > \frac{1}{R(\phi)}.$$

Case 2. $\tilde{p}_k = 0$.

By the construction of the regular continued fraction we have $\tilde{b}_i \in \mathbb{F}[x] \smallsetminus \mathbb{F}$, for all $1 \leq i \leq s$, then by (1.2), $|\tilde{p}_i|_\infty \geq 1$ for all $1 \leq i \leq s$. It follows that $k = 0$. Thus

$$R(\psi) = \frac{1}{|\tilde{q}_0|_\infty \, \|\tilde{q}_0 \psi\|} = |b_1|_\infty = \frac{1}{|\psi|_\infty}.$$

Therefore,

$$R(\psi) \leq \max\left\{ \left|\frac{1}{\psi}\right|_\infty, |ad - bc|_\infty R(\phi) \right\}.$$

□

2.2 Zaremba's conjecture for $2^s \cdot 3^t$

A famous conjecture attributed to Zaremba, see e.g., [20], states that for a positive integer $m \geq 2$ there exists a reduced fraction a/m such that

$$\max\{b_1, \ldots, b_n\} \leq 5,$$

where $[b_0; b_1, \ldots, b_n]$ is the regular continued fraction with $b_n > 1$ of a/m.

In this section, Zaremba's conjecture for the case $m = 2^s \cdot 3^t$ where s, t are non-negative integers verified by using the identity known as the Folding lemma similar to [30].

For an arbitrary field K and $\alpha, \beta \in K$, we adopt the notation

$$[\ldots, \alpha, 0, \beta, \ldots] = [\ldots, \alpha + \beta, \ldots]. \tag{2.24}$$

Lemma 2.2.1. *(Folding lemma) Let $s_1 \in \mathbb{R} \setminus \{0\}$, $n \geq 0$ and $\frac{p_n}{q_n}$ be the last convergent of a continued fraction $[b_0; b_1, \ldots, b_n]$ over $\mathbb{R}$. Then*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{s_1 q_n^2} = \begin{cases} [b_0; s_1 - 1, 1] & ; \quad \text{if } n = 0, \\ [b_0; b_1, \ldots, b_n, s_1 - 1, 1, b_n - 1, b_{n-1}, \ldots, b_1] & ; \quad \text{if } n \geq 1. \end{cases}$$

Proof. We have by (1.6) and (1.7) that

$$\left[b_0; b_1, \ldots, b_n, s_1 - \frac{q_{n-1}}{q_n}\right] = \frac{\left(s_1 - \frac{q_{n-1}}{q_n}\right) p_n + p_{n-1}}{\left(s_1 - \frac{q_{n-1}}{q_n}\right) q_n + q_{n-1}} = \frac{p_n}{q_n} + \frac{(-1)^n}{s_1 q_n^2}.$$

Hence the computations

$$s_1 - q_{n-1}/q_n = s_1 - 1 + (q_n - q_{n-1})/q_n$$

$$q_n/(q_n - q_{n-1}) = 1 + q_{n-1}/(q_n - q_{n-1})$$
$$(q_n - q_{n-1})/q_{n-1} = -1 + q_n/q_{n-1}$$

allow us to rewrite

$$\frac{p_n}{q_n} + \frac{(-1)^n}{s_1 q_n^2} = \left[b_0; b_1, \ldots, b_n, s_1 - 1, 1, -1, \frac{q_{n-1}}{q_n}\right].$$

Therefore, for the case $n = 0$, the desired result follows from the definition of q_{-1} and (1.6) and for the case $n \geq 1$, it follows from (1.5) and (2.24). $\square$

Theorem 2.2.2. *For any positive integer $m \geq 2$ of the form $m = 2^s \cdot 3^t$, where s, t are non-negative integers, there exists a reduced fraction a/m such that*

$$\max\{b_1, \ldots, b_n\} \leq 5,$$

where $[0; b_1, \ldots, b_n]$ is the regular continued fraction with $b_n > 1$ of a/m.

Proof. Starting from the following fractions

$$\tfrac{1}{2} = [0; 2]; \qquad \tfrac{1}{3} = [0; 3]; \qquad \tfrac{5}{2\cdot 3} = [0; 1, 5];$$
$$\tfrac{1}{2^2} = [0; 4]; \qquad \tfrac{2}{3^2} = [0; 4, 2]; \qquad \tfrac{8}{2^2\cdot 3^2} = [0; 3, 2, 1, 2];$$
$$\tfrac{3}{2^3} = [0; 2, 1, 2]; \qquad \tfrac{8}{3^3} = [0; 3, 2, 1, 2]; \qquad \tfrac{49}{2^3\cdot 3^3} = [0; 4, 2, 2, 4, 2];$$
$$\tfrac{5}{2\cdot 3^2} = [0; 3, 1, 1, 2]; \qquad \tfrac{5}{2^2\cdot 3} = [0; 2, 2, 2]; \qquad \tfrac{11}{2^3\cdot 3} = [0; 2, 5, 2];$$
$$\tfrac{17}{2\cdot 3^3} = [0; 3, 5, 1, 2]; \qquad \tfrac{29}{2^2\cdot 3^3} = [0; 3, 1, 2, 1, 1, 3]; \qquad \tfrac{11}{2^3\cdot 3^2} = [0; 4, 4, 4],$$

and the proof is then completed by showing the stronger statement that for any positive integer $k \geq 2$ and any positive integer $m \geq 2$ of the forms

$$m = 2^k \cdot 3^j \quad \text{or} \quad m = 2^j \cdot 3^k \qquad (0 \leq j \leq k),$$

there exists a reduced fraction a/m such that

$$1 < b_1, b_n < 5 \quad \text{and} \quad b_i \leq 5 \quad \text{for all } 2 \leq i \leq n-1,$$

where $[0; b_1, \ldots, b_n]$ is the regular continued fraction with $b_n > 1$ of a/m.

We will prove this stronger statement by using induction on k. By the above fractions, the stronger statement holds for $k = 2, 3$. Now assume that the stronger statement holds for $2 \le i \le k \quad (k \ge 3)$. Let $0 \le j \le k+1$.

If both $k+1$ and j are even, then by the hypothesis there exist reduced fractions $b/(2^{\frac{k+1}{2}} \cdot 3^{\frac{j}{2}})$ and $c/(2^{\frac{j}{2}} \cdot 3^{\frac{k+1}{2}})$ such that

$$1 < c_1, c_h < 5 \quad \text{and} \quad c_i \le 5 \quad \text{for all } 2 \le i \le h-1; \tag{2.25}$$

$$1 < d_1, d_r < 5 \quad \text{and} \quad d_i \le 5 \quad \text{for all } 2 \le i \le r-1, \tag{2.26}$$

where $[0; c_1, \ldots, c_h]$ and $[0; d_1, \ldots, d_r]$ are the regular continued fractions with the last partial quotient > 1 of $b/(2^{\frac{k+1}{2}} \cdot 3^{\frac{j}{2}})$ and $c/(2^{\frac{j}{2}} \cdot 3^{\frac{k+1}{2}})$, respectively. Applying Lemma 2.2.1 and then by (2.24), we have

$$\begin{aligned} \frac{b}{2^{\frac{k+1}{2}} \cdot 3^{\frac{j}{2}}} + \frac{(-1)^h}{2^{k+1} \cdot 3^j} &= [0; c_1, \ldots, c_h, 0, 1, c_h - 1, c_{h-1}, \ldots, c_1] \\ &= [0; c_1, \ldots, c_h + 1, c_h - 1, c_{h-1}, \ldots, c_1], \end{aligned}$$

and

$$\begin{aligned} \frac{c}{2^{\frac{j}{2}} \cdot 3^{\frac{k+1}{2}}} + \frac{(-1)^r}{2^j \cdot 3^{k+1}} &= [0; d_1, \ldots, d_r, 0, 1, d_r - 1, d_{r-1}, \ldots, d_1] \\ &= [0; d_1, \ldots, d_r + 1, d_r - 1, d_{r-1}, \ldots, d_1]. \end{aligned}$$

It is clear that

$$\gcd(b \cdot (2^{\frac{k+1}{2}} \cdot 3^{\frac{j}{2}}) + (-1)^h, 2^{k+1} \cdot 3^j) = 1,$$

and
$$\gcd(c \cdot (2^{\frac{j}{2}} \cdot 3^{\frac{k+1}{2}}) + (-1)^r, 2^j \cdot 3^{k+1}) = 1.$$

Hence the stronger statement is established by (2.25) and (2.26).

If at least one of $k+1$ and j is odd, then we can write

$$2^{k+1} \cdot 3^j = u_1 \cdot v_1^2 \quad \text{and} \quad 2^j \cdot 3^{k+1} = u_2 \cdot v_2^2$$

where $u_1, u_2 \in \{2,3,6\}$, $v_1 = 2^{n_1} \cdot 3^{n_2}$ for some $2 \leq n_1 < k+1$, $0 \leq n_2 \leq n_1$ and $v_2 = 2^{n_3} \cdot 3^{n_4}$ for some $2 \leq n_4 < k+1$, $0 \leq n_3 \leq n_4$. Then by the hypothesis there exist reduced fractions b/v_1 and c/v_2 such that

$$1 < c_1, c_h < 5 \quad \text{and} \quad c_i \leq 5 \quad \text{for all } 2 \leq i \leq h-1; \tag{2.27}$$

$$1 < d_1, d_r < 5 \quad \text{and} \quad d_i \leq 5 \quad \text{for all } 2 \leq i \leq r-1, \tag{2.28}$$

where $[0; c_1, \ldots, c_h]$ and $[0; d_1, \ldots, d_r]$ are the regular continued fractions with the last partial quotient > 1 of b/v_1 and c/v_2, respectively. Applying Lemma 2.2.1, we have

$$\frac{b}{v_1} + \frac{(-1)^h}{u_1 v_1^2} = [0; c_1, \ldots, c_h, u_1 - 1, 1, c_h - 1, c_{h-1}, \ldots, c_1],$$

and

$$\frac{c}{v_2} + \frac{(-1)^r}{u_2 v_2^2} = [0; d_1, \ldots, d_r, u_2 - 1, 1, d_r - 1, d_{r-1}, \ldots, d_1].$$

It is clear that

$$\gcd(bu_1v_1 + (-1)^h, u_1 v_1^2) = 1 \quad \text{and} \quad \gcd(cu_2v_2 + (-1)^r, u_2 v_2^2) = 1.$$

Hence the stronger statement is established by (2.27), (2.28) and the definitions of u_1 and u_2. □

CHAPTER III

CONTINUED FRACTIONS WITH SOME PATTERNS

In the chapter, we begin with a generalization of Theorem 2.3 in [9] which considered continued fractions over $\mathbb{R}$ to continued fractions over a general field K. This generalized theorem is then applied to continued fractions over the field $\mathbb{F}((x^{-1}))$ of formal series over a based field $\mathbb{F}$ to produce some interesting identities. Next, an identity for continued fractions with palindromic property is extended in the last section.

3.1 Identities for continued fractions with some patterns

Theorem 3.1.1. *Let K be an arbitrary field and $[b_0; b_1, \ldots, b_n]$ $(n \geq 0)$ be a continued fraction over K. Then for any $d \in K$ with $d \neq q_{n-1}$, we have*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{dq_n} = \left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{q_n}{d - q_{n-1}} \right].$$

Proof. It is obvious for $n = 0$ from $\frac{p_0}{q_0} + \frac{(-1)^0}{dq_0} = b_0 + \frac{1}{d}$. Now consider the case $n \geq 1$, by (1.6), (1.2) and (1.7), respectively, we have

$$\begin{aligned}\left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{q_n}{d - q_{n-1}} \right] &= \frac{\left(b_n + \frac{q_n}{d-q_{n-1}} \right) p_{n-1} + p_{n-2}}{\left(b_n + \frac{q_n}{d-q_{n-1}} \right) q_{n-1} + q_{n-2}} \\ &= \frac{d(b_n p_{n-1} + p_{n-2}) - q_{n-1}(b_n p_{n-1} + p_{n-2}) + q_n p_{n-1}}{d(b_n q_{n-1} + q_{n-2}) - q_{n-1}(b_n q_{n-1} + q_{n-2}) + q_n q_{n-1}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{dp_n - q_{n-1}p_n + q_np_{n-1}}{dq_n - q_{n-1}q_n + q_nq_{n-1}} \\
&= \frac{p_n}{q_n} + \frac{(-1)^n}{dq_n}
\end{aligned}
$$

as desired. □

Choosing d related to p_n, q_n, and q_{n-1}, many identities for continued fractions with some patterns are obtained as interesting applications of Theorem 3.1.1.

Corollary 3.1.2. *Let* $[b_0; b_1, \ldots, b_n]$ $(n \geq 0)$ *be a regular continued fraction over* $\mathbb{F}((x^{-1}))$. *Then for any* $s \in \mathbb{F}[x] \setminus \{0\}$, *we have*

$$
\frac{p_n}{q_n} + \frac{(-1)^n}{(sq_n + 2q_{n-1})q_n} = \begin{cases} [b_0; s] & ; \quad \textit{if} \ \ n = 0, \\ [b_0; b_1, \ldots, b_n, s, b_n \ldots, b_1] & ; \quad \textit{if} \ \ n \geq 1. \end{cases}
$$

Proof. It is obvious for $n = 0$, since $\frac{p_0}{q_0} + \frac{(-1)^0}{(sq_0+2q_{-1})q_0} = b_0 + \frac{1}{s} = [b_0; s]$. Now consider the case $n \geq 1$. Since $|q_{n-1}|_\infty < |q_n|_\infty \leq |sq_n|_\infty$, we have $sq_n + 2q_{n-1} \neq q_{n-1}$. By applying Theorem 3.1.1, we have

$$
\begin{aligned}
\frac{p_n}{q_n} + \frac{(-1)^n}{(sq_n + 2q_{n-1})q_n} &= \left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{q_n}{sq_n + q_{n-1}}\right] \\
&= \left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{1}{s + \frac{q_{n-1}}{q_n}}\right].
\end{aligned}
$$

Hence

$$
\frac{p_n}{q_n} + \frac{(-1)^n}{(sq_n + 2q_{n-1})q_n} = [b_0; b_1, \ldots, b_n, s, b_n \ldots, b_1]
$$

from (1.5). □

The symmetric pattern appearing in the case $n \geq 1$ of Corollary 3.1.2 is called *2-duplicate symmetry*, following Cohn [10]. It is obvious that a 2-duplicating symmetric continued fraction is palindromic.

The next corollary is the Folding lemma for the case of formal series.

Corollary 3.1.3. *Let* $[b_0; b_1, \dots, b_n]$ $(n \geq 0)$ *be a regular continued fraction over* $\mathbb{F}((x^{-1}))$*. Then for any* $s \in \mathbb{F}[x] \setminus \{0\}$*, we have*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{sq_n^2} = \begin{cases} [b_0; s] & ; \quad \text{if } n = 0, \\ [b_0; b_1, \dots, b_n, s, -b_n \dots, -b_1] & ; \quad \text{if } n \geq 1. \end{cases}$$

Proof. It is obvious for $n = 0$ from $\frac{p_0}{q_0} + \frac{(-1)^0}{sq_0^2} = b_0 + \frac{1}{s} = [b_0; s]$. Now consider the case $n \geq 1$. Since $|q_{n-1}|_\infty < |q_n|_\infty \leq |sq_n|_\infty$, $sq_n \neq q_{n-1}$. By Theorem 3.1.1, we have

$$\begin{aligned} \frac{p_n}{q_n} + \frac{(-1)^n}{(sq_n)q_n} &= \left[b_0; b_1, \dots, b_{n-1}, b_n + \frac{q_n}{sq_n - q_{n-1}}\right] \\ &= \left[b_0; b_1, \dots, b_{n-1}, b_n + \frac{1}{s - \frac{q_{n-1}}{q_n}}\right], \end{aligned}$$

and hence

$$\frac{p_n}{q_n} + \frac{(-1)^n}{sq_n^2} = [b_0; b_1, \dots, b_n, s, -b_n \dots, -b_1]$$

as required. □

Corollary 3.1.4. *Let* $[b_0; b_1, \dots, b_n]$ $(n \geq 0)$ *be a regular continued fraction over* $\mathbb{F}((x^{-1}))$*. Then for any* $s \in \mathbb{F}[x] \setminus \{0\}$*, we have*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{((s - b_0)q_n + q_{n-1} + p_n)q_n} = \begin{cases} [b_0; s] & ; \quad \text{if } n = 0, \\ [b_0; b_1, \dots, b_n, s, b_1 \dots, b_n] & ; \quad \text{if } n \geq 1, \end{cases} \tag{3.1}$$

and

$$\frac{p_n}{q_n} + \frac{(-1)^n}{((s+b_0)q_n + q_{n-1} - p_n)q_n} = \begin{cases} [b_0; s] & ; \textit{ if } \ n = 0, \\ [b_0; b_1, \ldots, b_n, s, -b_1 \ldots, -b_n] & ; \textit{ if } \ n \geq 1. \end{cases} \tag{3.2}$$

Proof. (3.1) and (3.2) are obvious for $n = 0$, because

$$\frac{p_0}{q_0} + \frac{(-1)^0}{((s-b_0)q_0 + q_{-1} + p_0)q_0} = \frac{p_0}{q_0} + \frac{(-1)^0}{((s+b_0)q_0 + q_{-1} - p_0)q_0} = b_0 + \frac{1}{s} = [b_0; s].$$

Now consider the case $n \geq 1$. Since $|q_{n-1}|_\infty < |q_n|_\infty \leq |sq_n|_\infty$ and $\frac{p_n - b_0 q_n}{q_n}$ is the fractional part of $\frac{p_n}{q_n}$, we have by (1.1) that

$$|(s-b_0)q_n + q_{n-1} + p_n|_\infty = |sq_n|_\infty = |(s+b_0)q_n + q_{n-1} - p_n|_\infty,$$

and so $(s-b_0)q_n + q_{n-1} + p_n$ and $(s+b_0)q_n + q_{n-1} - p_n$ are different from q_{n-1}. By applying Theorem 3.1.1, we have

$$\begin{aligned} \frac{p_n}{q_n} + \frac{(-1)^n}{((s-b_0)q_n + q_{n-1} + p_n)q_n} &= \left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{q_n}{(s-b_0)q_n + p_n} \right] \\ &= \left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{1}{s + \frac{p_n - b_0 q_n}{q_n}} \right], \end{aligned}$$

and

$$\begin{aligned} \frac{p_n}{q_n} + \frac{(-1)^n}{((s+b_0)q_n + q_{n-1} - p_n)q_n} &= \left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{q_n}{(s+b_0)q_n - p_n} \right] \\ &= \left[b_0; b_1, \ldots, b_{n-1}, b_n + \frac{1}{s - \frac{p_n - b_0 q_n}{q_n}} \right]. \end{aligned}$$

We finally reach

$$\frac{p_n}{q_n} + \frac{(-1)^n}{((s-b_0)q_n + q_{n-1} + p_n)q_n} = [b_0; b_1, \ldots, b_n, s, b_1 \ldots, b_n],$$

and

$$\frac{p_n}{q_n} + \frac{(-1)^n}{((s+b_0)q_n + q_{n-1} - p_n)q_n} = [b_0; b_1, \ldots, b_n, s, -b_1 \ldots, -b_n]$$

as desired. □

If we repeatedly apply (3.1) in Corollary 3.1.4 with the same s and with n equals 2 (n of the previous iteration)+1, we get an infinite series expansion for an irrational element in $\mathbb{F}((x^{-1}))$, which is a root of a quadratic equation with the coefficients are in $\mathbb{F}[x]$ whose regular continued fraction is $[b_0; \overline{b_1, b_2, \ldots, b_n, s}]$.

