ภาวะเอกฐานเชิงเส้นเฉพาะที่และการสลายตัวของผลการแปลงเชียร์เล็ตต่อเนื่อง

นาย จักรพงศ์ เตี้ยมมี

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตร์มหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2554

LOCAL LINEAR SINGULARITIES AND DECAY OF THE CONTINUOUS SHEARLET TRANSFORM

Mr. Jukrapong Tiammee

A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Mathematics

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2011

Thesis Title	LOCAL LINEAR SINGULARITIES AND DECAYS OF THE CONTINUOUS SHEARLET TRANSFORM
By	Mr. Jukrapong Tiammee
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Songkiat Sumetkijakan, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

. . . . Chairman

(Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.)

. . . . Thesis Advisor

(Assistant Professor Songkiat Sumetkijakan, Ph.D.)

. . . . Examiner

(Keng Wiboontan, Ph.D.)

. . . . External Examiner

(Associate Professor Sanoe Koonprasert, Ph.D.)

จักรพงศ์ เตี้ยมมี : ภาวะเอกฐานเชิงเส้นเฉพาะที่และอัตราการสลายตัวของผลการแปลงเชียร์เล็ตต่อเนื่อง
(LOCAL LINEAR SINGULARITIES AND DECAY OF THE CONTINUOUS SHEARLET TRANSFORM)

อ. ที่ปรึกษาวิทยานิพนธ์: ผศ. ดร.ทรงเกียรติ สุเมธกิจการ, 48 หน้า.

เราได้ความสัมพันธ์ระหว่างความเป็นเอกฐานเชิงเส้นเฉพาะที่ของฟังก์ชันสองตัวแปรและอัตราการสลายตัวของผลการแปลงเชียร์เล็ตต่อเนื่อง ซึ่งความเป็นเอกฐานเชิงเส้นของฟังก์ชันถูกนิยามด้วยความราบเรียบของโฮลเดอร์แบบมีทิศทาง เราพิจารณาสภาพของฟังก์ชันเป็นเอกฐานบนเส้นตรงเฉพาะที่ในทิศทางตั้งฉากที่น้อยกว่าในทิศทางขนานกับเส้นตรง

ภาควิชา คณิตศาสตร์ และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต..
สาขาวิชา คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..................
ปีการศึกษา 2554	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

5272244223 : MAJOR MATHEMATICS

KEYWORDS : SHEARLET TRANSFORM / DIRECTIONAL HÖLDER REGULARITY / LINEAR SINGULARITIES

JUKRAPONG TIAMMEE : LOCAL LINEAR SINGULARITIES AND DECAYS OF THE CONTINUOUS SHEARLET TRANSFORM. ADVISOR : ASST. PROF. SONGKIAT SUMETKIJAKAN, Ph.D., 48 pp.

We obtain a relationship between local linear singularity of a bivariate function and local decay rates of its continuous shearlet transform. This local linear singularity of a function is modeled by directional Hölder regularities. We then consider the situation where singularity on a line segment in a perpendicular direction is significantly lower than that in the direction along the line in a neighborhood.

Department	: Mathematics and Computer Science	Student's Signature : ....
Field of Study	: ....Mathematics....	Advisor's Signature : ....
Academic Year	: ....2011...........	Co-advisor's Signature : ....

ACKNOWLEDGEMENTS

First I would like to show my deep gratitude to Assistant Professor Dr. Songkiat Sumetkijakan, my thesis advisor, for his kindness and many good advices. I also would like to give thank to my thesis committees, Associate Professor Dr. Wicharn Lewkeeratiyutkul, Dr. Keng Wiboontan, and Associate Professor Dr. Sanoe Koonprasert.

In particular, I feel thankful to my friends for my good times at Chulalongkorn University. Moreover, I am indebted to all of my teachers who have taught me for my knowledge and skills.

Special appreciation is also given to Science Achievement Scholarship of Thailand (SAST) for long term financial support and for granting me great opportunity to study mathematics.

And last but not least, I would like to give thanks to my family, for their love and support in all my lifetime.

CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

For many years, wavelet transform has been shown to detect singularities. Independently, Jaffard(1991) and Holschneider and Tchamitchian(1991) gave a characterization of point singularities by using the continuous wavelet transform. In fact, while the continuous wavelet transform is able to detect singularities of function, it lacks the ability to capture directional and linear singularity. Since then, there are several transforms which are similar to wavelet transform, such as curvelet transform, Smith transform and shearlet transform. They all have parabolic scaling and are able to detect directional and linear singularity.

The shearlet transform is extended from the classical wavelet transform. The continuous shearlet transform is defined via a collection of scaling, shearing and translating of a single function. Moreover, the shearlet transform have a simple reconstruction formula. Kutyniok and Labate(2009) obtained the decay rates of continuous shearlet transform of distributions with point, linear or polygonal singularities. Recently, Lakhonchai, Sampo and Sumetkijakan(2010) proposed the linear singularity of functions satisfying a set of directional Holder regularities. They obtained the same decay rates as the results of Kutyniok et al.

This thesis is organized as follows. In the Preliminaries we recall the notations and definitions of Hölder regularities, the continuous wavelet transform, the continuous curvelet transform and then introduce the continous shearlet transform. In Chapter 2 we will investigate vanishing directional moments and decay properties of shearlet functions. In Chapter 3 we show a relationship of linear singularities of function and the decays of its continuous curvelet transform and continuous shearlet transform. The main result of this work is proved in Chapter 4, in which we give a local version of the Lakhonchai et al., where the singularity line is replaced by a singularity line segment. We then consider the situation where singularity on a line segment in a perpendicular direction is significantly lower than that in the direction along the line in a neighborhood.

1.2 Preliminaries

In this section, we recall necessary definitions and properties involving our work.

Notation

All through this thesis, we will consider

1. $x \in \mathbb{R}^2$ is a column vector, that is $x = (x_1, x_2)^T$.

2. $\xi \in \hat{\mathbb{R}}^2$ (the frequency domain) is a row vector, that is $\xi = (\xi_1, \xi_2)$.

3. For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $\nu = (\nu_1, \nu_2) \in \mathbb{N}_0^2$,

   - $x + y = (x_1 + y_1, x_2 + y_2)$
   - $ax = (ax_1, ax_2)$ where $a \in \mathbb{R}$
   - $\|x\| = \sqrt{x_1^2 + x_2^2}$
   - $|\nu| = \nu_1 + \nu_2$
   - $x^\nu = \prod_{i=1}^{2} x_i^{\nu_i}$
   - $\partial^\nu f = \partial_1^{\nu_1} f \partial_2^{\nu_2} f$

The L^2- spaces : Let f in $\mathbb{R}^2$ be a real measurable function on $(\mathbb{R}^2, \mathcal{L}, \lambda)$ (here $\mathcal{L}$ is the Lebesgue measurable sets, and λ is Lebesgue measure on $\mathbb{R}^2$).We shall denote the integral of f with respect to Lebesgue measure by $\int_{\mathbb{R}^2} f(x)dx$. Then

$$f \in L^2(\mathbb{R}^2) \quad \text{iff} \quad \int_{\mathbb{R}^2} |f(x)|^2 dx < \infty,$$

and $\|f\|_2 = \left(\int_{\mathbb{R}^2} |f(x)|^2 dx\right)^{\frac{1}{2}}$.

Theorem 1.1. (Fubini's theorem) If $f(x,y)$ is Lebesgue measurable on $\mathbb{R}^2$ and $\int_{\mathbb{R}}\int_{\mathbb{R}}|f(x,y)|dxdy < \infty$, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|f(x,y)|dxdy = \int_{\mathbb{R}}\left[\int_{\mathbb{R}}|f(x,y)|dy\right]dx$$
$$= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}|f(x,y)|dx\right]dy.$$

A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{F}$ is an inner product if

- $\langle v,u\rangle$ is the **complex conjugate** of $\langle u,v\rangle$:

$$\langle v,u\rangle = \overline{\langle u,v\rangle}$$

- $\langle u,v\rangle$ is a **linear** in its first argument

$$\langle au_1 + bu_2, v\rangle = a\langle u_1,v\rangle + b\langle u_2,v\rangle, \quad \text{for all } a,b \in \mathbb{C}.$$

- The inner product $\langle\cdot,\cdot\rangle$ is **positive definite** :

$$\langle x,x\rangle \geq 0,$$

  where the case of equality holds precisely when $x = 0$.

A standard example of a Hilbert space is $L^2(\mathbb{R}^2)$, with $\langle f,g\rangle = \int_{\mathbb{R}^2} f(x)\overline{g(x)}dx$. A standard inequality in a Hilbert space is the Cauchy-Schwarz inequality,

$$|\langle v,w\rangle| \leq \|v\|\|w\|.$$

Fourier Transform

The Fourier transform of $f \in L^1(\mathbb{R}^2)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i \xi x} dx$$

where $x = (x_1, x_2)^T$ is a column vector and $\xi = (\xi_1, \xi_2)$ is a row vector in $\mathbb{R}^2$. The Fourier transform of $f \in L^2(\mathbb{R}^2)$ is defined in the limit. So the Plancherel's formula becomes

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle.$$

Inversion of the Fourier transform is then given by

$$f(x) = \int_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

Hölder Regularities

Hölder regularities of a bivariate function is defined as follows.

