STABILITY OF MIXED-TYPE TRIGONOMETRIC AND QUADRATIC FUNCTIONAL EQUATIONS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2009
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เสถียรภาพของสมการเชิงฟังก์ชันแบบผสมตรีโกณมิติและกำลังสอง

นางสาวจรรยารักษ์ ทองสมพร

ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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ปีการศึกษา 2552

ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

Thesis Title	Stability of Mixed-Type Trigonometric and Quadratic Functional Equations
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จรรยารักษ์ ทองสมพร : เสถียรภาพของสมการเชิงฟังก์ชันแบบผสมตรีโกณมิติและกำลังสอง (STABILITY OF MIXED–TYPE TRIGONOMETRIC AND QUADRATIC FUNCTIONAL EQUATIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.ดร. พัฒนี อุดมกะวานิช, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม: ศ. ดร. วิเชียร เลาหโกศล, 45 หน้า.

เนื้อหาส่วนแรกของวิทยานิพนธ์แสดงการ พิสูจน์เสถียรภาพของสมการเชิงฟังก์ชันนัยทั่วไปของฟังก์ชันตรีโกณมิติ $\mathcal{F}(x+y)-\mathcal{G}(x-y)=2\mathcal{H}(x)\mathcal{K}(y)$ เหนือโดเมนของกลุ่มอาบีเลียนและเรนจ์ของฟิลด์ของจำนวนเชิงซ้อน ผลหลากหลายที่ตามมาโดยตรงครอบคลุมผลลัพธ์ที่รู้จักโดยทั่วไปก่อนหน้านี้ อาทิ ผลลัพธ์ที่เกี่ยวกับสมการเชิงฟังก์ชันไซน์ สมการเชิงฟังก์ชันดาเลมเบิร์ต และสมการเชิงฟังก์ชันวิลสัน นอกจากนี้ได้นำผลที่ได้ไปประยุกต์กับตัวดำเนินการในพีชคณิตบานาค

ส่วนที่สองของวิทยานิพนธ์เริ่มด้วยการหาผลเฉลยที่เป็นฟังก์ชันซึ่งหาอนุพันธ์ได้ของสมการเชิงฟังก์ชันแบบผสมตรีโกณมิติและกำลังสองในรูป $\mathcal{F}(x+y)+\mathcal{G}(x-y)=2\mathcal{H}(x)\mathcal{K}(y)+\mathcal{L}(x)+\mathcal{M}(y)$ เหนือโดเมนของฟิลด์ของจำนวนจริงและเรนจ์ของฟิลด์ของจำนวนเชิงซ้อน จากนั้นเป็นการพิสูจน์เสถียรภาพของสมการเชิงฟังก์ชันแบบผสมตรีโกณมิติและกำลังสองเหนือโดเมนของกลุ่มอาบีเลียนและเรนจ์ของฟิลด์ของจำนวนเชิงซ้อน

ภาควิชา คณิตศาสตร์ ลายมือชื่อนิสิต จรรยารักษ์ ทองสมพร

สาขาวิชา คณิตศาสตร์ ลายมือชื่ออ.ที่ปรึกษาวิทยานิพนธ์หลัก พัฒนี อุดมกะวานิช

ปีการศึกษา 2552 ลายมือชื่ออ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

4873812123 : MAJOR MATHEMATICS
KEYWORDS : STABILITY / FUNCTIONAL EQUATION / TRIGONOMETRIC FUNCTIONAL EQUATION / QUADRATRIC FUNCTIONAL EQUATION

JANYARAK TONGSOMPORN : STABILITY OF MIXED-TYPE TRIGONOMETRIC AND QUADRATIC FUNCTIONAL EQUATIONS. THESIS ADVISOR : ASSOC. PROF. PATANEE UDOMKAVANICH, Ph.D., THESIS CO-ADVISOR : PROF. VICHIAN LAOHAKOSOL, Ph.D., 45 pp.

The first part of the thesis proves the stability of a generalized trigonometric functional equation $\mathcal{F}(x+y) - \mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y)$ over the domain of an abelian group and the range of the complex field. Several related results extending a number of previously known ones, such as the ones dealing with the sine functional equation, the d'Alembert functional equation and Wilson functional equation, are derived as direct consequences and applications to operators in Banach algebra are derived.

The second part of the thesis first determines differentiable solution functions of a mixed-type trigonometric and quadratic functional equation of the form $\mathcal{F}(x+y)+\mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y) + \mathcal{L}(x) + \mathcal{M}(y)$ over the domain of the real number field and the range of the complex field. Then the stability of this latter functional equation over the domain of an abelian group and the range of the complex field, is proved.

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ACKNOWLEDGEMENTS

I would like to express my profound gratitude and deep appreciation to Associate Professor Dr. Patanee Udomkavanich and to Professor Dr. Vichian Laohakosol, my thesis advisor and co-advisor, respectively, for their advice, endless patience, encouragement and motivating personality. Without the constant encouragement and guidance from them, this dissertation would be impossible to exist. Sincere thanks and deep appreciation also extend to Associate Professor Dr. Ajchara Harnchoowong, the chairman, Associate Professor Dr. Paisan Nakmahachalasint, Dr. Tuangrat Chaichana and Associate Professor Dr. Utsanee Leerawat, committee members, for their comments and suggestions. Also, I thank all teachers who have taught me all along.

In particular, and most importantly, I wish to express my deep gratitude to my mother Aim Tongsomporn, my father Sian Tongsomporn, my brothers Monchai, Pisit and Damrongsak for their unconditional love and supports.

Finally, it is a pleasure to thank the staff in Department of Mathematics, Chulalongkorn University for their support, the Higher Education Commission for a Ph.d. scholarship and to thank all my friends for the brilliant cordiality.

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CHAPTER I

INTRODUCTION

1.1 Background

In 1940, Ulam, [15], proposed the following problem, which has since been referred to as a *stability* problem: let f be a mapping from a group $(G_1, +)$ to a metric group $(G_2, +)$ with metric $d(\cdot, \cdot)$ such that

$$d(f(x+y), f(x) + f(y)) \leq \epsilon. \tag{1.1}$$

Do there exist a group homomorphism $L : G_1 \to G_2$ and a constant $\delta_\epsilon > 0$ such that

$$d(f(x), L(x)) \leq \delta_\epsilon$$

for all $x \in G_1$? This problem was affirmatively solved one year later by Hyers, [7], under the assumption that G_2 is a Banach space with norm $\|\cdot\|$. In 1978, a generalized version of Hyers' result was proved by Rassias, [14], where $f : G_1 \to G_2$ satisfies, instead of (1.1), the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad (x, y \in G_1),$$

where $\theta \neq 0$ and $0 \leq p < 1$. In 1979, Baker, Lawrence, and Zorzitto, [4], showed that if f is a function from a vector space to $\mathbb{R}$ satisfying

$$|f(x+y) - f(x)f(y)| \leq \epsilon,$$

for some fixed $\epsilon > 0$, then either f is bounded or satisfies the exponential Cauchy functional equation

$$f(x+y) = f(x)f(y). \tag{1.2}$$

Such a result is referred to as the *superstability* of the functional equation (1.2).

In this dissertation, the stability question about a generalized trigonometric functional equation is investigated. To be systematic, we first list all the functional equations that are of interest here using the terminology of Kim, [13].

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)f(y) \qquad \text{(sine functional equation)}, \tag{S}$$

$$f(x+y) + f(x-y) = 2f(x)f(y) \qquad \text{(d'Alembert functional equation)}, \tag{A}$$

$$f(x+y) + f(x-y) = 2f(x)g(y) \qquad \text{(Wilson functional equation)}, \tag{A(fg)}$$

$$f(x+y) + f(x-y) = 2g(x)f(y), \tag{A(gf)}$$

$$f(x+y) + f(x-y) = 2g(x)g(y), \tag{A(gg)}$$

$$f(x+y) - f(x-y) = 2f(x)f(y), \tag{T}$$

$$f(x+y) - f(x-y) = 2g(x)f(y), \tag{T(gf)}$$

$$f(x+y) - f(x-y) = 2f(x)g(y), \tag{T(fg)}$$

$$f(x+y) - f(x-y) = 2g(x)g(y), \tag{T(gg)}$$

$$f(x+y) - f(x-y) = 2g(x)h(y), \tag{T(gh)}$$

$$f(x+y) - f(x-y) = 2f(y). \tag{Jy}$$

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Let us briefly review some relevant earlier works.