3.2 A generalization of continued fractions with 2-duplicate symmetry

As mentioned in the previous section, $[b_0; b_1, \ldots, b_n, s, b_n, \ldots, b_1]$ is said to be a 2-duplicating symmetric continued fraction. This notion is generalized as follows.

Definition 3.2.1. Let $k \geq 2$, $n \geq 1$ and K be an arbitrary field and $b_0, b_1, \ldots, b_n$, $s_1, \ldots, s_{k-1} \in K$. Denote the word $b_1 b_2 \ldots b_n$ by $\overrightarrow{w}$ and use $\overleftarrow{w}$ to denote $b_n b_{n-1} \ldots b_1$. We call

$$dS_k := \begin{cases} [b_0; \overrightarrow{w}, s_1, \overleftarrow{w}, s_2, \overrightarrow{w}, s_3, \overleftarrow{w}, \ldots, s_{k-1}, \overrightarrow{w}] & ; \text{ if } k \text{ is odd} \\ [b_0; \overrightarrow{w}, s_1, \overleftarrow{w}, s_2, \overrightarrow{w}, s_3, \overleftarrow{w}, \ldots, s_{k-1}, \overleftarrow{w}] & ; \text{ if } k \text{ is even,} \end{cases}$$

a *k-duplicating symmetric* continued fraction and denote $dS_1 := [b_0; \overrightarrow{w}]$.

In order to generalize 2-duplicating symmetric continued fractions in Corollary 3.1.2 to dS_k $(k \geq 2)$, we need a notation for the convergents of a continued fraction defined by segments of partial quotients from another.

Definition 3.2.2. Let $m \geq 0$, K be an arbitrary field and $[b_0; b_1, b_2, \ldots, b_m]$ be a continued fraction over K. For $0 \leq u \leq m$, define

$$P_{u,u-1} = 1, \quad Q_{u,u-1} = 0, \quad P_{u,u} = b_u, \quad Q_{u,u} = 1,$$

$$P_{u,v} = b_v P_{u,v-1} + P_{u,v-2} \quad \text{and} \quad Q_{u,v} = b_v Q_{u,v-1} + Q_{u,v-2} \quad (u < v \leq m). \tag{3.3}$$

Analogous to the formal definition of the numerators and denominators of continued fractions, we have the following lemma

Lemma 3.2.3. *Let $m \geq 0$, K be an arbitrary field and $[b_0; b_1, b_2, \ldots, b_m]$ be a continued fraction over K. Then*

$$\frac{P_{u,v}}{Q_{u,v}} = [b_u; b_{u+1}, \ldots, b_v] \qquad (0 \leq u \leq v \leq m), \tag{3.4}$$

$$\frac{Q_{u,v-1}}{Q_{u,v}} = [0; b_v, \ldots, b_{u+1}] \qquad (0 \leq u < v \leq m). \tag{3.5}$$

Lemma 3.2.4. *Let $m \geq 0$, K be an arbitrary field and $[b_0; b_1, b_2, \ldots, b_m]$ be a continued fraction over K. The following identities hold for $0 \leq u \leq v \leq m$:*

(1) $p_v q_u - q_v p_u = (-1)^u Q_{u+1,v}$,

(2) $q_v Q_{u,v-1} - Q_{u,v} q_{v-1} = (-1)^{v-u-1} q_{u-1}$,

(3) $q_u P_{u+1,v} + q_{u-1} Q_{u+1,v} = q_v$.

Proof. All three statements are proved by induction on $h = v - u$. The $h = 0$ case of each is a consequence of the definitions of $Q_{u,u-1}$ and $P_{u,u-1}$. Now we consider each of them for the $h \geq 1$ case.

(1) (1.7) and the definition of $Q_{u+1,u+1}$ lead to $p_{u+1} q_u - q_{u+1} p_u = (-1)^u = (-1)^u Q_{u+1,u+1}$, and hence the equation hold for $h = 1$. Now assume that the statement holds for all $0 \leq i \leq h - 1$ $(2 \leq h \leq m - u)$. By applying (1.2) and the hypothesis we have

$$p_{u+h}q_u - q_{u+h}p_u = (b_{u+h}p_{u+h-1} + p_{u+h-2})q_u - (b_{u+h}q_{u+h-1} + q_{u+h-2})p_u$$
$$= b_{u+h}(-1)^u Q_{u+1,u+h-1} + (-1)^u Q_{u+1,u+h-2} = (-1)^u Q_{u+1,u+h},$$

and so the equation is established.

(2) The definitions of $Q_{u,u}$ and $Q_{u,u+1}$ and (1.2) lead to $q_{u+1}Q_{u,u} - Q_{u,u+1}q_u = q_{u+1} - b_{u+1}q_u = q_{u-1} = (-1)^0 q_{u-1}$, and so the equation holds for $h = 1$. Now assume that the statement holds for all $0 \leq i \leq h-1 \quad (2 \leq h \leq m-u)$. By applying (1.2), (3.3) and the hypothesis, respectively, we have

$$q_{u+h}Q_{u,u+h-1} - Q_{u,u+h}q_{u+h-1}$$
$$= (b_{u+h}q_{u+h-1} + q_{u+h-2})Q_{u,u+h-1} - (b_{u+h}Q_{u,u+h-1} + Q_{u,u+h-2})q_{u+h-1}$$
$$= q_{u+h-2}Q_{u,u+h-1} - Q_{u,u+h-2}q_{u+h-1} = (-1)^{u+h-u-1}q_{u-1},$$

and hence the equation is established.

(3) The definitions of $P_{u+1,u+1}$ and $Q_{u+1,u+1}$ and (1.2) lead to $q_u P_{u+1,u+1} + q_{u-1}Q_{u+1,u+1} = q_u b_{u+1} + q_{u-1} = q_{u+1}$, and hence the equation hold for $h = 1$. Now assume the statement holds for all $0 \leq i \leq h-1 \quad (2 \leq h \leq m-u)$. By applying (3.3), the hypothesis and (1.2), respectively, we have

$$q_u P_{u+1,u+h} + q_{u-1}Q_{u+1,u+h}$$
$$= q_u(b_{u+h}P_{u+1,u+h-1} + P_{u+1,u+h-2}) + q_{u-1}(b_{u+h}Q_{u+1,u+h-1} + Q_{u+1,u+h-2})$$
$$= b_{u+h}(q_u P_{u+1,u+h-1} + q_{u-1}Q_{u+1,u+h-1}) + q_u P_{u+1,u+h-2} + q_{u-1}Q_{u+1,u+h-2}$$
$$= b_{u+h}q_{u+h-1} + q_{u+h-2} = q_{u+h},$$

and then the equation is established. □

Theorem 3.2.5. *Let* $r \geq 2$, $n \geq r-1$, K *be an arbitrary field and* $[b_0; b_1, b_2, \ldots, b_n]$ *be a continued fraction over* K *and let* $b_{n+1}, \ldots, b_{n+r} \in K$. *If* $b_{n-r+2+i} = b_{n+r-i}$ *for* *all* $0 \leq i \leq r-2$, *then*

$$\frac{p_{n+r}}{q_{n+r}} = \frac{p_n}{q_n} + \frac{(-1)^n Q_{n+1,n+r}}{(Q_{n-r+1,n-1} + P_{n+1,n+r})q_n^2 + (-1)^{r-1} q_{n-r} q_n},$$

where $\frac{p_{n+r}}{q_{n+r}}$ *is the last convergent of* $[b_0; b_1, \dots, b_n, b_{n+1}, \dots, b_{n+r}]$.

Proof. Upon using Lemma 3.2.4 (1), it suffices to show that

$$q_{n+r} = (Q_{n-r+1,n-1} + P_{n+1,n+r})q_n + (-1)^{r-1} q_{n-r}. \tag{3.6}$$

Recall Lemma 3.2.4 (3) and (2) that

$$q_n P_{n+1,n+r} + q_{n-1} Q_{n+1,n+r} = q_{n+r}, \tag{3.7}$$

$$q_{n+1} Q_{n-r+1,n} - Q_{n-r+1,n+1} q_n = (-1)^{r-1} q_{n-r}. \tag{3.8}$$

Subtracting (3.7) by (3.8), we obtain

$$q_n P_{n+1,n+r} + q_{n-1} Q_{n+1,n+r} - q_{n+1} Q_{n-r+1,n} + Q_{n-r+1,n+1} q_n + (-1)^{r-1} q_{n-r} = q_{n+r}.$$

Hence, by applying (1.2) to q_{n+1} and (3.3) to $Q_{n-r+1,n+1}$, we get

$$q_n P_{n+1,n+r} + q_{n-1} Q_{n+1,n+r} - q_{n-1} Q_{n-r+1,n} + Q_{n-r+1,n-1} q_n + (-1)^{r-1} q_{n-r} = q_{n+r}. \tag{3.9}$$

We have by (3.4) that $\frac{P_{n-r+1,n}}{Q_{n-r+1,n}} = [b_{n-r+1}; b_{n-r+2}, \dots, b_n]$ and by (3.5) that

$$\frac{Q_{n-r+1,n-1}}{Q_{n-r+1,n}} = [0; b_n, \dots, b_{n-r+2}].$$

Observe that the last denominator of $[0; b_n, \dots, b_{n-r+2}]$ equals the last denominator of $[b_{n+1}; b_n, \dots, b_{n-r+2}]$ and thus by the assumption $b_{n-r+2+i} = b_{n+r-i}$ $(0 \leq i \leq$

$r-2)$ which lead to

$$[b_{n+1}; b_n, \ldots, b_{n-r+2}] = [b_{n+1}; b_{n+2}, \ldots, b_{n+r}] = \frac{P_{n+1,n+r}}{Q_{n+1,n+r}}.$$

Hence we can conclude that $Q_{n-r+1,n} = Q_{n+1,n+r}$. Therefore, by putting $Q_{n-r+1,n} = Q_{n+1,n+r}$ into (3.9) we get (3.6) as desired. □

For $k \geq 2$, $k-1$ times repeated applications of Theorem 3.2.5 with $r = n+1$ produce a series representing a k-duplicating symmetric continued fraction, described as follows:

Corollary 3.2.6. *Let $k \geq 2$, $n \geq 1$ and K be an arbitrary field and $b_0, b_1, \ldots, b_n$, $s_1, \ldots, s_{k-1} \in K$. Then*

$$dS_k = \frac{p_n}{q_n}$$
$$+ \sum_{i=2}^{k} \frac{(-1)^{(i-1)n+i-2} Q_{(i-1)n+i-1,in+i-1}}{(Q_{(i-2)n+i-2,(i-1)n+i-3} + P_{(i-1)n+i-1,in+i-1}) q^2_{(i-1)n+i-2} + (-1)^n q_{(i-2)n+i-3} q_{(i-1)n+i-2}}.$$

Remark 3.2.7. Corollary 3.2.6 with the $k = 2$ case leads to

$$dS_2 = \frac{p_n}{q_n} + \frac{(-1)^n Q_{n+1,2n+1}}{(Q_{0,n-1} + P_{n+1,2n+1}) q_n^2}.$$

Here $\frac{P_{n+1,2n+1}}{Q_{n+1,2n+1}} = [s_1; b_n, \ldots, b_1]$ and the fact that the last denominator of $[s_1; b_n, \ldots, b_1]$ equals the last denominator of $[0; b_n, \ldots, b_1]$ combined with the last denominator of $[0; b_n, \ldots, b_1]$ is q_n, imply

$$Q_{n+1,n+2n+1} = q_n,$$

hence $P_{n+1,n+2n+1} = s_1 q_n + q_{n-1}$ and by the definition $Q_{0,n-1} = q_{n-1}$. Therefore this speacial case gives Corollary 3.1.2.

CHAPTER IV

EXPLICIT CONTINUED FRACTIONS RELATED TO CERTAIN SERIES

Explicit formulae for regular continued fractions representing real numbers expressed by certain series are provided in the first section of this chapter. Analogues of these results are also established for formal series in the latter.

4.1 Real number case

For $n \geq 0$, define $\theta_n(T; f)$ to be the series expressed as follows

$$\theta_n(T; f) = \sum_{m=0}^{n} \frac{(-1)^m}{f_0(T) f_1(T) \ldots f_m(T)},$$

where $f(T) \in \mathbb{Z}[T] \setminus \{0\}$; $f_0(T) = T$ and for all $i \geq 1$, $f_i(T) = f(f_{i-1}(T))$ with $T \in \mathbb{Z}$, and for those $T \in \mathbb{Z}$ for which the limit exists we define

$$\theta(T; f) = \lim_{n \to \infty} \theta_n(T; f).$$

Throughout this section, we put for any $f(T) \in \mathbb{Z}[T] \setminus \{0\}$,

$$A_n = A_n(T) = (-1)^n + \sum_{m=1}^{n} (-1)^{m+1} f_m(T) f_{m+1}(T) \ldots f_n(T), \quad (n \geq 1) \; ; \; A_0 = 1,$$

$$B_n = B_n(T) = f_0(T) f_1(T) \ldots f_n(T) \quad (n \geq 0).$$

Note that for any $f(T) \in \mathbb{Z}[T] \smallsetminus \{0\}$ and $n \geq 0$, A_n and B_n are the numerator and denominator of the series $\theta_n(T; f)$, respectively, and for $n \geq 1$,

$$\begin{aligned}
A_n(T) &= (-1)^n + \sum_{m=1}^{n}(-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_n(T) \\
&= (-1)^n + f_n(T)\left((-1)^{n-1} + \left(\sum_{m=1}^{n-1}(-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{n-1}(T)\right)\right) \\
&= (-1)^n + f_n(T) \cdot A_{n-1}(T), \qquad (4.1) \\
A_n^2(T) - 1 &= ((-1)^n + f_n(T) A_{n-1}^2(T))^2 - 1 \\
&= f_n(T)(f_n(T) A_{n-1}^2(T) + 2(-1)^n A_{n-1}(T)) \qquad (4.2) \\
&= f_n(T)((f_n(T) + 2) A_{n-1}^2(T) - 2A_{n-1}^2(T) + 2(-1)^n A_{n-1}(T)) \\
&= f_n(T)((f_n(T) + 2) A_{n-1}^2(T) - 2A_{n-1}(T)(A_{n-1}(T) + (-1)^{n-1})) \\
&= f_n(T)((f_n(T) + 2) A_{n-1}^2(T) - 2A_{n-1}(T)(f_{n-1}(T) A_{n-2}(T) + 2(-1)^{n-1})). \\
& \qquad (4.3)
\end{aligned}$$

Lemma 4.1.1. *For $f(T) \in \mathbb{Z}[T] \smallsetminus \{0\}$, we have for all $n, i \geq 0$,*

$$A_n(f_i(T)) = A_{n+i}(T) + D(T) f_i(T) \quad \text{or} \quad A_n(f_i(T)) = -A_{n+i}(T) + D(T) f_i(T),$$

for some $D(T) \in \mathbb{Z}[T]$.