Definition 1.2. Let $\alpha \in (0, \infty) \setminus \mathbb{N}$ and $u \in \mathbb{R}^2$. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be **pointwise Hölder regular with exponent** α **at** u, denote by $f \in C^\alpha(u)$, if there exists a polynomial $P = P_u$ of degree less than α and a constant $C = C_u$ such that for all x in a neighborhood of u

$$|f(x) - P(x - u)| \leq C \|x - u\|^\alpha. \tag{1}$$

Let Ω be an open subset of $\mathbb{R}^d$. If there exists a uniform constant C so that for all $u \in \Omega$ there is a polynomial P_u of a degree less than α such that (1) holds for all $x \in \Omega$, then we say that f is **uniformly Hölder regular with exponent** α **on** Ω or $f \in C^\alpha(\Omega)$.

For a fixed unit vector $v \in \mathbb{R}^d$, f is said to be **pointwise Hölder regular with exponent** α **at** u **in the direction** v, denoted by $f \in C^\alpha(u; v)$, if there exist a constant $C = C_{u,v}$ and a polynomial $P = P_{u,v}$ of a degree less than α such that (1) holds for all x in an open line segment that contains the point u and is parallel to v.

Let Ω_1 be a subset of $\mathbb{R}^d$ and Ω_2 be an open neighborhood of Ω_1. Then f is said to be in $C^\alpha(\Omega_1, \Omega_2; v)$ if there exists a constant $C = C_v$ so that for all $u \in \Omega_1$ there is a polynomial $P = P_{u,v}$ of a degree less than (1) holds for all $x \in \Omega_2$ on the line passing through u and parallel to v. If $\Omega_1 = \Omega_2$ is open, then we denote $C^\alpha(\Omega_1, \Omega_2; v)$ by $C^\alpha(\Omega_1; v)$.

Continuous Wavelet Transform

Let us recall the definition of the wavelet transform. Continuous wavelet transform (CWT) is an integral transform like the Fourier-transform defined in the previous section.

Definition 1.3. (Continuous Wavelet Transform) The continuous wavelet transform of an $L^2(\mathbb{R})$ function f is defined by

$$W_f(a,b) = \int_{\mathbb{R}} f(x)\overline{\psi_{a,b}(x)}dx = \frac{1}{a}\int_{\mathbb{R}} f(x)\overline{\psi\left(\frac{x-b}{a}\right)}dx$$

where this Lebesgue integral is well-defined for all $a \in (0, \infty)$, $b \in \mathbb{R}$, $\psi \in L^2(\mathbb{R})$ and $\psi_{a,b}(x) = a^{-1}\psi(a^{-1}(x-b))$. The parameter b is a position(translation) parameter and a is interpreted as a scale parameter. ψ is called *Mother wavelet* which satisfies vanishing moment, i.e.

$$\int_{\mathbb{R}} \psi(t)dt = 0.$$

This means that a wavelet should have a zero-order vanishing moment. Also higher order vanishing moments are demanded. For comparsion, curvelets and shearlets have directional vanishing moment defined in Definition 1.6, which wavelets do not necessary have. Moreover ψ satisfies admissible condition that ensure the existence of inverse transform, i.e.

$$0 < c_\psi = \int_0^\infty |\hat{\psi}(a\xi)| \frac{da}{a^2} < \infty \quad \text{for a.e. } \xi \in \mathbb{R}^2.$$

Then (see Daubechies [1]) for $f, g \in L^2(\mathbb{R})$

$$\int_0^\infty \int_{\mathbb{R}} W_f(a,b)\overline{W_g(a,b)} \frac{dbda}{a^3} = c_\psi \langle f, g \rangle.$$

We have Parseval's Formula that for $f \in L^2(\mathbb{R})$,

$$\int_0^\infty \int_{\mathbb{R}} |W_f(a,b)|^2 \frac{dbda}{a^3} = c_\psi \|f\|_2^2.$$

The local regularity of a function implies an equivalent local decrease of its wavelet coefficients at small scale as shown by the following theorem.

Theorem 1.4. Let f be a bounded and $f \in C^\alpha(u)$ at some point $u \in \mathbb{R}$. Then its wavelet transform with respect to a wavelet ψ satisfies

$$W_f(a,b) \leq C(a^\alpha + |b-u|^\alpha) \qquad \alpha \in (0,1]$$

where C independent of a, b. The wavelet is supposed to satisfy $\psi \in L^1(\mathbb{R})$, $x^\alpha \psi \in L^1(\mathbb{R})$ and $\int \psi = 0$.

Continuous Curvelet Transform

There exists different constructions of curvelets, we choose definition in Candes and Donoho(2005). Continuous Curvelet Transform (CCT) is defined in the polar coordinates (r, w) of the Fourier/frequency domain. Let W be a positive real-valued function supported inside $(1/2, 2)$, called a *radial window*, and let V be a real-valued function supported on $[-1, 1]$, called an *angular window*. Functions W and V have the following admissibility conditions:

$$\int_0^\infty W(r)^2 \frac{dr}{r} = 1 \quad \text{and} \quad \int_{-1}^1 V(w)^2 dw = 1.$$

At each scale a, $0 < a < a_0$, γ_{a00} is defined by

$$\widehat{\gamma_{a00}}(r\cos(w), r\sin(w)) = a^{\frac{3}{4}} W(ar) V(w/\sqrt{a}) \quad \text{for } r \geq 0 \text{ and } w \in [0, 2\pi).$$

For each $0 < a < a_0$, $b \in \mathbb{R}^2$ and $\theta \in [0, 2\pi)$, a curvelet $\gamma_{ab\theta}$ is defined by

$$\gamma_{ab\theta}(x) = \gamma_{a00}(R_\theta(x - b)) \,, \quad \text{for } x \in \mathbb{R}^2.$$

Notice that now curvelets have a little bit different generating function γ for each scale. This is different from wavelet transform. Also, because of definition of radial window, $\gamma_{ab\theta}$ are high frequency functions.

The continuous curvelet transform is defined as, an integral transform

$$\Gamma_f(a, b, \theta) := \langle \gamma_{ab\theta}, f \rangle \quad \text{where } 0 < a < a_0,\ b \in \mathbb{R}^2 \text{ and } \theta \in [0, 2\pi).$$

Notice that Candes and Donoho(2005) assume V and W are C^∞, even though we can assume only C^N for N large enough for which curvelets and their derivatives up to desired order decay fast enough. Lemma 1.6 will show that curvelets have vanishing directional moments with increasing number of directions when a decreases.

The admissibility conditions and the polar coordinate design of curvelets yields the following reconstruction formula for all $f \in L^2(\mathbb{R}^2)$.

Theorem 1.5. There exists a bandlimited purely radial function Φ such that for all $f \in L^2(\mathbb{R}^2)$,

$$f = \int_{\mathbb{R}^2} \langle \Phi_b, f \rangle \Phi_b db + \int_0^{a_0} \int_0^{2\pi} \int_{\mathbb{R}^2} \langle \gamma_{ab\theta}, f \rangle dbd\theta \frac{da}{a^3},$$

where $\Phi_b(x) = \Phi(x - b)$.

For analysis of singularities, the low frequency part $\int_{\mathbb{R}^2} \langle \Phi_b, f \rangle \Phi_b db$ is not a problem as it is always C^∞.

Properties of Curvelet Transform

For any vectors v and v' in $\mathbb{R}^2$, let us denote the angle from v to v' in clockwise direction by $\angle(v, v')$.

Lemma 1.6. There exists $C < \infty$ (independent of a, b and θ) such that the curvelet functions $\gamma_{ab\theta}$ have directional vanishing moments of any order $L < \infty$ along all directions v that satisfy $\pi/2 \geq |\angle(v_\theta, v)| \geq Ca^{1/2}$.

Some results on the decay of $\gamma_{ab\theta}$ are given below.

Lemma 1.7. Suppose that the windows V and W in the definition of CCT are C^∞ and have compact supports. Then for each $N = 1, 2, ...$ there is a constant C_N such that

$$\forall x \in \mathbb{R}^2 \quad |\partial^v \gamma_{ab\theta}(x)| \leq \frac{C_N a^{-3/4-|v|}}{1 + \|x - b\|_{a,\theta}^{2N}}$$

where $\|x - b\|_{a,\theta} = \|D_{1/a} R_{-\theta}(x - b)\|$.

Continuous Shearlet Transform

We choose definition in Kutyniok and Labate(2009).

Definition 1.8. Given ψ_1 and $\psi_2 \in L^2(\mathbb{R})$, let $\psi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_2}{\xi_1}) \qquad \text{for } \xi = (\xi_1, \xi_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}.$$

Then ψ is called a *continuous shearlet function* if:

i. $\psi_1 \in L^2(\mathbb{R})$ satisfies the admissibility condition, that is

$$\int_{-\infty}^{\infty} |\hat{\psi}_1(a\xi)|^2 \frac{da}{a} = 1 \quad \text{for a.e. } \xi \in \mathbb{R},$$

and $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ with supp $\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$;

ii. $\|\psi_2\|_2 = 1$, $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ with supp $\hat{\psi}_2 \subset [-1, 1]$ and $\hat{\psi}_2 > 0$ on $(-1, 1)$.

