CHAPTER 3

SOLUTIONS OF FINITE PIEZOELECTRIC CYLINDER

In this chapter, the derivation of analytical solutions for a homogeneous piezoelectric cylinder under axisymmetric end loading and electric field is presented. Figure 3.1 shows a piezoelectric solid cylinder of diameter D $(2R)$ and length H $(2h)$. A cylindrical polar coordinate system (r, θ, z) as shown in Figure 3.1 is used with the z-axis along the axis of symmetry of the cylinder. The cylinder is made of a piezoelectric material with the poling direction parallel to the z-axis. Both mechanical and electrical loads are assumed to be axisymmetric and applied at the top and bottom surfaces of the cylinder.

3.1 Basic Equations of Piezoelectric Materials

The constitutive relations of a piezoelectric material undergoing axisymmetric deformation can be expressed as follows (Parton and Kudryavtsev, 1988)

$$\sigma_{rr} = c_{11}\epsilon_{rr} + c_{12}\epsilon_{\theta\theta} + c_{13}\epsilon_{zz} - e_{31}E_z \tag{3.1}$$

$$\sigma_{\theta\theta} = c_{12}\epsilon_{rr} + c_{11}\epsilon_{\theta\theta} + c_{13}\epsilon_{zz} - e_{31}E_z \tag{3.2}$$

$$\sigma_{zz} = c_{13}\epsilon_{rr} + c_{13}\epsilon_{\theta\theta} + c_{33}\epsilon_{zz} - e_{33}E_z \tag{3.3}$$

$$\sigma_{rz} = 2c_{44}\epsilon_{rz} - e_{15}E_r \tag{3.4}$$

$$D_r = 2e_{15}\epsilon_{rz} + \varepsilon_{11}E_r \tag{3.5}$$

$$D_z = e_{31}\epsilon_{rr} + e_{31}\epsilon_{\theta\theta} + e_{33}\epsilon_{zz} + \varepsilon_{33}E_z \tag{3.6}$$

where σ_{ij}, ϵ_{ij}, D_i, and E_i are the components of stress, strain, electric displacement and electric field, respectively. c_{11}, c_{12}, c_{13}, c_{33} and c_{44} are elastic constants under zero or constant electric field. e_{15}, e_{31} and e_{33} are piezoelectric coefficients. ε_{11} and ε_{33}

are dielectric constants under zero or constant strain. The components of strain and electric field can be expressed as

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{3.7}$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r} \tag{3.8}$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{3.9}$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{3.10}$$

where $u_r(r,z)$ and $u_z(r,z)$ denote the mechanical displacements in the $r-$ and z - direction, respectively. In addition, the relationship between the electric field E_i ($i = r,\ z$) and the electric potential $\phi(r,z)$ is given by

$$E_r = -\frac{\partial \phi}{\partial r} \tag{3.11}$$

$$E_z = -\frac{\partial \phi}{\partial z} \tag{3.12}$$

The equilibrium equations for a piezoelectric cylinder subjected to axisymmetric end loading can be expressed as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{3.13}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{3.14}$$

$$\frac{\partial D_r}{\partial r} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} = 0 \tag{3.15}$$

Substitution of equations (3.1) to (3.12) into equations (3.13) to (3.15) yields

$$c_{11}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}u_r\right) + c_{44}\frac{\partial^2 u_r}{\partial z^2} + (c_{13} + c_{44})\frac{\partial^2 u_z}{\partial r \partial z} + (e_{15} + e_{31})\frac{\partial^2 \phi}{\partial r \partial z} = 0 \tag{3.16}$$

$$(c_{13}+c_{44})\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+c_{44}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+c_{33}\frac{\partial^2 u_z}{\partial z^2}+e_{15}\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}\right)+e_{33}\frac{\partial^2 \phi}{\partial z^2}=0 \qquad (3.17)$$

$$(e_{31}+e_{15})\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+e_{15}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+e_{33}\frac{\partial^2 u_z}{\partial z^2}-\varepsilon_{11}\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}\right)-\varepsilon_{33}\frac{\partial^2 \phi}{\partial z^2}=0 \qquad (3.18)$$

The above equations are the governing equations of a transversely isotropic piezoelectric cylinder undergoing axisymmetric deformations.

The fact that the elastic constants, the piezoelectric constants and the dielectric constant have different orders may lead to precision problem and numerical instability of numerical solutions. For example, for PZT-4, the orders of material constants are as follows:

$$c_{ij}\sim 10^{10}\ N/m^2; \quad e_{ij}\sim 10^{0}\ C/m^2; \quad \varepsilon_{ij}\sim 10^{-9}\ F/m$$

Therefore, all variables will be non-dimensionalized before solving the equations (3.16) to (3.18). The coordinates r and z and the displacements u_r and u_z are non-dimensionalized by the radius of the cylinder R; the stresses and elastic constants are non-dimensionalized by c_{44}; and the electric displacements and piezoelectric constants are non-dimensionalized by e_{31}. For convenience, the non-dimensional, coordinate, mechanical displacements, stresses, electric displacements, elastic constants and piezoelectric constants are denoted by the same symbols. In addition, the following non-dimensional parameters of coordinates, electric potential and the dielectric constants are introduced.

$$\tilde{\phi}=\frac{e_{31}}{c_{44}R}\phi \qquad (3.19)$$

$$\tilde{\varepsilon}_{11}=\frac{\varepsilon_{11}c_{44}}{e_{31}^2} \qquad (3.20)$$

$$\tilde{\varepsilon}_{33}=\frac{\varepsilon_{33}c_{44}}{e_{31}^2} \qquad (3.21)$$

By using the above non-dimensional quantities, equations (3.16) to (3.18) then becomes

$$c_{11}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}u_r\right)+\frac{\partial^2 u_r}{\partial z^2}+\left(1+c_{13}\right)\frac{\partial^2 u_z}{\partial r\partial z}+(1+e_{15})\frac{\partial^2\tilde{\phi}}{\partial r\partial z}=0 \tag{3.22}$$

$$\left(1+c_{13}\right)\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+c_{33}\frac{\partial^2 u_z}{\partial z^2}+e_{15}\left(\frac{\partial^2\tilde{\phi}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\phi}}{\partial r}\right)+e_{33}\frac{\partial^2\tilde{\phi}}{\partial z^2}=0 \tag{3.23}$$

$$\left(1+e_{15}\right)\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+e_{15}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+e_{33}\frac{\partial^2 u_z}{\partial z^2}-\tilde{\varepsilon}_{11}\left(\frac{\partial^2\tilde{\phi}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\phi}}{\partial r}\right)-\tilde{\varepsilon}_{33}\frac{\partial^2\tilde{\phi}}{\partial z^2}=0 \tag{3.24}$$

