CHAPTER 3

Feynman's Method

In this chapter we present Feynman's method for calculating the harmonic-oscillator path integral. In the first section we introduce the form of the harmonic-oscillator path integral and demonstrate the mathematical technique used by Feynman for separating the harmonic-oscillator propagator into the prefactor and the exponent of the classical action. In sec. 3.2 and 3.3 we evaluate the classical action and the prefactor of the harmonic-oscillator propagator respectively. In the last section we discuss some of the mathematical difficulties involved but were avoided by Feynman.

3.1 Harmonic-Oscillator Path Integral

Let us consider the one-dimensional harmonic oscillator problem where the lagrangian of the system is given by

$$L(\dot{x},x,t) = \frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}m\omega^2x^2(t) \tag{3.1}$$

For the sake of simplicity, let the particle moves from the point x_a at time $t = 0$ to the point x_b at time $t = T$. In the path integral formalism, the propagator can be written as

$$K(x_b,T;x_a,0) = \int_{x(0)=x_a}^{x(T)=x_b} \mathcal{D}[x(t)] \exp\left\{\frac{i}{\hbar}S[x]\right\} \tag{3.2}$$

where $$S[x] = \int_0^T L(\dot{x},x,t)\,dt \tag{3.3}$$

In order to separate the propagator in eq. (3.2) into two parts, the prefactor and the exponent of classical action, Feynman described the path $x(t)$ by means of the classical path $\bar{x}(t)$ and the deviation from the classical path $y(t)$ as

$$x(t) = \bar{x}(t) + y(t) \tag{3.4}$$

By this trick, the path differential $x(t)$ can be represented by $y(t)$ and the action in the eq. (3.3) becomes

$$S[x] = \int_0^T L(\dot{\bar{x}},\bar{x},t)\,dt + \int_0^T L(\dot{y},y,t)\,dt$$

$$= S_{cl} + S[y] \tag{3.5}$$

where S_{cl} is the classical action and the resulting integral for all terms which contain $y(t)$ as linear factor in the integrand vanishes. Feynman called this simple integral the "the Gaussian Path Integral".

On substituting eq. (3.5) into eq. (3.2) we obtain

$$K(x_b,T;x_a,0) = F(T)\exp\left\{\frac{i}{\hbar}S_{cl}\right\} \tag{3.6}$$

where $$F(T) = \int_0^0 \mathcal{D}[y]\exp\left\{\frac{i}{\hbar}S[y]\right\} \tag{3.7}$$

is only a function of the time at the end point. Since all paths $y(t)$ start from and return to the point $y = 0$. The path integrand $F(T)$ is called "the prefactor".

3.2 The Evaluation of the Classical Action

To calculate the classical action we must first solve the classical equation of motion

$$\ddot{\bar{x}}(t) + \omega^2 \bar{x}(t) = 0 \tag{3.8}$$

with the boundary conditions $x(o) = x_a$, $x(T) = x_b$. The standard method for solving the above equation gives

$$\bar{x}(t) = x_a \cos\omega t + \frac{1}{\sin\omega T}\left[x_b - x_a\cos\omega T\right]\sin\omega t \tag{3.9}$$

Since the classical action of the harmonic oscillator can be written as

$$S_{cl} = \frac{m}{2}\int_0^T \dot{\bar{x}}^2(t)\,dt - \frac{m\omega^2}{2}\int_0^T \bar{x}^2(t)\,dt \tag{3.10}$$

we can integrate by part to the first term and by using the classical equation (3.9), we obtain

$$S_{cl} = \frac{m\omega}{2\sin\omega T}\left[(x_a^2 + x_b^2)\cos\omega T - 2x_a x_b\right] \tag{3.11}$$

In the next section we shall evaluate the prefactor in eq. (3.7) using Feynman's method.

3.3 Harmonic-Oscillator Prefactor

To evaluate the prefactor Feynman represented the paths $y(t)$ as a Fourier sine series with a fundamental period of T, i.e.

$$y(t) = \sum_m a_m \sin \frac{m\pi t}{T} \tag{3.12}$$

Using this mathematical representation, Feynman considered the paths as functions of the coefficients of a_n instead of y at any particular value of t. This is a linear transformation whose jacobian J is a constant, which is obviously independent of ω, m, and $\hbar$.