A *continuous shearlet system* is the set of functions generated by ψ, namely,

$$\{\psi_{ast} = a^{-\frac{3}{4}}\psi(M_{as}^{-1}(\cdot - t)) : a \in I \subset \mathbb{R}^+, s \in S \subset \mathbb{R}, t \in \mathbb{R}^2\},$$

where $M_{as} = B_s D_a$, B_s is the shear matrix $\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$ and D_a is the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$. The *continuous shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is then defined by

$$\mathcal{SH}_\psi f(a, s, t) = \langle f, \psi_{ast} \rangle, \text{ for } a \in (0, 1), s \in [-2, 2] \text{ and } t \in \mathbb{R}^2.$$

Here $I = (0, 1)$ (a set of parabolic scales) and $S = [-2, 2]$ (a set of shear parameters). A direct computational shows that

$$\hat{\psi}_{ast}(\xi) = a^{-\frac{3}{4}} e^{-2\pi i \xi t} \hat{\psi}(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1))$$

$$= a^{-\frac{3}{4}} e^{-2\pi i \xi t} \hat{\psi}_1(a\xi_1)\hat{\psi}_2(\frac{1}{\sqrt{a}}(\frac{\xi_2}{\xi_1} - s)).$$

Then, each function $\hat{\psi}_{ast}$ is supported on the set

$$\text{supp}\hat{\psi}_{ast} \subseteq \{(\xi_1, \xi_2) : \xi_1 \in \left[-\frac{2}{a}, -\frac{1}{2a}\right] \cup \left[\frac{1}{2a}, \frac{2}{a}\right], \left|\frac{\xi_2}{\xi_1} - s\right| \leq \sqrt{a}\}.$$

Let $E \subset \widehat{\mathbb{R}^2}$ be given by $E = \{(\xi_1, \xi_2) \in \widehat{\mathbb{R}^2} : |\xi_1| \geq 2$ and $\left|\frac{\xi_2}{\xi_1}\right| \leq 1\}$ and define $L^2(E)^{\vee} = \{f \in L^2(\mathbb{R}^2) : \text{supp}\hat{f} \subset E\}$. Then, there is a reconstruction formula for functions in this proper subspace.

Theorem 1.9. Let $\psi \in L^2(\mathbb{R}^2)$ be a shearlet function. Then, for all $f \in L^2(E)^{\vee}$,

$$f = \int_{\mathbb{R}^2} \int_{-2}^{2} \int_{0}^{1} \langle \psi_{ast}, f \rangle \psi_{ast} \frac{da}{a^3} ds dt \qquad \text{in } L^2(E^{\vee}).$$

Moreover, we obtain a reproducing formula for all $f \in L^2(\mathbb{R}^2)$ by defining a vertical shearlet function $\psi^{(v)}$ by

$$\hat{\psi}^{(v)}(\xi) = \hat{\psi}^{(v)}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_2)\hat{\psi}_2(\frac{\xi_1}{\xi_2})$$

where $\hat{\psi}_1, \hat{\psi}_2$ are defined as in Definition 1.8 above.

The shearlets $\psi^{(v)}_{ast}$ are defined by $\psi^{(v)}_{ast} = a^{-3/4}\psi((M^v_{as})^{-1}(\cdot - t))$, where $M^{(v)}_{as} = B^{(v)}_s D^{(v)}_a$ such that $B^{(v)}_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$ and $D^{(v)}_a = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & a \end{pmatrix}$. Therefore $\{\psi^v_{ast}\}$ is the continuous shearlet system for $L^2(E^{(\text{v})})^{\vee}$ where $E^{(\text{v})} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \geq 2$ and $|\xi_1/\xi_2| \leq 1\}$ and the associated vertical continuous shearlet transform is $SH^{(\text{v})}_{\psi} f(a, s, t) = \langle f, \psi^{(\text{v})}_{ast} \rangle$.

Properties of shearlet functions

Definition 1.10. The function f of two variables is said to have L - *order vanishing directional moments along the direction* $v = (v_1, v_2)^T \neq 0$ if

$$\int_{\mathbb{R}} b^n f(bv + w) db = 0, \quad for\ all\ w \in \mathbb{R}^2\ and\ 0 \leq n < L.$$

Lakhonchai et al.(2010) proved vanishing directional moments and decay of shearlet functions, which are used frequently in our regularity analysis and shown below.

Lemma 1.11. For all $a \in (0,1), s \in [-2,2]$ and $t \in \mathbb{R}^2$, the following hold.

1. The shearlet function ψ_{ast} has vanishing directional moments of any order $L < \infty$ along any direction $v = (v_1, v_2)^T$ satisfying $|v_2 s + v_1| > |v_2|\sqrt{a}$.

2. For each $N = 1, 2, ...$, there is a constant C_N independent of a, s and t such that

$$|\partial^\nu \psi_{ast}(x)| \leq \frac{C_N a^{-3/4-|\nu|}(\sqrt{a} + |s|)^{\nu_2}}{1 + \|D_{1/a} B_{-s}(x - t)\|^{2N}}$$

   for all $x \in \mathbb{R}^2$ and $\nu \in \mathbb{N}_0^2$.

CHAPTER II

VANISHING DIRECTIONAL MOMENTS AND DECAY PROPERTIES

In this section we will investigate vanishing directional moments and decay properties of ψ_{ast}. This properties will be needed in proving theorems in Chapter III.

Vanishing Directional Moments and Properties of Shearlet Function

We define definition of an L - order vanishing directional moments along a direction v.

Definition 2.1. The function f of two variables is said to have L - o*rder vanishing directional moments along the direction* $v = (v_1, v_2)^T \neq 0$ if

$$\int_{\mathbb{R}} b^n f(bv + w)db = 0, \quad for\ all\ w \in \mathbb{R}^2\ and\ 0 \leq n < L.$$

The above definition mean essentially that any 1-D slices of the function have vanishing moments of order L. Notice from the definition that f has vanishing directional moment along direction v if and only if the same holds along direction $-v$.

In the following Lemma, we found a condition under which ψ_{ast} have vanishing directional moments of any order $L < \infty$ along the direction v.

Lemma 2.2. The shearlet function ψ_{ast} has vanishing directional moments of any order $L < \infty$ along any direction $v = (v_1, v_2)^T$ satisfying $|v_2 s + v_1| > |v_2|\sqrt{a}$.

Proof. Because $\text{supp}(\widehat{\psi_{a0t}}) \subseteq \{(\xi_1, \xi_2) : \frac{1}{2a} \leq |\xi_1| \leq \frac{2}{a} \text{ and } \xi_2 \leq \sqrt{a}|\xi_1|\}$ and $|s + \frac{v_1}{v_2}| > \sqrt{a}$, it follows that $\text{supp}(\widehat{\psi_{a0t}}) \cap \{(\xi_1, \xi_2) : \xi_2 = (s + \frac{v_1}{v_2})\xi_1\} = \varnothing$. Consequently, we have that all partial derivatives of $\widehat{\psi_{a(s+\frac{v_1}{v_2})t}}$ vanish on the ξ_1-axis. Next, we show that $\psi_{a(s+\frac{v_1}{v_2})t}$ has vanishing directional moments along the direction of x_2-axis of any order L. Let $g(x_1) := \int x_2^n \psi_{a(s+\frac{v_1}{v_2})t}(x_1, x_2)dx_2$. For each $\xi_1 \in \hat{\mathbb{R}}$,

$$\begin{aligned}
\hat{g}(\xi_1) &= \int g(x_1)e^{-2\pi x_1 \xi_1} dx_1 \\
&= \int \int x_2^n \psi_{a(s+\frac{v_1}{v_2})t}(x_1, x_2)e^{-2\pi x_1 \xi_1} dx_1 dx_2 \\
&= (x_2^n \widehat{\psi_{a(s+\frac{v_1}{v_2})t}})(\xi_1, 0) \\
&= (-2\pi i)^{-n} \partial_2^n \widehat{\psi_{a(s+\frac{v_1}{v_2})t}}(\xi_1, 0) \\
&= 0,
\end{aligned}$$

so $g(x_1) \equiv 0$. Therefore $\psi_{a(s+\frac{v_1}{v_2})t}$ has vanishing moments along the direction $(0, 1)^T$. Hence ψ_{ast} has vanishing moments along the direction $B_{-\frac{v_1}{v_2}}(0, 1)^T = \begin{pmatrix} 1 & \frac{v_1}{v_2} \\ 0 & 1 \end{pmatrix} (0, 1)^T = (\frac{v_1}{v_2}, 1)^T$, i.e. ψ_{ast} has vanishing moments along the direction v. Finally, If $v_2 = 0$, then we use the fact that, for all $a > 0$, $t \in \mathbb{R}^2$ and $s \in [-2, 2]$, $\text{supp}(\widehat{\psi_{ast}}) \cap \{(\xi_1, \xi_2) : \xi_1 = 0\} = \varnothing\}$ and hence, by the same line of proof, we have ψ_{ast} has vanishing moments along the direction $(1, 0)^T$. $\square$

In the following lemma we obtain a decay property of all partial derivatives of shearlet functions.