3.2 General Solution of Piezoelectric Cylinder

It seems to be extremely difficult to obtain the solution by means of direct integration due to the complexity of equations (3.22) to (3.24). But the problem may become more tractable if we introduce a following set of potential functions that can transform equations (3.22) to (3.24) into familiar differential equations.

$$u_r=\frac{\partial\psi}{\partial r} \tag{3.25}$$

$$u_z=k_1\frac{\partial\psi}{\partial z} \tag{3.26}$$

$$\tilde{\phi}=k_2\frac{\partial\psi}{\partial z} \tag{3.27}$$

where $\psi(r,z)$ is the potential function, and k_1 and k_2 are the unknown constants. Substituting equations (3.25) to (3.27) into equations (3.22) to (3.24) leads to

$$c_{11}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\left[1+\left(1+c_{13}\right)k_1+(1+e_{15})k_2\right]\frac{\partial^2\psi}{\partial z^2}=0 \tag{3.28}$$

$$\left(1+c_{13}+k_1+e_{15}k_2\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\left[c_{33}k_1+e_{33}k_2\right]\frac{\partial^2\psi}{\partial z^2}=0 \tag{3.29}$$

$$\left(1+e_{15}+e_{15}k_1-\tilde{\varepsilon}_{11}k_2\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\left[e_{33}k_1-\tilde{\varepsilon}_{33}k_2\right]\frac{\partial^2\psi}{\partial z^2}=0 \tag{3.30}$$

In equations (3.28) to (3.30), the terms $\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}$ and $\frac{\partial^2\psi}{\partial z^2}$ are not identically equal to zero. Under this condition, a nontrivial solution of equations (3.28) to (3.30) will exist only if they are identical equations, i.e.,

$$\frac{1+(1+c_{13})k_1+(1+e_{15})k_2}{c_{11}}=\frac{c_{33}k_1+e_{33}k_2}{1+c_{13}+k_1+e_{15}k_2}=\Omega \tag{3.31}$$

$$\frac{1+(1+c_{13})k_1+(1+e_{15})k_2}{c_{11}}=\frac{e_{33}k_1-\tilde{\varepsilon}_{33}k_2}{1+e_{15}+e_{15}k_1-\tilde{\varepsilon}_{11}k_2}=\Omega \tag{3.32}$$

By eliminating k_1 and k_2 in equations (3.31) and (3.32), a cubic algebra equation of Ω is obtained as

$$A\Omega^3+B\Omega^2+C\Omega+D=0 \tag{3.33}$$

where

$$A=c_{11}e_{15}^2+c_{11}\tilde{\varepsilon}_{11} \tag{3.34}$$

$$B=2c_{13}e_{15}^2-1+2c_{13}e_{15}-2c_{11}e_{15}e_{33}+c_{13}^2\tilde{\varepsilon}_{11}+2c_{13}\tilde{\varepsilon}_{11}-c_{11}c_{33}\tilde{\varepsilon}_{11}-c_{11}\tilde{\varepsilon}_{33} \tag{3.35}$$

$$\begin{aligned}C=&\,c_{33}e_{15}^2+2c_{33}e_{15}+c_{33}-2c_{13}e_{15}e_{33}-2c_{13}e_{33}-2e_{33}+\\&c_{33}\tilde{\varepsilon}_{11}+c_{11}e_{33}^2-c_{13}^2\tilde{\varepsilon}_{33}-2c_{13}\tilde{\varepsilon}_{33}+c_{11}c_{33}\tilde{\varepsilon}_{33}\end{aligned} \tag{3.36}$$

$$D=-e_{33}^2-c_{33}\tilde{\varepsilon}_{33} \tag{3.37}$$

The three roots of equation (3.33) can be denoted by $\Omega_i(i=1,\,2,\,3)$, in which Ω_1 is assumed to be a positive real number, whereas Ω_2 and Ω_3 are either positive real numbers or a pair of conjugate complex roots with positive real parts. Corresponding to these three roots, there are three potential functions $\psi_i(i=1,\,2,\,3)$ in which each of them must satisfy one of the following equations.

$$\frac{\partial^2\psi_i}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_i}{\partial r}+\Omega_i\frac{\partial^2\psi_i}{\partial z^2}=0 \qquad (i=1,\,2,\,3) \tag{3.38}$$

Therefore, the mechanical displacements and electric potential in equations (3.25) to (3.27) can be written as

$$u_r = \frac{\partial}{\partial r}\left(\psi_1 + \psi_2 + \psi_3\right) \tag{3.39}$$

$$u_z = k_{11}\frac{\partial \psi_1}{\partial z} + k_{12}\frac{\partial \psi_2}{\partial z} + k_{13}\frac{\partial \psi_3}{\partial z} \tag{3.40}$$

$$\tilde{\phi} = k_{21}\frac{\partial \psi_1}{\partial z} + k_{22}\frac{\partial \psi_2}{\partial z} + k_{23}\frac{\partial \psi_3}{\partial z} \tag{3.41}$$

By substituting equations (3.39) to (3.41) into equations (3.1) to (3.6), the stresses and electric displacements can be obtained in term of ψ_i as follows:

$$\sigma_{rr} = \left(c_{11}\frac{\partial^2}{\partial r^2} + c_{12}\frac{1}{r}\frac{\partial}{\partial r}\right)\psi_i + \left(c_{13}k_{1i} + k_{2i}\right)\frac{\partial^2 \psi_i}{\partial z^2} \tag{3.42}$$

$$\sigma_{\theta\theta} = \left(c_{12}\frac{\partial^2}{\partial r^2} + c_{11}\frac{1}{r}\frac{\partial}{\partial r}\right)\psi_i + \left(c_{13}k_{1i} + k_{2i}\right)\frac{\partial^2 \psi_i}{\partial z^2} \tag{3.43}$$

$$\sigma_{zz} = c_{13}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\psi_i + \left(c_{33}k_{1i} + e_{33}k_{2i}\right)\frac{\partial^2 \psi_i}{\partial z^2} \tag{3.44}$$

$$\sigma_{rz} = \left(1 + k_{1i} + e_{15}k_{2i}\right)\frac{\partial^2 \psi_i}{\partial r \partial z} \tag{3.45}$$

$$D_z = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\psi_i + \left(e_{33}k_{1i} - \tilde{\varepsilon}_{33}k_{2i}\right)\frac{\partial^2 \psi_i}{\partial z^2} \tag{3.46}$$

$$D_r = \left(e_{15} + e_{15}k_{1i} - \tilde{\varepsilon}_{11}k_{2i}\right)\frac{\partial^2 \psi_i}{\partial r \partial z} \tag{3.47}$$