Here and throughout, $(G,+)$ *always denotes an abelian group. In certain cases, it may satisfy additional hypothesis of being a abelian* 2*-divisible group. By an additive (respectively, exponential) function* A *(respectively,* E*) we refer to a function* A *(respectively,* E*) satisfying the additive (respectively, exponential) Cauchy functional equation*

$$A(x+y) = A(x) + A(y) \quad (\textit{respectively, } E(x+y) = E(x)E(y))$$

for all x, y *belonging to the domain of* A *(respectively,* E*).*

The superstability of the cosine functional equation (A), was investigated by Baker, [3], in 1980 with the following result: let $\delta > 0$. If $f : G \to \mathbb{C}$ satisfies

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \delta, \tag{1.3}$$

then either

$$|f(x)| \leq \left(1 + \sqrt{1+2\delta}\right)/2 \qquad (x \in G)$$

or f is a solution of the equation (A). In 1983, Cholewa, [6], investigated the superstability of the sine functional equation (S), with the following result: let $\delta > 0$ and let $(G,+)$ be a 2-divisible abelian group. If an unbounded function $f : G \to \mathbb{C}$ satisfies

$$|f(x+y)f(x-y) - f(x)^2 + f(y)^2| \leq \delta,$$

then it satisfies (S). The superstability of the generalized sine functional equation was treated by Kim, [11], with the following result: let $\epsilon > 0$. If the functions

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$f, g, h : G \to \mathbb{C}$ satisfy

$$\left| g(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leq \epsilon,$$

then either g is bounded or h satisfies (S); moreover, if g satisfies $g(0) = 0$ or if f satisfies $f(x)^2 = f(-x)^2$, then either h is bounded or g satisfies (S). Later in 2007, Kim, [12], investigated the superstability related to the d'Alembert and the Wilson functional equations, with the following results:

Theorem 1.1.1. *Let* $\varphi : G \to \mathbb{R}$.

I. *If* $f, g : G \to \mathbb{C}$ *satisfy*

$$|f(x+y) - f(x-y) - 2g(x)f(y)| \leq \varphi(x),$$

then either f *is bounded or* g *satisfies* (A).

II. *If* $f, g : G \to \mathbb{C}$ *satisfy*

$$|f(x+y) - f(x-y) - 2g(x)f(y)| \leq \varphi(y),$$

and if g *is unbounded, then* g *satisfies* (A)*, or* f *and* g *satisfy* (T(gf))*, or* f *and* g *satisfy* (A(fg)).

In the same work, Kim also considered the superstability of the functional equation (T(fg)) with the following results:

Theorem 1.1.2. *Let* $\varphi : G \to \mathbb{R}$.

I. *Suppose* $f, g : G \to \mathbb{C}$ *satisfy*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \varphi(y). \tag{1.4}$$

If f is unbounded, then

(i) g *satisfies* (S) *when* G *is a* 2*-divisible group;*

(ii) f *satisfies* (A) *and* f, g *are solutions of* $g(x+y) - g(x-y) = 2f(x)g(y)$.

II. *Suppose* $f, g : G \to \mathbb{C}$ *satisfy*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \varphi(x). \tag{1.5}$$

If g is unbounded, then

(i) f, g *are solutions of* (T(fg))*;*

(ii) when G *is a* 2*-divisible group and either* $f(0) = 0$ *or* $f(x) = f(-x)$*, we have* f *satisfies* (S)*;*

(iii) g *satisfies* (A) *or* (T)*, and* f, g *are solutions of* (A(fg)).

Recently, Kim, [13], investigated the superstability of the pexiderized trigonometric functional equation (T(gh)) and proved the next theorem.

Theorem 1.1.3. *Let* $\varphi : G \to \mathbb{R}$.

I. *Suppose that* $f, g, h : G \to \mathbb{C}$ *satisfy*

$$|f(x+y) - f(x-y) - 2g(x)h(y)| \leq \varphi(y). \tag{1.6}$$

If g is unbounded, then

(i) h *satisfies* (S)*;*

(ii) g *satisfies* (A) *and* g, h *are solutions of* (T(gh)).

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II. *Suppose* $f, g, h : G \to \mathbb{C}$ *satisfy*

$$|f(x+y) - f(x-y) - 2g(x)h(y)| \leq \varphi(x). \tag{1.7}$$

If h *is unbounded, then*

(i) g *satisfies* (S) *when* $g(0) = 0$ *or* $f(x) = f(-x)$*;*

(ii) h *satisfies* (A) *or* (T) *and* g, h *are solutions of* (A(fg)).

Our first objective is to investigate the stability of a generalized trigonometric functional equation

$$\mathcal{F}(x+y) - \mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y),$$

where $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ are nonzero functions from an abelian group $(G, +)$ to the complex field $\mathbb{C}$, which encompass all the functional equations elaborated in the above list.

In another direction, Jung, [8], in 2000 established the stability of the quadratic functional equation of pexider type

$$f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y) \tag{1.8}$$

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where f_1, f_2, f_3 and f_4 are functions from a norm space E_1 to a Banach space E_2.

In [10], Kannappan gave the general solution of (1.8) which states that:

Theorem 1.1.4. *Let* G *be a 2-divisible group and* $\mathbb{F}$ *be a field of characteristic different from* 2*. The general solution of* (1.8) *with* $f_i(x+y+z) = f_i(x+z+y)$ $(i = 1, 2)$

for any $x, y, z \in G$ is given by

$$\begin{aligned} f_1(x) &= B(x,x) - (A_1 - A_2)(x) + b_1, \\ f_2(x) &= B(x,x) - (A_1 + A_2)(x) + b_2, \\ f_3(x) &= 2B(x,x) - 2A_1(x) + b_3, \\ f_4(x) &= 2B(x,x) + 2A_2(x) + b_4, \end{aligned} \tag{1.9}$$

with $b_1 + b_2 = b_3 + b_4$, where $B : G \times G \to \mathbb{F}$ is a symmetric biadditive function and $A_i : G \to \mathbb{F}$ $(i = 1, 2)$ are additive.

Our second objective is to investigate the stability of a mixed-type trigonometric and quadratic functional equation of the form

$$\mathcal{F}(x+y) + \mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y) + \mathcal{L}(x) + \mathcal{M}(y),$$

where $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{L}$ and $\mathcal{M}$ are nonzero functions from an abelian group $(G, +)$ to the complex field $\mathbb{C}$.

1.2 Objectives and structure of the dissertation

The two objectives of this dissertaion are:

1. to investigate the stability of a generalized trigonometric functional equation

   $$\mathcal{F}(x+y) - \mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y),$$

   where $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ are nonzero functions from an abelian group $(G, +)$ to the complex field $\mathbb{C}$;

2. to investigate the stability of a mixed-type trigonometric and quadratic functional equation

$$\mathcal{F}(x+y)+\mathcal{G}(x-y)=2\mathcal{H}(x)\mathcal{K}(y)+\mathcal{L}(x)+\mathcal{M}(y),$$

where $\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{L}$ and $\mathcal{M}$ are nonzero functions from an abelian group $(G,+)$ to the complex field $\mathbb{C}$.

There are two principal tools used in this dissertation. The stability of both types of functional equations (generalized trigonometric funtional equation and mixed-type trigonometric and quadratic functional equation) is dealt with by a method due originally to Kim, [12]. The finding of differentiable solution functions of a mixed-type trigonometric and quadratic functional equation is based on a classical method due to Levi-Civita , [1].

We now outline the structure of this dissertation. The stability of the equation

$$\mathcal{F}(x+y)-\mathcal{G}(x-y)=2\mathcal{H}(x)\mathcal{K}(y),$$

which encompass all the functional equations elaborated in the above list is established in Chapter II. After stating and proving the main theorem in the first section, several consequences are elaborated and applications to operators in Banach algebra are derived in the last section. In Chapter III, we solve for differentiable solution functions of the mixed-type trigonometric and quadratic functional equation

$$\mathcal{F}(x+y)+\mathcal{G}(x-y)=2\mathcal{H}(x)\mathcal{K}(y)+\mathcal{L}(x)+\mathcal{M}(y),$$

in the first section, while its stability is investigated in the last section.

CHAPTER II

GENERALIZED TRIGONOMETRIC FUNCTIONAL EQUATIONS

We fix the following terminology throughout Chapter II:

- $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, f, g, h$ nonzero functions from an abelian group $(G, +)$ to the complex field $\mathbb{C}$;
- $\varphi : G \to \mathbb{R}^+$, the set of positive real numbers.

2.1 The main theorem and its proof

The following theorem is our main result.

Theorem 2.1.1. *Suppose $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}$ satisfy*

$$|\mathcal{F}(x+y) - \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y)| \le \varphi(x). \tag{2.1}$$

Then either

(i) $\mathcal{K}$ is bounded, or

(ii) there is a sequence $\{y_n\} \subset G$ such that

$$\ell_{\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{K}(y_n + y) + \mathcal{K}(y_n - y)}{\mathcal{K}(y_n)}$$

exists for each $y \in G$*, and* $\mathcal{H}$ *satisfies*

$$\mathcal{H}(x+y) + \mathcal{H}(x-y) = \mathcal{H}(x)\ell_{\mathcal{K}}(y) \quad (x, y \in G).$$

Assume (ii) holds.