Lemma 2.3. For each $N = 1, 2, ...$, there is a constant C_N independent of a, s and t such that

$$|\partial^\nu \psi_{ast}(x)| \leq \frac{C_N a^{-3/4-|\nu|}(\sqrt{a}+|s|)^{\nu_2}}{1+\|D_{1/a}B_{-s}(x-t)\|^{2N}}$$

for all $x \in \mathbb{R}^2$ and $\nu \in \mathbb{N}_0^2$.

Proof. We restrict first to the case $s = 0$ and $t = 0$. Fix an index vector $\nu := (\nu_1, \nu_2)$ and define

$$h_a(x) := \psi_{a00}(D_a x) \quad \text{and} \quad g_a(x) := \partial^\nu h_a(x) = a^{\nu_1+\frac{\nu_2}{2}}(\partial^\nu \psi_{a00})(D_a x).$$

By a straightforward computation we have

$$\begin{aligned} \hat{g_a}(x) &= (2\pi\xi)^\nu \hat{h_a}(\xi) \\ &= (2\pi\xi)^\nu a^{-3/2} \hat{\psi_{a00}}(D_{1/a}\xi) \\ &= (2\pi\xi)^\nu a^{-3/2} a^{-3/4} \hat{\psi}(\xi) a^{3/2} \\ &= (2\pi\xi)^\nu a^{-3/4} \hat{\psi}(\xi). \end{aligned}$$

Now, replacing x by $D_{1/a}x$ in the equation $(-4\pi^2\|x\|^2)^k g_a(x) = \int_{\mathbb{R}^2} \Delta^k \hat{g_a}(\xi) e^{2\pi i x\xi} d\xi$, where Δ is the Laplacian, yields

$$\begin{aligned} \left|(-4\pi^2\|D_{1/a}x\|)^k (\partial^\nu \psi_{a00})(x)\right| &= \left|(-4\pi^2\|D_{1/a}x\|)^k a^{-(\nu_1+\nu_2/2)} g_a(D_{1/a}x)\right| \\ &= a^{-(\nu_1+\nu_2/2)} \left|\int_{\mathbb{R}^2} (\Delta^k \hat{g_a})(\xi) e^{2\pi i x\xi} d\xi\right| \\ &\leq a^{-(3/4+\nu_1+\nu_2/2)} \int_{\mathbb{R}^2} \left|\Delta^k((2\pi\xi)^\nu \hat{\psi}(\xi))\right| d\xi \\ &\leq C_k a^{-(3/4+\nu_1+\nu_2/2)}. \end{aligned}$$

In the last step we used the notation that $\int_{\mathbb{R}^2} \left|\Delta^k((2\pi\xi)^\nu \hat{\psi}(\xi))\right| d\xi \leq C_k$ where C_k

is in fact independent of k. Consequently, if $k = 0$ then we have inequality

$$|(\partial^{\nu}\psi_{a00})(x)| \leq Ca^{-(3/4+\nu_1+\nu_2/2)}.$$

Since

$$(1+(2\pi)^{2k})\|D_{1/a}x\|^{2k})|\partial^{\nu}(\psi_{a00})(x)| = |\partial^{\nu}(\psi_{a00})(x)| + \left|(4\pi^2\|D_{1/a}x\|^2)^k(\partial^{\nu}\psi_{a00})(x)\right|$$
$$\leq Ca^{-(3/4+\nu_1+\nu_2/2)},$$

We have

$$|\partial^{\nu}(\psi_{a00})(x)| \leq \frac{Ca^{-3/4-\nu_1-\nu_2/2}}{1+(2\pi)^{2k}\|D_{1/a}x\|^{2k}} \leq \frac{Ca^{-3/4-\nu_1-\nu_2/2}}{1+\|D_{1/a}x\|^{2k}}.$$

Next, we show how to estimate for general $s \in \mathbb{R}$:

$$\begin{aligned}
\partial_2^{\nu_2}\partial_1^{\nu_1}\psi_{as0}(x) &= \partial_2^{\nu_2}\partial_1^{\nu_1-1}(\partial_1\psi_{as0}(x)) \\
&= \partial_2^{\nu_2}\partial_1^{\nu_1-1}(\partial_1\psi_{as0}(x_1+sx_2,x_2)) \\
&= \partial_2^{\nu_2}(\partial_1^{\nu_1}\psi_{as0}(x_1+sx_2,x_2)) \\
&= \sum_{l=0}^{\nu_2}\begin{pmatrix}\nu_2\\ l\end{pmatrix} s^{\nu_2-l}\partial_1^{\nu_1+\nu_2-l}\partial_2^{l}\psi_{a00}(x_1+sx_2,x_2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|\partial_2^{\nu_2}\partial_1^{\nu_1}\psi_{as0}(x)| &\leq \sum_{l=0}^{\nu_2}\begin{pmatrix}\nu_2\\ l\end{pmatrix} |s|^{\nu_2-l}|\partial_1^{\nu_1+\nu_2-l}\partial_2^{l}\psi_{a00}(x_1+sx_2,x_2)| \\
&\leq \sum_{l=0}^{\nu_2}\begin{pmatrix}\nu_2\\ l\end{pmatrix} |s|^{\nu_2-l}\frac{Ca^{-3/4-\nu_1-\nu_2+l-\frac{l}{2}}}{1+\|D_{1/a}(x_1+sx_2,x_2)^T\|^{2k}} \\
&= \frac{Ca^{-3/4-\nu_1-\nu_2}}{1+\|D_{1/a}x\|^{2k}}\sum_{l=0}^{\nu_2}\begin{pmatrix}v_{\nu}\\ l\end{pmatrix}|s|^{\nu_2-l}a^{\frac{l}{2}}
\end{aligned}$$

$$= \frac{Ca^{-3/4-\nu_1-\nu_2}(\sqrt{a}+|s|)^{\nu_2}}{1+\|D_{1/a}x\|^{2k}}$$
$$= \frac{Ca^{-3/4-|\nu_1|}(\sqrt{a}+|s|)^{\nu_2}}{1+\|D_{1/a}x\|^{2k}}.$$

It is clear that all above hold also for a general t because translation does not change regularity properties. $\square$

CHAPTER III

LINEAR SINGULARITIES AND DECAY OF TRANSFORMS

In this chapter, we show a relationship between linear singularities of bivariate function and decay rates of its continuous curvelet transform (and continuous shearlet transform).

Continuous Curvelet Transform

Theorem 3.1 (Sampo and Sumetkijakan(2009))**.** If a bounded function $f \in C^{\alpha}(\mathbb{R}^2)$, then there exist a constant C and a fixed coarsest scale a_0 for which

$$|\langle \gamma_{ab\theta}, f \rangle| \leq C a^{\alpha + \frac{3}{4}}$$

for all $0 < a < a_0$, $b \in \mathbb{R}^2$ and $\theta \in [0, 2\pi)$.

Pointwise Hölder regularity estimates are harder to obtain than those for uniform regularity. The following theorem gives decay of curvelet transform for pointwise Hölder regularity.

Theorem 3.2 (Sampo and Sumetkijakan(2009))**.** Let $f \in C^{\alpha}(u)$ then there exists $C < \infty$ such that

$$|\langle \gamma_{ab\theta}, f \rangle| \leq C a^{\frac{\alpha}{2} + \frac{3}{2}} (1 + \|\frac{b-u}{a^{1/2}}\|)^{\alpha}$$

for all $0 < a < a_0$, $b \in \mathbb{R}^2$ and $\theta \in [0, 2\pi)$.

Next, the following theorem be the case of directional Hölder regularities and decay of its.

Theorem 3.3 (Sampo and Sumetkijakan(2009))**.** Let f be bounded with local Hölder exponent $\alpha \in (0,1]$ at points u and $f \in C^{2\alpha+1+\epsilon}(\mathbb{R}^2, v_\theta)$ for some $\theta_0 \in [0, 2\pi)$ with any fixed $\epsilon > 0$. Then there exist $\alpha^{'} \in [\alpha-\epsilon, \alpha]$ and $A, C < \infty$ such that for $a > 0$ and $b \in \mathbb{R}^2$,

$$|\langle \gamma_{ab\theta}, f \rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } \theta \notin \theta_0 + Aa^{1/2}[-1,1], \\ Ca^{\alpha^{'}+\frac{3}{4}}(1+\|\frac{b-u}{a}\|^{\alpha^{'}}), & \text{if } \theta \notin \theta_0 + Aa^{1/2}[-1,1]. \end{cases}$$

Continuous Shearlet Transform

In the following, we will examine the behavior of the continuous shearlet transform of several distributions containing different types of singularities. This will be useful to illustrate the basic properties of the shearlet transform, before stating a more general result in the next section. Indeed, the rate of decay of the continuous shearlet transform exactly describes the location and orientation of the singularities. Interestingly, despite the different mathematical structure, the decay rates found for the continuous shearlet transform are consistent with those found using the continuous curvelet transform in by Candes and Donoho(2005).

In order to state our results, let us recall computations of decay rates of the continuous shearlet transform of some distributions with point and linear singularities by Kutyniok and Labate(2009). They extended the definition of continuous shearlet transforms to the tempered distributions, so that it is defined for the Dirac δ and the linear delta distribution $\nu_p(x_1, x_2) = \delta(x_1 + px_2),\ p \in \mathbb{R}$. They showed that

$$\text{for } t = 0, \quad \mathcal{SH}_\psi \delta(a, s, t) \sim a^{-\frac{3}{4}} \quad \text{as} \ \ a \to 0$$

and for $t \neq 0$,

$$\mathcal{SH}_\psi \delta(a, s, t) \text{ decays repidly as } a \to 0.$$

And if $t_1 = -pt_2$ and $s = p$, we have

$$\mathcal{SH}_\psi \nu_p(a, s, t) \sim a^{-\frac{1}{4}} \quad \text{as} \ \ a \to 0.$$

In all other cases, $\mathcal{SH}_\psi \nu_p(a, s, t)$ decays rapidly as $a \to 0$.