Thereafter, the general solutions for a piezoelectric cylinder undergoing axisymmetric deformations can be obtained by solving equation (3.38) for a function ψ_i and then substituting the solution of ψ_i into equations (3.39) to (3.47). To obtain the solution for ψ_i, first consider equations (3.38) in the following form

$$\frac{\partial^2 \psi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_i}{\partial r} + \frac{\partial^2 \psi_i}{\partial z_i^2} = 0, \qquad (i = 1,\ 2,\ 3) \tag{3.48}$$

where $z_i = z/\sqrt{\Omega_i}$. The solution of equation (3.48) can be written as

$$\psi_i = \left[CJ_0(\zeta r) + GY_0(\zeta r)\right]\left[A_i \cosh(\zeta z_i) + B_i \sinh(\zeta z_i)\right] \tag{3.49}$$

where $J_0(\zeta r)$ and $Y_0(\zeta r)$ are Bessel functions of the first and second kind of zero order (Abramowitz and Stegun, 1972), respectively, and ζ, A_i, B_i, C and G are arbitrary constants to be determined.

On the other hand, equation (3.38) can also be expressed in the following form

$$\frac{\partial^2 \psi_i}{\partial r_i^2} + \frac{1}{r_i}\frac{\partial \psi_i}{\partial r_i} + \frac{\partial^2 \psi_i}{\partial z^2} = 0, \qquad (i = 1, 2, 3) \tag{3.50}$$

where $r_i = \sqrt{\Omega_i} r$. The solutions of the above Laplace equations are in the form of

$$\psi_i = \left[H_i I_0(\lambda r_i) + L_i K_0(\lambda r_i)\right]\left[M \cos(\lambda z) + N \sin(\lambda z)\right] \tag{3.51}$$

where $I_0(\lambda r_i)$ and $K_0(\lambda r_i)$ are the modified Bessel functions of the first and second kind of zero order (Abramowitz and Stegun, 1972), respectively, and λ, H_i, L_i, M and N are arbitrary constants to be determined.

In equations (3.42) to (3.45), all normal stresses σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} must be an even function of z, while σ_{rz} must be an odd function of z. The linear combination of the two general solutions yields the solution for ψ as

$$\psi = A_{01} \ln r + \sum_{i=1}^{3} \left[\begin{array}{l} B_{i0}\left(r^2 - 2q_i^2 z^2\right) - \sum_{n=1}^{\infty} \frac{\cos(\zeta_n z)}{\zeta_n^2}\left[A_{in} I_0(p_i \zeta_n r) + B_{in} K_0(p_i \zeta_n r)\right] \\ -\sum_{s=1}^{\infty} \frac{\cosh(q_i \lambda_s z)}{\lambda_s^2}\left[C_{is} J_0(\lambda_s r) + D_{is} Y_0(\lambda_s r)\right] \end{array} \right] \tag{3.52}$$

where λ_s is the s^{th} root of $J_1(\lambda_s) = 0$, $q_i = 1/\sqrt{\Omega_i}$, $p_i = \sqrt{\Omega_i}$, and $\zeta_n = n\pi/h$. Note that the first three terms with constant coefficients A_{01} and B_{io} of the potential function contribute to the case where only constant normal stresses are applied on the surfaces of the cylinder.

The solutions for mechanical displacements, electric potential, electric fields, stresses and electric displacements of piezoelectric cylinder are obtained by substituting the function $\psi(r,z)$ given by equation (3.52) into equations (3.39) to (3.47). The details of derivation for these solutions are given in Appendix A. First, the solution for displacements and electric potential can be expressed as

$$u_r = \frac{A_{01}}{r} + \sum_{i=1}^{\infty} \begin{bmatrix} 2\rho B_{0i} + \sum_{n=1}^{\infty} \frac{p_i \cos(\zeta_n z)}{\zeta_n} \left[-A_{in} I_1(p_i \zeta_n r) + B_{in} K_1(p_i \zeta_n r) \right] \\ + \sum_{s=1}^{\infty} \frac{\cosh(q_i \lambda_s z)}{\lambda_s} \left[C_{is} J_1(\lambda_s r) + D_{is} Y_1(\lambda_s r) \right] \end{bmatrix} \tag{3.53}$$

$$u_z = \sum_{i=1}^{\infty} k_{1i} \begin{bmatrix} -4q_i^2 z B_{0i} + \sum_{n=1}^{\infty} \frac{\sin(\zeta_n z)}{\zeta_n} \left[A_{in} I_0(p_i \zeta_n r) + B_{in} K_0(p_i \zeta_n r) \right] \\ + \sum_{s=1}^{\infty} \frac{q_i \sinh(q_i \lambda_s z)}{\lambda_s} \left[-C_{is} J_0(\lambda_s r) - D_{is} Y_0(\lambda_s r) \right] \end{bmatrix} \tag{3.54}$$

$$\tilde{\phi} = \sum_{i=1}^{\infty} k_{2i} \begin{bmatrix} -4q_i^2 z B_{0i} + \sum_{n=1}^{\infty} \frac{\sin(\zeta_n z)}{\zeta_n} \left[A_{in} I_0(p_i \zeta_n r) + B_{in} K_0(p_i \zeta_n r) \right] \\ + \sum_{s=1}^{\infty} \frac{q_i \sinh(q_i \lambda_s z)}{\lambda_s} \left[-C_{is} J_0(\lambda_s r) - D_{is} Y_0(\lambda_s r) \right] \end{bmatrix} \tag{3.55}$$

The electric fields can be determined by substituting electric potential given by equation (3.55) into equations (3.11) and (3.12). Then

$$E_r = \sum_{i=1}^{\infty} k_{2i} \begin{bmatrix} \sum_{n=1}^{\infty} \sin(\zeta_n z) \left[-A_{in} p_i I_1(p_i \zeta_n r) + B_{in} p_i K_1(p_i \zeta_n r) \right] \\ + \sum_{s=1}^{\infty} q_i \sinh(q_i \lambda_s z) \left[-C_{is} J_1(\lambda_s r) - D_{is} Y_1(\lambda_s r) \right] \end{bmatrix} \tag{3.56}$$

$$E_z = \sum_{i=1}^{\infty} k_{2i} \begin{bmatrix} 4q_i^2 B_{io} + \sum_{n=1}^{\infty} \cos(\zeta_n r) \left[-A_{in} I_0(p_i \zeta_n r) - B_{in} K_0(p_i \zeta_n r) \right] \\ + \sum_{s=1}^{\infty} q_i^2 \cosh(q_i \lambda_s z) \left[C_{is} J_0(\lambda_s r) + D_{is} Y_0(\lambda_s r) \right] \end{bmatrix} \tag{3.57}$$