(a) If $\mathcal{K}$ *satisfies the equation* (A)*, then* $\mathcal{H}, \mathcal{K}$ *are solutions of the equation* (A(fg)) *and are given by*

$$\mathcal{K}(x) = \frac{E(x) + E^*(x)}{2}, \quad \mathcal{H}(x) = \frac{k(E(x) + E^*(x))}{2} + c(E(x) - E^*(x)),$$

where $k, c \in \mathbb{C}$*,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$*;*

(b) If $\mathcal{H}(0) = 0$ *and* G *is a 2-divisible group, then* $\mathcal{H}$ *satisfies* (S) *and is of the form*

$$\mathcal{H}(x) = A(x) \quad \text{or} \quad \mathcal{H}(x) = c(E(x) - E^*(x)),$$

where $A : G \to \mathbb{C}$ *is an additive function,* c, E *and* $E^*(x)$ *are as in (a).*

Remark 2.1.2. *Let us make the following important remarks.*

- *The result of Theorem 2.1.1 involves only information about the functions* $\mathcal{H}$ *and* $\mathcal{K}$*, but is independent of the functions* $\mathcal{F}$ *and* $\mathcal{G}$*. The possibility (i) where the function* $\mathcal{H}$ *(or* $\mathcal{K}$*) is constant has been previously treated by Jung, [8], in 2000, and this explains why our main result deals mostly with possibility (ii).*
- *Theorem 2.1.1 continues to hold when* $\varphi(x) = \epsilon$*, a positive constant, which includes a number of earlier known results.*

Proof. Assume that $\mathcal{K}$ is unbounded. Then there is a sequence $\{y_n\}$ in G such that

$$0 \neq |\mathcal{K}(y_n)| \to \infty \quad (n \to \infty).$$

Substituting y_n for y in (2.1) we have

$$\left| \frac{\mathcal{F}(x+y_n) - \mathcal{G}(x-y_n)}{2\mathcal{K}(y_n)} - \mathcal{H}(x) \right| \leq \frac{\varphi(x)}{2|\mathcal{K}(y_n)|} \to 0 \quad (n \to \infty),$$

i.e.,

$$\lim_{n\to\infty} \frac{\mathcal{F}(x+y_n) - \mathcal{G}(x-y_n)}{2\mathcal{K}(y_n)} = \mathcal{H}(x). \tag{2.2}$$

Replacing y by $y_n + y$ and $y_n - y$ in the equation (2.1), we respectively get

$$|\mathcal{F}(x+(y_n+y)) - \mathcal{G}(x-(y_n+y)) - 2\mathcal{H}(x)\mathcal{K}(y_n+y)| \leq \varphi(x), \tag{2.3}$$

and

$$|\mathcal{F}(x+(y_n-y)) - \mathcal{G}(x-(y_n-y)) - 2\mathcal{H}(x)\mathcal{K}(y_n-y)| \leq \varphi(x). \tag{2.4}$$

Using (2.3), (2.4) and the triangle inequality, we have

$$\begin{aligned} 2\varphi(x) &\geq |\mathcal{F}(x+(y_n+y)) - \mathcal{G}(x-(y_n+y)) - 2\mathcal{H}(x)\mathcal{K}(y_n+y)| \\ &\qquad + |\mathcal{F}(x+(y_n-y)) - \mathcal{G}(x-(y_n-y)) - 2\mathcal{H}(x)\mathcal{K}(y_n-y)| \\ &\geq |\mathcal{F}(x+(y_n+y)) - \mathcal{G}(x-(y_n+y)) - 2\mathcal{H}(x)\mathcal{K}(y_n+y) \\ &\qquad + \mathcal{F}(x+(y_n-y)) - \mathcal{G}(x-(y_n-y)) - 2\mathcal{H}(x)\mathcal{K}(y_n-y)|. \end{aligned} \tag{2.5}$$

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Thus,

$$\left|\frac{\mathcal{F}(x+(y_n+y))-\mathcal{G}(x-(y_n-y))}{2\mathcal{K}(y_n)}+\frac{\mathcal{F}(x+(y_n-y))-\mathcal{G}(x-(y_n+y))}{2\mathcal{K}(y_n)}\right.$$
$$\left.-2\mathcal{H}(x)\frac{\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)}{2\mathcal{K}(y_n)}\right|\le\frac{2\varphi(x)}{2|\mathcal{K}(y_n)|}\to 0\quad(n\to\infty).$$

Combining with (2.2), we get

$$\mathcal{H}(x+y)+\mathcal{H}(x-y)=\mathcal{H}(x)\lim_{n\to\infty}\frac{\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)}{\mathcal{K}(y_n)}, \tag{2.6}$$

which proves (ii).

Assume now that (ii) holds.

(a) If $\mathcal{K}$ satisfies the equation (A), then

$$\lim_{n\to\infty}\frac{\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)}{\mathcal{K}(y_n)}=2\mathcal{K}(y).$$

This relation together with (2.6) show that $\mathcal{H},\mathcal{K}$ satisfy

$$\mathcal{H}(x+y)+\mathcal{H}(x-y)=2\mathcal{H}(x)\mathcal{K}(y),$$

which is (A(fg)). The given explicit solutions are taken from [9] and [10, p. 148].

(b) From its definition, we see that $\ell_{\mathcal{K}}$ satisfies

$$\mathcal{H}(x+y)+\mathcal{H}(x-y)=\mathcal{H}(x)\ell_{\mathcal{K}}(y). \tag{2.7}$$

Since $\mathcal{H}(0)=0$, we have $\mathcal{H}(y)+\mathcal{H}(-y)=0$, i.e., $\mathcal{H}$ is an odd function. Observe also

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that

$$\begin{aligned}\mathcal{H}(x+y)^2 - \mathcal{H}(x-y)^2 &= \{\mathcal{H}(x+y) + \mathcal{H}(x-y)\}\{\mathcal{H}(x+y) - \mathcal{H}(x-y)\} \\ &= \mathcal{H}(x)\{\ell_{\mathcal{K}}(y)\mathcal{H}(x+y) - \ell_{\mathcal{K}}(y)\mathcal{H}(x-y)\} = \mathcal{H}(x)\{\mathcal{H}(x+2y) - \mathcal{H}(x-2y)\} \\ &= \mathcal{H}(x)\{\mathcal{H}(2y+x) + \mathcal{H}(2y-x)\} = \mathcal{H}(x)\mathcal{H}(2y)\ell_{\mathcal{K}}(x).\end{aligned}$$

Replacing y by x in (2.7), we have

$$\mathcal{H}(2x) = \mathcal{H}(x)\ell_{\mathcal{K}}(x),$$

and so the last relations become

$$\mathcal{H}(x+y)^2 - \mathcal{H}(x-y)^2 = \mathcal{H}(2x)\mathcal{H}(2y), \tag{2.8}$$

i.e., $\mathcal{H}$ satisfies the equation (S) when G is 2-divisible. Appealing to the solutions of (S) in [10, p. 153], explicit shapes of $\mathcal{H}$ are as stated in the statement of the theorem. This completes the proof of Theorem 2.1.1. □

Remark 2.1.3. For later usage, let us mention that in the step of the proof after the equation (2.2), if we substitute y by $y + y_n$ and $y - y_n$ in the equation (2.1) and proceed as before, we end up with

$$\lim_{n\to\infty}\left\{\frac{\mathcal{F}((x+y)+y_n) - \mathcal{F}((x+y)-y_n)}{2\mathcal{K}(y_n)} + \frac{\mathcal{G}((x-y)+y_n) - \mathcal{G}((x-y)-y_n)}{2\mathcal{K}(y_n)} - 2\mathcal{H}(x)\frac{\mathcal{K}(y+y_n) - \mathcal{K}(y-y_n)}{2\mathcal{K}(y_n)}\right\} = 0.$$

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2.2 Corollaries

In this section, we apply Theorem 2.1.1 to derive almost all of the above-mentioned previous results. Since the functions $\mathcal{F}$ and $\mathcal{G}$ do not appear in the conclusion of Theorem 2.1.1, interchanging x with y and re-defining the functions $\mathcal{F}$ and $\mathcal{G}$ accordingly, we have:

Corollary 2.2.1. *If* $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}$ *satisfy*

$$|\mathcal{F}(x+y) - \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y)| \leq \varphi(y), \tag{2.9}$$

then

(i) $\mathcal{H}$ *is bounded, or*

(ii) there is a sequence $\{y_n\} \subset G$ *such that*

$$\ell_{\mathcal{H}}(y) := \lim_{n\to\infty} \frac{\mathcal{H}(y_n + y) + \mathcal{H}(y_n - y)}{\mathcal{H}(y_n)}$$

exists for each $y \in G$*, and* $\mathcal{K}$ *satisfies*

$$\mathcal{K}(x+y) + \mathcal{K}(x-y) = \mathcal{K}(x)\ell_{\mathcal{H}}(y) \quad (x, y \in G).$$

Assume (ii) holds.

(a) If $\mathcal{H}$ *satisfies the equation* (A)*, then* $\mathcal{H}, \mathcal{K}$ *are solutions of the equation* (A(fg)) *and are given by*

$$\mathcal{H}(x) = \frac{E(x) + E^*(x)}{2}, \quad \mathcal{K}(x) = \frac{k(E(x) + E^*(x))}{2} + c(E(x) - E^*(x)),$$

where $k, c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$*;*

(b) If $\mathcal{K}(0) = 0$, and G is a 2-divisible group, then $\mathcal{K}$ satisfies the equation (S) *and is given by*

$$\mathcal{K}(x) = A(x) \quad \text{or} \quad \mathcal{K}(x) = c(E(x) - E^*(x)),$$

where $A : G \to \mathbb{C}$ is an additive function, c, E and $E^*(x)$ are as in (a).

Our next corollary is an extension of Kim's result, [13], corresponding to (1.7).

Corollary 2.2.2. *If f, g, h satisfy*

$$|f(x+y) - f(x-y) - 2g(x)h(y)| \leq \varphi(x), \tag{2.10}$$

then

(i) h is bounded, or

(ii) there is a sequence $\{y_n\} \subset G$ such that

$$\ell_h(y) := \lim_{n\to\infty} \frac{h(y_n + y) + h(y_n - y)}{h(y_n)}$$

exists for each $y \in G$, and g satisfies

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$$g(x+y) + g(x-y) = g(x)\ell_h(y) \quad (x, y \in G).$$

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Assume (ii) holds.