Proof. It is obvious for the case $i = 0$. If $i > 0$ and $n = 0$, then the desired result follows from the definition of A_0 and (4.1). Now consider for $n, i \geq 1$,

$$\begin{aligned}
A_{n+i}(T) &= (-1)^{n+i} + \sum_{m=1}^{n+i}(-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{n+i}(T) \\
&= (-1)^{n+i} + \sum_{m=1}^{i}(-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{n+i}(T) \\
&\quad + \sum_{m=i+1}^{n+i}(-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{n+i}(T).
\end{aligned}$$

Case n, i are even.

$$\begin{aligned}A_{n+i}(T) &= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(-1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)\\ &\quad +1+\sum_{m=1}^{n}(-1)^{m+1}f_{m+i}(T)f_{m+1+i}(T)\dots f_{n+i}(T)\\ &= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(-1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)\\ &\quad +A_n(f_i(T)).\end{aligned}$$

Case n is even, i is odd.

$$\begin{aligned}A_{n+i}(T) &= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)\\ &\quad -1-\sum_{m=1}^{n}(-1)^{m+1}f_{m+i}(T)f_{m+1+i}(T)\dots f_{n+i}(T)\\ &= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)\\ &\quad -A_n(f_i(T)).\end{aligned}$$

Case n is odd, i is even

$$\begin{aligned}A_{n+i}(T) &= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(-1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)\\ &\quad -1+\sum_{m=1}^{n}(-1)^{m+1}f_{m+i}(T)f_{m+1+i}(T)\dots f_{n+i}(T)\\ &= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(-1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)\\ &\quad +A_n(f_i(T)).\end{aligned}$$

Case n, i are odd.

$$A_{n+i}(T) = f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)$$
$$1-\sum_{m=1}^{n}(-1)^{m+1}f_{m+i}(T)f_{m+1+i}(T)\dots f_{n+i}(T)$$
$$= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(1+\sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)$$
$$- A_n(f_i(T)).$$

Therefore, for all $n, i \geq 0$,

$$A_n(f_i(T)) = A_{n+i}(T) + D(T)f_i(T) \text{ or } A_n(f_i(T)) = -A_{n+i}(T) + D(T)f_i(T),$$

for some $D(T) \in \mathbb{Z}[T]$. □

In this section, two main theorems which give some classes of real numbers represented by palindromic regular continued fractions are proved.

The first main theorem reads:

Theorem 4.1.2. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T(T+2)(T-2)g(T) - T^2 + 2, \tag{4.4}$$

$g(T) \in \mathbb{Z}[T]$, where the leading coefficient of $g(T)$ is positive, and let $T_1 = T_1(f) \geq 3$ be the smallest integer such that $2s-2 < f(s)$ for all integers $s \geq T_1$. If $\tilde{T}$ $(\geq T_1(f))$ is an integer, then

$$\theta_0(\tilde{T}; f) = [0; \tilde{T}],$$

and for all $n \geq 0$, $\theta_{n+1}(\tilde{T}; f)$ is given recursively by the following regular continued *fraction*

$$\theta_{n+1}(\tilde{T};f)=\begin{cases}[0;b_1,\dots,b_k-1,1,d_{n+1}(\tilde{T}),1,b_k-1,\dots,b_1] & ;\ \textit{if}\ \ n \textit{ is odd}\\[2ex] [0;b_1,\dots,b_k,d_{n+1}(\tilde{T}),b_k,\dots,b_1] & ;\ \textit{if}\ \ n \textit{ is even,}\end{cases} \tag{4.5}$$

if the regular continued fraction which the last partial quotient is different from 1 *of* $\theta_n(\tilde{T};f)$ *is* $[0;b_1,\dots,b_k]$, *where*

$$d_{n+1}(\tilde{T})=\begin{cases}\left[\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}\right]-1 & ;\ \textit{if}\ \ n \textit{ is odd}\\[2ex] \left[\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}\right] & ;\ \textit{if}\ \ n \textit{ is even.}\end{cases}$$

In particular,

$$\theta(\tilde{T};f)=[0;\tilde{T},d_1(\tilde{T}),\tilde{T}-1,1,d_2(\tilde{T}),1,\tilde{T}-1,d_1(\tilde{T}),\tilde{T},d_3(\tilde{T}),\dots].$$

To prove Theorem 4.1.2, we make use of the following Lemma 4.1.3 to Lemma 4.1.7.

Lemma 4.1.3. *Let* $f(T)$ *be the polynomial of the form* (4.4). *If* $\tilde{T}$ $(\neq 0)$ *is an integer, then* $\tilde{T} \mid (A_n^2(\tilde{T})-1)$ $(n \geq 0)$.

Proof. Since $f(T)=T(T+2)(T-2)g(T)-T^2+2$,

$$f_1(0)=2 \quad \text{and} \quad f_n(0)=-2 \quad (n \geq 2).$$

Because $A_0(T)=1$, then by (4.1) we have

$$A_1(0)=(-1)^1+f_1(0)\cdot A_0(0)=-1+2\cdot 1=1,$$

$$A_2(0)=(-1)^2+f_2(0)\cdot A_1(0)=1+(-2)\cdot 1=-1,$$

$$A_3(0)=(-1)^3+f_3(0)\cdot A_2(0)=-1+(-2)\cdot(-1)=1,$$

proceed inductively we get

$$A_n(0) = (-1)^{n+1} \quad (n \geq 1),$$

Hence we obtain for all $n \geq 0$,

$$A_n^2(T) - 1 = T \cdot D(T), \quad \text{for some} \;\; D(T) \in \mathbb{Z}[T],$$

and so the desired result follows. □

Lemma 4.1.4. *Let $f(T)$ be the polynomial of the form (4.4), and let $T_1 = T_1(f) \geq 3$ be the smallest integer such that $2s - 2 < f(s)$ for all integers $s \geq T_1$. If $\tilde{T}$ $(\geq T_1)$ is an integer, then $B_n(\tilde{T}) \neq 0$ and $B_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1)$ $(n \geq 0)$.*

Proof. If $\tilde{T}$ $(\geq T_1)$, then from the definition of T_1,

$$3 \leq \tilde{T} < 2\tilde{T} - 2 < f(\tilde{T}) < 2f(\tilde{T}) - 2 < f_2(\tilde{T}) < \ldots,$$

so that

$$3 \leq f_0(\tilde{T}) < f_1(\tilde{T}) < f_2(\tilde{T}) < \ldots. \tag{4.6}$$

Therefore $B_n(\tilde{T}) \neq 0$ for all $n \geq 0$. Now from Lemma 4.1.3, we get

$$f_n(\tilde{T}) \mid (A_n^2(f_n(\tilde{T})) - 1), \quad \text{for all} \;\; n \geq 0. \tag{4.7}$$

But we have from Lemma 4.1.1 that for any non-negative integers n, i,

$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) + 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

or

$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) - 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

for some $D \in \mathbb{Z}$. Thus by (4.7)

$$f_i(\tilde{T}) \mid (A_{n+i}^2(\tilde{T}) - 1) \quad \text{for all} \;\; n, i \geq 0.$$

More precisely,

$$f_i(\tilde{T}) \mid (A^2_{(n-i)+i}(\tilde{T}) - 1) = (A^2_n(\tilde{T}) - 1) \quad ; \quad i = 0, 1, \ldots n. \tag{4.8}$$

It remains to prove that

$$B_n(\tilde{T}) = f_0(\tilde{T})f_1(\tilde{T}) \ldots f_n(\tilde{T}) \mid (A^2_n(\tilde{T}) - 1). \tag{4.9}$$

Since for any non-nengative integers j, k such that $j < k$

$$f_k(\tilde{T}) = f_{k-j}(f_j(\tilde{T})) \equiv f_{k-j}(0) \pmod{f_j(\tilde{T})},$$

noticing here, from the proof of Lemma 4.1.3, that

$$f_{k-j}(0) = \begin{cases} 2 & \text{for } k = j+1 \\ -2 & \text{for } k > j+1, \end{cases}$$

we obtain

$$\gcd(f_j(\tilde{T}), f_k(\tilde{T})) = \gcd(f_j(\tilde{T}), 2) = 1 \text{ or } 2. \tag{4.10}$$

We consider the following cases

(1°) $\tilde{T}$ is odd, $g(\tilde{T})$ is even,

(2°) $\tilde{T}$ is odd, $g(\tilde{T})$ is odd,

(3°) $\tilde{T}$ is even.

Case (1°) $\tilde{T}$ is odd, $g(\tilde{T})$ is even.

Claim that for all $i \geq 1$ $f_i(\tilde{T})$ is odd. Since $f_1(\tilde{T}) = \tilde{T}(\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}^2 + 2$, $f_1(\tilde{T})$ is odd. Assume that $f_m(\tilde{T})$ $(m \geq 1)$ is odd. Then $g(f_m(\tilde{T}))$ is even. Hence $f_{m+1}(\tilde{T}) = f_m(\tilde{T})(f_m(\tilde{T})+2)(f_m(\tilde{T})-2)g(f_m(\tilde{T})) - f^2_m(\tilde{T}) + 2$ is odd. Thus we have the claim. Therefore, (4.9) follows from (4.8) and (4.10).

For the cases (2°) and (3°), we make use of the following identity for $n \geq 2$,

$$\begin{aligned}
f_n &= f(f_{n-1}) = f_{n-1}(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-1}^2 + 2 \\
&= f_{n-1}(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) \\
&- (f_{n-2}((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}) - f_{n-2}) + 2)^2 + 2 \\
&= f_{n-1}(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-2}^2((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}) - f_{n-2})^2 \\
&- 4f_{n-2}((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}) - f_{n-2}) - 2, \\
f_n + 2 &= f_{n-1}(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-2}^2((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}))^2 \\
&+ 2f_{n-2}^3(f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}) - 4f_{n-2}(f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}) \\
&- f_{n-2}^4 + 4f_{n-2}^2 \\
&= f_{n-1}(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-2}(f_{n-2}+2)(f_{n-2}-2)\times \\
&\quad \big(f_{n-2}(f_{n-2}+2)(f_{n-2}-2)g^2(f_{n-2}) - 2f_{n-2}^2 g(f_{n-2}) + 4g(f_{n-2}) + f_{n-2}\big)\,. \qquad (4.11)
\end{aligned}$$

Case (2°) $\tilde{T}$ is odd, $g(\tilde{T})$ is odd.

Then $f(\tilde{T}) = \tilde{T}(\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}^2 + 2$ is even. Let u be the positive integer such that $2^u \mid f(\tilde{T})$ and $2^u \nmid f(\tilde{T})$, denoted by $2^u \parallel f(\tilde{T})$. Hence

$$f_2(\tilde{T}) \equiv 2 \pmod{2^{u+1}}, \qquad (4.12)$$

since $f_2 - 2 = f(f_1) - 2 = f_1(f_1+2)(f_1-2)g(f_1) - f_1^2 = f_1((f_1+2)(f_1-2)g(f_1) - f_1)$. By (4.11), we have

$$\begin{aligned}
f_3 + 2 &= f_2(f_2+2)(f_2-2)g(f_2) - f_1(f_1+2)(f_1-2)\times \\
&\quad (f_1(f_1+2)(f_1-2)g^2(f_1) - 2f_1^2 g(f_1) + 4g(f_1) + f_1)\,,
\end{aligned}$$

then, by using (4.12), $f_3(\tilde{T}) \equiv -2 \pmod{2^{u+2}}$, and so by induction, (4.11) and

(4.12) we obtain

$$f_n(\tilde{T}) \equiv -2 \pmod{2^{u+n-1}} \qquad (n \geq 3). \tag{4.13}$$

Claim that $2^{u+n} \mid (A_n^2(\tilde{T}) - 1)$ for all $n \geq 0$. We prove the claim by induction. It is clear that $2^u \mid 0 = A_0^2(\tilde{T}) - 1$ and $A_1^2 - 1 = (f_1 - 1)^2 - 1 = f_1(f_1 - 2)$, and then $2^{u+1} \mid (A_1^2(\tilde{T}) - 1)$. From (4.1), we get

$$\begin{aligned} A_2^2 - 1 = (1 + f_2A_1)^2 - 1 &= f_2(f_2A_1^2 + 2A_1) = f_2((f_2 - 2)A_1^2 + 2A_1^2 + 2A_1) \\ &= f_2((f_2 - 2)A_1^2 + 2A_1(A_1 + 1)) = f_2((f_2 - 2)A_1^2 + 2A_1(f_1 - 1 + 1)), \end{aligned}$$

so by (4.12), we obtain $2^{u+2} \mid (A_2^2(\tilde{T}) - 1)$. Now assume $2^{u+k} \mid (A_k^2(\tilde{T}) - 1)$ for all $k = 0, 1, \ldots, n-1 \quad (n \geq 3)$. By the hypothesis

$$2^{u+n-1} \mid (A_{n-1}^2(\tilde{T}) - 1) \text{ and } 2^{u+n-2} \mid (A_{n-2}^2(\tilde{T}) - 1),$$

and so the latter leads to $2 \nmid A_{n-2}(\tilde{T})$. Hence we have

$$2^{u+n-2} \mid (f_{n-1}(\tilde{T})A_{n-2}(\tilde{T}) + 2(-1)^{n-1}), \tag{4.14}$$

since, by (4.2), $A_{n-1}^2 - 1 = f_{n-1}A_{n-2}(f_{n-1}A_{n-2} + 2(-1)^{n-1})$ and, by (4.12) and (4.13), $2 \| f_{n-1}$. Thus the claim follows from (4.3), (4.13) and (4.14).

But (4.12) and (4.13) lead to $2^{u+n-1} \| f_0f_1 \ldots f_n$. Therefore, (4.9) follows from (4.8) and (4.10).

Case (3°) $\tilde{T}$ is even.

Let u be the positive integer such that $2^u \| \tilde{T}$. Since $f_1(\tilde{T}) - 2 = \tilde{T}((\tilde{T} - 2)(\tilde{T} + 2)g(\tilde{T}) - \tilde{T})$,

$$f_1(\tilde{T}) \equiv 2 \pmod{2^{u+1}}. \tag{4.15}$$

By (4.11), we have

$$f_2(\tilde{T})+2 = f_1(\tilde{T})(f_1(\tilde{T})+2)(f_1(\tilde{T})-2)g(f_1(\tilde{T})) - \tilde{T}(\tilde{T}+2)(\tilde{T}-2)\times \left(\tilde{T}(\tilde{T}+2)(\tilde{T}-2)g^2(\tilde{T}) - 2\tilde{T}^2 g(\tilde{T}) + 4g(\tilde{T}) + \tilde{T}\right),$$

then by using (4.15), $f_2(\tilde{T}) \equiv -2 \pmod{2^{u+2}}$, and so by induction, (4.11) and (4.15) we obtain

$$f_n(\tilde{T}) \equiv -2 \pmod{2^{u+n-1}} \qquad (n \geq 2). \tag{4.16}$$

Claim that $2^{u+n} \mid (A_n^2(\tilde{T})-1)$ for all $n \geq 0$. We prove the claim by induction. It is clear that $2^u \mid 0 = A_0^2(\tilde{T})-1$ and $A_1^2 = f_1(f_1-2)$, and then $2^{u+1} \mid (A_1^2(\tilde{T})-1)$. Now assume $2^{u+k} \mid (A_k^2(\tilde{T})-1)$ for all $k = 0, 1, \ldots, n-1$ $(n \geq 2)$. By the hypothesis

$$2^{u+n-1} \mid (A_{n-1}^2(\tilde{T})-1) \text{ and } 2^{u+n-2} \mid (A_{n-2}^2(\tilde{T})-1),$$

and so the latter leads to $2 \nmid A_{n-2}(\tilde{T})$. Hence we have

$$2^{u+n-2} \mid (f_{n-1}(\tilde{T})A_{n-2}(\tilde{T}) + 2(-1)^{n-1}), \tag{4.17}$$

since, by (4.2), $A_{n-1}^2 - 1 = f_{n-1}A_{n-2}(f_{n-1}A_{n-2} + 2(-1)^{n-1})$ and, by (4.15) and (4.16), $2 \| f_{n-1}$. Thus the claim follows from (4.3), (4.16) and (4.17).

But (4.15) and (4.16) lead to $2^{u+n} \| f_0 f_1 \ldots f_n$. Therefore, (4.9) follows from (4.8) and (4.10). □

Lemma 4.1.5. *Let $f(T)$ be the polynomial of the form (4.4), and let $T_1 = T_1(f) \geq 3$ be the smallest integer such that $2s-2 < f(s)$ for all integers $s \geq T_1$. If $\tilde{T}$ $(\geq T_1)$ is an integer, then*

$$0 < \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} = 2\theta_n(\tilde{T}; f) < 1 \qquad (n \geq 0).$$

Proof. It is clear by (4.6) that for all $n \geq 0$, $B_n(\tilde{T}) > 0$, then we will show for all $n \geq 0$, $\theta_n(\tilde{T}; f) > 0$ by proving that for all $n \geq 0$, $A_n(\tilde{T}) > 0$. It is obvious by the definition that

$$A_0(\tilde{T}) = 1 > 0,$$

and, by (4.1) and (4.6), we have

$$A_1(\tilde{T}) = f_1(\tilde{T}) - 1 > 2 > 0,$$
$$A_2(\tilde{T}) = 1 + f_2(\tilde{T})A_1(\tilde{T}) > 11 > 0.$$

Assume that $A_{k-1}(\tilde{T}) > 0$ $(k \geq 3)$. Then by (4.1), (4.6), and the hypothesis

$$A_k(\tilde{T}) = (-1)^k + f_k(\tilde{T}) \cdot A_{k-1}(\tilde{T}) > 11 > 0.$$

It remains to prove that $\theta_n(\tilde{T}; f) \leq \frac{1}{3}$ for all $n \geq 0$.