For f be a distribution on $\mathbb{R}^2, \mathcal{SH}_\psi f(a, s, t)$ be defined as in Definition 1.7, and let $r \in \mathbb{R}$. Then $\mathcal{SH}_\psi f(a, s, t)$ *decays rapidly* as $a \to 0$, if

$$\mathcal{SH}_\psi f(a, s, t) = O(a^k) \quad \text{as } a \to 0 \ \text{ for every } k \geq 0.$$

We use the notation: $\mathcal{SH}_\psi f(a, s, t) \sim a^r$ as $a \to 0$, if there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha a^r \leq \mathcal{SH}_\psi f(a, s, t) \leq \beta a^r \quad \text{as} \ \ a \to 0.$$

The following theorem gives decay of shearlet transforms for Hölder regularity.

Theorem 3.4 (Lakhonchai et al.(2010))**.** If a bounded function $f \in C^\alpha(\mathbb{R}^2)$, then there exists a constant C such that

$$|\langle \psi_{ast}, f \rangle| \leq C a^{\alpha + \frac{3}{4}}$$

for all $0 < a < 1$, $s \in [-2, 2]$ and $t \in \mathbb{R}^2$.

Proof. Since uniform regularity is translation invariant, we can without loss of generality assume that $t = 0$. By assumption $f \in C^{\alpha}(\mathbb{R}^2)$, there exists a constant $C > 0$ such that for each $x \in \mathbb{R}$, there exists a polynomial $P_{B_s(0,x_2)^T}$ such that for all $x_1 \in \mathbb{R}$,

$$\begin{aligned}|f(B_s x) - P_{B_s(0,x_2)^T}(B_s x - B_s(0,x_2)^T)| &\leq C\|B_s x - B_s(0,x_2)^T\|^{\alpha} \\ &\leq C\|B_s(x_1,0)^T\|^{\alpha} \\ &\leq Cc(s)^{\alpha}\|(x_1,0)\|^{\alpha} = C|x_1|^{\alpha},\end{aligned}$$

when $c(s) = \|B_s\|_{op} = \left(a + s^2/2 + (s^2 + s^4/4)^{1/2}\right)^{1/2} \leq \sqrt{3+\sqrt{8}} = 1 + \sqrt{2}$ (By Lakhonchai, Sampo, Sumetkijakan (2010)). By the rapid decay of shearlets ψ_{ast} the integral

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |P_{B_s(0,x_2)^T}(B_s x - B_s(x_1,0)^T)\psi_{a00}(x_1,x_2)|dx_1 dx_2 < \infty$$

So by the assumption that ψ_{a00} has vanishing directional moments of any order along the x_1 - axis for $a \in (0,1)$ and Fubini's theorem, we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |P_{B_s(0,x_2)^T}(B_s x - B_s(x_1,0)^T)\psi_{a00}(x_1,x_2)|dx_1 dx_2 \\ &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |P_{B_s(0,x_2)^T}(B_s x - B_s(x_1,0)^T)\psi_{a00}(x_1,x_2)|dx_1\right) dx_2 \\ &= \int_{\mathbb{R}} 0 dx_2 = 0.\end{aligned}$$

Therefore

$$\begin{aligned}|\langle\psi_{as0} f\rangle| &= \left|\int_{\mathbb{R}^2} f(B_s x) - P_{B_s(0,x_2)^T}(B_s x - B_s(x_1,0)^T)\psi_{a00}(x)dx\right| \\ &\leq \int_{\mathbb{R}^2} |f(B_s x) - P_{B_s(0,x_2)^T}(B_s x - B_s(x_1,0)^T)||\psi_{a00}(x)|dx\end{aligned}$$

$$\leq C \int_{\mathbb{R}^2} |x_1|^\alpha \left| \frac{a^{-3/4}}{1 + \|D_{1/a}x\|^{2N}} \right| dx$$

$$= C \int_{\mathbb{R}^2} |ay_1|^\alpha \left| \frac{a^{-3/4}}{1 + \|y\|^{2N}} \right| dx$$

$$\leq C a^{\alpha+3/4}.$$

$\square$

Next, a function is a pointwise regularity estimates on the shearlet transform.

Theorem 3.5 (Lakhonchai et al.(2010))**.** If a bounded function $f \in C^\alpha(u)$ then there exists $C < \infty$ such that

$$|\langle \psi_{ast}, f \rangle| \leq C a^{\frac{\alpha}{2}+\frac{3}{4}} (1 + \| \frac{t-u}{a^{1/2}} \|^\alpha)$$

for all $0 < a < 1$, $s \in [-2, 2]$ and $t \in \mathbb{R}^2$.

Proof. By definition, the polynomial approximation property holds only in some neighborhood of point u but f is bounded and so this property holds in all $\mathbb{R}^2$. Since

$$\int_{\mathbb{R}^2} \psi_{ast}(x) P_u(x-u) dx = \int_{\mathbb{R}^2} \psi_{a00} P_u(B_s(x-u)+t) dx = 0.$$

Therefore

$$|\langle \psi_{ast}, f \rangle| \leq \int_{\mathbb{R}^2} |\psi_{ast}(x)||f(x) - P_u(x-u)| dx$$

$$\leq C a^{-3/4} \int_{\mathbb{R}^2} \frac{\|x-u\|^\alpha}{1 + \|D_{1/a}B_{-s}(x-t)\|^{2N}} dx$$

$$\leq C a^{-3/4+3/2} \int_{\mathbb{R}^2} \frac{\|B_s D_a y + t - u\|^\alpha}{1 + \|y\|^{2N}} dy$$

$$\leq C a^{3/4} \int_{\mathbb{R}^2} \frac{\|B_s D_a y\|^\alpha + \|t-u\|^\alpha}{1 + \|y\|^{2N}} dy$$

$$
\begin{aligned}
&= Ca^{3/4}\int_{\mathbb{R}^2}\frac{c(s)^\alpha a^{\alpha/2}\|y\|^\alpha+\|t-u\|^\alpha}{1+\|y\|^{2N}}dy\\
&\le Ca^{3/4+\alpha/2}\left(1+\|\frac{t-u}{a^{1/2}}\|^\alpha\right),
\end{aligned}
$$

since we can choose N as large enough so that the last integral is finite. We have also used the fact that B_sD_a is a bounded linear operator with norm $\|B_sD_a\| = c(s)a^{1/2}$ and $c(s)\le 1+\sqrt{2}$. □

DIRECTION OF SINGULARITY

In the following theorems, for any given $L>0$, $s_0\in[-2,2]$ and $u=(u_1,u_2)\in\mathbb{R}^2$, let Γ_u denote the vertical line passing through u and let Γ_{u,s_0} denote the line passing through u with slope $-\frac{1}{s_0}$. Observe that we may write $\Gamma_u=\Gamma_{u,0}$ so that $(x_1,x_2)\in\Gamma_{u,s_0}$ if and only if $x_1=-s_0(x_2-u_2)+u_1$. Recall that for a subset N of $\mathbb{R}^2$, $N(L)$ denotes the L-neighborhood of N , i.e. the set of all points whose distance to N is less than L.

Theorem 3.6 (Lakhonchai et al.(2010))**.** Let f be bounded with $f\in C^\alpha(\Gamma_{(u_1,0)},\mathbb{R}^2;(1,0))$ when $\alpha\in(0,1]$ and $f\in C^{2\alpha+1+\epsilon}(\mathbb{R}^2;(0,1))$ for any fixed $\epsilon>0$ and $u_1\in\mathbb{R}$. Then there exist $C<\infty$ such that for $0<a<1$, $t=(t_1,t_2)\in\mathbb{R}^2$, and $s\in[-2,2]$,

$$
|\langle\psi_{ast},f\rangle|\le\begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s|>\sqrt{a},\\[2ex] Ca^{\alpha+\frac{3}{4}}(1+|\frac{t_1-u_1}{a}|)^\alpha, & \text{if } |s|\le\sqrt{a}.\end{cases}
$$

Proof. For $u_1\in\mathbb{R}$ and $|s|\le\sqrt{a}$, we have that

$$
\begin{aligned}
|\langle\psi_{ast}f\rangle| &= \left|\int_{\mathbb{R}^2}(f(x)-f(u_1,x_2))\psi_{ast}(x)dx\right|\\
&\le C\int_{\mathbb{R}^2}|x_1-u_1|^\alpha\left(\frac{a^{-3/4}}{1+\|D_{1/a}B_{-s}(x-t)\|^{2N}}\right)dx
\end{aligned}
$$

$$= C \int_{\mathbb{R}^2} |ay_1 - s\sqrt{a}y_2 + t_1 - u_1|^\alpha \left(\frac{a^{3/4}}{1 + \|y\|^{2N}} \right) dy$$
$$\leq C \int_{\mathbb{R}^2} (|ay_1|^\alpha + (|a||y_2|)^\alpha + |t_1 - u_1|^\alpha) \left(\frac{a^{3/4}}{1 + \|y\|^{2N}} \right) dy$$
$$= Ca^{\alpha+3/4} \left(1 + \left| \frac{t_1 - u_1}{a} \right|^\alpha \right).$$