Finally, substitution of $\psi(r,z)$ into equations (3.42) to (3.47) yields the following expressions for stresses and electric displacements.

$$\sigma_{rr} = (c_{12} - c_{11})\frac{1}{r^2}A_{01} + \sum_{i=1}^{3} B_{i0}\left[2(c_{12} + c_{11}) - 4(c_{13}k_{1i} + k_{2i})q_i^2\right]$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} A_{in}\left[(-c_{11}p_i^2 + c_{13}k_{1i} + k_{2i})I_0(p_i\zeta_n r) + (c_{11} - c_{12})p_i\frac{I_1(p_i\zeta_n r)}{\zeta_n r}\right]\cos(\zeta_n z)$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} B_{in}\left[(-c_{11}p_i^2 + c_{13}k_{1i} + k_{2i})K_0(p_i\zeta_n r) + (c_{12} - c_{11})p_i\frac{K_1(p_i\zeta_n r)}{\zeta_n r}\right]\cos(\zeta_n z) \quad (3.58)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} C_{is}\left[(c_{11} - c_{13}k_{1i}q_i^2 - k_{2i}q_i^2)J_0(\lambda_s r) + (c_{12} - c_{11})\frac{J_1(\lambda_s r)}{\lambda_s r}\right]\cosh(q_i\lambda_s z)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} D_{is}\left[(c_{11} - c_{13}k_{1i}q_i^2 - k_{2i}q_i^2)Y_0(\lambda_s r) + (c_{12} - c_{11})\frac{Y_1(\lambda_s r)}{\lambda_s r}\right]\cosh(q_i\lambda_s z)$$

$$\sigma_{\theta\theta} = \sum_{i=1}^{3} B_{i0}\left[2(c_{12} + c_{11}) - 4(c_{13}k_{1i} + k_{2i})q_i^2\right]$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} A_{in}\left[(-c_{12}p_i^2 + c_{13}k_{1i} + k_{2i})I_0(p_i\zeta_n r) + (c_{12} - c_{11})p_i\frac{I_1(p_i\zeta_n r)}{\zeta_n r}\right]\cos(\zeta_n z)$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} B_{in}\left[(-c_{12}p_i^2 + c_{13}k_{1i} + k_{2i})K_0(p_i\zeta_n r) + (c_{11} - c_{12})p_i\frac{K_1(p_i\zeta_n r)}{\zeta_n r}\right]\cos(\zeta_n z) \quad (3.59)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} C_{is}\left[(c_{12} - c_{13}k_{1i}q_i^2 - k_{2i}q_i^2)J_0(\lambda_s r) + (c_{11} - c_{12})\frac{J_1(\lambda_s r)}{\lambda_s r}\right]\cosh(q_i\lambda_s z)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} D_{is}\left[(c_{12} - c_{13}k_{1i}q_i^2 - k_{2i}q_i^2)Y_0(\lambda_s r) + (c_{11} - c_{12})\frac{Y_1(\lambda_s r)}{\lambda_s r}\right]\cosh(q_i\lambda_s z)$$

$$\sigma_{zz} = \sum_{i=1}^{3} B_{i0}\left[4c_{13} - 4(c_{33}k_{1i} + e_{33}k_{2i})q_i^2\right]$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} A_{in}\left[(-c_{13}p_i^2 + c_{33}k_{1i} + e_{33}k_{2i})I_0(p_i\zeta_n r)\right]\cos(\zeta_n z)$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} B_{in}\left[(-c_{13}p_i^2 + c_{33}k_{1i} + e_{33}k_{2i})K_0(p_i\zeta_n r)\right]\cos(\zeta_n z) \quad (3.60)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} C_{is}\left[(c_{13} - c_{33}k_{1i}q_i^2 - e_{33}k_{2i}q_i^2)J_0(\lambda_s r)\right]\cosh(q_i\lambda_s z)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} D_{is}\left[(c_{13} - c_{33}k_{1i}q_i^2 - e_{33}k_{2i}q_i^2)Y_0(\lambda_s r)\right]\cosh(q_i\lambda_s z)$$

$$\sigma_{rz} = \sum_{i=1}^{3}\sum_{n=1}^{\infty} A_{in}\left(1 + k_{1i} + e_{15}k_{2i}\right)p_i I_1(p_i\zeta_n r)\sin(\zeta_n z)$$
$$-\sum_{i=1}^{3}\sum_{n=1}^{\infty} B_{in}\left(1 + k_{1i} + e_{15}k_{2i}\right)p_i K_1(p_i\zeta_n r)\sin(\zeta_n z)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} C_{is}\left(1 + k_{1i} + e_{15}k_{2i}\right)q_i J_1(\lambda_s r)\sinh(q_i\lambda_s z) \quad (3.61)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} D_{is}\left(1 + k_{1i} + e_{15}k_{2i}\right)q_i Y_1(\lambda_s r)\sinh(q_i\lambda_s z)$$

$$D_r = \sum_{i=1}^{3}\sum_{n=1}^{\infty} A_{in}\left(e_{15} + e_{15}k_{1i} - \tilde{\varepsilon}_{11}k_{2i}\right) p_i I_1(p_i\zeta_n r)\sin(\zeta_n z)$$
$$-\sum_{i=1}^{3}\sum_{n=1}^{\infty} B_{in}\left(e_{15} + e_{15}k_{1i} - \tilde{\varepsilon}_{11}k_{2i}\right) p_i K_1(p_i\zeta_n r)\sin(\zeta_n z)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} C_{is}\left(e_{15} + e_{15}k_{1i} - \tilde{\varepsilon}_{11}k_{2i}\right) q_i J_1(\lambda_s r)\sinh(q_i\lambda_s z) \quad (3.62)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} D_{is}\left(e_{15} + e_{15}k_{1i} - \tilde{\varepsilon}_{11}k_{2i}\right) q_i Y_1(\lambda_s r)\sinh(q_i\lambda_s z)$$

$$D_z = \sum_{i=1}^{3} B_{i0}\left[4e_{31} - 4(e_{33}k_{1i} - \tilde{\varepsilon}_{33}k_{2i})q_i^2\right]$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} A_{in}\left[(-p_i^2 + e_{33}k_{1i} - \tilde{\varepsilon}_{33}k_{2i})I_0(p_i\zeta_n r)\right]\cos(\zeta_n z)$$
$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} B_{in}\left[(-p_i^2 + e_{33}k_{1i} - \tilde{\varepsilon}_{33}k_{2i})K_0(p_i\zeta_n r)\right]\cos(\zeta_n z) \quad (3.63)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} C_{is}\left[(1 - e_{33}k_{1i}q_i^2 + \tilde{\varepsilon}_{33}k_{2i}q_i^2)J_0(\lambda_s r)\right]\cosh(q_i\lambda_s z)$$
$$+\sum_{i=1}^{3}\sum_{s=1}^{\infty} D_{is}\left[(1 - e_{33}k_{1i}q_i^2 + \tilde{\varepsilon}_{33}k_{2i}q_i^2)Y_0(\lambda_s r)\right]\cosh(q_i\lambda_s z)$$

3.3 Piezoelectric Cylinder under Surface Load and Electric Field

In the previous section, mechanical displacements, electric potential, electric fields, stresses and electric displacements are expressed in terms of arbitrary constants. Those constants have to be determined from the appropriate boundary conditions corresponding to a given boundary value problem. In this section, boundary value problems involving a piezoelectric under mechanical loads and electric fields are considered.