Case n is even.

$$\begin{aligned}\theta_n((\tilde{T}; f)) &= \frac{1}{f_0} - \left(\frac{1}{f_0 f_1} - \frac{1}{f_0 f_1 f_2} + \ldots - \frac{1}{f_0 f_1 \ldots f_n}\right) \\ &= \frac{1}{f_0} - \left(\frac{f_2 - 1}{f_0 f_1 f_2} + \frac{f_4 - 1}{f_0 f_1 f_2 f_3 f_4} + \ldots + \frac{f_n - 1}{f_0 f_1 \ldots f_n}\right).\end{aligned}$$

Case n is odd.

$$\begin{aligned}\theta_n(\tilde{T}; f) &= \frac{1}{f_0} - \left(\frac{1}{f_0 f_1} - \frac{1}{f_0 f_1 f_2} + \ldots - \frac{1}{f_0 f_1 \ldots f_{n-1}}\right) - \frac{1}{f_0 f_1 \ldots f_n} \\ &= \frac{1}{f_0} - \left(\frac{f_2 - 1}{f_0 f_1 f_2} + \frac{f_4 - 1}{f_0 f_1 f_2 f_3 f_4} + \ldots + \frac{f_{n-1} - 1}{f_0 f_1 \ldots f_n}\right) - \frac{1}{f_0 f_1 \ldots f_n}.\end{aligned}$$

Thus, by (4.6)

$$\theta_n(\tilde{T}; f) = \frac{1}{f_0(\tilde{T})} < \frac{1}{\tilde{T}} \leq \frac{1}{3}, \qquad \text{for all } n \geq 0.$$

Therefore, the lemma is established. □

Lemma 4.1.6. *Let $f(T)$ be the polynomial of the form (4.4), and let $T_1 = T_1(f) \geq 3$ be the smallest integer such that $2s-2 < f(s)$ for all integers $s \geq T_1$. If $\tilde{T}$ $(\geq T_1)$ is an integer, then for $n \geq 0$,*

$$\left[\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}\right] = \begin{cases} \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} + 2\frac{A_n(\tilde{T})}{B_n(\tilde{T})} - 1 & ; \text{ if } n \text{ is odd} \\ \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - 2\frac{A_n(\tilde{T})}{B_n(\tilde{T})} & ; \text{ if } n \text{ is even.} \end{cases}$$

Proof. Let $n \geq 0$. Combining Lemma 4.1.4, $B_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1)$ and (4.2) leads to

$$\begin{aligned} \frac{f_{n+1}(\tilde{T})A_n^2(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T})}{B_n(\tilde{T})} &= \frac{f_{n+1}(\tilde{T})(f_{n+1}(\tilde{T})A_n^2(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T}))}{B_{n+1}(\tilde{T})} \\ &= \frac{A_{n+1}^2(\tilde{T}) - 1}{B_{n+1}(\tilde{T})} \in \mathbb{Z}. \end{aligned}$$

Since $A_n^2(\tilde{T}) = D \cdot B_n(\tilde{T}) + 1$ for some $D \in \mathbb{Z}$, we have that

$$\frac{f_{n+1}(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T})}{B_n(\tilde{T})} \in \mathbb{Z},$$

i.e.,

$$\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} = E - \frac{2(-1)^{n+1}A_n(\tilde{T})}{B_n(\tilde{T})} \quad \text{for some } E \in \mathbb{Z}. \tag{4.18}$$

But we have from Lemma 4.1.5

$$0 < \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} = 2\theta_n(\tilde{T}; f) < 1.$$

Hence we obtain by (4.18) that

$$\left[\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}\right] = \begin{cases} E - 1 = \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} + 2\frac{A_n(\tilde{T})}{B_n(\tilde{T})} - 1 & ; \text{ if } n \text{ is odd} \\ E = \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - 2\frac{A_n(\tilde{T})}{B_n(\tilde{T})} & ; \text{ if } n \text{ is even.} \end{cases}$$

□

Lemma 4.1.7. *Let $f(T)$ be the polynomial of the form (4.4), and let $T_1 = T_1(f) \geq 3$ be the smallest integer such that $2s-2 < f(s)$ for all integers $s \geq T_1$. If $\tilde{T}$ $(\geq T_1)$ is an integer, then for each $n \geq 0$, $d_{n+1}(\tilde{T})$ defined by*

$$d_{n+1}(\tilde{x}) = \begin{cases} \left[\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}\right] - 1 & ; \text{ if } n \text{ is odd} \\ \left[\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}\right] & ; \text{ if } n \text{ is even,} \end{cases}$$

is a positive integer.

Proof. From Lemma 4.1.5 - 4.1.6, it suffices to show that

$$\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} > 2, \qquad \text{for all } n \geq 0.$$

We proceed by induction. For $n = 0$, Lemma 4.1.6 and $3 \leq \tilde{T} < f_1(\tilde{T})$ lead to

$$d_1(\tilde{T}) = \left[\frac{f_1(\tilde{T})}{B_0(\tilde{T})}\right] = \frac{f_1(\tilde{T})}{B_0(\tilde{T})} - 2\frac{A_0(\tilde{T})}{B_0(\tilde{T})} = (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T} \geq 2,$$

and then by Lemma 4.1.5 we get $\frac{f_1(\tilde{T})}{B_0(\tilde{T})} > 2$. Now assume that $\frac{f_{k+1}(\tilde{T})}{B_k(\tilde{T})} > 2$ $(k \geq 0)$. Since $f_{k+2}(\tilde{T}) = f_{k+1}(\tilde{T})(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}^2(\tilde{T}) + 2$ and, by (4.6), $f_{k+1}(\tilde{T}) \geq 4$ and $f_{k+2}(\tilde{T}) \geq 5$, $g(f_{k+1}(\tilde{T})) \geq 1$. Then

$$(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}(\tilde{T}) > 2f_{k+1}(\tilde{T}) - f_{k+1}(\tilde{T}) = f_{k+1}(\tilde{T}).$$

Hence

$$\begin{aligned} 2 &< \frac{f_{k+1}(\tilde{T}) \cdot f_{k+1}(\tilde{T})}{B_k(\tilde{T}) \cdot f_{k+1}(\tilde{T})} \\ &< \frac{f_{k+1}(\tilde{T}) \cdot ((f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}(\tilde{T}))}{B_{k+1}(\tilde{T})} < \frac{f_{k+2}(\tilde{T})}{B_{k+1}(\tilde{T})} \end{aligned}$$

as desired. □

Now by making use of the above lemmas, we are ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2 For any non-negative integer r, denoted by $k(r)$ the length of the regular continued fraction which the last partial quotient is different from 1 representing $\theta_r(\tilde{T}; f) = \frac{A_r(\tilde{T})}{B_r(\tilde{T})}$.

The proof will be completed by induction. We have by a direct calculation

$$\theta_0(\tilde{T}; f) = [0; \tilde{T}] \quad \text{and} \quad \theta_1(\tilde{T}; f) = [0; \tilde{T}, (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}, \tilde{T}],$$

which, by Lemma 4.1.6, $d_1(\tilde{T}) = \frac{f_1(\tilde{T})}{B_0(\tilde{T})} - \frac{2A_0(\tilde{T})}{B_0(\tilde{T})} = (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}$, and hence the statement (4.5) holds for $n = 0$. Now assume for $n \geq 1$ that the regular continued fraction which the last partial quotient is different from 1 of $\theta_n(\tilde{T}; f)$ is expressed as

$$\theta_n(\tilde{T}; f) = \begin{cases} [0; \alpha_1, \ldots, \alpha_{k(n-1)} - 1, 1, d_n(\tilde{T}), 1, \alpha_{k(n-1)} - 1, \ldots, \alpha_1] & ; \text{ if } n\text{-}1 \text{ is odd} \\ [0; \alpha_1, \ldots, \alpha_{k(n-1)}, d_n(\tilde{T}), \alpha_{k(n-1)}, \ldots, \alpha_1] & ; \text{ if } n\text{-}1 \text{ is even,} \end{cases}$$

if the regular continued fraction which the last partial quotient is different from 1 of $\theta_{n-1}(\tilde{T}; f)$ is $[0; \alpha_1, \ldots, \alpha_{k(n-1)}]$. Then we have $k(n)$ is odd, and we write

$$\theta_n(\tilde{T}; f) = [0; b_1, \ldots, b_{k(n)}] = \frac{p_{k(n)}}{q_{k(n)}}.$$

By using Lemma 4.1.4 we have $A_n(\tilde{T})$ and $B_n(\tilde{T})$ are relatively prime, so that

$$A_n(\tilde{T}) = p_{k(n)} \quad \text{and} \quad B_n(\tilde{T}) = q_{k(n)}.$$

Since $k(n)$ is odd, we have by (1.7),

$$q_{k(n)} p_{k(n)-1} = p_{k(n)} q_{k(n)-1} - 1, \tag{4.19}$$

and hence by the hypothesis $[0; b_1, \ldots, b_{k(n)}]$ is palindromic we have by Remark 1.2.6

$$p_{k(n)} = q_{k(n)-1}. \tag{4.20}$$

Case n is odd.

By (1.5) we have $\frac{q_{k(n)}}{q_{k(n)-1}} = [b_{k(n)}; b_{k(n)-1}, \ldots, b_1]$ and then a simple manipulation leads to

$$-(d_{n+1}(\tilde{T}) + 2) + \tfrac{q_{k(n)-1}}{q_{k(n)}} = [0; -1, 1, d_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1].$$

Hence by (2.24) and (1.6)

$$\begin{aligned}
&[0; b_1, \ldots, b_{k(n)} - 1, 1, d_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1] \\
&\qquad = [0; b_1, \ldots, b_{k(n)}, 0, -1, 1, d_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1] \\
&\qquad = \frac{\left(-d_{n+1}(\tilde{T}) - 2 + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(-d_{n+1}(\tilde{T}) - 2 + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}}.
\end{aligned}$$

From Lemma 4.1.6 we have

$$-d_{n+1}(\tilde{T}) - 2 = -\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} = -\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}}.$$

Thus we get

$$\begin{aligned}
&[0; b_1, \ldots, b_{k(n)} - 1, 1, d_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1] \\
&\qquad = \frac{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}},
\end{aligned}$$

and so we obtain by (4.19) and (4.20) that

$$\begin{aligned}
\frac{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}} &= \frac{f_{n+1}(\tilde{T})p_{k(n)} + 1}{f_{n+1}(\tilde{T})q_{k(n)}} \\
&= \frac{p_{k(n)}}{q_{k(n)}} + \frac{1}{f_{n+1}(\tilde{T})q_{k(n)}} = \frac{A_n(\tilde{T})}{B_n(\tilde{T})} + \frac{1}{B_{n+1}(\tilde{T})} = \theta_{n+1}(\tilde{T}; f).
\end{aligned}$$

Case n is even.

We have by (1.5) that $d_{n+1} + \frac{q_{k-1}}{q_k} = [d_{n+1}; b_k, b_{k-1}, \ldots, b_1]$. Hence by (1.6)

$$[0; b_1, \ldots, b_{k(n)}, d_{n+1}(\tilde{T}), b_{k(n)}, b_{k(n)-1}, \ldots, b_1] = \frac{\left(d_{n+1}(\tilde{T}) + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(d_{n+1}(\tilde{T}) + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}}.$$

From Lemma 4.1.6 we have

$$d_{n+1}(\tilde{T}) = \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} = \frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}}.$$

Thus we get

$$[0; b_1, \ldots, b_{k(n)}, d_{n+1}(\tilde{T}), b_{k(n)}, b_{k(n)-1}, \ldots, b_1]$$
$$= \frac{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}},$$

and so we obtain by (4.19) and (4.20) that

$$\frac{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}} = \frac{f_{n+1}(\tilde{T})p_{k(n)} - 1}{f_{n+1}(\tilde{T})q_{k(n)}}$$

$$= \frac{p_{k(n)}}{q_{k(n)}} - \frac{1}{f_{n+1}(\tilde{T})q_{k(n)}} = \frac{A_n(\tilde{T})}{B_n(\tilde{T})} - \frac{1}{B_{n+1}(\tilde{T})} = \theta_{n+1}(\tilde{T}; f).$$

Therefore,

$$\theta_{n+1}(\tilde{T}; f) = \begin{cases} [0; b_1, \ldots, b_{k(n)} - 1, 1, d_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, \ldots, b_1] & ; \text{ if } n \text{ is odd} \\ [0; b_1, \ldots, b_{k(n)}, d_{n+1}(\tilde{T}), b_{k(n)}, \ldots, b_1] & ; \text{ if } n \text{ is even,} \end{cases}$$

which Lemma 4.1.7 leads to these continued fractions which the last partial quotients are different from 1 are regular. □

The following theorem is the second main theorem.

Theorem 4.1.8. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T^2(T+2)(T-2)g(T) - T^2 + 2, \tag{4.21}$$

$g(T) \in \mathbb{Z}[T]$, where the leading coefficient of $g(T)$ is positive, and let $T_2 = T_2(f) \geq 3$ be the smallest integer such that $2s - 2 < f(s)$ for all integers $s \geq T_2$. If $\tilde{T}$ $(\geq T_2(f))$ is an integer, then

$$\theta_0(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2],$$

and for all $n \geq 0$, $\theta_{n+1}(\tilde{T}; f)/\tilde{T}$ is given recursively by the following regular continued fraction

$$\theta_{n+1}(\tilde{T}; f)/\tilde{T} = \begin{cases} [0; b_1, \ldots, b_k - 1, 1, c_{n+1}(\tilde{T}), 1, b_k - 1, \ldots, b_1] & ; \text{ if } n \text{ is odd} \\ [0; b_1, \ldots, b_k, c_{n+1}(\tilde{T}), b_k, \ldots, b_1] & ; \text{ if } n \text{ is even,} \end{cases} \tag{4.22}$$

if the regular continued fraction which the last partial quotient is different from 1 of $\theta_n(\tilde{T}; f)/\tilde{T}$ is $[0; b_1, \ldots, b_k]$, where

$$c_{n+1}(\tilde{T}) = \begin{cases} \left[\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right] - 1 & ; \text{ if } n \text{ is odd} \\ \left[\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right] & ; \text{ if } n \text{ is even.} \end{cases}$$

In particular,

$$\theta(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2, c_1(\tilde{T}), \tilde{T}^2 - 1, 1, c_2(\tilde{T}), 1, \tilde{T}^2 - 1, c_1(\tilde{T}), \tilde{T}^2, c_3(\tilde{T}), \ldots].$$

The following Lemma 4.1.9 to Lemma 4.1.13 are built to establish the proof of Theorem 4.1.8

Lemma 4.1.9. *Let $f(T)$ be the polynomial of the form (4.21). If $\tilde{T}$ $(\neq 0)$ is an integer, then $\tilde{T}^2 \mid (A_n^2(\tilde{T}) - 1)$ $(n \geq 0)$.*

Proof. Similar to the proof of Lemma 4.1.3, we obtain

$$f_1(0) = 2, \qquad f_n(0) = -2 \qquad (n \geq 2), \tag{4.23}$$

and for all $n \geq 0$,

$$A_n^2(T) - 1 = T \cdot D(T) \quad \text{for some} \;\; D(T) \in \mathbb{Z}[T].$$

Hence we will prove this lemma by showing that

$$\frac{d}{dT}(A_n^2(T) - 1) = 0 \quad \text{at} \;\; T = 0, \qquad \text{for all } n \geq 0. \tag{4.24}$$

It is obvious for the case $n = 0$. Now we consider the cases $n \geq 1$. By (4.2), we have

$$\begin{aligned}\frac{d}{dT}(A_n^2(T) - 1) &= 2f_n(T)f_n'(T)A_{n-1}^2(T) + 2f_n^2(T)A_{n-1}(T)A_{n-1}'(T) \\ &\quad + 2(-1)^n f_n'(T)A_{n-1}(T) + 2(-1)^n f_n(T)A_{n-1}'(T),\end{aligned}$$

then, to prove (4.24), it suffices to show that

$$f_n'(0) = 0 \quad \text{and} \quad A_{n-1}'(0) = 0, \qquad \text{for all} \;\; n \geq 1.$$

Since $f_1(T) = T^2(T+2)(T-2)g(T) - T^2 + 2$,

$$f_1'(T) = 2T(T+2)(T-2)g(T) + T^2((T+2)(T-2)g(T))' - 2T,$$

and so $f_1'(0) = 0$. Now assume that $f_k'(0) = 0$ $(k \geq 1)$. From the definition of f_{k+1}, we get

$$\begin{aligned}f_{k+1}'(T) &= 2f_k(T)f_k'(T)(f_k(T)+2)(f_k(T)-2)g(f_k(T)) \\ &\quad + f_k^2(T)(f_k(T)+2)(f_k(T)-2)(g(f_k(T)))' \\ &\quad + f_k^2(T)\left(f_k'(T)(f_k(T)-2) + (f_k(T)+2)f_k'(T)\right)g(f_k(T)) + 2f_k(T)f_k'(T).\end{aligned}$$

Hence the induction hypothesis and (4.23) lead to $f'_{k+1}(0) = 0$. Thus for all $n \geq 1$ we have that $f'_n(0) = 0$, and so by the mathematical induction, the definition of A_0 and (4.1) we also have for all $n \geq 1$, $\quad A'_{n-1}(0) = 0$. □

Lemma 4.1.10. *Let $f(T)$ be the polynomial of the form* (4.21), *and let $T_2 = T_2(f) \geq 3$ be the smallest integer such that $2s - 2 < f(s)$ for all integers $s \geq T_2$. If $\tilde{T}$ $(\geq T_2)$ is an integer, then $\tilde{T}B_n(\tilde{T}) \neq 0$ and $\tilde{T}B_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1)$ $(n \geq 0)$*

Proof. If $\tilde{T}$ $(\geq T_2)$, then from the definition of T_2,

$$3 \leq \tilde{T} < 2\tilde{T} - 2 < f(\tilde{T}) < 2f(\tilde{T}) - 2 < f_2(\tilde{T}) < \ldots,$$

so that

$$3 \leq f_0(\tilde{T}) < f_1(\tilde{T}) < f_2(\tilde{T}) < \ldots. \tag{4.25}$$

Therefore $\tilde{T}B_n(\tilde{T}) \neq 0$ for all $n \geq 0$. Now from Lemma 4.1.9, we get for all $n \geq 0$,

$$f_n(\tilde{T}) \mid (A_n^2(f_n(\tilde{T})) - 1). \tag{4.26}$$

But from Lemma 4.1.1, we have for any non-negative integers n, i,

$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) + 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

or

$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) - 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

for some $D \in \mathbb{Z}$. Thus (4.26) implies

$$f_i(\tilde{T}) \mid (A_{n+i}^2(\tilde{T}) - 1) \qquad \text{for all } \ n, i \geq 0.$$

More precisely,

$$f_i(\tilde{T}) \mid (A_{(n-i)+i}^2(\tilde{T}) - 1) = (A_n^2(\tilde{T}) - 1) \quad ; \quad i = 0, 1, \ldots n. \tag{4.27}$$

Also, we have from Lemma 4.1.9 that

$$\tilde{T}^2 \mid (A_n^2(\tilde{T}) - 1) \tag{4.28}$$

It remains to prove that

$$\tilde{T}B_n(\tilde{T}) = \tilde{T}^2 f_1(\tilde{T}) \ldots f_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1). \tag{4.29}$$

Since for any non-nengative integers j, k such that $j < k$

$$f_k(\tilde{T}) = f_{k-j}(f_j(\tilde{T})) \equiv f_{k-j}(0) \pmod{f_j(\tilde{T})},$$

noticing here, from (4.23), that

$$f_{k-j}(0) = \begin{cases} 2 & \text{for } k = j+1 \\ -2 & \text{for } k > j+1, \end{cases}$$

we obtain

$$\gcd(f_j(\tilde{T}), f_k(\tilde{T})) = \gcd(f_j(\tilde{T}), 2) = 1 \text{ or } 2. \tag{4.30}$$

We consider the following cases

(1°) $\tilde{T}$ is odd, $g(\tilde{T})$ is even,

(2°) $\tilde{T}$ is odd, $g(\tilde{T})$ is odd,

(3°) $\tilde{T}$ is even.