Next, let $|s| > \sqrt{a}$. We denote the rectangle $R_a := [-a^{-c}, a^{-c}]^2$ for some $0 < c < 1/2$, to be determined later. We notice that $B_s D_a R_a$ is sheared similar to the essential support of ψ_{ast} and $R_a \to \mathbb{R}^2$ while $B_s D_a R_a \to 0$ when $a \to 0$. We will also use here the notation $v(x) := (x_1, \frac{x_1}{|s|})$. Since the line $x_2 = \frac{x_1}{|s|}$ is parallel to major axis of $B_s D_a R_a$, $v(x)$ lies on major axis of $B_s D_a R_a$ and $v(x) - x$ is always parallel to x_2 - axis. Let $h_t(y) := f(y + t)$. By assumption if f we have $h_t \in C^{2\alpha+1+\epsilon}(\mathbb{R}^2, (0,1))$.

$$|\langle \psi_{ast} f \rangle| = |\langle \psi_{as0}, h_t \rangle|$$
$$= \left| \int_{\mathbb{R}^2} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x) dx \right|$$
$$\leq \left| \int_{\mathbb{R}^2 \setminus B_s D_a R_a} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x) dx \right|$$
$$+ \left| \int_{B_s D_a R_a} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x) dx \right|.$$

Since h_t is bounded by M. So the first integral can be bounded by

$$\left| \int_{\mathbb{R}^2 \setminus B_s D_a R_a} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x) dx \right|$$
$$\leq Ca^{-3/4} \int_{\mathbb{R}^2 \setminus B_s D_a R_a} \frac{|h_t(x) - P_{v(x)}(x - v(x))|}{1 + \|D_{1/a} B_{-s} x\|^{2N}} dx$$

$$\leq Ca^{3/4} \int_{\mathbb{R}^2 \setminus R_a} \frac{M + P_{y'}(C'\|y\|)}{1 + \|y\|^{2N}} dy \quad \text{(for some } C' > 0\text{)}$$
$$= Ca^{3/4 + c(2N - 1 - \text{degree} P_{y'})}$$

where $y' = v(B_s D_a y)$. Since c is fixed, we can choose N such that $3/4 + c(2N - 1 - \text{degree} P_{y'})$ as large as necessary.

Assume that $f \in C^{2\alpha+1+\epsilon}(\mathbb{R}^2, (0,1))$. Thus, for every $y \in \mathbb{R}^2$, there exists a polynomial P_y such that for x in a neighborhood of y such that $(x-y)||(0,1)$

$$|f(x) - P_y(x-y)| \leq C\|x-y\|^{2\alpha+1+\epsilon}$$

Since a is small enough, we can choose $r > 0$ with $B_s D_a R_a \subset B(0,r)$. Therefore, for $x \in B_s D_a R_a$, $\|x - v(x)\|$ is less than the length l of the part of the line parallel to the x_2 - axis lying inside the rectangle $B_s D_a R_a$. Observe that $|l| \leq 2a^{1/2-c}$, hence

$$|f(x) - P_{v(x)}(x - v(x))| \leq C\|x - v(x)\|^{2\alpha+1+\epsilon} \leq C|l|^{2\alpha+1+\epsilon} \leq Ca^{(1/2-c)(2\alpha+1+\epsilon)}$$

Note that we can choose any $c \in (0, 1/2)$ and hence for any small ϵ we can choose $c = \frac{\epsilon}{4\alpha+2+2\epsilon}$. With this c, we obtain an estimate for the second integral

$$\begin{aligned}
&\left| \int_{B_s D_a R_a} (f(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x)dx \right| \\
&\leq \int_{B_s D_a R_a} |f(x) - P_{v(x)}(x - v(x))||\psi_{as0}(x)|dx \\
&\leq C \int_{B_s D_a R_a} \frac{a^{(1/2-c)(2\alpha+1+\epsilon)} a^{-3/4}}{1 + \|D_{1/a} B_{-s} x\|^{2N}} dx \\
&\leq C \int_{R_a} \frac{a^{(1/2-c)(2\alpha+1+\epsilon)} a^{3/4}}{1 + \|y\|^{2N}} dy \\
&\leq Ca^{(2\alpha+1+\epsilon)(1/2-c)+3/4} \leq Ca^{\alpha+5/4}.
\end{aligned}$$

□

Theorem 3.7 (Lakhonchai et al.(2010)). Let $u_1 \in \mathbb{R}$ and f be bounded with $f \in C^\alpha(\Gamma_{(u_1,0)}, \Gamma_{(u_1,0)}(L); (1,0))$ when $\alpha \in (0,1]$, $L > 1$ and $f \in C^{2\alpha+1+\epsilon}(\Gamma_{(u_1,0)}(L); (0,1))$

for any fixed $\epsilon > 0$ and $u_1 \in \mathbb{R}$. Then there exists $C < \infty$ is that, for $0 < a < 1$ and $a_0 < 1$, if $0 < a < a_0$ and $t = (t_1, t_2) \in \Gamma_{(u_1,0)}(r)$ with $r < L/2$ and $s \in [-2, 2]$, we have

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s| > \sqrt{a}, \\ \\ Ca^{\alpha+\frac{3}{4}}(1 + |\frac{t_1-u_1}{a}|)^{\alpha}, & \text{if } |s| \leq \sqrt{a}. \end{cases}$$

Proof. We assume that $u_1 = 0$, the general case follows by the simple translation. Let $|s| \leq \sqrt{a}$. Since

$$\begin{aligned} |\langle \psi_{ast} f \rangle| &= \left| \int_{\mathbb{R}^2} (f(x) - f(0, x_2))\psi_{ast}(x)dx \right| \\ &\leq C \int_{\mathbb{R}^2} |x_1|^{\alpha} \left(\frac{a^{-3/4}}{1 + \|D_{1/a}B_{-s}(x-t)\|^{2N}} \right) dx \\ &= C \int_{\mathbb{R}^2} |ay_1 - s\sqrt{a}y_2 + t_1|^{\alpha} \left(\frac{a^{3/4}}{1 + \|y\|^{2N}} \right) dy \\ &\leq C \int_{\mathbb{R}^2} (|ay_1|^{\alpha} + (|a||y_2|)^{\alpha} + |t_1|^{\alpha}) \left(\frac{a^{3/4}}{1 + \|y\|^{2N}} \right) dy \\ &\leq C \int_{\mathbb{R}^2} ((2a\|y\|)^{\alpha} + |t_1|^{\alpha}) \left(\frac{a^{3/4}}{1 + \|y\|^{2N}} \right) dy \\ &= Ca^{\alpha+3/4} \left(1 + \left| \frac{t_1}{a} \right|^{\alpha} \right). \end{aligned}$$

Next, let $|s| > \sqrt{a}$. We denote the rectangle $R_a := [-a^{-c}, a^{-c}]^2$ for some $0 < c < 1/2$, to be determined later. We notice that $B_s D_a R_a$ is sheared similar to the essential support of ψ_{ast} and $R_a \to \mathbb{R}^2$ while $B_s D_a R_a \to 0$ when $a \to 0$. We will also use here the notation $v(x) := (x_1, \frac{x_1}{|s|})$. Since the line $x_2 = \frac{x_1}{|s|}$ is parallel to major axis of $B_s D_a R_a$, $v(x)$ lies on major axis of $B_s D_a R_a$ and $v(x) - x$ is always parallel to x_2 - axis. Let $t \in \Gamma_{(u_1,0)}(r)$ and $h_t(y) := f(y + t)$. By assumption of f we have $h_t \in C^{2\alpha+1+\epsilon}(\Gamma_{(u_1,0)}(r), (0, 1))$ and h_t is bounded.

$$|\langle \psi_{ast} f \rangle| = |\langle a^{-3/4}\psi(D_{1/a}B_{-s}(\cdot - t)), f(\cdot) \rangle|$$

$$= |\langle a^{-3/4}\psi(D_{1/a}B_{-s}(\cdot)), f(\cdot + t)\rangle|$$

$$= |\langle a^{-3/4}\psi(D_{1/a}B_{-s}(\cdot)), h_t(\cdot)\rangle|$$

$$= |\langle \psi_{as0}, h_t\rangle|$$

$$= \left|\int_{\mathbb{R}^2} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x)dx\right|$$

$$\leq \left|\int_{\mathbb{R}^2 \setminus B_sD_aR_a} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x)dx\right|$$

$$+ \left|\int_{B_sD_aR_a} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x)dx\right|.$$

By the proof of previous Theorem, we have that the first integral can be bounded by

$$\left|\int_{\mathbb{R}^2 \setminus B_sD_aR_a} (h_t(x) - P_{v(x)}(x - v(x)))\psi_{as0}(x)dx\right| \leq Ca^K$$

where $y' = v(B_sD_aR_ay)$ and K can be chosen arbitrary large.