3.3.1 Cylinder Subjected to Mechanical Loading

The boundary conditions of a piezoelectric cylinder subjected to the mechanical loading applied to the top and bottom surfaces as shown in Figure 3.2 can be expressed as follows:

- For the curved surface, i.e. $r = 1$ and $-h \le z \le h$

$$\sigma_{rr}(1, z) = 0 \quad (3.64)$$

$$\sigma_{rz}(1, z) = 0 \quad (3.65)$$

$$D_r(1,z) \quad = \quad 0 \tag{3.66}$$

- For the top and bottom surfaces, i.e. $z = \pm h$ and $0 \le r \le 1$

$$\sigma_{zz}(r,\pm h) \quad = \quad P(r) \tag{3.67}$$

$$\sigma_{rz}(r,\pm h) \quad = \quad 0 \tag{3.68}$$

$$D_z(r,\pm h) \quad = \quad 0 \tag{3.69}$$

where $P(r)$ denotes the non-dimensional normal pressure applied at the top and bottom surfaces of the cylinder. The boundary conditions given by equations (3.58) to (3.63) will be used to determine the arbitrary constants A_{in}, B_{in}, C_{is} and D_{is} $(i = 1,\ 2,\ 3; n,\ s = 1,\ 2,\ 3...\infty)$. In the case of a solid cylinder, all terms related to $\ln r$, $K_0(p_i\zeta_n r)$ and $Y_0(\lambda_s r)$ must be discarded because the stress field at the center of a solid cylinder must be finite, therefore, $A_{01} = B_{in} = D_{is} = 0$.

First, consider the expression for the radial stress σ_{rr} on the curved surface which can be obtained by setting $r = 1$ in equation (3.58), then

$$\sigma_{rr}(1,z) = \sum_{i=1}^{3}\left\{\begin{array}{l} B_{i0}\left[2(c_{12}+c_{11}) - 4(c_{13}k_{1i}+k_{2i})q_i^2\right] \\ +\sum_{n=1}^{\infty} A_{in}\left[(-c_{11}p_i^2 + c_{13}k_{1i} + k_{2i})I_0(p_i\zeta_n) + (c_{11}-c_{12})p_i\dfrac{I_1(p_i\zeta_n)}{\zeta_n}\right]\cos(\zeta_n z) \\ +\sum_{s=1}^{\infty} C_{is}\left[(c_{11} - c_{13}k_{1i}q_i^2 - k_{2i}q_i^2)J_0(\lambda_s)\right]\cosh(q_i\lambda_s z) \end{array}\right\} \tag{3.70}$$

By expressing the hyperbolic cosine functions in the following form

$$\cosh(q_i\lambda_s z) = \frac{\sinh(q_i\lambda_s h)}{q_i\lambda_s} + \sum_{n=1}^{\infty}\frac{2(-1)^n q_i\gamma_s \sinh(q_i\lambda_s h)}{q_i^2\gamma_s^2 + \zeta_n^2}\cos(\zeta_n z) \tag{3.71}$$

and substituting equation (3.70) into the boundary condition, equation (3.64), the following set of the relationship between the arbitrary functions can be established

$$\sum_{i=1}^{3}\left\{\begin{array}{l} B_{i0}\left[2(c_{12}+c_{11}) - 4(c_{13}k_{1i}+k_{2i})q_i^2\right] + \\ \sum_{s=1}^{\infty} C_{is}\left[(c_{11} - c_{13}k_{1i}q_i^2 - k_{2i}q_i^2)J_0(\lambda_s)\right]\dfrac{\sinh(q_i\lambda_s h)}{q_i\gamma_s} \end{array}\right\} = 0 \tag{3.72}$$

$$\sum_{i=1}^{3}\left\{\begin{array}{l} A_{in}\left[(-c_{11}p_i^2+c_{13}k_{1i}+k_{2i})I_0(p_i\zeta_n)+(c_{11}-c_{12})p_i\dfrac{I_1(p_i\zeta_n)}{\zeta_n}\right] \\ +\sum\limits_{s=1}^{\infty}C_{is}\left[(c_{11}-c_{13}k_{1i}q_i^2-k_{2i}q_i^2)J_0(\lambda_s)\right]\dfrac{2(-1)^n q_i\lambda_s\sinh(q_i\lambda_s h)}{q_i^2\lambda_s^2+\zeta_n^2} \end{array}\right\}=0 \tag{3.73}$$

The condition of zero shear stress, i.e. $\sigma_{rz}=0$ on the curved surface $r=1$, equation (3.65), and the two end surface $z=\pm h$, equation (3.68), leads to the following equations, respectively,

$$\sum_{i=1}^{3}A_{in}\left(1+k_{1i}+e_{15}k_{2i}\right)p_iI_1(p_i\zeta_n)=0 \tag{3.74}$$

$$\sum_{i=1}^{3}C_{is}\left(1+k_{1i}+e_{15}k_{2i}\right)q_i\sinh(q_i\lambda_s h)=0 \tag{3.75}$$

The condition of vertical stress $\sigma_{zz}=P(r)$ at both end surfaces, equation (3.67), can be written as

$$\sigma_{zz}(r,\pm h)=\sum_{i=1}^{3}\left\{\begin{array}{l} B_{i0}\left[4c_{13}-4(c_{33}k_{1i}+e_{33}k_{2i})q_i^2\right] \\ +\sum\limits_{n=1}^{\infty}A_{in}\left[(-c_{13}p_i^2+c_{33}k_{1i}+e_{33}k_{2i})I_0(p_i\zeta_n r)\right]\cos(\zeta_n h) \\ +\sum\limits_{s=1}^{\infty}C_{is}\left[(c_{13}-c_{33}k_{1i}q_i^2-e_{33}k_{2i}q_i^2)J_0(\lambda_s r)\right]\cosh(q_i\lambda_s h) \end{array}\right\}=P(r) \tag{3.76}$$