Case (1°) $\tilde{T}$ is odd, $g(\tilde{T})$ is even.

Since $f(\tilde{T}) = \tilde{T}^2(\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}^2 + 2$, $f_i(\tilde{T})$ $(i = 0, 1, 2, \ldots)$ is odd.

Thus (4.29) follows from (4.27), (4.28) and (4.30).

For the cases (2°) and (3°), we make use of the following identity for $n \geq 2$,

$$
\begin{aligned}
f_n = f(f_{n-1}) &= f_{n-1}^2(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-1}^2 + 2 \\
&= f_{n-1}^2(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) \\
&\quad - \left(f_{n-2}^2((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2})-1)+2\right)^2 + 2 \\
&= f_{n-1}^2(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-2}^4((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2})-1)^2 \\
&\quad - 4f_{n-2}^2((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2})-1) - 2, \\
f_n + 2 &= f_{n-1}^2(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-2}^4((f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}))^2 \\
&\quad + 2f_{n-2}^4(f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}) - 4f_{n-2}^2(f_{n-2}+2)(f_{n-2}-2)g(f_{n-2}) \\
&\quad - f_{n-2}^4 + 4f_{n-2}^2 \\
&= f_{n-1}^2(f_{n-1}+2)(f_{n-1}-2)g(f_{n-1}) - f_{n-2}^2(f_{n-2}+2)(f_{n-2}-2)\times \\
&\quad \left(f_{n-2}^2(f_{n-2}+2)(f_{n-2}-2)g^2(f_{n-2}) - 2f_{n-2}^2 g(f_{n-2}) + 4g(f_{n-2}) + 1\right). \qquad (4.31)
\end{aligned}
$$

Case (2°) $\tilde{T}$ is odd, $g(\tilde{T})$ is odd.

Then $f(\tilde{T}) = \tilde{T}^2(\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}^2 + 2$ is even. Let u be the positive integer such that $2^u \mid f(\tilde{T})$ and $2^u \nmid f(\tilde{T})$. Hence

$$f_2(\tilde{T}) \equiv 2 \pmod{2^{2u}}, \qquad (4.32)$$

since $f_2 - 2 = f(f_1) - 2 = f_1^2(f_1+2)(f_1-2)g(f_1) - f_1^2 = f_1^2((f_1+2)(f_1-2)g(f_1) - 1)$.

By (4.31), $$f_3 + 2 = f_2^2(f_2+2)(f_2-2)g(f_2) - f_1^2(f_1+2)(f_1-2)\times$$

$$(f_1^2(f_1+2)(f_1-2)g^2(f_1) - 2f_1^2 g(f_1) + 4g(f_1) + 1),$$

then, from (4.32), $f_3(\tilde{T}) \equiv -2 \pmod{2^{2u+2}}$, and so by induction, (4.31) and (4.32) we obtain

$$f_n(\tilde{T}) \equiv -2 \pmod{2^{2u+n-1}} \qquad (n \geq 3). \tag{4.33}$$

Claim that $2^{u+n} \mid (A_n^2(\tilde{T}) - 1)$ for all $n \geq 0$. We prove the claim by the mathematical induction. It is clear that $2^u \mid 0 = A_0^2(\tilde{T}) - 1$ and $A_1^2 - 1 = (f_1 - 1)^2 - 1 = f_1(f_1 - 2)$, and then $2^{u+1} \mid (A_1^2(\tilde{T}) - 1)$. From (4.1), we get

$$\begin{aligned} A_2^2 - 1 &= (1 + f_2 A_1)^2 - 1 = f_2(f_2 A_1^2 + 2A_1) = f_2((f_2 - 2)A_1^2 + 2A_1^2 + 2A_1) \\ &= f_2((f_2 - 2)A_1^2 + 2A_1(A_1 + 1)) = f_2((f_2 - 2)A_1^2 + 2A_1(f_1 - 1 + 1)), \end{aligned}$$

so by (4.32), we obtain $2^{u+2} \mid (A_2^2(\tilde{T}) - 1)$. Now assume $2^{u+k} \mid (A_k^2(\tilde{T}) - 1)$ for all $k = 0, 1, \ldots, n-1 \quad (n \geq 3)$. By the hypothesis

$$2^{u+n-1} \mid (A_{n-1}^2(\tilde{T}) - 1) \text{ and } 2^{u+n-2} \mid (A_{n-2}^2(\tilde{T}) - 1),$$

and so the latter leads to $2 \nmid A_{n-2}(\tilde{T})$. Hence we have

$$2^{u+n-2} \mid (f_{n-1}(\tilde{T}) A_{n-2}(\tilde{T}) + 2(-1)^{n-1}), \tag{4.34}$$

since, by (4.2), $A_{n-1}^2 - 1 = f_{n-1} A_{n-2}(f_{n-1} A_{n-2} + 2(-1)^{n-1})$ and, by (4.32) and (4.33), $2 \| f_{n-1}$. Thus the claim follows from (4.3), (4.33) and (4.34).

But (4.32) and (4.33) lead to $2^{u+n-1} \| \tilde{T} f_0 f_1 \ldots f_n$. Therefore, (4.29) follows from (4.27), (4.28) and (4.30).

Case (3°) $\tilde{T}$ is even.

Let u be the positive integer such that $2^u \| \tilde{T}$. Since $f_1(\tilde{T}) - 2 = \tilde{T}^2((\tilde{T} - 2)(\tilde{T} + 2)g(\tilde{T}) - 1)$,

$$f_1(\tilde{T}) \equiv 2 \pmod{2^{2u}}. \tag{4.35}$$

By (4.31), we have

$$f_2(\tilde{T})+2 = f_1^2(\tilde{T})(f_1(\tilde{T})+2)(f_1(\tilde{T})-2)g(f_1(\tilde{T})) - \tilde{T}^2(\tilde{T}+2)(\tilde{T}-2)\times \left(\tilde{T}^2(\tilde{T}+2)(\tilde{T}-2)g^2(\tilde{T}) - 2\tilde{T}^2 g(\tilde{T}) + 4g(\tilde{T}) + 1\right),$$

then, by using (4.35), $f_2(\tilde{T}) \equiv -2 \pmod{2^{2u+2}}$, and so by induction, (4.31) and (4.35) we obtain

$$f_n(\tilde{T}) \equiv -2 \pmod{2^{2u+n}} \qquad (n \geq 2). \tag{4.36}$$

Claim that $2^{2u+n} \mid (A_n^2(\tilde{T})-1)$ for all $n \geq 0$. We prove the claim by the mathematical induction. It is clear that $2^{2u} \mid 0 = A_0^2(\tilde{T})-1$ and $A_1^2 = f_1(f_1-2)$, then by (4.35) $2^{2u+1} \mid (A_1^2(\tilde{T})-1)$. Now assume $2^{2u+k} \mid (A_k^2(\tilde{T})-1)$ for all $k = 0, 1, \ldots, n-1$ $(n \geq 2)$. By the hypothesis

$$2^{2u+n-1} \mid (A_{n-1}^2(\tilde{T})-1) \text{ and } 2^{2u+n-2} \mid (A_{n-2}^2(\tilde{T})-1),$$

and so the latter leads to $2 \nmid A_{n-2}(\tilde{T})$. Hence we have

$$2^{2u+n-2} \mid (f_{n-1}(\tilde{T})A_{n-2}(\tilde{T}) + 2(-1)^{n-1}), \tag{4.37}$$

since, by (4.2), $A_{n-1}^2 - 1 = f_{n-1}A_{n-2}(f_{n-1}A_{n-2} + 2(-1)^{n-1})$ and, by (4.35) and (4.36), $2 \| f_{n-1}$. Thus the claim follows from (4.3), (4.36) and (4.37).

But (4.35) and (4.36) lead to $2^{2u+n} \| \tilde{T} f_0 f_1 \ldots f_n$. Therefore, (4.29) follows from (4.27), (4.28) and (4.30). $\square$

Lemma 4.1.11. *Let $f(T)$ be the polynomial of the form* (4.21)*, and let $T_2 = T_2(f) \geq 3$ be the smallest integer such that $2s-2 < f(s)$ for all integers $s \geq T_2$. If $\tilde{T}$ $(\geq T_2)$ is an integer, then*

$$0 < \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} = 2\theta_n(\tilde{T}; f) < 1 \qquad (n \geq 0).$$

Proof. By using (4.25), the proof is same as that of Lemma 4.1.5. □

Lemma 4.1.12. *Let $f(T)$ be the polynomial of the form (4.21), and let $T_2 = T_2(f) \geq 3$ be the smallest integer such that $2s-2 < f(s)$ for all integers $s \geq T_2$. If $\tilde{T}$ $(\geq T_2)$ is an integer, then for $n \geq 0$,*

$$\left[\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right] = \begin{cases} \frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} + 2\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - 1 & ; \text{ if } n \text{ is odd} \\ \frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - 2\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} & ; \text{ if } n \text{ is even.} \end{cases}$$

Proof. Let $n \geq 0$. Combining Lemma 4.1.10, $B_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1)$ and (4.2) leads to

$$\frac{f_{n+1}(\tilde{T})A_n^2(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} = \frac{f_{n+1}(\tilde{T})(f_{n+1}(\tilde{T})A_n^2(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T}))}{\tilde{T}B_{n+1}(\tilde{T})}$$

$$= \frac{A_{n+1}^2(\tilde{T}) - 1}{\tilde{T}B_{n+1}(\tilde{T})} \in \mathbb{Z}.$$

Since $A_n^2(\tilde{T}) = D \cdot \tilde{T}B_n(\tilde{T}) + 1$ for some $D \in \mathbb{Z}$, we have that

$$\frac{f_{n+1}(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} \in \mathbb{Z},$$

i.e.,
$$\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} = E - \frac{2(-1)^{n+1}A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} \quad \text{for some } E \in \mathbb{Z}. \tag{4.38}$$

But we have from Lemma 4.1.11 that

$$0 < \frac{2A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} < 1. \tag{4.39}$$

Hence we obtain by (4.38) that

$$\left[\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right] = \begin{cases} E-1 = \frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} + 2\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - 1 & ; \text{ if n is odd} \\ \quad E \quad = \frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - 2\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} & ; \text{ if n is even.} \end{cases}$$

□

Lemma 4.1.13. *Let $f(T)$ be the polynomial of the form (4.21), and let $T_2 = T_2(f) \geq 3$ be the smallest integer such that $2s-2 < f(s)$ for all integers $s \geq T_2$. If $\tilde{T}$ $(\geq T_2)$ is an integer, then for each $n \geq 0$, $c_{n+1}(\tilde{T})$ defined by*

$$c_{n+1}(\tilde{T}) = \begin{cases} \left[\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right] - 1 & ; \textit{ if } n \textit{ is odd} \\ \left[\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right] & ; \textit{ if } n \textit{ is even.} \end{cases}$$

is a positive integer.

Proof. From Lemma 4.1.12 and (4.39), it suffices to show that

$$\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} > 2, \qquad \text{for all } n \geq 0.$$

We proceed by induction. For $n = 0$, Lemma 4.1.12 and $3 \leq \tilde{T} < f_1(\tilde{T})$ lead to

$$c_1(\tilde{T}) = \left[\frac{f_1(\tilde{T})}{\tilde{T}B_0(\tilde{T})}\right] = \frac{f_1(\tilde{T})}{\tilde{T}B_0(\tilde{T})} - 2\frac{A_0(\tilde{T})}{\tilde{T}B_0(\tilde{T})} = (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - 1 \geq 2,$$

and then by (4.39) we get $\frac{f_1(\tilde{T})}{\tilde{T}B_0(\tilde{T})} > 2$. Now assume that $\frac{f_{k+1}(\tilde{T})}{\tilde{T}B_k(\tilde{T})} > 2$ $(k \geq 0)$. Since $f_{k+2}(\tilde{T}) = f_{k+1}^2(\tilde{T})(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}^2(\tilde{T}) + 2$ and, by (4.25), $f_{k+1}(\tilde{T}) \geq 4$ and $f_{k+2}(\tilde{T}) \geq 5$, $g(f_{k+1}(\tilde{T})) \geq 1$. Then

$$f_{k+1}(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}(\tilde{T}) > 12f_{k+1}(\tilde{T}) - f_{k+1}(\tilde{T}) > f_{k+1}(\tilde{T}).$$

Hence

$$\begin{aligned} 2 &< \frac{f_{k+1}(\tilde{T}) \cdot f_{k+1}(\tilde{T})}{\tilde{T}B_k(\tilde{T}) \cdot f_{k+1}(\tilde{T})} \\ &< \frac{f_{k+1}(\tilde{T}) \cdot (f_{k+1}(\tilde{T})(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}(\tilde{T}))}{\tilde{T}B_{k+1}(\tilde{T})} \\ &< \frac{f_{k+2}(\tilde{T})}{\tilde{T}B_{k+1}(\tilde{T})} \end{aligned}$$

as required □

Now by making use of the above lemmas, we are ready to prove Theorem 4.1.8.

Proof of Theorem 4.1.8 For any non-negative integer r, denoted by $k(r)$ the length of the regular continued fraction which the last partial quotient is different from 1 of $\theta_r(\tilde{T};f)/\tilde{T} = \frac{A_r(\tilde{T})}{\tilde{T}B_r(\tilde{T})}$.

The proof will be completed by induction. We have by a direct calculation

$$\theta_0(\tilde{T};f)/\tilde{T} = [0;\tilde{T}^2] \quad \text{and} \quad \theta_1(\tilde{T};f)/\tilde{T} = [0;\tilde{T}^2,(\tilde{T}+2)(\tilde{T}-2)g(\tilde{T})-1,\tilde{T}^2],$$

which, by Lemma 4.1.12, $c_1(\tilde{T}) = \frac{f_1(\tilde{T})}{\tilde{T}B_0(\tilde{T})} - \frac{2A_0(\tilde{T})}{\tilde{T}B_0(\tilde{T})} = (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T})-1$, and hence the statement (4.22) holds for $n=0$. Now assume for $n \geq 1$ that the regular continued fraction which the last partial quotient is different from 1 of $\theta_n(\tilde{T};f)/\tilde{T}$ is expressed as

$$\theta_n(\tilde{T};f)/\tilde{T} = \begin{cases} [0;\alpha_1,\ldots,\alpha_{k(n-1)}-1,1,c_n(\tilde{T}),1,\alpha_{k(n-1)}-1,\ldots,\alpha_1] & ;\text{ if } n\text{-1 is odd} \\ [0;\alpha_1,\ldots,\alpha_{k(n-1)},c_n(\tilde{T}),\alpha_{k(n-1)},\ldots,\alpha_1] & ;\text{ if } n\text{-1 is even,} \end{cases}$$

if the regular continued fraction which the last partial quotient is different from 1 of $\theta_{n-1}(\tilde{T};f)/\tilde{T}$ is $[0;\alpha_1,\ldots,\alpha_{k(n-1)}]$. Then we have $k(n)$ is odd, and we write

$$\theta_n(\tilde{T};f)/\tilde{T} = [0;b_1,\ldots,b_{k(n)}] = \frac{p_{k(n)}}{q_{k(n)}}.$$

By using Lemma 4.1.10 we have $A_n(\tilde{T})$ and $B_n(\tilde{T})$ are relatively prime, so that

$$A_n(\tilde{T}) = p_{k(n)} \quad \text{and} \quad \tilde{T}B_n(\tilde{T}) = q_{k(n)}.$$

Since $k(n)$ is odd, we have by (1.7),

$$q_{k(n)}p_{k(n)-1} = p_{k(n)}q_{k(n)-1} - 1, \tag{4.40}$$

and hence by the hypothesis $[0; b_1, \ldots, b_{k(n)}]$ is palindromic we have by Remark 1.2.6

$$p_{k(n)} = q_{k(n)-1}. \tag{4.41}$$

Case n is odd.