Let $a_0 < 1$ be such that, for all $0 < a < a_0$ and $s \in [-2, 2]$, $B_sD_aR_a \subseteq \Gamma_{(u_1,0)}(r) \subseteq \Gamma_{(u_1,0)}(L)$. Since $h_t \in C^{2\alpha+1+\epsilon}(\Gamma_{(u_1,0)}(r), (0, 1))$ i.e. for every $y \in \mathbb{R}^2$ there exists a polynomial P_y such that

$$|h_t(x) - P_y(x - y)| \leq C\|x - y\|^{2\alpha+1+\epsilon} \quad , \quad \text{when } (x - y)||(0, 1)$$

for all x in some neighborhood of y. Hence, for $x \in B_sD_aR_a$, $\|x - v(x)\|$ less than the length l of the part of the line parallel to the x_2 - axis lying inside the rectangle $B_sD_aR_a$ is at most $|l| \leq Ca^{1/2-c}$ and so

$$|h_t(x) - P_{v(x)}(x - v(x))| \leq C\|x - v(x)\| \leq C|l|^{2\alpha+1+\epsilon} \leq Ca^{(1/2-c)(2\alpha+1+\epsilon)}$$

We can choose any $c \in (0, 1/2)$ and hence for any small ϵ we can choose $c =$

$\frac{\epsilon}{4\alpha+2+2\epsilon}$. With this c, we obtain an estimate for the second integral

$$
\begin{aligned}
&\left|\int_{B_sD_aR_a}(f(x)-P_{v(x)}(x-v(x)))\psi_{as0}(x)dx\right| \\
&\leq \int_{B_sD_aR_a}|f(x)-P_{v(x)}(x-v(x))||\psi_{as0}(x)|dx \\
&\leq C\int_{B_sD_aR_a}\frac{a^{(1/2-c)(2\alpha+1+\epsilon)}a^{-3/4}}{1+\|D_{1/a}B_{-s}x\|^{2N}}dx \\
&\leq C\int_{R_a}\frac{a^{(1/2-c)(2\alpha+1+\epsilon)}a^{3/4}}{1+\|y\|^{2N}}dy \\
&\leq Ca^{(2\alpha+1+\epsilon)(1/2-c)+3/4}\leq Ca^{\alpha+5/4}.
\end{aligned}
$$

□

Lemma 3.8 (Lakhonchai et al.(2010))**.** Let $L>0$ and f be bounded with $f\in C^{\alpha}(\Gamma_{u_1,s_0},\mathbb{R}^2;(1,0))$ for some $s_0\in[-2,2]$ and $u=(u_1,u_2)\in\mathbb{R}^2$. Then $f\circ B_{s_0}\in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)},\mathbb{R}^2;(1,0))$.

Moreover, if $f\in C^{\alpha}(\Gamma_{u_1,s_0},\Gamma_{u_1,s_0}(L);(1,0))$ for some $s_0\in[-2,2]$ and $u=(u_1,u_2)\in\mathbb{R}^2$. then $f\circ B_{s_0}\in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)},\Gamma_{(u_1+s_0u_2,0)}(L);(1,0))$

Proof. Assume that $f\in C^{\alpha}(\Gamma_{u_1,s_0},\mathbb{R}^2;(1,0))$. Then, for each $x\in\Gamma_{u_1,s_0}$ there exists a polynomial P_x and a constant $C>0$ such that

$$|f(y)-P_x(y-x)|\leq C\|y-x\|^{\alpha}\ \ ,\quad \text{when } (y-x)||(1,0).$$

We have that $B_{s_0}\Gamma_{(u_1+s_0u_2,0)}=\Gamma_{u_1,s_0}$ and $B_{s_0}(1,0)=(1,0)$. Then, for $x'\in\Gamma_{(u_1+s_0u_2,0)}$ and $y'\in\mathbb{R}^2$ with $B_{s_0}(y'-x')||(1,0)$. Therefore

$$|f\circ B_{s_0}(y')-P_{x'}\circ B_{s_0}(y'-x')|\leq C\|B_{s_0}(y'-x')\|^{\alpha}\leq C\|y'-x'\|^{\alpha}.$$

So we have that $f\circ B_{s_0}\in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)},(1,0))$. The latter part of the Lemma can be proved in a similar way. □

Lemma 3.9 (Lakhonchai et al.(2010))**.** Let f be bounded with $f \in C^\alpha(\mathbb{R}^2; B_{s_0}(0,1))$ for some $s_0 \in [-2,2]$. Then $f \circ B_{s_0} \in C^\alpha(\mathbb{R}^2; (0,1))$.

Moreover, if $f \in C^\alpha(\Gamma_{u,s_0}(L); B_{s_0}(0,1))$ for some $s_0 \in [-2,2]$ and $u = (u_1, u_2) \in \mathbb{R}^2$. then $f \circ B_{s_0} \in C^\alpha(\Gamma_{(u_1+s_0u_2,0)}(L); (0,1))$.

Proof. Assume $f \in C^\alpha(\mathbb{R}^2; B_{s_0}(0,1))$. Then, for each $y \in \mathbb{R}^2$, there exist a polynomial P_y and constant $C > 0$ such that

$$|f(x) - P_y(x-y)| \leq C\|x-y\|^\alpha \quad , \quad \text{when } (x-y)||B_{s_0}(0,1).$$

Let $(x-y)||(0,1)$. For $s_0 \in [-2,2]$, we have $B_{s_0}(x-y)||B_{s_0}(0,1)$. So

$$|f \circ B_{s_0}(x) - P_y \circ B_{s_0}(x-y)| \leq C\|B_{s_0}(x-y)\|^\alpha \leq C\|x-y\|^\alpha.$$

From this inequality we have $f \circ B_{s_0} \in C^\alpha(\mathbb{R}^2; (0,1))$. The latter part of the Lemma can be proved in a similar way. □

Theorem 3.10 (Lakhonchai et al.(2010))**.** Let f be bounded with $f \in C^\alpha(\Gamma_{u_1,s_0}, \mathbb{R}^2; (1,0))$ when $\alpha \in (0,1]$ and $f \in C^{2\alpha+1+\epsilon}(\mathbb{R}^2; B_{s_0}(0,1))$ for some $s_0 \in [-2,2]$ with any fixed $\epsilon > 0$ and $u = (u_1, u_2) \in \mathbb{R}^2$. Then there exists $C < \infty$ such that for $0 < a < 1$, $t = (t_1, t_2) \in \mathbb{R}^2$ and $s \in [-2,2]$,

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s-s_0| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}\left(1 + |\dfrac{t_1 + s_0t_2 - u_1 - s_0u_2}{a}|^\alpha\right), & \text{if } |s-s_0| \leq \sqrt{a}. \end{cases}$$

Proof. Consider

$$\langle \psi_{ast}, f \rangle = a^{-3/4}\langle \psi(D_{1/a}B_{-s}(\cdot - t)), f(\cdot) \rangle$$

$$= a^{-3/4}\langle \psi(D_{1/a}B_{-s}B_{s_0}B_{-s_0}(\cdot - t)), f(B_{s_0}B_{-s_0}\cdot)\rangle$$

$$= a^{-3/4}\langle \psi(D_{1/a}B_{-s}B_{s_0}(B_{-s_0}\cdot - B_{-s_0}t)), f(B_{s_0}B_{-s_0}\cdot)\rangle$$

$$= a^{-3/4}\langle \psi(D_{1/a}B_{-s}B_{s_0}(\cdot - B_{-s_0}t)), f(B_{s_0}\cdot)\rangle$$

$$= a^{-3/4}\langle \psi(D_{1/a}B_{-(s-s_0)}(\cdot - B_{-s_0}t)), f(B_{s_0}\cdot)\rangle$$

$$= \langle \psi_{a(s-s_0)B_{-s_0}t}, f \circ B_{s_0}\rangle.$$

By the two previous Lemmas, $f \circ B_{s_0} \in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)}, \mathbb{R}^2; (1,0))$ and $f \circ B_{s_0} \in C^{2\alpha+1+\epsilon}(\mathbb{R}^2; (0,1))$. Using Theorem 1.9 with above equation we have

$$|\langle \psi_{ast}, f\rangle| = \langle \psi_{a(s-s_0)B_{-s_0}t}, f \circ B_{s_0}\rangle \leq \begin{cases} Ca^{\alpha+\frac{5}{4}} & \text{if } |s-s_0| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}\left(a + \left|\frac{t_1+s_0t_2-u_1-s_0u_2}{a}\right|^{\alpha}\right) & \text{if } |s-s_0| \geq \sqrt{a}. \end{cases}$$

□

Theorem 3.11 (Lakhonchai et al.(2010))**.** Let f be bounded with $f \in C^{\alpha}(\Gamma_{(u_1,u_2),s_0}, \Gamma_{(u_1,u_2),s_0}(L); (1$ when $\alpha \in (0,1]$, $L > 1$ and $f \in C^{2\alpha+1+\epsilon}(\Gamma_{(u_1,u_2),s_0}(L); B_{s_0}(0,1))$ for some $s_0 \in [-2,2]$ with any fixed $\epsilon > 0$ and $u = (u_1,u_2) \in \mathbb{R}^2$. Then there exists $C < \infty$ is that, for $0 < a < 1$ and $a_0 < 1$, if $0 < a < a_0$ and $t = (t_1,t_2) \in \Gamma_{u_1,s_0}(r)$ with $r < L/2$ and $s \in [-2,2]$, we have

$$|\langle \psi_{ast}, f\rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s-s_0| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}(1 + |\frac{t_1+s_0t_2-u_1-s_0u_2}{a}|^{\alpha}), & \text{if } |s-s_0| \leq \sqrt{a}. \end{cases}$$

Proof. By Lemma 3.8, Lemma 3.9 and the same way of the proof of theorem 3.10, the proof is complete. □

Theorem 3.10 says essentially that, a bounded function f has low regularity on L in the horizontal direction ($f \in C^{\alpha}(\Gamma_{u_1,s_0}, \mathbb{R}^2; (1,0))$ is that the continuous shearlet transform $\langle \psi_{ast}, f \rangle$ decays like $a^{\alpha+\frac{5}{4}}$ in directions away from the direction of L and that needed decay rate in directions near the line is half an order lower and depends also on the horizontal distance from the line to the parallel line containing the t. Theorem 3.11 can be considered as the same result with weakened conditions where only regularity information on a neighborhood of the singularity line is assumed.