(a) If h satisfies the equation (A) *or* (T)*, then g, h are solutions of the equation* (A(fg)) *and are given by*

$$h(x) = \frac{E(x) + E^*(x)}{2}, \quad g(x) = \frac{k(E(x) + E^*(x))}{2} + c(E(x) - E^*(x)),$$

where $k, c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$;

(b) *If* $g(0) = 0$ *or* $f(x) = f(-x)$ *and* G *is a 2-divisible group, then* g *satisfies the equation* (S) *and is given by*

$$g(x) = A(x) \quad or \quad g(x) = c(E(x) - E^*(x)),$$

where $c \in \mathbb{C}$, $A : G \to \mathbb{C}$ *is an additive function,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$.

Proof. All the resuts, except two places, follow from Theorem 2.1.1 by taking

$$\mathcal{G}(x) = \mathcal{F}(x) = f(x),\ \mathcal{H}(x) = g(x),\ \mathcal{K}(y) = h(y).$$

The first of the two new assertions is in (a) where h is supposed to satisfy (T). Applying Remark 2.1.3, we get $g(x+y) + g(x-y) = 2g(x)h(y)$, so that g, h are solutions of the equation (A(fg)). The second of the two new assertions is in (b) where we assume $f(x) = f(-x)$. To get the desired result, it suffices to show that $g(0) = 0$. This is achieved by first putting $x = 0$ in (2.10) to get

$$|2g(0)h(y)| = |f(y) - f(-y) - 2g(0)h(y)| \leq \varphi(x).$$

Then replacing y by y_n, dividing by $|2h(y_n)|$ and letting $n \to \infty$. □

Taking

$$\mathcal{G}(x) = \mathcal{F}(x) = f(x),\ \mathcal{H}(x) = g(x),\ \mathcal{K}(y) = h(y),$$

Corollary 2.2.1 yields the following result, which is an extension of Kim's result, [13], corresponding to (1.6).

Corollary 2.2.3. *If f, g, h satisfy*

$$|f(x+y) - f(x-y) - 2g(x)h(y)| \leq \varphi(y), \tag{2.11}$$

then

(i) *g is bounded, or*

(ii) *there is a sequence $\{y_n\} \subset G$ such that*

$$\ell_g(y) := \lim_{n\to\infty} \frac{g(y_n + y) + g(y_n - y)}{g(y_n)}$$

exists for each $y \in G$, and h satisfies

$$h(x+y) + h(x-y) = h(x)\ell_g(y).$$

Assume (ii) holds.

(a) *If g satisfies the equation* (A)*, then h, g are solutions of the equation* (A(fg)) *given by*

$$g(x) = \frac{E(x) + E^*(x)}{2}, \quad h(x) = \frac{k(E(x) + E^*(x))}{2} + c(E(x) - E^*(x)),$$

where $k, c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ is a homomorphism and $E^*(x) = 1/E(x)$;

(b) *If $h(0) = 0$ and G is a 2-divisible group, then h satisfies the equation* (S) *and is given by*

$$h(x) = A(x) \quad \text{or} \quad h(x) = c(E(x) - E^*(x)),$$

where $c \in \mathbb{C}$, $A : G \to \mathbb{C}$ *is an additive function,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$.

The next corollary is an extension of Kim's result, [12], corresponding to (1.5).

Corollary 2.2.4. *If* f, g *satisfy*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \varphi(x), \tag{2.12}$$

then

(*i*) g *is bounded, or*

(*ii*) *there is a sequence* $\{y_n\} \subset G$ *such that*

$$\ell_g(y) := \lim_{n\to\infty} \frac{g(y_n + y) + g(y_n - y)}{g(y_n)}$$

exists for each $y \in G$, *and* f *satisfies*

$$f(x+y) + f(x-y) = f(x)\ell_g(y).$$

Assume (ii) holds.

(*a*) *If* g *satisfies the equation* (A) *or* (T), *then* f, g *are solutions of the equation* (A(fg)) *given by*

$$g(x) = \frac{E(x) + E^*(x)}{2}, \quad f(x) = \frac{k(E(x) + E^*(x))}{2} + c(E(x) - E^*(x)),$$

where $k, c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$;

(b) If $f(0) = 0$ *or* $f(x) = f(-x)$ *and* G *is a* 2*-divisible group, then* f *satisfies the equation* (S) *and is given by*

$$f(x) = A(x) \quad or \quad f(x) = c(E(x) - E^*(x)),$$

where $c \in \mathbb{C}$, $A : G \to \mathbb{C}$ *is an additive function,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$.

(iii) f, g *are solutions of the equation* (T(fg)).

Proof. All the resuts for Parts (i) and (ii), except one place, follow from Theorem 2.1.1 by taking

$$\mathcal{G}(x) = \mathcal{F}(x) = f(x),\ \mathcal{H}(x) = f(x),\ \mathcal{K}(y) = g(y).$$

The only new assertion is in (a) where g is assumed to satisfy (T), which is dealt with by making use of Remark 2.1.3 as in the proof of Corollary 2.2.2.

There remains to prove Part *(iii)*. Taking

$$-\mathcal{G}(x) = \mathcal{F}(x) = f(x),\ \mathcal{H}(x) = f(x),\ \mathcal{K}(y) = g(y),$$

in Theorem 2.1.1 and using Remark 2.1.3, we deduce that $f(x+y) - f(x-y) = 2g(y)f(x)$, i.e., f, g are solutions of the equation (T(fg)) as desired. □

Taking $\mathcal{F}(x) = \mathcal{G}(x) = \mathcal{H}(x) = f(x)$, $\mathcal{K}(y) = g(y)$, Corollary 2.2.1 yields an extension of Kim's result, [12], corresponding to (1.4).

Corollary 2.2.5. *Let* f, g *satisfy*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \varphi(y), \tag{2.13}$$

then

(i) f *is bounded, or*

(ii) *there is a sequence* $\{y_n\} \subset G$ *such that*

$$\ell_f(y) := \lim_{n\to\infty} \frac{f(y_n + y) + f(y_n - y)}{f(y_n)}$$

exists for each $y \in G$*, and* g *satisfies*

$$g(x + y) + g(x - y) = g(x)\ell_f(y).$$

Assume (ii) holds.

(a) *If* f *satisfies the equation* (A)*, then* f, g *are solutions of the equation* (A(fg)) *given by*

$$f(x) = \frac{E(x) + E^*(x)}{2}, \quad g(x) = \frac{k(E(x) + E^*(x))}{2} + c(E(x) - E^*(x)),$$

where $k, c \in \mathbb{C}$*,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$*;*

(b) *If* G *is a* 2*-divisible group, then* g *satisfies the equation* (S) *and is given by*

$$g(x) = A(x) \quad \textit{or} \quad g(x) = c(E(x) - E^*(x)),$$

where $c \in \mathbb{C}$*,* $A : G \to \mathbb{C}$ *is an additive function,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$*.*

Regarding Baker's result, [3], corresponding to (1.3), we have:

Corollary 2.2.6. ***A.*** *If f satisfies*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \leq \varphi(x), \tag{2.14}$$

then

(*i*) *f is bounded, or*

(*ii*) *there is a sequence $\{y_n\} \subset G$ such that*

$$\ell_f(y) := \lim_{n\to\infty} \frac{f(y_n + y) + f(y_n - y)}{f(y_n)}$$

exists for each $y \in G$, and f satisfies

$$f(x+y) - f(x-y) = f(x)\ell_f(y).$$

Assume (ii) holds. If $f(0) = 0$ and G is a 2-divisible group, then f satisfies the equation (S) *given by*

$$f(x) = A(x) \quad or \quad f(x) = c(E(x) - E^*(x)),$$

where $c \in \mathbb{C}$, $A : G \to \mathbb{C}$ is an additive function, $E : G \to \mathbb{C}^*$ is a homomorphism and $E^*(x) = 1/E(x)$.

B. *Let $\epsilon > 0$. If f satisfies*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \epsilon. \tag{2.15}$$

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Then either f is bounded or f satisfies the equation (A) *and is given by*

$$f(x) = \frac{E(x) + E^*(x)}{2},$$

where $E : G \to \mathbb{C}^*$ is a homomorphism and $E^*(x) = 1/E(x)$.

Proof. For the part A, put $\mathcal{F} = \mathcal{G} = \mathcal{H} = \mathcal{K} = f$ in Theorem 2.1.1. For the part B, put $\mathcal{F}(x) = \mathcal{H}(x) = \mathcal{K}(x) = f(x) = -\mathcal{G}(x)$ and $\varphi(x) = \epsilon$ in Theorem 2.1.1.