The integral for the action can be written in terms of the Fourier series of eq. (3.12). The kinetic-energy term becomes

$$\int_0^T \dot{y}^2(t)\,dt = \frac{T}{2} \sum_m a_m^2 \left(\frac{m\pi}{T}\right)^2 \tag{3.13}$$

and similarly the potential-energy term becomes

$$\int_0^T y^2(t)\,dt = \frac{T}{2} \sum_m a_m^2 \tag{3.14}$$

On the assumption of dividing the time T into N discrete steps of length ϵ, eq. (3.7) can be written as

$$F(T) = \lim_{N\to\infty} \frac{J}{A} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ \frac{im}{2\hbar} \sum_{m=1}^{N-1} \left[\left(\frac{m\pi}{T}\right)^2 - \omega^2 \right] a_m^2 \right\} \frac{da_1}{A} \cdots \frac{da_{N-1}}{A} \tag{3.15}$$

Since the exponent can be separated into factors, the integral over each a_n can be done separately. The result of one such integration is

$$\int_{-\infty}^{\infty} \exp\left\{ \frac{im}{2\hbar}\left[\left(\frac{m\pi}{T}\right)^2 - \omega^2\right] a_m^2 \right\} \frac{da_m}{A} = \left[\left(\frac{m\pi}{T}\right)^2 - \omega^2\right]^{-1/2} \tag{3.16}$$

Thus the path integral is proportional to

$$\prod_{m=1}^{N-1} \left[\left(\frac{m\pi}{T}\right)^2 - \omega^2 \right]^{-1/2} = \prod_{m=1}^{N-1} \left(\frac{T}{m\pi}\right) \prod_{m=1}^{N-1} \left[1 - \left(\frac{\omega T}{m\pi}\right)^2 \right]^{-1/2} \tag{3.17}$$

The first product does not depend on ω and on combining it with the jacobian and other factors, we get a constant. The second factor has the limit $[\sin \omega T/\omega T]^{-\frac{1}{2}}$ as $N \to \infty$, or as $\epsilon \to 0$. Thus

$$F(T) = C\left[\frac{\sin \omega T}{\omega T}\right]^{-1/2} \tag{3.18}$$

where C is independent of ω. For $\omega = 0$ our integral is that for a free particle which we know to be (chapter 3, Ref. 10)

$$F(T) = \left[\frac{m}{2\pi i\hbar T}\right]^{1/2} \tag{3.19}$$

Hence for the harmonic oscillator we have

$$F(T) = \left[\frac{m\omega}{2\pi i\hbar \sin \omega T}\right]^{1/2} \tag{3.20}$$

Substituting the eqs. (3.20) and (3.11) into the eq. (3.6) we obtain the harmonic-oscillator propagator in the following form :

$$K(x_b,T;x_a,0) = \left[\frac{m\omega}{2\pi i\hbar \sin \omega T}\right]^{1/2} \exp\left\{ \frac{im\omega}{2\hbar \sin \omega T}\left[(x_a^2 + x_b^2)\cos \omega T - 2x_a x_b \right] \right\} \tag{3.21}$$

3.4 Conclusion and Discussion

In calculating the harmonic-oscillator propagator, Feynman expressed all possible paths $x(t)$ as the sum of the classical path $\bar{x}(t)$ and the deviation from the classical path, $y(t)$. He separated the harmonic-oscillator path integral into two parts, the prefactor and the exponent of the classical action.

Since the prefactor can be expressed as the path integral with vanishing boundary points, Feynman expressed $y(t)$ as the Fourier sine series and transformed this path integral to be the multiple integrals of the Fourier coefficients. Instead of using a systematic method of performing direct calculation, he used the free-particle limit to evaluate the integrals.

At this point, one can see that if one wants to perform direct calculation one has to solve for the jacobian of transformation and take care of all factors arising in the integrations. In the following chapters we shall show that by using the present technique the propagator can be determined of systematically.