The function $P(r)$ and $I_0(p_i\zeta_n r)$ $(i=1,\,2,\,3)$ in the above equation can be expressed in terms of Fourier-Bessel series as follows

$$P(r)=a_0+\sum_{s=1}^{\infty}a_sJ_0(\lambda_s r) \tag{3.77}$$

and

$$I_0(p_i\zeta_n r)=\frac{2I_1(p_i\zeta_n)}{p_i\zeta_n}+\sum_{s=1}^{\infty}\frac{2p_i\zeta_nI_1(p_i\zeta_n)}{\left(\lambda_s^2+p_i^2\zeta_n^2\right)J_0(\lambda_s)}J_0(\lambda_s r) \tag{3.78}$$

where

$$a_0 = 2\int_0^1 rP(r)dr \tag{3.79}$$

$$a_s = \frac{2}{J_0^2(\lambda_s)}\int_0^1 rP(r)J_0(\lambda_s r)dr \tag{3.80}$$

The equation (3.76) can then be written as

$$\sum_{i=1}^{3}\left\{\begin{aligned}&B_{i0}\left[2(c_{13}+c_{13})-4(c_{33}k_{1i}+e_{33}k_{2i})q_i^2\right]\\&+\sum_{n=1}^{\infty}A_{in}\left[(-c_{13}p_i^2+c_{33}k_{1i}+e_{33}k_{2i})\frac{2(-1)^n I_1(p_i\zeta_n)}{p_i\zeta_n}\right]\end{aligned}\right\}=a_0 \tag{3.81}$$

$$\sum_{i=1}^{3}\left\{\begin{aligned}&\sum_{n=1}^{\infty}A_{in}\left[(-c_{13}p_i^2+c_{33}k_{1i}+e_{33}k_{2i})\frac{2(-1)^n p_i\zeta_n I_1(p_i\zeta_n)}{\left(\lambda_s^2+p_i^2\zeta_n^2\right)J_0(\lambda_s)}\right]\\&+C_{is}\left[(c_{13}-c_{33}k_{1i}q_i^2-e_{33}k_{2i}q_i^2)\right]\cosh(q_i\lambda_s h)\end{aligned}\right\}=a_n \tag{3.82}$$

The final set of equations representing the boundary condition for electric displacement along the cylinder surfaces is given by

$$\sum_{i=1}^{3}A_{in}\left(e_{15}+e_{15}k_{1i}-\tilde{\varepsilon}_{11}k_{2i}\right)p_i^2 I_1(p_i\zeta_n)=0 \tag{3.83}$$

for $D_r(1,z)=0$, i.e., equation (3.66), and

$$\sum_{i=1}^{3}\left\{\begin{aligned}&B_{i0}\left[4-4(e_{33}k_{1i}-\tilde{\varepsilon}_{33}k_{2i})q_i^2\right]+\\&\sum_{n=1}^{\infty}A_{in}\left[(-p_i^2+e_{33}k_{1i}-\tilde{\varepsilon}_{33}k_{2i})\frac{2(-1)^n I_1(p_i\zeta_n)}{p_i\zeta_n}\right]\end{aligned}\right\}=0 \tag{3.84}$$

$$\sum_{i=1}^{3}\left\{\begin{aligned}&\sum_{n=1}^{\infty}A_{in}\left[(-p_i^2+e_{33}k_{1i}-\tilde{\varepsilon}_{33}k_{2i})\frac{2(-1)^n p_i\zeta_n I_1(p_i\zeta_n)}{\left(\lambda_s^2+p_i^2\zeta_n^2\right)J_0(\lambda_s)}\right]\\&+C_{is}\left[(1-e_{33}k_{1i}q_i^2+\tilde{\varepsilon}_{33}k_{2i}q_i^2)\right]\cosh(q_i\lambda_s h)\end{aligned}\right\}=0 \tag{3.85}$$

for $D_z(r,\pm h)=0$, i.e., equation (3.69).

The equations (3.73), (3.74), (3.75), (3.82), (3.83) and (3.85) yield a system of linear algebraic equations of order $3n+3s$ with arbitrary constants A_{in} and C_{is} (i=1, 2, 3; $n,s=1, 2, 3,\ldots,\infty$). The three remaining boundary conditions B_{0i} $(i=1, 2, 3)$ can be obtained by solving equations (3.72), (3.81) and (3.84). All arbitrary constants are obtained numerically and the procedure of their numerical evaluation is discussed in Chapter 4.

3.3.2 Cylinder under Electric Field

Consider a piezoelectric cylinder subjected to electric potential applied over a circular area of radius r_0 at the top and bottom surfaces as shown in Figure 3.3. The boundary conditions for this mixed boundary value problem at the curved surface are the same as those of a cylinder under mechanical loading given by equations (3.64) to (3.66). For the top and bottom surfaces of the cylinder, the boundary conditions depend on the rigidity of the electrodes and the contact condition between the electrodes and the cylinder. In this thesis, two extreme cases are considered, i.e., a very flexible electrode in smooth contact with the cylinder and a rigid electrode perfectly bonded to the cylinder. First, let consider the case of the electrodes of negligible stiffness with smooth contact surface. In this case, the boundary conditions at the top and bottom surfaces are

$$\sigma_{zz}(r,\pm h) = 0 \tag{3.86}$$

$$\sigma_{rz}(r,\pm h) = 0 \tag{3.87}$$

$$\phi(r,h) = \phi_0 \quad \text{for} \quad 0<r<r_0 \tag{3.88}$$

$$\phi(r,-h) = -\phi_0 \quad \text{for} \quad 0<r<r_0 \tag{3.89}$$

$$D_z(r,\pm h) = 0 \quad \text{for} \quad r_0<r<1 \tag{3.90}$$

where ϕ_0 and $-\phi_0$ are non-dimensional electric potentials of the top and bottom electrodes, respectively. The first two boundary conditions lead to

$$\sum_{i=1}^{3}\left\{\begin{aligned}&B_{i0}\left[2(c_{13}+c_{13})-4(c_{33}k_{1i}+e_{33}k_{2i})q_i^2\right]+\\&\sum_{n=1}^{\infty}A_{in}\left[(-c_{13}p_i^2+c_{33}k_{1i}+e_{33}k_{2i})\frac{2(-1)^n I_1(p_i\zeta_n)}{p_i\zeta_n}\right]\end{aligned}\right\}=0 \tag{3.91}$$

$$\sum_{i=1}^{3}\left\{\begin{aligned}&\sum_{n=1}^{\infty}A_{in}\left[(-c_{13}p_i^2+c_{33}k_{1i}+e_{33}k_{2i})\frac{2(-1)^n p_i\zeta_n I_1(p_i\zeta_n)}{\left(\lambda_s^2+p_i^2\zeta_n^2\right)J_0(\lambda_s)}\right]\\&+C_{is}\left[(c_{13}-c_{33}k_{1i}q_i^2-e_{33}k_{2i}q_i^2)\right]\cosh(q_i\lambda_s h)\end{aligned}\right\}=0 \qquad (3.92)$$