□

Taking $\mathcal{F}(x) = \mathcal{G}(x) = \mathcal{H}(x) = \mathcal{K}(x) = f(x)$, Corollary 2.2.1 yields another extension of Baker's result, [3], corresponding to (1.3), which is

Corollary 2.2.7. *If f satisfies*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \le \varphi(y), \tag{2.16}$$

then

(*i*) *f is bounded, or*

(*ii*) *there is a sequence $\{y_n\} \subset G$ such that*

$$\ell_f(y) := \lim_{n\to\infty} \frac{f(y_n + y) + f(y_n - y)}{f(y_n)}$$

exists for each $y \in G$, and f satisfies

$$f(x+y) + f(x-y) = f(x)\ell_f(y).$$

Assume (ii) holds. If G is a 2-divisible group, then f satisfies the equation (S) *and* *is given by*

$$f(x) = A(x) \quad or \quad f(x) = c(E(x) - E^*(x)),$$

where $c \in \mathbb{C}$, $A : G \to \mathbb{C}$ *is an additive function,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^*(x) = 1/E(x)$.

2.3 Application to Banach algebra

Applying Theorem 2.1.1 and its corollaries to semisimple commutative Banach algebra, interesting results about operators can be obtained. We illustrate here just one instance.

Theorem 2.3.1. *Let* $(X, \|.\|)$ *be a semisimple commutative Banach algebra.*

A. *Assume that* $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K} : G \to X$ *and* $\varphi : G \to \mathbb{R}^+$ *satisfy*

$$\|\mathcal{F}(x+y) - \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y)\| \leq \varphi(x). \tag{2.17}$$

Let $x^* \in X^* : X \to \mathbb{C}$ *be an arbitrary linear multiplicative functional. Suppose that the* $x^* \circ \mathcal{K}$ *is not bounded.*

- (i) *If* $x^* \circ \mathcal{K}$ *satisfies the equation* (A)*, then* $\mathcal{H}$ *and* $\mathcal{K}$ *are solutions of the equation* (A(fg)).
- (ii) *If* $x^* \circ \mathcal{H}(0) = 0$ *and* G *is a* 2*-divisible group, then* $\mathcal{H}$ *satisfies the equation* (S).

B. *Assume that* $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K} : G \to X$ *and* $\varphi : G \to \mathbb{R}^+$ *satisfy*

$$\|\mathcal{F}(x+y) - \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y)\| \leq \varphi(y). \tag{2.18}$$

Let $x^* \in X^* : X \to \mathbb{C}$ *be an arbitrary linear multiplicative functional. Suppose that the* $x^* \circ \mathcal{H}$ *is not bounded.*

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(*i*) *If* $x^* \circ \mathcal{H}$ *satisfies the equation* (A)*, then* $\mathcal{H}$ *and* $\mathcal{K}$ *are solutions of the equation* (A(fg)).

(*ii*) *If* $x^* \circ \mathcal{K}(0) = 0$ *and* G *is a 2-divisible group, then* $\mathcal{K}$ *satisfies the equation* (S).

Proof. We give only the proof of Part A, as that of Part B is similar. For case (i) of Part A, fix an arbitrary linear multiplicative functional $x^* \in X^*$. Since $\|x^*\| = 1$, we claim that

$$|(x^* \circ \mathcal{F})(x+y) - (x^* \circ \mathcal{G})(x-y) - 2(x^* \circ \mathcal{H})(x)(x^* \circ \mathcal{K})(y)| \leq \varphi(x).$$

This follows easily from

$$\begin{aligned}\varphi(x) &\geq \|\mathcal{F}(x+y) - \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y)\| \\ &= \sup_{\|x^*\|=1} |x^*(\mathcal{F}(x+y) - \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y))| \\ &\geq |x^*(\mathcal{F}(x+y) - \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y))| \\ &= |(x^* \circ \mathcal{F})(x+y) - (x^* \circ \mathcal{G})(x-y) - 2(x^* \circ \mathcal{H})(x)(x^* \circ \mathcal{K})(y))|.\end{aligned}$$

Supposing $x^* \circ \mathcal{K}$ is unbounded, Theorem 2.1.1 shows that if $x^* \circ \mathcal{K}$ satisfies the equation (A), then $x^* \circ \mathcal{H}$ and $x^* \circ \mathcal{K}$ are solutions of

$$\begin{aligned}0 &= (x^* \circ \mathcal{H})(x+y) + (x^* \circ \mathcal{H})(x-y) - 2(x^* \circ \mathcal{H})(x)(x^* \circ \mathcal{K})(y) \\ &= x^*(\mathcal{H}(x+y) + \mathcal{H}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y)),\end{aligned}$$

i.e., $\mathcal{H}(x+y) + \mathcal{H}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y) \in \mathrm{Ker}(x^*)$. Since X is semisimple, we have

$$\mathcal{H}(x+y) + \mathcal{H}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y) \in \bigcap_{x^* \in X^*} \mathrm{Ker}(x^*) = \{0\},$$

i.e., $\mathcal{H}(x+y)+\mathcal{H}(x-y)-2\mathcal{H}(x)\mathcal{K}(y)=0$. Since $x^*\circ\mathcal{K}$ satisfies the equation (A) and X is semisimple, $\mathcal{K}$ also satifies the equation (A).

For case (ii) of Part A, Theorem 2.1.1 shows that if $x^*\circ\mathcal{H}(0)=0$ and G is a 2-divisible group, then $x^*\circ\mathcal{H}$ satisfies the equation (S), i.e.,

$$0=(x^*\circ\mathcal{H})\left(\frac{x+y}{2}\right)^2-(x^*\circ\mathcal{H})\left(\frac{x-y}{2}\right)^2-(x^*\circ\mathcal{H})(x)(x^*\circ\mathcal{H})(y).$$

Since x^* is a linear multiplicative functional, we get

$$x^*\left(\mathcal{H}\left(\frac{x+y}{2}\right)^2-\mathcal{H}\left(\frac{x-y}{2}\right)^2-\mathcal{H}(x)\mathcal{H}(y)\right)=0,$$

i.e., $\mathcal{H}\left(\frac{x+y}{2}\right)^2-\mathcal{H}\left(\frac{x-y}{2}\right)^2-\mathcal{H}(x)\mathcal{H}(y)\in\operatorname{Ker}(x^*)$. Since X is semisimple, we have

$$\mathcal{H}\left(\frac{x+y}{2}\right)^2-\mathcal{H}\left(\frac{x-y}{2}\right)^2-\mathcal{H}(x)\mathcal{H}(y)\in\bigcap_{x^*\in X^*}\operatorname{Ker}(x^*)=\{0\},$$

i.e., $\mathcal{H}\left(\frac{x+y}{2}\right)^2-\mathcal{H}\left(\frac{x-y}{2}\right)^2-\mathcal{H}(x)\mathcal{H}(y)=0$, showing that $\mathcal{H}$ satisfies (S). $\square$

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CHAPTER III

MIXED-TYPE TRIGONOMETRIC AND QUADRATIC FUNCTIONAL EQUATIONS

In this chapter, we deal with a mixed-type trigonometric and quadratic functional equation of the form

$$\mathcal{F}(x+y)+\mathcal{G}(x-y)=2\mathcal{H}(x)\mathcal{K}(y)+\mathcal{L}(x)+\mathcal{M}(y). \tag{3.1}$$

In order to ascertain the existence of solution, we start by determining its general set of differentiable solution functions using a method of Levi-Civita as expounded in [1] in the next section and proceed to consider its stability in the subsequent section.

Observe that solutions of (3.1) are plentiful as seen from the following examples.

Example 3.0.2.

1) If $\mathcal{G}(x)=0$, $\mathcal{H}(x)\mathcal{K}(y)=0$ and $\mathcal{L}(x)=\mathcal{M}(x)=\mathcal{F}(x)$, the general solution of (3.1) *is $\mathcal{F}(x)=A(x)$, additive function.*

2) If $\mathcal{G}(x)=\mathcal{L}(x)=\mathcal{M}(y)=0$, $\mathcal{H}(x)=\mathcal{K}(x)=\frac{\mathcal{F}(x)}{\sqrt{2}}$, the general solution of (3.1) *is $\mathcal{F}(x)=E(x)$, exponential function.*

3) $\mathcal{H}(x)=\mathcal{K}(x)=e^x$, $\mathcal{F}(x)=2e^x$, $\mathcal{L}(x)=-\mathcal{M}(x)=x$, $G(x)=x$ is a solution of (3.1).

From such observation, to find a general solution of (3.1) is quite a tedious matter.

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3.1 Levi-Civita's method

We are content have to use a method due to Levi-Civita in solving a simple case of (3.1) with additional hypothesis. It is to be noted that solutions of (3.1) are mostly exponential polynomials.