By (1.5) we have $\frac{q_{k(n)}}{q_{k(n)-1}} = [b_{k(n)}; b_{k(n)-1}, \ldots, b_1]$ and a simple manipulation leads to

$$-(c_{n+1}(\tilde{T}) + 2) + \tfrac{q_{k(n)-1}}{q_{k(n)}} = [0; -1, 1, c_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1].$$

Hence by (2.24) and (1.6)

$$\begin{aligned}
&[0; b_1, \ldots, b_{k(n)} - 1, 1, c_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1] \\
&\qquad = [0; b_1, \ldots, b_{k(n)}, 0, -1, 1, c_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1] \\
&\qquad = \frac{\left(-c_{n+1}(\tilde{T}) - 2 + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(-c_{n+1}(\tilde{T}) - 2 + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}}.
\end{aligned}$$

From Lemma 4.1.12 we have

$$-c_{n+1}(\tilde{T}) - 2 = -\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} = -\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}}.$$

Thus we get

$$\begin{aligned}
&[0; b_1, \ldots, b_{k(n)} - 1, 1, c_{n+1}(\tilde{T}), 1, b_{k(n)} - 1, b_{k(n)-1}, \ldots, b_1] \\
&\qquad = \frac{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}},
\end{aligned}$$

and so we obtain by (4.40) and (4.41) that

$$\frac{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}}-\frac{2p_{k(n)}}{q_{k(n)}}+\frac{q_{k(n)-1}}{q_{k(n)}}\right)p_{k(n)}+p_{k(n)-1}}{\left(-\frac{f_{n+1}(\tilde{T})}{q_{k(n)}}-\frac{2p_{k(n)}}{q_{k(n)}}+\frac{q_{k(n)-1}}{q_{k(n)}}\right)q_{k(n)}+q_{k(n)-1}}=\frac{f_{n+1}(\tilde{T})p_{k(n)}+1}{f_{n+1}(\tilde{T})q_{k(n)}}$$
$$=\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})}+\frac{1}{\tilde{T}B_{n+1}(\tilde{T})}=\theta_{n+1}(\tilde{T};f)/\tilde{T}.$$

Case n is even.

We have by (1.5) that $c_{n+1}(\tilde{T})+\frac{q_{k(n)-1}}{q_{k(n)}}=[c_{n+1}(\tilde{T});b_{k(n)},b_{k(n)-1},\dots,b_1]$.

Hence by (1.6)

$$[0;b_1,\dots,b_{k(n)},c_{n+1}(\tilde{T}),b_{k(n)},b_{k(n)-1},\dots,b_1]=\frac{\left(c_{n+1}(\tilde{T})+\frac{q_{k(n)-1}}{q_{k(n)}}\right)p_{k(n)}+p_{k(n)-1}}{\left(c_{n+1}(\tilde{T})+\frac{q_{k(n)-1}}{q_{k(n)}}\right)q_{k(n)}+q_{k(n)-1}}.$$

From Lemma 4.1.12 we have

$$c_{n+1}(\tilde{T})=\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}-\frac{2A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})}=\frac{f_{n+1}(\tilde{T})}{q_{k(n)}}-\frac{2p_{k(n)}}{q_{k(n)}}.$$

Thus we get

$$[0;b_1,\dots,b_{k(n)},c_{n+1}(\tilde{T}),b_{k(n)},b_{k(n)-1},\dots,b_1]$$
$$=\frac{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}}-\frac{2p_{k(n)}}{q_{k(n)}}+\frac{q_{k(n)-1}}{q_{k(n)}}\right)p_{k(n)}+p_{k(n)-1}}{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}}-\frac{2p_{k(n)}}{q_{k(n)}}+\frac{q_{k(n)-1}}{q_{k(n)}}\right)q_{k(n)}+q_{k(n)-1}},$$

and so we obtain by (4.19) and (4.20) that

$$\frac{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}}-\frac{2p_{k(n)}}{q_{k(n)}}+\frac{q_{k(n)-1}}{q_{k(n)}}\right)p_{k(n)}+p_{k(n)-1}}{\left(\frac{f_{n+1}(\tilde{T})}{q_{k(n)}}-\frac{2p_{k(n)}}{q_{k(n)}}+\frac{q_{k(n)-1}}{q_{k(n)}}\right)q_{k(n)}+q_{k(n)-1}}=\frac{f_{n+1}(\tilde{T})p_{k(n)}-1}{f_{n+1}(\tilde{T})q_{k(n)}}$$
$$=\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})}-\frac{1}{\tilde{T}B_{n+1}(\tilde{T})}=\theta_{n+1}(\tilde{T};f)/\tilde{T}.$$

Therefore,

$$\theta_{n+1}(\tilde{T};f)/\tilde{T}=\begin{cases}[0;b_1,\dots,b_k-1,1,c_{n+1}(\tilde{T}),1,b_k-1,\dots,b_1] & ;\ \text{if } n \text{ is odd}\\ [0;b_1,\dots,b_k,c_{n+1}(\tilde{T}),b_k,\dots,b_1] & ;\ \text{if } n \text{ is even,}\end{cases}$$

which Lemma 4.1.13 leads to these continued fractions which the last partial quotients are different from 1 are regular. □

4.2 Formal series case

Throughout this section, we let $\mathbb{F}$ be a field of characteristic zero.

Analogues of Theorems 4.1.2 and 4.1.8 are investigated for continued fractions in the field of formal series over a field $\mathbb{F}$. We begin with the following analogous setup.

For $n \geq 0$, define $\theta_n(T;f)$ to be the series expressed as follows

$$\theta_n(T;f) = \sum_{m=0}^{n} \frac{(-1)^m}{f_0(T)f_1(T)\ldots f_m(T)}, \tag{4.42}$$

where $f(T) \in (\mathbb{F}[x])\,[T] \setminus \{0\}$; $f_0(T) = T$ and for all $i \geq 1$, $f_i(T) = f(f_{i-1}(T))$ with $T \in \mathbb{F}[x] \setminus \{0\}$, and for those $T \in \mathbb{F}[x] \setminus \{0\}$ for which the limit exists we define

$$\theta(T;f) = \lim_{n\to\infty} \theta_n(T;f).$$

For any $f(T) \in (\mathbb{F}[x])[T] \setminus \{0\}$, we put

$A_n = A_n(T) = (-1)^n + \sum_{m=1}^{n}(-1)^{m+1} f_m(T)f_{m+1}(T)\ldots f_n(T)$, $(n \geq 1)$; $A_0 = 1$,

$B_n = B_n(T) = f_0(T)f_1(T)\ldots f_n(T) \quad (n \geq 0)$.

Similar to the classical case, for any $f(T) \in (\mathbb{F}[x])\,[T] \setminus \{0\}$ and $n \geq 0$, A_n and B_n are the numerator and denominator of the series $\theta_n(T;f)$ given by (4.42), respectively, and for $n \geq 1$,

$$\begin{aligned} A_n(T) &= (-1)^n + \sum_{m=1}^{n}(-1)^{m+1} f_m(T)f_{m+1}(T)\ldots f_n(T) \\ &= (-1)^n + f_n(T)\left((-1)^{n-1} + \left(\sum_{m=1}^{n-1}(-1)^{m+1} f_m(T)f_{m+1}(T)\ldots f_{n-1}(T)\right)\right) \\ &= (-1)^n + f_n(T)\cdot A_{n-1}(T), \end{aligned} \tag{4.43}$$

$$\begin{aligned} A_n^2(T) - 1 &= ((-1)^n + f_n(T)A_{n-1}^2(T))^2 - 1 \\ &= f_n(T)(f_n(T)A_{n-1}^2(T) + 2(-1)^n A_{n-1}(T)) \end{aligned} \tag{4.44}$$

Lemma 4.2.1. *For any $f(T) \in (\mathbb{F}[x])[T] \setminus \{0\}$, we have for all $n, i \geq 0$,*

$$A_n(f_i(T)) = A_{n+i}(T) + D(T)f_i(T) \quad \textit{or} \quad A_n(f_i(T)) = -A_{n+i}(T) + D(T)f_i(T),$$

for some $D(T) \in (\mathbb{F}[x])[T]$.

Proof. It is obvious for the case $i = 0$. If $i > 0$ and $n = 0$, then the desired result follows from the definition of A_0 and (4.43). Now consider for $n, i \geq 1$,

$$\begin{aligned} A_{n+i}(T) &= (-1)^{n+i} + \sum_{m=1}^{n+i} (-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{n+i}(T) \\ &= (-1)^{n+i} + \sum_{m=1}^{i} (-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{n+i}(T) \\ &\qquad + \sum_{m=i+1}^{n+i} (-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{n+i}(T). \end{aligned}$$

Case n, i are even.

$$\begin{aligned} A_{n+i}(T) &= f_i(T) f_{i+1}(T) \dots f_{n+i}(T) \left(-1 + \sum_{m=1}^{i-1} (-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{i-1}(T) \right) \\ &\quad 1 + \sum_{m=1}^{n} (-1)^{m+1} f_{m+i}(T) f_{m+1+i}(T) \dots f_{n+i}(T) \\ &= f_i(T) f_{i+1}(T) \dots f_{n+i}(T) \left(-1 + \sum_{m=1}^{i-1} (-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{i-1}(T) \right) \\ &\quad + A_n(f_i(T)). \end{aligned}$$

Case n is even, i is odd.

$$\begin{aligned} A_{n+i}(T) &= f_i(T) f_{i+1}(T) \dots f_{n+i}(T) \left(1 + \sum_{m=1}^{i-1} (-1)^{m+1} f_m(T) f_{m+1}(T) \dots f_{i-1}(T) \right) \\ &\quad - 1 - \sum_{m=1}^{n} (-1)^{m+1} f_{m+i}(T) f_{m+1+i}(T) \dots f_{n+i}(T) \end{aligned}$$

$$= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(1 + \sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)$$

$$- A_n(f_i(T)).$$

Case n is odd, i is even

$$A_{n+i}(T) = f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(-1 + \sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)$$

$$- 1 + \sum_{m=1}^{n}(-1)^{m+1}f_{m+i}(T)f_{m+1+i}(T)\dots f_{n+i}(T)$$

$$= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(-1 + \sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)$$

$$+ A_n(f_i(T)).$$

Case n, i are odd.

$$A_{n+i}(T) = f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(1 + \sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)$$

$$1 - \sum_{m=1}^{n}(-1)^{m+1}f_{m+i}(T)f_{m+1+i}(T)\dots f_{n+i}(T)$$

$$= f_i(T)f_{i+1}(T)\dots f_{n+i}(T)\left(-1 + \sum_{m=1}^{i-1}(-1)^{m+1}f_m(T)f_{m+1}(T)\dots f_{i-1}(T)\right)$$

$$- A_n(f_i(T)).$$

Therefore, for all $n, i \geq 0$,

$$A_n(f_i(T)) = A_{n+i}(T) + D(T)f_i(T) \quad \text{or} \quad A_n(f_i(T)) = -A_{n+i}(T) + D(T)f_i(T),$$

for some $D(T) \in (\mathbb{F}[x])\,[T]$. $\square$

Lemma 4.2.2. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) \in (\mathbb{F}[x])\,[T]$. If $\tilde{T} \in \mathbb{F}[x] \setminus \{0\}$, then for all $n \geq 0$,

$$\tilde{T} \mid (A_n^2(\tilde{T}) - 1).$$

Proof. Since $f(T) = T(T+2)(T-2)g(T) - T^2 + 2$,

$$f_1(0) = 2 \quad \text{and} \quad f_n(0) = -2 \quad (n \geq 2).$$

Because $A_0(T) = 1$, then by (4.43) we have

$$\begin{aligned} A_1(0) &= (-1)^1 + f_1(0) \cdot A_0(0) = -1 + 2 \cdot 1 = 1, \\ A_2(0) &= (-1)^2 + f_2(0) \cdot A_1(0) = 1 + (-2) \cdot 1 = -1, \\ A_3(0) &= (-1)^3 + f_3(0) \cdot A_2(0) = -1 + (-2) \cdot (-1) = 1, \end{aligned}$$

proceed inductively we get

$$A_n(0) = (-1)^{n+1} \qquad (n \geq 1).$$

Hence we obtain for all $n \geq 0$,

$$A_n^2(T) - 1 = T \cdot D(T) \qquad \text{for some } \; D(T) \in (\mathbb{F}[x])\,[T],$$

and so the desired result follows. □

Lemma 4.2.3. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) \in (\mathbb{F}[x])\,[T]$. If $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$, then for all $n \geq 0$,

$$|B_n(\tilde{T})|_\infty \neq 0 \;\; \textit{and} \;\; B_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1).$$

Proof. Since $f(\tilde{T}) = \tilde{T}(\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}^2 + 2$ and $\tilde{T} \in \mathbb{F}[x] \smallsetminus \mathbb{F}$,

$$2 \leq |f_0(\tilde{T})|_\infty < |f_1(\tilde{T})|_\infty < |f_2(\tilde{T})|_\infty < \dots . \tag{4.45}$$

Thus $|B_n(\tilde{T})|_\infty \neq 0$ for all n. Now from Lemma 4.2.2, we get

$$f_n(\tilde{T}) \mid (A_n^2(f_n(\tilde{T})) - 1), \quad \text{for all } n \geq 0. \tag{4.46}$$

But we have from Lemma 4.2.1 that for any non-negative integers n, i,

$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) + 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

or

$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) - 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

for some $D \in \mathbb{F}[x]$. Thus by (4.46)

$$f_i(\tilde{T}) \mid (A_{n+i}^2(\tilde{T}) - 1) \quad \text{for all } n, i \geq 0.$$

More precisely,

$$f_i(\tilde{T}) \mid (A_{(n-i)+i}^2(\tilde{T}) - 1) = (A_n^2(\tilde{T}) - 1) \quad ; \quad i = 0, 1, \dots n. \tag{4.47}$$

It remains to prove that

$$B_n(\tilde{T}) = f_0(\tilde{T})f_1(\tilde{T}) \dots f_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1). \tag{4.48}$$

Since for any non-negative integers j, k such that $j < k$

$$f_k(\tilde{T}) = f_{k-j}(f_j(\tilde{T})) \equiv f_{k-j}(0) \pmod{f_j(\tilde{T})},$$

noticing here, from the proof of Lemma 4.2.2, that

$$f_{k-j}(0) = \begin{cases} 2 & \text{for } k = j+1 \\ -2 & \text{for } k > j+1, \end{cases}$$

we obtain

$$\gcd(f_j(\tilde{T}), f_k(\tilde{T})) = \gcd(f_j(\tilde{T}), 2) = 1. \tag{4.49}$$

Therefore, (4.48) follows from (4.47) and (4.49). $\square$

Lemma 4.2.4. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) \in (\mathbb{F}[x])[T]$. If $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$, then for all $n \geq 0$,

$$0 < \left|\frac{A_n(\tilde{T})}{B_n(\tilde{T})}\right|_\infty < 1 \quad \text{and} \quad \left|\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}\right|_\infty \geq 2.$$

Proof. It is obvious by the definition that

$$\left|\frac{A_0(\tilde{T})}{B_0(\tilde{T})}\right|_\infty = \frac{1}{|f_0(\tilde{T})|_\infty} = \frac{1}{|\,\tilde{T}\,|_\infty}.$$

For $n \geq 1$, we have by (4.45)

$$2 \leq |f_0(\tilde{T})|_\infty < |f_1(\tilde{T})|_\infty < |f_2(\tilde{T})|_\infty < \ldots,$$

and hence by (1.1)

$$\left|\frac{A_n(\tilde{T})}{B_n(\tilde{T})}\right|_\infty = \left|\frac{(-1)^n + \sum_{m=1}^{n}(-1)^{m+1} f_m(\tilde{T}) f_{m+1}(\tilde{T}) \ldots f_n(\tilde{T})}{f_0(\tilde{T}) f_1(\tilde{T}) \ldots f_n(\tilde{T})}\right|_\infty$$

$$= \left| \frac{f_1(\tilde{T})f_2(\tilde{T})\dots f_n(\tilde{T})}{f_0(\tilde{T})f_1(\tilde{T})\dots f_n(\tilde{T})} \right|_\infty = \frac{1}{|f_0(\tilde{T})|_\infty} = \frac{1}{|\ \tilde{T}\ |_\infty}.$$

Thus we get for all $n \geq 0$,

$$0 < \left| \frac{A_n(\tilde{T})}{B_n(\tilde{T})} \right|_\infty < 1,$$

since $\tilde{T} \in \mathbb{F}[x] \smallsetminus \mathbb{F}$.

Next, we will show $\left| \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} \right|_\infty \geq 2$ by using the induction. Since

$$\left| \frac{f_1(\tilde{T})}{B_0(\tilde{T})} \right|_\infty = \left| \frac{\tilde{T}(\tilde{T}+2)(\tilde{T}-1)g(\tilde{T}) - \tilde{T}^2 + 2}{\tilde{T}} \right|_\infty = \left| \frac{\tilde{T}(\tilde{T}+2)(\tilde{T}-1)g(\tilde{T}) - \tilde{T}^2}{\tilde{T}} \right|_\infty$$

$$= \left| (\tilde{T}+2)(\tilde{T}-1)g(\tilde{T}) - \tilde{T} \right|_\infty = \max \left\{ \left| (\tilde{T}+2)(\tilde{T}-1)g(\tilde{T}) \right|_\infty, \left| \tilde{T} \right|_\infty \right\}$$

$$\geq 2,$$

the statement holds for $n = 0$. Now assume $\left| \frac{f_{k+1}(\tilde{T})}{B_k(\tilde{T})} \right|_\infty \geq 2 \quad (k \geq 0)$. Since

$$|f_{k+1}(\tilde{T})|_\infty < |f_{k+1}(\tilde{T})|_\infty |(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}(\tilde{T})|_\infty,$$

we obtain by the hypothesis

$$2 \leq \left| \frac{f_{k+1}(\tilde{T})}{B_k(\tilde{T})} \right|_\infty = \frac{|f_{k+1}(\tilde{T})|_\infty |f_{k+1}(\tilde{T})|_\infty}{|f_{k+1}(\tilde{T})|_\infty |B_k(\tilde{T})|_\infty}$$

$$< \frac{|f_{k+1}(\tilde{T})|_\infty |(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}|_\infty}{|B_{k+1}(\tilde{T})|_\infty} = \frac{|f_{k+2}(\tilde{T})|_\infty}{|B_{k+1}(\tilde{T})|_\infty}. \qquad \square$$

Now we are ready to state the analogue of Theorem 4.1.2.