To obtain the electric displacement under the electrodes, the contact surface between the electrode and the cylinder which is a circular area of radius r_0 is discretized into a total number of N_e ring elements as shown in Figure 3.4. In addition, D_z within each ring element is assumed to be constant. The relationship between unknown electric displacement and the applied electric potential on the contact surface can be expressed as

$$\left[G_{ij}\right]\left\{D_{zj}\right\} \quad = \quad \left\{\phi_j\right\} \qquad i,\ j=1,2,3,\ldots,\ N_e \qquad (3.93)$$

In equation (3.93), G_{ij} denotes the influence function which is the normalized electric potential at the center of the i^{th} ring element due to a uniform electric displacement over the j^{th} ring element. The required electric displacement D_{zj} can be obtained by solving equation (3.93) with ϕ_j being equal to the applied electric potential ϕ_0. Then, expanding D_{zj} into a Fourier Bessel series as follows:

$$D_{zj}=a_{0,Dzj}+\sum_{s=1}^{\infty}a_{s,Dzj}J_0(\lambda_s r) \qquad (3.94)$$

where

$$a_{0,Dzj}=2\int_{r_j}^{r_{j+1}}rD_{zj}dr \qquad (3.95)$$

$$a_{s,Dzj}=\frac{2}{J_0^2(\lambda_s)}\int_{r_j}^{r_{j+1}}rD_{zj}J_0(\lambda_s r)dr \qquad (3.96)$$

and r_j and r_{j+1} are the inner and outer radii, respectively, of the j^{th} ring element. The boundary conditions governing the electric displacement at the top and bottom surfaces can be written as

$$\sum_{i=1}^{3}\left\{\begin{array}{l} B_{i0}\left[4-4(e_{33}k_{1i}-\tilde{\varepsilon}_{33}k_{2i})q_i^2\right]+ \\ \sum_{n=1}^{\infty}A_{in}\left[(-p_i^2+e_{33}k_{1i}-\tilde{\varepsilon}_{33}k_{2i})\dfrac{2(-1)^n I_1(p_i\zeta_n)}{p_i\zeta_n}\right]\end{array}\right\}=a_{0,Dzj} \quad (3.97)$$

$$\sum_{i=1}^{3}\left\{\begin{array}{l} \sum_{n=1}^{\infty}A_{in}\left[(-p_i^2+e_{33}k_{1i}-\tilde{\varepsilon}_{33}k_{2i})\dfrac{2(-1)^n p_i\zeta_n I_1(p_i\zeta_n)}{\left(\lambda_s^2+p_i^2\zeta_n^2\right)J_0(\lambda_s)}\right] \\ +C_{is}\left[(1-e_{33}k_{1i}q_i^2+\tilde{\varepsilon}_{33}k_{2i}q_i^2)\right]\cosh(q_i\lambda_s h)\end{array}\right\}=a_{s,Dzj} \quad (3.98)$$

Therefore, for the case a piezoelectric cylinder subjected to electric field with very flexible and smooth electrodes, the arbitrary constants A_{in} and C_{is} (i=1, 2, 3, n,s=1, 2, 3,…,∞) can be obtained from equations (3.73), (3.74), (3.75), (3.83), (3.92) and (3.98), and the arbitrary constant B_{0i} (i=1, 2, 3) can be determined from equations (3.72), (3.91) and (3.97).

Next, let consider the other extreme case in which the electrode is rigid and it is perfectly bonded to the cylinder surface. In this case, the boundary conditions along the curved surface at r=1 and $-h \le z \le h$ remain unchanged, i.e., they are given by equations (3.64) to (3.66). For the boundary conditions at the top and bottom surfaces of the cylinder, since the electrodes are rigid and perfectly bonded with the cylinder surfaces, then under the contact area, $z=\pm h$ and $0 \le r \le 1$,

$$u_r(r,\pm h) = 0 \quad (3.99)$$

$$u_z(r,\pm h) = 0 \quad (3.100)$$

$$\phi(r,h) = \phi_0 \quad (3.101)$$

$$\phi(r,-h) = -\phi_0 \quad (3.102)$$

and for the region outside the electrodes,

$$\sigma_{zz}(r,\pm h) = 0 \quad (3.103)$$

$$\sigma_{rz}(r,\pm h) = 0 \quad (3.104)$$

$$D_z(r,\pm h) = 0 \quad (3.105)$$

Similar to the case of a flexible electrode in smooth contact with the cylinder, the circular contact area is discretized into a number of N_e ring elements to determine the unknown tractions T_r and T_z and the unknown electric displacement D_z under the electrodes. In addition, T_r, T_z and D_z are assumed to be constant within each ring element. Then, the following matrix equation can be established

$$\begin{bmatrix} \mathbf{G}^{rr} & \mathbf{G}^{zr} & \mathbf{G}^{\phi r} \\ \mathbf{G}^{rz} & \mathbf{G}^{zz} & \mathbf{G}^{\phi z} \\ \mathbf{G}^{r\phi} & \mathbf{G}^{z\phi} & \mathbf{G}^{\phi\phi} \end{bmatrix} \begin{Bmatrix} \mathbf{T}_r \\ \mathbf{T}_z \\ \mathbf{D}_z \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_r \\ \mathbf{u}_z \\ \boldsymbol{\phi} \end{Bmatrix} \tag{3.106}$$

In above equation, G_{ij}^{kl}, where $i, j = 1, 2, 3, ..., N_e$, denotes the influence function which is the non-dimensional radial displacement ($k = r$), vertical displacement ($k = z$) and electric potential ($k = \phi$) at the center of the i^{th} ring element due to the unit radial traction ($l = r$), unit vertical traction ($l = z$) and unit electric displacement ($l = \phi$) applied over the j^{th} ring element. Note that all elements of $\mathbf{u}_r$ and $\mathbf{u}_z$ are zero and those of $\boldsymbol{\phi}$ are equal to ϕ_0.