Theorem 3.1.1. *Let* $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}, \mathcal{M} : \mathbb{R} \to \mathbb{C}$ *be differentiable functions satisfying* (3.1)*. If* $\mathcal{L}'$*,* $\mathcal{H}'$ *are* $\mathbb{C}$*-multiple of a non-constant function* $\mathcal{H}$*, then*

$$\begin{aligned}
\mathcal{H}(x) &= e^{c_1 x};\\
\mathcal{L}(x) &= \frac{c_2}{c_1}e^{c_1 x} + \alpha;\\
\mathcal{K}(y) &= \gamma_1 e^{c_1 y} + \gamma_2 e^{-c_1 y} - \frac{c_2}{2c_1};\\
\mathcal{M}(y) &= \delta_1 y + \delta_2;\\
\mathcal{F}(x) &= 2\gamma_1 e^{c_1 x} + \frac{\delta_1}{2}x + \beta_1;\\
\mathcal{G}(x) &= 2\gamma_2 e^{c_1 x} - \frac{\delta_1}{2}x + \alpha + \delta_2 - \beta_1,
\end{aligned}$$

where $c_1(\neq 0), c_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \alpha, \beta_1 \in \mathbb{C}$*.*

Proof. Differentiating (3.1) with repect to x, we obtain

$$\mathcal{F}'(x+y) + \mathcal{G}'(x-y) = 2\mathcal{H}'(x)\mathcal{K}(y) + \mathcal{L}'(x). \tag{3.2}$$

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Differentiating (3.1) with repect to y, we have

$$\mathcal{F}'(x+y) - \mathcal{G}'(x-y) = 2\mathcal{H}(x)\mathcal{K}'(y) + \mathcal{M}'(y). \tag{3.3}$$

Adding and substracting (3.2) and (3.3), we get

$$2\mathcal{F}'(x+y) = 2\mathcal{H}'(x)\mathcal{K}(y) + 2\mathcal{H}(x)\mathcal{K}'(y) + \mathcal{L}'(x) + \mathcal{M}'(y); \qquad (3.4)$$

$$2\mathcal{G}'(x-y) = 2\mathcal{H}'(x)\mathcal{K}(y) - 2\mathcal{H}(x)\mathcal{K}'(y) + \mathcal{L}'(x) - \mathcal{M}'(y). \qquad (3.5)$$

Let $\mathcal{H}'(x) = c_1\mathcal{H}(x)$ where $c_1 \in \mathbb{C}$. Then $\mathcal{H}(x) = e^{c_1x} + \sigma$ where $\sigma \in \mathbb{C}$. Since $\mathcal{H}$ is a non-constant function, we must have $c_1 \neq 0$ and $\sigma = 0$. So

$$\mathcal{H}(x) = e^{c_1x}$$

and $\mathcal{L}'(x) = c_2e^{c_1x}$ where $c_2 \in \mathbb{C}$. Hence

$$\mathcal{L}(x) = \frac{c_2}{c_1}e^{c_1x} + \alpha$$

for some $\alpha \in \mathbb{C}$. Substituting $\mathcal{H}, \mathcal{H}'$ and $\mathcal{L}'$ in (3.4), we obtain

$$2\mathcal{F}'(x+y) = 2c_1e^{c_1x}\mathcal{K}(y) + 2e^{c_1x}\mathcal{K}'(y) + c_2e^{c_1x} + \mathcal{M}'(y). \qquad (3.6)$$

Differentiating (3.6) with repect to x, we have

$$2\mathcal{F}''(x+y) = 2c_1^2e^{c_1x}\mathcal{K}(y) + 2c_1e^{c_1x}\mathcal{K}'(y) + c_2c_1e^{c_1x}. \qquad (3.7)$$

Differentiating (3.6) with repect to y, we have

$$2\mathcal{F}''(x+y) = 2c_1e^{c_1x}\mathcal{K}'(y) + 2e^{c_1x}\mathcal{K}''(y) + \mathcal{M}''(y). \qquad (3.8)$$

From (3.7) and (3.8), we get

$$2c_1^2 e^{c_1 x}\mathcal{K}(y) + c_2 c_1 e^{c_1 x} = 2e^{c_1 x}\mathcal{K}''(y) + \mathcal{M}''(y). \tag{3.9}$$

Substituting $x = 0$ and $x = 1$ into (3.9), we obtain

$$2c_1^2\mathcal{K}(y) + c_2 c_1 = 2\mathcal{K}''(y) + \mathcal{M}''(y); \tag{3.10}$$

$$2c_1^2 e^{c_1}\mathcal{K}(y) + c_2 c_1 e^{c_1} = 2e^{c_1}\mathcal{K}''(y) + \mathcal{M}''(y). \tag{3.11}$$

Substracting (3.10) and (3.11), we have

$$2(e^{c_1} - 1)\mathcal{K}''(y) - 2c_1^2(e^{c_1} - 1)\mathcal{K}(y) = c_2 c_1(e^{c_1} - 1).$$

Since $c_1 \neq 0$,

$$\mathcal{K}''(y) - c_1^2\mathcal{K}(y) = \frac{c_2 c_1}{2}. \tag{3.12}$$

By solving a second order differential equation, we obtain

$$\mathcal{K}(y) = \gamma_1 e^{c_1 y} + \gamma_2 e^{-c_1 y} - \frac{c_2}{2c_1}$$

where $\gamma_1, \gamma_2 \in \mathbb{C}$. Putting $\mathcal{K}''(y)$ from (3.12) into (3.10), we obtain $\mathcal{M}''(y) = 0$. Then

$$\mathcal{M}(y) = \delta_1 y + \delta_2$$

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for some $\delta_1, \delta_2 \in \mathbb{C}$. Substituting $\mathcal{K}, \mathcal{K}', \mathcal{M}'$ into (3.6), we get

$$2\mathcal{F}'(x+y) = 4c_1\gamma_1 e^{c_1x}e^{c_1y} + \delta_1. \tag{3.13}$$

Then

$$\mathcal{F}(x+y) = 2\gamma_1 e^{c_1(x+y)} + \frac{\delta_1}{2}(x+y) + \beta_1$$

where $\beta_1 \in \mathbb{C}$. Substituting $\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}', \mathcal{L}'$ and $\mathcal{M}'$ into (3.5), we have

$$2\mathcal{G}'(x-y) = 4c_1\gamma_2 e^{c_1x}e^{-c_1y} - \delta_1. \tag{3.14}$$

Thus

$$\mathcal{G}(x-y) = 2\gamma_2 e^{c_1(x-y)} - \frac{\delta_1}{2}(x-y) + \beta_2$$

where $\beta_2 \in \mathbb{C}$. Substituting $\mathcal{H}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{F}$ and $\mathcal{G}$ into (3.1), we obtain $\beta_2 = \alpha + \delta_2 - \beta_1$. □

In solving the equation (3.1), we now consider (3.4) by setting $f(x) := 2\mathcal{F}'(x)$. Then it becomes a equation of the form

$$f(x+y) = \sum_{k=1}^{n} f_k(x)g_k(y). \tag{3.15}$$

After that we follow here the idea of T. Levi-Civita with some modification. It may obviously be assumed that the functions $f_k(x)$ on the one hand and the functions $g_k(y)$ on the other hand are linearly independent. Otherwise, we replace in (3.15) the linearly dependent functions with their representations as linear combinations relative to a basis of linearly independent functions, the result of which is an equation of similar form. If (3.15) is partially differentiated with respect first to x and then to

y, we obtain

$$\sum_{k=1}^{n} f_k'(x) g_k(y) = f'(x+y) = \sum_{k=1}^{n} f_k(x) g_k'(y). \tag{3.16}$$

The functions $g_k(y)$ being linearly independent, there exist constants $y_1, \ \ldots, \ y_n$ such that the determinant $\det\left(g_k(m)\right) \neq 0 \quad (k, m = 1, \ \ldots, n)$. substituting these into (3.16), we can express the $f_k'(x)$ with aid of the $f_j(x)$ $(j, k = 1, \ \ldots, n)$:

$$f_k'(x) = \sum_{j=1}^{n} a_{kj} f_j(x) \qquad (k = 1, \ 2, \ \ldots, n). \tag{3.17}$$

On the other hand, it follows from (3.15), again with y held constant at 0, that

$$f(x) = \sum_{k=1}^{n} c_k f_k(x), \tag{d0}$$

and by successive differentiation of this equation, taking (3.17) into consideration, we find

$$f'(x) = \sum_{k=1}^{n} c_{1k} f_k(x), \tag{d1}$$

$$f''(x) = \sum_{k=1}^{n} c_{2k} f_k(x), \tag{d2}$$

$$\vdots$$

$$f^{(n)}(x) = \sum_{k=1}^{n} c_{nk} f_k(x), \tag{dn}$$

from which the $f_k(x)$ can be eliminated. This yield for $f(x)$ a homogeneous linear differential equation of nth order with constant coefficients. In fact, if $\det\left(c_{mk}\right) =$

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0 $(m,\ k = 1,\ \ldots,\ n)$, then from (d1), ..., (dn)

$$\sum_{m=1}^{n} a_m f^{(m)}(x) = 0.$$

follows with at least one $a_k \neq 0$, while if $\det c_{mk} \neq 0$, then the $f_k(x)$ $(k = 1,\ \ldots,\ n)$ can be expressed from these equations as linear combinations of the $f^{(m)}(x)$ $(m = 1,\ \ldots,\ n)$, which substituted in (d0) give again

$$f(x) + \sum_{m=1}^{n} a_m f^{(m)}(x) = 0.$$

Thus, as a general n-times differrentiable solution of (3.15), we obtain

$$f(x) = \sum_{k=1}^{m} P_k(x) \exp \omega_k x,$$

where the $P_k(x)$ are polynomials of $(n_k - 1)$st degree

$$\sum_{k=1}^{m} n_k = n$$

and the ω_k are, in general, complex constants, namely, the roots of the characteristic equation. All these functions do satisfy equations of form (3.15), with suitably chosen $f_k(x),\ g_k(y)$ $(k = 1,\ 2,\ \ldots,\ n)$. Therefore we obtain that $\mathcal{F}(x)$ is an exponential polynomial. In a similar manner, if we consider (3.5), then we also obtain that $\mathcal{G}(x)$ is an exponential polynomial.