Theorem 4.2.5. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) \in (\mathbb{F}[x])\,[T]$. If $\tilde{T} \in \mathbb{F}[x] \smallsetminus \mathbb{F}$, then

$$\theta_0(\tilde{T}; f) = [0; \tilde{T}],$$

and for all $n \geq 0$,

$$\theta_{n+1}(\tilde{T}; f) = [0; b_1, \ldots, b_k, u_{n+1}(\tilde{T}), b_k, \ldots, b_1], \tag{4.50}$$

if $[0; b_1, \ldots, b_k]$ *is a palindromic continued fraction representing* $\theta_n(\tilde{T}; f)$ *and*

$$u_{n+1}(\tilde{T}) = (-1)^n \delta_n^2 \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - 2\frac{A_n(\tilde{T})}{B_n(\tilde{T})},$$

where δ_n *is the element in* $\mathbb{F} \setminus \{0\}$ *such that* $A_n(\tilde{T}) = \delta_n p_k$ *and* $B_n(\tilde{T}) = \delta_n q_k$ *provided* $\frac{p_k}{q_k}$ *is the* k^{th} *(last) convergent of* $\theta_n(\tilde{T}; f)$ *respect to* $[0; b_1, \ldots, b_k]$.
In particular,

$$\theta(\tilde{T}; f) = [0; \tilde{T}, u_1(\tilde{T}), \tilde{T}, u_2(\tilde{T}), \tilde{T}, u_1(\tilde{T}), \tilde{T}, u_3(\tilde{T}), \ldots].$$

Proof. For any non-negative integer r, denoted by $k(r)$ the length of a continued fraction representing $\theta_r(\tilde{T}; f) = \frac{A_r(\tilde{T})}{B_r(\tilde{T})}$ which all partial numerators are 1.

The proof will be completed by induction. We have by a direct calculation

$$\theta_0(\tilde{T}; f) = [0; \tilde{T}] \quad \text{and} \quad \theta_1(\tilde{T}; f) = [0; \tilde{T}, (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}, \tilde{T}],$$

which $u_1(\tilde{T}) = \frac{(-1)^0 \delta_0^2 f_1(\tilde{T})}{B_0(\tilde{T})} - \frac{2A_0(\tilde{T})}{B_0(\tilde{T})} = \frac{f_1(\tilde{T})}{\tilde{T}} - \frac{2}{\tilde{T}} = (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}$, and hence the statement (4.50) holds for $n = 0$. Now suppose for $n \geq 1$ that

$$\theta_n(\tilde{T}; f) = [0; \alpha_1, \ldots, \alpha_{k(n-1)}, u_n(\tilde{T}), \alpha_{k(n-1)}, \ldots, \alpha_1],$$

if $[0; \alpha_1, \ldots, \alpha_{k(n-1)}]$ is a palindromic continued fraction representing $\theta_{n-1}(\tilde{T}; f)$ and

$$u_n(\tilde{T}) = (-1)^{n-1} \delta_{n-1}^2 \frac{f_n(\tilde{T})}{B_{n-1}(\tilde{T})} - 2\frac{A_{n-1}(\tilde{T})}{B_{n-1}(\tilde{T})},$$

We write

$$\theta_n(\tilde{T}; f) = [0; b_1, \ldots, b_{k(n)}] = \frac{p_{k(n)}}{q_{k(n)}},$$

and then $k(n)$ is odd. By using Lemma 4.2.3, we have $A_n(\tilde{T})$ and $B_n(\tilde{T})$ are relatively prime, so that there exists $\delta_n \in \mathbb{F} \setminus \{0\}$ such that

$$A_n(\tilde{T}) = \delta_n p_{k(n)} \quad \text{and} \quad B_n(\tilde{T}) = \delta_n q_{k(n)}.$$

Since $k(n)$ is odd, we have by (1.7),

$$q_{k(n)} p_{k(n)-1} = p_{k(n)} q_{k(n)-1} - 1, \tag{4.51}$$

and hence by the hypothesis $[0; b_1, \ldots, b_{k(n)}]$ is palindromic we have by Remark 1.2.6

$$p_{k(n)} = q_{k(n)-1}. \tag{4.52}$$

Case n is odd.

We have by (1.5) that

$$\begin{aligned}\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}} &= \frac{-\delta_n^2 f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} + \frac{q_{k(n)-1}}{q_{k(n)}} \\ &= [u_{n+1}(\tilde{T}); b_{k(n)}, \ldots, b_1].\end{aligned}$$

Hence by (1.6)

$$\begin{aligned}&[0; b_1, \ldots, b_{k(n)}, u_{n+1}(\tilde{T}), b_{k(n)}, \ldots, b_1] \\ &= \frac{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}},\end{aligned}$$

and so we obtain by (4.51) and (4.52) that

$$\begin{aligned}\frac{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}} &= \frac{p_{k(n)}}{q_{k(n)}} + \frac{1}{\delta_n f_{n+1}(\tilde{T}) q_{k(n)}} \\ &= \frac{A_n(\tilde{T})}{B_n(\tilde{T})} + \frac{1}{B_{n+1}(\tilde{T})} = \theta_{n+1}(\tilde{T}).\end{aligned}$$

Case n is even.

We have by (1.5) that

$$\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}} = \frac{\delta_n^2 f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} + \frac{q_{k(n)-1}}{q_{k(n)}}$$
$$= [u_{n+1}(\tilde{T}); a_{k(n)}, \ldots, a_1].$$

Hence by (1.6)

$[0; b_1, \ldots, b_{k(n)}, u_{n+1}(\tilde{T}), b_{k(n)}, \ldots, b_1]$

$$= \frac{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}},$$

and so we obtain by (4.51) and (4.52) that

$$\frac{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}} = \frac{p_{k(n)}}{q_{k(n)}} + \frac{-1}{\delta_n f_{n+1}(\tilde{T}) q_{k(n)}}$$
$$= \frac{A_n(\tilde{T})}{B_n(\tilde{T})} + \frac{-1}{B_{n+1}(\tilde{T})} = \theta_{n+1}(\tilde{T}).$$

Therefore, the theorem is established. □

Remark 4.2.6. Different from the case of real numbers, we cannot assure that the continued fractions produced by the above theorem are regular. Because for each $n \geq 0$, u_{n+1}, that we added into a given palindromic continued fraction of θ_n to produce a continued fraction of θ_{n+1}, is in $\mathbb{F}[x]$ only the case $\delta_n = \pm 1$. This fact is proven in the following lemma.

Lemma 4.2.7. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) \in (\mathbb{F}[x])[T]$. Let $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$. Then for $n \geq 0$, $u_{n+1}(\tilde{T})$, defined as in Theorem 4.2.5, is in $\mathbb{F}[x]$ if and only if $\delta_n = \pm 1$.

Proof. Let n be a non-negative integer. From Lemma 4.2.3 and (4.44), we have

$$\begin{aligned}\frac{f_{n+1}(\tilde{T})A_n^2(\tilde{T})+2(-1)^{n+1}A_n(\tilde{T})}{B_n(\tilde{T})} &= \frac{f_{n+1}(\tilde{T})(f_{n+1}(\tilde{T})A_n^2(\tilde{T})+2(-1)^{n+1}A_n(\tilde{T}))}{B_{n+1}(\tilde{T})} \\ &= \frac{A_{n+1}^2(\tilde{T})-1}{B_{n+1}(\tilde{T})} \in \mathbb{F}[x] \qquad (4.53)\end{aligned}$$

Since, by Lemma 4.2.3, $B_n(\tilde{T}) \mid (A_n^2(\tilde{T})-1)$,

$$A_n^2(\tilde{T}) = D\cdot B_n(\tilde{T})+1 \quad \text{for some } D\in\mathbb{F}[x].$$

Then from (4.53), we have

$$\frac{f_{n+1}(\tilde{T})+2(-1)^{n+1}A_n(\tilde{T})}{B_n(\tilde{T})} \in \mathbb{F}[x],$$

i.e., $$\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} = E - \frac{2(-1)^{n+1}A_n(\tilde{T})}{B_n(\tilde{T})} \quad \text{for some } E\in\mathbb{F}[x]. \qquad (4.54)$$

We have by Lemma 4.2.4 that

$$0 \le \left|\frac{2(\delta_n^2-1)A_n(\tilde{T})}{B_n(\tilde{T})}\right|_\infty < 1. \qquad (4.55)$$

Hence the lemma is established by considering

$$\begin{aligned}u_{n+1}(\tilde{T}) &= \frac{(-1)^n\delta_n^2 f_{n+1}(\tilde{T})}{B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{B_n(\tilde{T})} \\ &= (-1)^n\delta_n^2\left(\frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})}+\frac{2(-1)^{n+1}A_n(\tilde{T})}{B_n(\tilde{T})}\right)+\frac{2(\delta_n^2-1)A_n(\tilde{T})}{B_n(\tilde{T})},\end{aligned}$$

with (4.54) and (4.55). □

From the above lemma, if there exist $n \ge 0$ such that $\delta_n \ne \pm 1$, then continued fractions produced by Theorem 4.2.5 are not regular. In this case, it is natural to worry about the convergence of $[0;\tilde{T},u_1(\tilde{T}),\tilde{T},u_2(\tilde{T}),\tilde{T},u_1(\tilde{T}),\tilde{T},u_3(\tilde{T}),\ldots]$.

This problem is treated by using a classical theorem of Pringsheim:
For each $i \geq 1$, denoted by a_i and b_i the i^{th} partial numerator and denominator of $[0;\tilde{T},u_1(\tilde{T}),\tilde{T},u_2(\tilde{T}),\tilde{T},u_1(\tilde{T}),\tilde{T},u_3(\tilde{T}),\ldots]$, respectively. By using Lemma 4.2.4 we have that

$$|a_i|_\infty = 1 \qquad \text{and} \qquad |b_i|_\infty \geq 2 \qquad \text{for all } i \geq 1.$$

Therefore, the convergence of $[0;\tilde{T},u_1(\tilde{T}),\tilde{T},u_2(\tilde{T}),\tilde{T},u_1(\tilde{T}),\tilde{T},u_3(\tilde{T}),\ldots]$ is guaranteed by Theorem 1.2.3.

Next, the following lammas are prepared to organize an analogue of Theorem 4.1.8.

Lemma 4.2.8. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T^2(T+2)(T-2)g(T) - T^2 + 2,$$

$g(T) \in (\mathbb{F}[x])[T] \setminus \{0\}$. *If $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$, then for all $n \geq 0$,*

$$\tilde{T}^2 \mid (A_n^2(\tilde{T}) - 1).$$

Proof. Similar to the proof of Lemma 4.2.2, we obtain

$$f_1(0) = 2, \qquad f_n(0) = -2 \qquad (n \geq 2), \tag{4.56}$$

and for all $n \geq 0$,

$$A_n^2(T) - 1 = T \cdot D(T) \qquad \text{for some } D(T) \in (\mathbb{F}[x])[T]. \tag{4.57}$$

Hence we will prove this lemma by showing that

$$\frac{d}{dT}(A_n^2(T) - 1) = 0 \quad \text{at } T = 0, \qquad \text{for all } n \geq 0. \tag{4.58}$$

It is obvious for the case $n = 0$. Now we consider the cases $n \geq 1$. By (4.44), we have

$$\begin{aligned}\frac{d}{dT}(A_n^2(T) - 1) &= 2f_n(T)f_n'(T)A_{n-1}^2(T) + 2f_n^2(T)A_{n-1}(T)A_{n-1}'(T) \\ &\quad + 2(-1)^n f_n'(T)A_{n-1}(T) + 2(-1)^n f_n(T)A_{n-1}'(T),\end{aligned}$$

then, to prove (4.58), it suffices to show that

$$f_n'(0) = 0 \qquad \text{and} \qquad A_{n-1}'(0) = 0, \qquad \text{for all } \ n \geq 1.$$

Since $f_1(T) = T^2(T+2)(T-2)g(T) - T^2 + 2$,

$$f_1'(T) = 2T(T+2)(T-2)g(T) + T^2((T+2)(T-2)g(T))' - 2T,$$

and so $f_1'(0) = 0$. Now assume that $f_k'(0) = 0 \quad (k \geq 1)$. From the definition of f_{k+1}, we get

$$\begin{aligned}f_{k+1}'(T) &= 2f_k(T)f_k'(T)(f_k(T)+2)(f_k(T)-2)g(f_k(T)) \\ &\quad + f_k^2(T)(f_k(T)+2)(f_k(T)-2)(g(f_k(T)))' \\ &\quad + f_k^2(T)\left(f_k'(T)(f_k(T)-2) + (f_k(T)+2)f_k'(T)\right)g(f_k(T)) + 2f_k(T)f_k'(T).\end{aligned}$$

Hence the induction hypothesis and (4.56) lead to $f_{k+1}'(0) = 0$. Thus for all $n \geq 1$, we have that $f_n'(0) = 0$, and so by the mathematical induction, the definition of A_0 and (4.43) we also have for all $n \geq 1$, $\quad A_{n-1}'(0) = 0$. □

Lemma 4.2.9. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T^2(T+2)(T-2)g(T) - T^2 + 2,$$

$g(T) \in (\mathbb{F}[x])[T] \setminus \{0\}$. *If $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$, then for all $n \geq 0$,*

$$|\tilde{T}B_n(\tilde{T})|_\infty \neq 0 \ \ \textit{and} \ \ \tilde{T}B_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1).$$

Proof. Since $f(\tilde{T}) = \tilde{T}^2(\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - \tilde{T}^2 + 2$ and $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$,

$$2 \leq |f_0(\tilde{T})|_\infty < |f_1(\tilde{T})|_\infty < |f_2(\tilde{T})|_\infty < \ldots.$$

Thus $|\tilde{T}B_n(\tilde{T})|_\infty \neq 0$ for all n. Now from Lemma (4.57), we get for all $n \geq 0$,

$$f_n(\tilde{T}) \mid (A_n^2(f_n(\tilde{T})) - 1). \tag{4.59}$$

But from Lemma 4.2.1, we have for any non-negative integers n, i,

$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) + 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

or
$$A_n^2(f_i(\tilde{T})) = A_{n+i}^2(\tilde{T}) - 2Df_i(\tilde{T})A_{n+i}(\tilde{T}) + D^2 f_i^2(\tilde{T}),$$

for some $D \in \mathbb{F}[x]$ and so (4.59) implies

$$f_i(\tilde{T}) \mid (A_{n+i}^2(\tilde{T}) - 1) \qquad \text{for all } \ n, i \geq 0.$$

More precisely,

$$f_i(\tilde{T}) \mid (A_{(n-i)+i}^2(\tilde{T}) - 1) = (A_n^2(\tilde{T}) - 1) \quad ; \quad i = 0, 1, \ldots n. \tag{4.60}$$

Also, we have from Lemma 4.2.8 that

$$\tilde{T}^2 \mid (A_n^2(\tilde{T}) - 1). \tag{4.61}$$

It remains to prove that

$$\tilde{T}B_n(\tilde{T}) = \tilde{T}^2 f_1(\tilde{T}) \ldots f_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1). \tag{4.62}$$

Since for any non-nengative integers j, k such that $j < k$

$$f_k(\tilde{T}) = f_{k-j}(f_j(\tilde{T})) \equiv f_{k-j}(0) \pmod{f_j(\tilde{T})},$$

noticing here, from (4.56), that

$$f_{k-j}(0) = \begin{cases} 2 & \text{for } k = j+1 \\ -2 & \text{for } k > j+1, \end{cases}$$

we obtain

$$\gcd(f_j(\tilde{T}), f_k(\tilde{T})) = \gcd(f_j(\tilde{T}), 2) = 1. \tag{4.63}$$

Therefore, (4.62) follows from (4.60), (4.61) and (4.63). □

Lemma 4.2.10. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T^2(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) = w_m T^m + \ldots + w_1 T + w_0$ with $m \geq 0$, $w_i \in \mathbb{F}[x]$ $(0 \leq i \leq m)$, $w_m \neq 0$ and $|w_m|_\infty \geq |w_i|_\infty$ for all $0 \leq i \leq m-1$. If $\tilde{T} \in \mathbb{F}[x] \smallsetminus \mathbb{F}$, then for all $n \geq 0$,

$$0 < \left| \frac{A_n(\tilde{T})}{B_n(\tilde{T})} \right|_\infty < 1 \quad \text{and} \quad \left| \frac{f_{n+1}(\tilde{T})}{B_n(\tilde{T})} \right|_\infty \geq 2.$$

Proof. It is obvious that

$$\left| \frac{A_0(\tilde{T})}{\tilde{T} B_0(\tilde{T})} \right|_\infty = \frac{1}{|f_0(\tilde{T})|_\infty} = \frac{1}{|\ \tilde{T}^2\ |_\infty}.$$

The definitions of $f(T)$, $g(T)$ and $\tilde{T}$ lead to

$$2 \leq |f_0(\tilde{T})|_\infty < |f_1(\tilde{T})|_\infty < |f_2(\tilde{T})|_\infty < \ldots.$$

Then we obtain, by (1.1), for $n \geq 1$,

$$\left|\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right|_\infty = \left|\frac{(-1)^n + \sum_{i=1}^{n}(-1)^{i+1} f_i(\tilde{T}) f_{i+1}(\tilde{T}) \dots f_n(\tilde{T})}{\tilde{T} f_0(\tilde{T}) f_1(\tilde{T}) \dots f_n(\tilde{T})}\right|_\infty$$

$$= \left|\frac{f_1(\tilde{T}) f_2(\tilde{T}) \dots f_n(\tilde{T})}{\tilde{T} f_0(\tilde{T}) f_1(\tilde{T}) \dots f_n(\tilde{T})}\right|_\infty = \frac{1}{|f_0(\tilde{T})|_\infty} = \frac{1}{|\ \tilde{T}^2\ |_\infty}.$$

Thus we get for all $n \geq 0$,

$$0 < \left|\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right|_\infty < 1,$$

since $\tilde{T} \in \mathbb{F}[x] \smallsetminus \mathbb{F}$.