By expanding T_{rj} and T_{zj} into a Fourier Bessel series, respectively, as

$$T_{rj} = \sum_{s=1}^{\infty} a_{s,Trj} J_1(\lambda_s r) \tag{3.107}$$

and

$$T_{zj} = a_{0,Tzj} + \sum_{s=1}^{\infty} a_{s,Tzj} J_0(\lambda_s r) \tag{3.108}$$

where

$$a_{s,Trj} = \frac{2}{J_2^2(\lambda_s)} \int_{r_j}^{r_{j+1}} r T_{rj} J_1(\lambda_s r) dr \tag{3.109}$$

$$a_{0,Tzj} = 2 \int_{r_j}^{r_{j+1}} r T_{zj} dr \tag{3.110}$$

$$a_{s,Tzj} = \frac{2}{J_0^2(\lambda_s)} \int_{r_j}^{r_{j+1}} r T_{zj} J_0(\lambda_s r) dr \tag{3.111}$$

and D_{zj} as given by equation (3.94), the following equations can be established

$$\sum_{i=1}^{3} C_{is}\left(1 + k_{1i} + e_{15}k_{2i}\right) q_i \sinh(q_i \lambda_s h) = a_{s,Trj} \tag{3.112}$$

$$\sum_{i=1}^{3} \left\{ \begin{array}{l} B_{i0}\left[4c_{13} - 4(c_{33}k_{1i} + e_{33}k_{2i})q_i^2\right] + \\ \sum_{n=1}^{\infty} A_{in}\left[(-c_{13}p_i^2 + c_{33}k_{1i} + e_{33}k_{2i}) \dfrac{2(-1)^n I_1(p_i\zeta_n)}{p_i\zeta_n}\right] \end{array} \right\} = a_{0,Tzj} \tag{3.113}$$

$$\sum_{i=1}^{3} \left\{ \begin{array}{l} \sum_{n=1}^{\infty} A_{in}\left[(-c_{13}p_i^2 + c_{33}k_{1i} + e_{33}k_{2i}) \dfrac{2(-1)^n p_i\zeta_n I_1(p_i\zeta_n)}{\left(\lambda_s^2 + p_i^2\zeta_n^2\right) J_0(\lambda_s)}\right] \\ + C_{is}\left[(c_{13} - c_{33}k_{1i}q_i^2 - e_{33}k_{2i}q_i^2)\right]\cosh(q_i\gamma_s) \end{array} \right\} = a_{s,Tzj} \tag{3.114}$$

Equations (3.112) to (3.114) together with equations (3.72) to (3.74), (3.83), (3.97) and (3.98) constitute the equation system of order $3n + 3s + 3$ to determine the arbitrary constants A_{in}, B_{oi} and C_{is} for the case of applied electric field with rigid electrodes perfectly bonded to the cylinder surfaces.

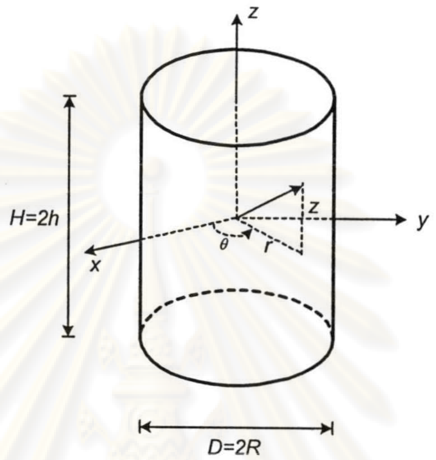

Figure 3.1 A piezoelectric solid cylinder considered in this thesis.

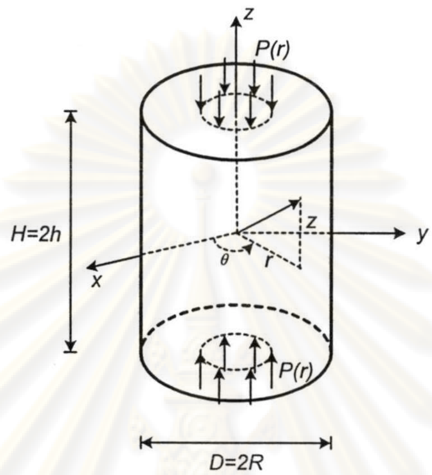

Figure 3.2 A piezoelectric cylinder under applied mechanical loading.

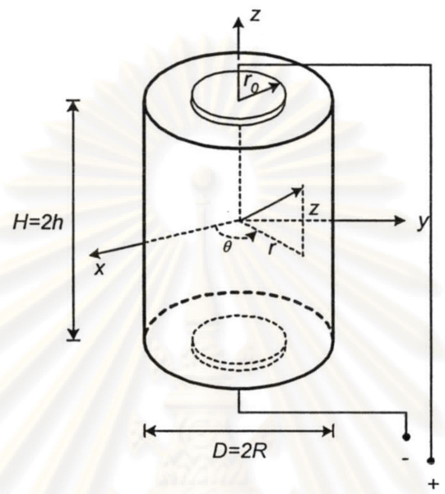

Figure 3.3 A piezoelectric cylinder under applied electric potential.

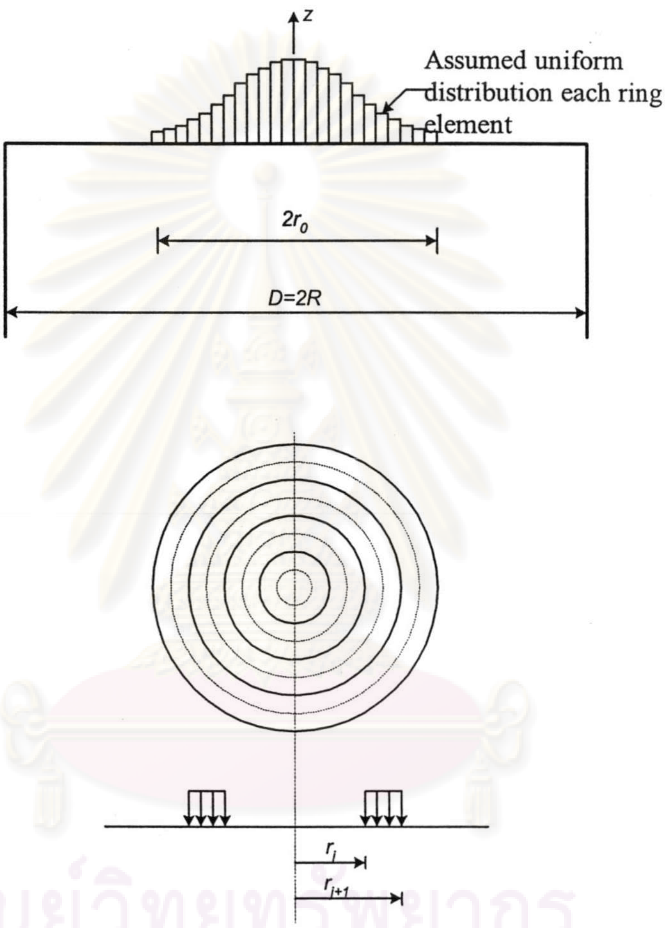

Figure 3.4 Discretization of tractions and electric displacement under electrodes for applied electric field.