3.2 Stability by Kim's method

In this section, we use Kim's method in [12] to study the stability of the mixed-type trigonometric and quadratic functional equations (3.1). We start with some preliminaries.

Lemma 3.2.1. *Let $\mathcal{H}, K, \ell : \mathbb{R} \to \mathbb{C}$ be differentiable functions with K being non-constant. If $K'(0) \neq 1$ and $\mathcal{H}, K, \ell$ satisfy*

$$\mathcal{H}(x+y) + \mathcal{H}(x-y) = \mathcal{H}(x)K(y) + \ell(y) \tag{3.18}$$

for all $x, y \in \mathbb{R}$, then $\mathcal{H}(x) \equiv h \in \mathbb{C}$.

Moreover,

(i) if $h = 0$, then $\ell(y) \equiv 0$ and K is an arbitrary function;

(ii) if $h \neq 0$, then $\ell(y) = h(2 - K(y))$ and K is an arbitrary function.

Proof. Differentiating (3.18) with repect to x and to y, we obtain respectively,

$$\mathcal{H}'(x+y) + \mathcal{H}'(x-y) = \mathcal{H}'(x)K(y). \tag{3.19}$$

$$\mathcal{H}'(x+y) - \mathcal{H}'(x-y) = \mathcal{H}(x)K'(y) + \ell'(y). \tag{3.20}$$

Adding and subtracting (3.19) and (3.20), we get respectively

$$2\mathcal{H}'(x+y) = \mathcal{H}'(x)K(y) + \mathcal{H}(x)K'(y) + \ell'(y). \tag{3.21}$$

$$2\mathcal{H}'(x-y) = \mathcal{H}'(x)K(y) - \mathcal{H}(x)K'(y) - \ell'(y). \tag{3.22}$$

Putting $y = 0$ in (3.21) and (3.22), we respectively get

$$2\mathcal{H}'(x) = \mathcal{H}'(x)K(0) + \mathcal{H}(x)K'(0) + \ell'(0). \tag{3.23}$$

$$2\mathcal{H}'(x) = \mathcal{H}'(x)K(0) - \mathcal{H}(x)K'(0) - \ell'(0). \tag{3.24}$$

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Adding (3.23) and (3.24) leads to

$$4\mathcal{H}'(x) = 2\mathcal{H}'(x)K(0). \tag{3.25}$$

We now consider two separate cases. If $K(0) \neq 2$, then, by (3.25), $\mathcal{H}'(x) \equiv 0$. If $K(0) = 2$, then, by (3.23), $\mathcal{H}(x)K'(0) = -\ell'(0)$. The both cases of $K(0)$ imply that $\mathcal{H}$ is a constant function, namely $h \in \mathbb{C}$. By (3.18), if $h = 0$, then $\ell(y) \equiv 0$ and K is an arbitrarily function. If $h \neq 0$, then $\ell(y) = h(2 - K(y))$ and K is an arbitrary function.

□

We shall need the following corollary of Theorem 1.1.4.

Corollary 3.2.2. *Let $\mathcal{H}, \ell$ be functions from a 2-divisible abelian group G to a field $\mathbb{F}$ of characteristic different from 2 and $c \in \mathbb{F}$.*

If

$$\mathcal{H}(x+y) + \mathcal{H}(x-y) = c\mathcal{H}(x) + \ell(y) \quad (x, y \in G),$$

then (i) for c=2, we have

$$\mathcal{H}(x) = B(x,x) + A(x) + r, \;\; \ell(x) = 2B(x,x),$$

where $B : G \times G \to \mathbb{F}$ is a symmetric biadditive function, $A : G \to \mathbb{F}$ is an additive function and $r \in \mathbb{F}$;
(ii) for $c \neq 2$, we have

$$\mathcal{H}(x) \equiv r, \;\; \ell(x) \equiv r(2-c)$$

are constant functions, where $r \in \mathbb{F}$.

Proof. Substituing $f_1(x) = f_2(x) = \mathcal{H}(x), \;\; f_3(x) = c\mathcal{H}(x), \;\; f_4(x) = \ell(y)$ in Theorem

1.1.4, we get

$$\mathcal{H}(x) = B(x,x) - (A_1 - A_2)(x) + b_1, \tag{3.26}$$

$$\mathcal{H}(x) = B(x,x) - (A_1 + A_2)(x) + b_2, \tag{3.27}$$

$$c\mathcal{H}(x) = 2B(x,x) - 2A_1(x) + b_3, \tag{3.28}$$

$$\ell(x) = 2B(x,x) + 2A_2(x) + b_4 \tag{3.29}$$

with $b_1 + b_2 = b_3 + b_4$, where $B : G \times G \to \mathbb{F}$ is a symmetric biadditive function and $A_1, A_2 : G \to \mathbb{F}$ are an additive function. Equating (3.26) and (3.27), we get $2A_2(x) = b_2 - b_1$, a contant function. Since A_2 is additive, we must have $A_2(x) \equiv 0$ and so $b_1 = b_2$. Putting this information back into (3.26), we get

$$\mathcal{H}(x) = B(x,x) - A_1(x) + b_1,$$

and multiplying by c to get

$$c\mathcal{H}(x) = cB(x,x) - cA_1(x) + cb_1. \tag{3.30}$$

Equating (3.30) and (3.28), and using $b_3 = 2b_1 - b_4$,

$$c(B(x,x) - A_1(x) + b_1) = 2B(x,x) - 2A_1(x) + 2b_1 - b_4. \tag{3.31}$$

Consider now two distinct cases.

Case 1: $c = 2$. By (3.31), we have $b_4 = 0$ and so

$$\ell(x) = 2B(x,x).$$

Case 2: $c \neq 2$. By (3.31), we obtain

$$B(x,x) - A_1(x) = \frac{b_4}{2-c} - b_1. \tag{3.32}$$

Thus,

$$B(x+y,x+y) - A_1(x+y) = \frac{b_4}{2-c} - b_1. \tag{3.33}$$

Since B is symmetric biadditive and A is additve, (3.33) yields

$$\begin{aligned}
\frac{b_4}{2-c} - b_1 &= B(x+y,x+y) - A_1(x+y) \\
&= B(x,x) + 2B(x,y) + B(y,y) - A_1(x) - A_1(y) \\
&= (B(x,x) - A_1(x)) + (B(y,y) - A_1(y)) + 2B(x,y) \\
&= \frac{b_4}{2-c} - b_1 + \frac{b_4}{2-c} - b_1 + 2B(x,y). \qquad (3.34)
\end{aligned}$$

The relation (3.34) shows that B is a constant function and so $B \equiv 0$ because B is symmetric biadditive. Thus, (3.32) shows that A_1 is a constant function and so $A_1 \equiv 0$ because A_1 is additive. Consequently, $b_4 = b_1(2-c)$ and $\mathcal{H}(x) = b_1$, $\ell(x) = b_1(2-c)$. □

We now state and prove our main result in this section.

Theorem 3.2.3. *Let G be an abelian group. Assume that $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}, \mathcal{M} : G \to \mathbb{C}$ satisfy*

$$|\mathcal{F}(x+y) + \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y) - \mathcal{L}(x) - \mathcal{M}(y)| \leq \varphi(x) \tag{3.35}$$

for all $x, y \in G$, where $\varphi : G \to \mathbb{R}^+$.

If $\mathcal{K}$ is unbounded, then there is a sequence $\{y_n\} \subset G$ such that

$$\Im_{\mathcal{K},\mathcal{M}}(x,y) := \lim_{n\to\infty} \frac{2\mathcal{H}(x)\left(\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)\right)+\left(\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)\right)}{2\mathcal{K}(y_n)} \tag{3.36}$$

exists for all $x, y \in G$, and the function $\mathcal{H}$ satisfies the functional equation

$$\mathcal{H}(x+y)+\mathcal{H}(x-y) = \Im_{\mathcal{K},\mathcal{M}}(x,y). \tag{3.37}$$

Moreover, if either the limit

$$\ell_{\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)}{\mathcal{K}(y_n)}$$

or the limit

$$\ell_{\mathcal{M},\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)}{2\mathcal{K}(y_n)}$$

exists for all $y \in G$, then the functional equation (3.37) simplifies to

$$\mathcal{H}(x+y)+\mathcal{H}(x-y) = \mathcal{H}(x)\ell_{\mathcal{K}}(y)+\ell_{\mathcal{M},\mathcal{K}}(y) \tag{3.38}$$

for all $x, y \in G$.

Assume that (3.38) holds.

I. When $G = \mathbb{R}$ and $\mathcal{H}, \ell_{\mathcal{K}}, \ell_{\mathcal{M},\mathcal{K}}$ are differentiable, if $\ell_{\mathcal{K}}$ is a non-constant function and $\ell'_{\mathcal{K}}(0) \neq 0$, then $\mathcal{H}(x) \equiv h \in \mathbb{C}$ is a constant function. Furthermore,

(Ia) if $h = 0$, then $\ell_{\mathcal{M},\mathcal{K}}(y) \equiv 0$ and $\ell_{\mathcal{K}}$ is an arbitrary function;

(Ib) if $h \neq 0$, then $\ell_{\mathcal{M},\mathcal{K}}(y) = h(2-\ell_{\mathcal{K}}(y))$ and $\ell_{\mathcal{K}}$ is an arbitrary function.