Next, we will show $\left|\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})}\right|_\infty \geq 2$ by using the induction. We have by the definition of $g(T)$ that

$$|g(S)|_\infty \geq 1 \quad \text{for all } S \in \mathbb{F}[x] \smallsetminus \mathbb{F}.$$

Hence

$$\left|\frac{f_1(\tilde{T})}{\tilde{T}B_0(\tilde{T})}\right|_\infty = \left|\frac{\tilde{T}^2(\tilde{T}+2)(\tilde{T}-1)g(\tilde{T}) - \tilde{T}^2 + 2}{\tilde{T}^2}\right|_\infty$$

$$= \left|(\tilde{T}+2)(\tilde{T}-1)g(\tilde{T}) - 1\right|_\infty > 2,$$

and so the statement holds for $n = 0$. Now assume $\left|\frac{f_{k+1}(\tilde{T})}{\tilde{T}B_k(\tilde{T})}\right|_\infty \geq 2 \quad (k \geq 0)$. Since

$$|f_{k+1}(\tilde{T})|_\infty < |f_{k+1}(\tilde{T})|_\infty |f_{k+1}(\tilde{T})(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}(\tilde{T})|_\infty,$$

we obtain by the hypothesis

$$2 \leq \left|\frac{f_{k+1}(\tilde{T})}{\tilde{T}B_k(\tilde{T})}\right|_\infty = \frac{|f_{k+1}(\tilde{T})|_\infty |f_{k+1}(\tilde{T})|_\infty}{|f_{k+1}(\tilde{T})|_\infty |\tilde{T}B_k(\tilde{T})|_\infty}$$

$$< \frac{|f_{k+1}(\tilde{T})|_\infty |f_{k+1}(\tilde{T})(f_{k+1}(\tilde{T})+2)(f_{k+1}(\tilde{T})-2)g(f_{k+1}(\tilde{T})) - f_{k+1}(\tilde{T})|_\infty}{|\tilde{T}B_{k+1}(\tilde{T})|_\infty}$$

$$= \frac{|f_{k+2}(\tilde{T})|_\infty}{|\tilde{T}B_{k+1}(\tilde{T})|_\infty}.$$

□

Theorem 4.2.11. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T^2(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) = w_mT^m + \ldots + w_1T + w_0$ with $m \geq 0$, $w_i \in \mathbb{F}[x]$ $(0 \leq i \leq m)$, $w_m \neq 0$ and $|w_m|_\infty \geq |w_i|_\infty$ for all $0 \leq i \leq m-1$. If $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$, then

$$\theta_0(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2],$$

and for all $n \geq 0$,

$$\theta_{n+1}(\tilde{T}; f)/\tilde{T} = [0; b_1, \ldots, b_k, v_{n+1}(\tilde{T}), b_k, \ldots, b_1], \tag{4.64}$$

if $[0; b_1, \ldots, b_k]$ is a palindromic continued fraction representing $\theta_n(\tilde{T}; f)/\tilde{T}$ and

$$v_{n+1}(\tilde{T}) = (-1)^n \delta_n^2 \frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - 2\frac{A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})},$$

where δ_n is the element in $\mathbb{F} \setminus \{0\}$ such that $A_n(\tilde{T}) = \delta_n p_k$ and $\tilde{T}B_n(\tilde{T}) = \delta_n q_k$ provided $\frac{p_k}{q_k}$ is the k^{th} convergent of $\theta_n(\tilde{T}; f)/\tilde{T}$ respect to $[0; b_1, \ldots, b_k]$.
In particular,

$$\theta(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2, v_1(\tilde{T}), \tilde{T}^2, v_2(\tilde{T}), \tilde{T}^2, v_1(\tilde{T}), \tilde{T}^2, v_3(\tilde{T}), \ldots].$$

Proof. For any non-negative integer r, denoted by $k(r)$ the length of a continued fraction representing $\theta_r(\tilde{T}; f) = \frac{A_r(\tilde{T})}{B_r(\tilde{T})}$ which all partial numerators are 1.

The proof will be completed by induction. We have by a direct calculation

$$\theta_0(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2] \quad \text{and} \quad \theta_1(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2, (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - 1, \tilde{T}^2],$$

which $v_1(\tilde{T}) = \frac{(-1)^0\delta_0^2 f_1(\tilde{T})}{\tilde{T}B_0(\tilde{T})} - \frac{2A_0(\tilde{T})}{\tilde{T}B_0(\tilde{T})} = \frac{f_1(\tilde{T})}{\tilde{T}^2} - \frac{2}{\tilde{T}^2} = (\tilde{T}+2)(\tilde{T}-2)g(\tilde{T}) - 1$, and hence the statement (4.64) holds for $n = 0$. Now suppose for $n \geq 1$ that

$$\theta_n(\tilde{T}; f)/\tilde{T} = [0; \alpha_1, \ldots, \alpha_{k(n-1)}, v_n(\tilde{T}), \alpha_{k(n-1)}, \ldots, \alpha_1],$$

if $[0;\alpha_1,\ldots,\alpha_{k(n-1)}]$ is a palindromic continued fraction representing $\theta_{n-1}(\tilde{T};f)/\tilde{T}$ and

$$v_n(\tilde{T}) = (-1)^{n-1}\delta_{n-1}^2\frac{f_n(\tilde{T})}{\tilde{T}B_{n-1}(\tilde{T})} - 2\frac{A_{n-1}(\tilde{T})}{\tilde{T}B_{n-1}(\tilde{T})}.$$

We write

$$\theta_n(\tilde{T};f)/\tilde{T} = [0;b_1,\ldots,b_{k(n)}] = \frac{p_{k(n)}}{q_{k(n)}},$$

and then $k(n)$ is odd. By using Lemma 4.2.9, we have $A_n(\tilde{T})$ and $\tilde{T}B_n(\tilde{T})$ are relatively prime, so that there exists $\delta_n \in \mathbb{F}\setminus\{0\}$ such that

$$A_n(\tilde{T}) = \delta_n p_{k(n)} \quad \text{and} \quad \tilde{T}B_n(\tilde{T}) = \delta_n q_{k(n)}.$$

Since $k(n)$ is odd, we have by (1.7),

$$q_{k(n)}p_{k(n)-1} = p_{k(n)}q_{k(n)-1} - 1, \tag{4.65}$$

and hence by the hypothesis $[0;b_1,\ldots,b_{k(n)}]$ is palindromic, we have by Remark 1.2.6

$$p_{k(n)} = q_{k(n)-1}, \tag{4.66}$$

Case n is odd.

We have by (1.5) that

$$\begin{aligned}\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}} &= \frac{-\delta_n^2 f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} + \frac{q_{k(n)-1}}{q_{k(n)}}\\ &= [v_{n+1}(\tilde{T});b_{k(n)},\ldots,b_1].\end{aligned}$$

Hence by (1.6)

$$\begin{aligned}&[0;b_1,\ldots,b_{k(n)},v_{n+1}(\tilde{T}),b_{k(n)},\ldots,b_1]\\ &\qquad = \frac{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right)p_{k(n)} + p_{k(n)-1}}{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right)q_{k(n)} + q_{k(n)-1}},\end{aligned}$$

and so we obtain by (4.65) and (4.66) that

$$\frac{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{-\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}} = \frac{p_{k(n)}}{q_{k(n)}} + \frac{1}{\delta_n f_{n+1}(\tilde{T}) q_{k(n)}}$$

$$= \frac{A_n(\tilde{T})}{\tilde{T} B_n(\tilde{T})} + \frac{1}{\tilde{T} B_{n+1}(\tilde{T})} = \theta_{n+1}(\tilde{T})/\tilde{T}.$$

Case n is even.

We have by (1.5) that

$$\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}} = \frac{\delta_n^2 f_{n+1}(\tilde{T})}{\tilde{T} B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{\tilde{T} B_n(\tilde{T})} + \frac{q_{k(n)-1}}{q_{k(n)}}$$

$$= [v_{n+1}(\tilde{T}); b_{k(n)}, \ldots, b_1].$$

Hence by (1.6)

$$[0; b_1, \ldots, b_{k(n)}, v_{n+1}(\tilde{T}), b_{k(n)}, \ldots, b_1] = \frac{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}},$$

and so we obtain by (4.51) and (4.52) that

$$\frac{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) p_{k(n)} + p_{k(n)-1}}{\left(\frac{\delta_n f_{n+1}(\tilde{T})}{q_{k(n)}} - \frac{2p_{k(n)}}{q_{k(n)}} + \frac{q_{k(n)-1}}{q_{k(n)}}\right) q_{k(n)} + q_{k(n)-1}} = \frac{p_{k(n)}}{q_{k(n)}} + \frac{-1}{\delta_n f_{n+1}(\tilde{T}) q_{k(n)}}$$

$$= \frac{A_n(\tilde{T})}{\tilde{T} B_n(\tilde{T})} + \frac{-1}{\tilde{T} B_{n+1}(\tilde{T})} = \theta_{n+1}(\tilde{T})/\tilde{T}.$$

Therefore, the theorem is established. □

Similar to Theorem 4.2.5, the continued fractions produced by Theorem 4.2.11 may not be regular described by the following lemma. The problem about the convergence of $[0; \tilde{T}^2, v_1(\tilde{T}), \tilde{T}^2, v_2(\tilde{T}), \tilde{T}^2, v_1(\tilde{T}), \tilde{T}^2, v_3(\tilde{T}), \ldots]$ is handled by using Lemma 4.2.10 and then Theorem 1.2.3.

Lemma 4.2.12. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T^2(T+2)(T-2)g(T) - T^2 + 2,$$

where $g(T) = w_mT^m + \ldots + w_1T + w_0$ *with* $m \geq 0$, $w_i \in \mathbb{F}[x]$ $(0 \leq i \leq m)$, $w_m \neq 0$ *and* $|w_m|_\infty \geq |w_i|_\infty$ *for all* $0 \leq i \leq m-1$. *Let* $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$. *Then for* $n \geq 0$, $v_{n+1}(\tilde{T})$, *defined as in Theorem 4.2.11, is in* $\mathbb{F}[x]$ *if and only if* $\delta_n = \pm 1$.

Proof. Let n be a non-negative integer. From Lemma 4.2.9 and (4.44), we have

$$\begin{aligned}\frac{f_{n+1}(\tilde{T})A_n^2(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} &= \frac{f_{n+1}(\tilde{T})(f_{n+1}(\tilde{T})A_n^2(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T}))}{\tilde{T}B_{n+1}(\tilde{T})} \\ &= \frac{A_{n+1}^2(\tilde{T}) - 1}{\tilde{T}B_{n+1}(\tilde{T})} \in \mathbb{F}[x]. \qquad (4.67)\end{aligned}$$

Since, by Lemma 4.2.9, $\tilde{T}B_n(\tilde{T}) \mid (A_n^2(\tilde{T}) - 1)$,

$$A_n^2(\tilde{T}) = D \cdot \tilde{T}B_n(\tilde{T}) + 1 \qquad \text{for some } D \in \mathbb{F}[x].$$

Then from (4.67), we have

$$\frac{f_{n+1}(\tilde{T}) + 2(-1)^{n+1}A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} \in \mathbb{F}[x],$$

i.e.,

$$\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} = E - \frac{2(-1)^{n+1}A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} \qquad \text{for some } E \in \mathbb{F}[x]. \qquad (4.68)$$

We have by Lemma 4.2.10 that

$$0 \leq \left| \frac{2(\delta_n^2 - 1)A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} \right|_\infty < 1. \qquad (4.69)$$

Hence the lemma is established by combining

$$\begin{aligned} v_{n+1}(\tilde{T}) &= \frac{(-1)^n \delta_n^2 f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} - \frac{2A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} \\ &= (-1)^n \delta_n^2 \left(\frac{f_{n+1}(\tilde{T})}{\tilde{T}B_n(\tilde{T})} + \frac{2(-1)^{n+1}A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})} \right) + \frac{2(\delta_n^2 - 1)A_n(\tilde{T})}{\tilde{T}B_n(\tilde{T})}, \end{aligned}$$

with (4.68) and (4.69). □

Using the same proof as in Theorem 4.2.5 and Theorem 4.2.11, analogues of Theorem 1 and Theorem 2 of Tamura [27] can also be established in Theorem 4.2.13 and Theorem 4.2.14, respectively.

For $n \geq 0$, define $\tilde{\theta}_n(T; f)$ to be the series expressed as follows

$$\tilde{\theta}_n(T; f) = \sum_{m=0}^{n} \frac{1}{f_0(T)f_1(T)\ldots f_m(T)},$$

where $f(T) \in (\mathbb{F}[x])\,[T] \setminus \{0\}$; $f_0(T) = T$ and for all $i \geq 1$, $f_i(T) = f(f_{i-1}(T))$ with $T \in \mathbb{F}[x] \setminus \{0\}$, and for those $T \in \mathbb{F}[x] \setminus \{0\}$ for which the limit exists we define

$$\tilde{\theta}(T; f) = \lim_{n\to\infty} \tilde{\theta}_n(T; f).$$

For any $f(T) \in (\mathbb{F}[x])[T] \setminus \{0\}$, we put

$\tilde{A}_n = \tilde{A}_n(T) = 1 + \sum_{m=1}^{n} f_m(T)f_{m+1}(T)\ldots f_n(T), \quad (n \geq 1)$; $\tilde{A}_0 = 1,$

$\tilde{B}_n = \tilde{B}_n(T) = f_0(T)f_1(T)\ldots f_n(T) \quad (n \geq 0).$

Theorem 4.2.13. *Let $f(T)$ be the polynomial of the form*

$$f(T) = T(T+2)(T-2)g(T) + T^2 - 2,$$

where $g(T) \in (\mathbb{F}[x])\,[T]$. If $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$, then

$$\tilde{\theta}_0(\tilde{T}; f) = [0; \tilde{T}],$$

and for all $n \geq 0$,

$$\tilde{\theta}_{n+1}(\tilde{T}; f) = [0; b_1, \ldots, b_k, y_{n+1}(\tilde{T}), b_k, \ldots, b_1],$$

if $[0; b_1, \ldots, b_k]$ *is a palindromic continued fraction representing* $\tilde{\theta}_n(\tilde{T}; f)$ *and*

$$y_{n+1}(\tilde{T}) = -\delta_n^2 \frac{f_{n+1}(\tilde{T})}{\tilde{B}_n(\tilde{T})} - 2\frac{\tilde{A}_n(\tilde{T})}{\tilde{B}_n(\tilde{T})},$$

where δ_n *is the element in* $\mathbb{F} \setminus \{0\}$ *such that* $\tilde{A}_n(\tilde{T}) = \delta_n p_k$ *and* $\tilde{B}_n(\tilde{T}) = \delta_n q_k$ *provided* $\frac{p_k}{q_k}$ *is the* k^{th} *(last) convergent of* $\tilde{\theta}_n(\tilde{T}; f)$ *respect to* $[0; b_1, \ldots, b_k]$.
In particular,

$$\tilde{\theta}(\tilde{T}; f) = [0; \tilde{T}, y_1(\tilde{T}), \tilde{T}, y_2(\tilde{T}), \tilde{T}, y_1(\tilde{T}), \tilde{T}, y_3(\tilde{T}), \ldots].$$

Theorem 4.2.14. *Let* $f(T)$ *be the polynomial of the form*

$$f(T) = T^2(T+2)(T-2)g(T) + T^2 - 2,$$

where $g(T) = w_m T^m + \ldots + w_1 T + w_0$ *with* $m \geq 0$, $w_i \in \mathbb{F}[x]$ $(0 \leq i \leq m)$, $w_m \neq 0$ *and* $|w_m|_\infty \geq |w_i|_\infty$ *for all* $0 \leq i \leq m-1$. *If* $\tilde{T} \in \mathbb{F}[x] \setminus \mathbb{F}$, *then*

$$\tilde{\theta}_0(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2],$$

and for all $n \geq 0$,

$$\tilde{\theta}_{n+1}(\tilde{T}; f)/\tilde{T} = [0; b_1, \ldots, b_k, z_{n+1}(\tilde{T}), b_k, \ldots, b_1],$$

if $[0; b_1, \ldots, b_k]$ *is a palindromic continued fraction representing* $\tilde{\theta}_n(\tilde{T}; f)/\tilde{T}$ *and*

$$z_{n+1}(\tilde{T}) = -\delta_n^2 \frac{f_{n+1}(\tilde{T})}{\tilde{T}\tilde{B}_n(\tilde{T})} - 2\frac{\tilde{A}_n(\tilde{T})}{\tilde{T}\tilde{B}_n(\tilde{T})},$$

where δ_n *is the element in* $\mathbb{F} \setminus \{0\}$ *such that* $\tilde{A}_n(\tilde{T}) = \delta_n p_k$ *and* $\tilde{T}\tilde{B}_n(\tilde{T}) = \delta_n q_k$ *provided* $\frac{p_k}{q_k}$ *is the* k^{th} *convergent of* $\tilde{\theta}_n(\tilde{T}; f)/\tilde{T}$ *respect to* $[0; b_1, \ldots, b_k]$.
In particular,

$$\tilde{\theta}(\tilde{T}; f)/\tilde{T} = [0; \tilde{T}^2, z_1(\tilde{T}), \tilde{T}^2, z_2(\tilde{T}), \tilde{T}^2, z_1(\tilde{T}), \tilde{T}^2, z_3(\tilde{T}), \ldots].$$

Remark 4.2.15. Similar to Theorem 4.2.5 and Theorem 4.2.11, the continued fractions produced by Theorem 4.2.13 and Theorem 4.2.14 are regular if and only if $\delta_n = \pm 1$ and we can guarantee convergences for infinite continued fractions.

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