II. When G is a 2-divisible abelian group, if $\ell_{\mathcal{K}}(y) \equiv c \in \mathbb{C}$ is a constant function,

then

(IIa) *for* $c = 2$*, we have*

$$\mathcal{H}(x) = B(x, x) + A(x) + r, \quad \ell_{\mathcal{M},\mathcal{K}}(x) = 2B(x, x),$$

where $B : G \times G \to \mathbb{C}$ *is a symmetric biadditive function,* $A : G \to \mathbb{C}$ *is an additive function and* $r \in \mathbb{C}$*;*

(IIb) *for* $c \neq 2$*, we have*

$$\mathcal{H}(x) \equiv r, \quad \ell_{\mathcal{M},\mathcal{K}}(x) \equiv r(2 - c)$$

are constant functions, where $r \in \mathbb{C}$*.*

Proof. Assume that $\mathcal{K}$ is unbounded. Then there is a sequence $\{y_n\} \subset G$ such that

$$0 \neq |\mathcal{K}(y_n)| \to \infty \qquad (n \to \infty).$$

Substituing $y = y_n$ in (3.35) and dividing by $|\mathcal{K}(y_n)|$ we get

$$\left| \frac{\mathcal{F}(x + y_n) + \mathcal{G}(x - y_n) - \mathcal{M}(y_n)}{2\mathcal{K}(y_n)} - \frac{\mathcal{L}(x)}{2\mathcal{K}(y_n)} - \mathcal{H}(x) \right| \leq \frac{\varphi(x)}{|2\mathcal{K}(y_n)|}.$$

Thus,

$$\lim_{n \to \infty} \frac{\mathcal{F}(x + y_n) + \mathcal{G}(x - y_n) - \mathcal{M}(y_n)}{2\mathcal{K}(y_n)} = \mathcal{H}(x). \tag{3.39}$$

Put $y = y_n \pm y$ in (3.35), we respectively obtain

$$|\mathcal{F}((x+y)+y_n) + \mathcal{G}((x-y)-y_n) - 2\mathcal{H}(x)\mathcal{K}(y_n+y) - \mathcal{L}(x) - \mathcal{M}(y_n+y)| \leq \varphi(x). \tag{3.40}$$

$$|\mathcal{F}((x-y)+y_n) + \mathcal{G}((x+y)-y_n) - 2\mathcal{H}(x)\mathcal{K}(y_n-y) - \mathcal{L}(x) - \mathcal{M}(y_n-y)| \leq \varphi(x). \tag{3.41}$$

From (3.40) and (3.41), we have

$$\begin{aligned} &\Bigg|\frac{\mathcal{F}((x+y)+y_n) + \mathcal{G}((x+y)-y_n) - \mathcal{M}(y_n)}{2\mathcal{K}(y_n)} \\ &+ \frac{\mathcal{F}((x-y)+y_n) + \mathcal{G}((x-y)-y_n) - \mathcal{M}(y_n)}{2\mathcal{K}(y_n)} \\ &- \frac{2\mathcal{H}(x)\left(\mathcal{K}(y_n+y) + \mathcal{K}(y_n-y)\right) + \left(\mathcal{M}(y_n+y) + \mathcal{M}(y_n-y) - 2\mathcal{M}(y_n)\right)}{2\mathcal{K}(y_n)} \\ &- \frac{\mathcal{L}(x)}{\mathcal{K}(y_n)}\Bigg| \leq \frac{\varphi(x)}{|\mathcal{K}(y_n)|} \end{aligned} \tag{3.42}$$

and so

$$\Im_{\mathcal{K},\mathcal{M}}(x,y) := \lim_{n\to\infty} \frac{2\mathcal{H}(x)\left(\mathcal{K}(y_n+y) + \mathcal{K}(y_n-y)\right) + \left(\mathcal{M}(y_n+y) + \mathcal{M}(y_n-y) - 2\mathcal{M}(y_n)\right)}{2\mathcal{K}(y_n)}$$

exists, and by (3.39) we deduce that

$$\Im_{\mathcal{K},\mathcal{M}}(x,y) = \mathcal{H}(x+y) + \mathcal{H}(x-y). \tag{3.43}$$

Because $\Im_{\mathcal{K},\mathcal{M}}(x,y)$ exists, we note that if the existence of either

$$\ell_{\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{K}(y_n+y) + \mathcal{K}(y_n-y)}{\mathcal{K}(y_n)}$$

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or

$$\ell_{\mathcal{M},\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)}{2\mathcal{K}(y_n)}$$

implies that of the other and yields at once

$$\mathcal{H}(x+y)+\mathcal{H}(x-y)=\mathcal{H}(x)\ell_{\mathcal{K}}(y)+\ell_{\mathcal{M},\mathcal{K}}(y). \tag{3.44}$$

Taking $G=\mathbb{R}, K=\ell_{\mathcal{K}}, \ell=\ell_{\mathcal{M},\mathcal{K}}$ in Lemma 3.2.1, we obtain Part I. Similarly, Part II follows from taking $\ell=\ell_{\mathcal{M},\mathcal{K}}, K\equiv c$ and G a 2-divisible abelian group in Corollary 3.2.2. □

Since the functions $\mathcal{F}$ and $\mathcal{G}$ do not appear in the conclusion of Theorem 3.2.3, interchanging x with y and re-defining the functions $\mathcal{F}$ and $\mathcal{G}$ accordingly, we have:

Corollary 3.2.4. *Let G be an abelian group. Assume that $\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{K},\mathcal{L},\mathcal{M}:G\to\mathbb{C}$ satisfy*

$$|\mathcal{F}(x+y)+\mathcal{G}(x-y)-2\mathcal{H}(x)\mathcal{K}(y)-\mathcal{L}(x)-\mathcal{M}(y)|\le\varphi(y) \tag{3.45}$$

for all $x,y\in G$ and where $\varphi:G\to\mathbb{R}^+$.

If $\mathcal{H}$ is unbounded, then there is a sequence $\{y_n\}\subset G$ such that

$$\Im_{\mathcal{H},\mathcal{L}}(x,y) := \lim_{n\to\infty} \frac{2\mathcal{K}(x)\left(\mathcal{H}(y_n+y)+\mathcal{H}(y_n-y)\right)+\left(\mathcal{L}(y_n+y)+\mathcal{L}(y_n-y)-2\mathcal{L}(y_n)\right)}{2\mathcal{H}(y_n)} \tag{3.46}$$

exists for all $x,y\in G$, and the function $\mathcal{K}$ satisfies the functional equation

$$\mathcal{K}(x+y)+\mathcal{K}(x-y)=\Im_{\mathcal{H},\mathcal{L}}(x,y). \tag{3.47}$$

Moreover, if either the limit

$$\ell_{\mathcal{H}}(y) := \lim_{n\to\infty} \frac{\mathcal{H}(y_n + y) + \mathcal{H}(y_n - y)}{\mathcal{H}(y_n)}$$

or the limit

$$\ell_{\mathcal{L},\mathcal{H}}(y) := \lim_{n\to\infty} \frac{\mathcal{L}(y_n + y) + \mathcal{L}(y_n - y) - 2\mathcal{L}(y_n)}{2\mathcal{H}(y_n)}$$

exists for all $y \in G$, then the functional equation (3.47) *simplifies to*

$$\mathcal{K}(x+y) + \mathcal{K}(x-y) = \mathcal{K}(x)\ell_{\mathcal{H}}(y) + \ell_{\mathcal{L},\mathcal{H}}(y) \tag{3.48}$$

for all $x, y \in G$.

Assume that (3.48) *holds.*

I. When $G = \mathbb{R}$ and $\mathcal{K}, \ell_{\mathcal{H}}, \ell_{\mathcal{L},\mathcal{H}}$ are differentiable, if $\ell_{\mathcal{H}}$ is a non-constant function and $\ell'_{\mathcal{H}}(0) \neq 0$, then $\mathcal{K}(x) \equiv k \in \mathbb{C}$ is a constant function. Furthermore,

(Ia) if $k = 0$, then $\ell_{\mathcal{L},\mathcal{H}}(y) \equiv 0$ and $\ell_{\mathcal{H}}$ is an arbitrary function;

(Ib) if $k \neq 0$, then $\ell_{\mathcal{L},\mathcal{H}}(y) = k(2 - \ell_{\mathcal{H}}(y))$ and $\ell_{\mathcal{H}}$ is an arbitary function.

II. When G is a 2-divisible abelian group, if $\ell_{\mathcal{H}}(y) \equiv c \in \mathbb{C}$ is a constant function, then

(IIa) for $c = 2$, we have

$$\mathcal{K}(x) = B(x,x) + A(x) + r, \quad \ell_{\mathcal{L},\mathcal{H}}(x) = 2B(x,x),$$

where $B : G \times G \to \mathbb{C}$ is a symmetric biadditive function, $A : G \to \mathbb{C}$ is an additive function and $r \in \mathbb{C}$, *(IIb) for* $c \neq 2$, *we have*

$$\mathcal{K}(x) \equiv r, \quad \ell_{\mathcal{L},\mathcal{H}}(x) \equiv r(2-c)$$

are constant functions, where $r \in \mathbb{C}$.

Remark 3.2.5. *Theorem 3.2.3 and Corollary 3.2.4 continue to hold when* $\varphi(x)$, *respectively* $\varphi(y)$, *are replaced by a positive constant.*

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