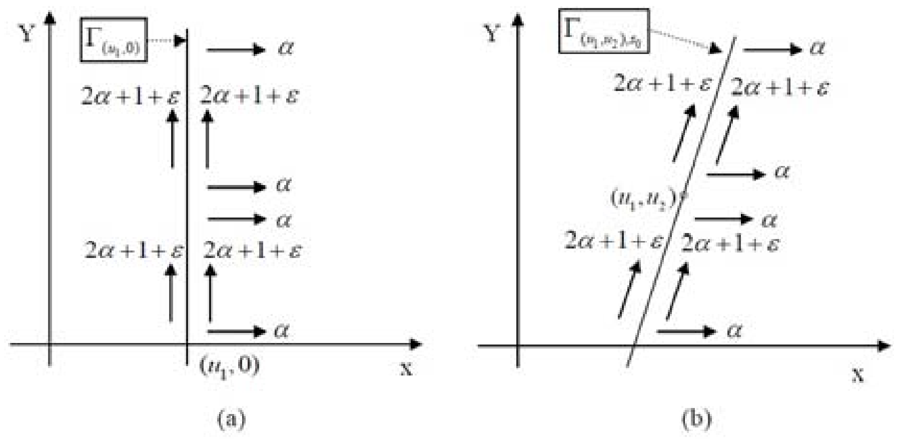


Figure 3.1: (a) An illustration of regularity in Theorem 3.6. (b) An illustration of regularity in Theorem 3.10

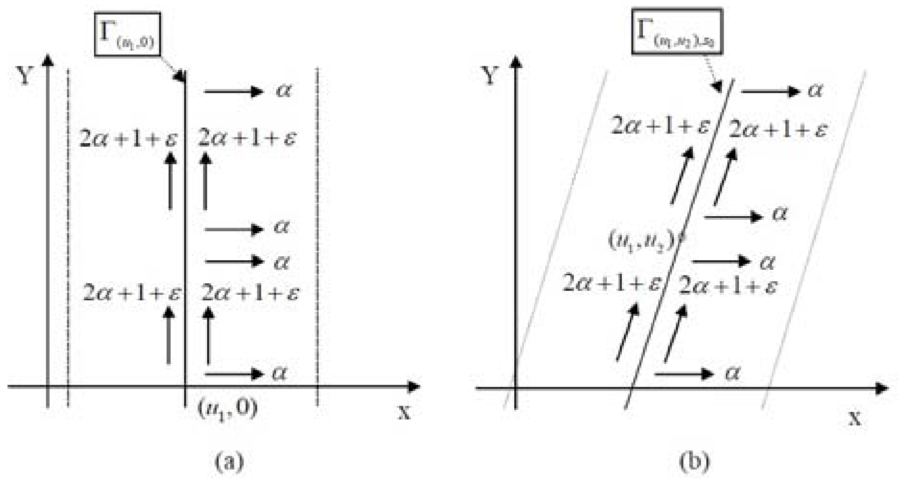


Figure 3.2: (a) An illustration of regularity in Theorem 3.7. (b) An illustration of regularity in Theorem 3.11

CHAPTER IV

LOCAL LINEAR SINGULARITIES AND DECAY OF THE CONTINUOUS SHEARLET TRANSFORM

In this chapter, we will prove the main result that singularity on a line segment of a function in a perpendicular direction is significantly lower than that in the direction along the line in a neighborhood. These results are similar to those in [10].

Notation of line segment

First, we give notations of a line segment and its neighborhood. Let $s_0 \in [-2, 2]$ be given and $P(u_1, u_2)$ and $Q(v_1, v_2)$ be points in $\mathbb{R}^2$ for which the slope of line segment $\overline{PQ}$ joining P and Q is $-\frac{1}{s_0}$ i.e $\frac{v_1-u_1}{v_2-u_2} = -s_0$. Observe that $(x_1, x_2) \in \overline{PQ}$ if and only if $x_1 = -s_0(x_2 - u_2) + u_1$. For $L > 0$ and a vector $(w_1, w_2)^T$ not parallel to $\overline{PQ}$, let $\overline{PQ}(L : (w_1, w_2))$ denote the set of all points whose distance to $\overline{PQ}$ in the direction $(w_1, w_2)^T$ is less than L. It is easy to see that $\overline{PQ}(L : (w_1, w_2))$ is the interior of the parallelogram whose the four corners are $(u_1 + m\rho, u_2 + m\rho)$, $(u_1 - m\rho, u_2 - m\rho)$, $(v_1 + m\rho, v_2 + m\rho)$ and $(v_1 - m\rho, v_2 - m\rho)$ where $\rho = \frac{L}{\sqrt{(w_2/w_1)^2+1}}$ and $m = \frac{w_2}{w_1}$. Note that such a parallelogram can be written as a shear (B_{s_0}) of a corresponding parallelogram whose vertices are $P'(u_1 + s_0 u_2, u_2)$ and $Q'(u_1 + s_0 u_2, v_2)$ that is $B_{s_0}\overline{P'Q'}(L' : B_{-s_0}(1, s_0)) = \overline{PQ}(L : (1, s_0))$ where $L' = L\sqrt{\frac{s_0^4+3s_0^3+1}{s_0^2+1}}$.

Denote $\overline{P^*Q^*} \subseteq \overline{PQ}$ where $P^*((u_1 + (u_1 + v_1)/2)/2, (u_2 + (u_2 + v_2)/2)/)$ and $Q^*((v_1 + (u_1 + v_1)/2)/2, (v_2 + (u_2 + v_2)/2)/2)$.Denote $D = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$ and $0 < \delta < 1$, let $\overline{P^\delta Q^\delta} \subseteq \overline{PQ}$ defined by $P^\delta(u_1 \pm \frac{\delta D s_0}{2\sqrt{s_0^2+1}}, u_2 \pm \frac{\delta D}{2\sqrt{s_0^2+1}})$ and $Q^\delta(v_1 \pm \frac{\delta D s_0}{2\sqrt{s_0^2+1}}, v_2 \pm \frac{\delta D}{2\sqrt{s_0^2+1}})$ depend on P and Q.

Decay of the Continuous Shearlet Transform

Let us first consider the case where the line $\overline{PQ}$ is vertical ($u_1 = v_1$).

Theorem 4.1. Let $0 < \alpha \leq 1$, $0 < \delta < 1$ and $r \in [-2, 2]$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and let $P(u_1, u_2), Q(u_1, v_2) \in \mathbb{R}^2$ and $L > 0$ be given.

1. If $f \in C^\alpha(\overline{PQ}, \overline{PQ}(L : (1, r)); (1, r))$ then there exist a constant $C < \infty$ such that for all $a \in (0, 1)$, $t \in \overline{P^\delta Q^\delta}((1 - \delta)L : (1, r))$ and $s \in [-2, 2]$ if $|s + 1/r| \geq \sqrt{a}$ and $|s| \leq \sqrt{a}$, then
$$|\langle \psi_{ast}, f \rangle| \leq C a^{3/4}(a^\alpha + |t_1 - u_1|^\alpha)$$
2. If $f \in C^{(1+\epsilon)\beta}(\overline{PQ}(L : (1, r); (0, 1)))$ then there exists a constant $C < \infty$ and a fixed coarse scale $a_0 \in (0, 1)$ such that for all $a \in (0, a_0)$, $t \in \overline{P^*Q^*}(L/2 : (1, r))$ and $s \in [-2, 2]$ if $|s| \geq \sqrt{a}$, then
$$|\langle \psi_{ast}, f \rangle| \leq C a^{\frac{\beta}{2} + \frac{3}{4}}$$
3. If $f \in C^{(1+\epsilon)\beta}(\overline{PQ}(L : (1, r); (0, 1)))$ and $0 < \gamma < \frac{\epsilon}{1+\epsilon} < \frac{1}{2}$ then there exists $C < \infty$ and a fixed coarse scale $a_0 \in (0, 1)$ such that for all $a \in (0, a_0)$, $t \in \overline{P^*Q^*}(L/2 : (1, r))$ and $s \in [-2, 2]$ if $|s| \geq a^\gamma$, then
$$|\langle \psi_{ast}, f \rangle| \leq C a^{\beta + \frac{3}{4}}$$

Proof. Let $u_1 \in \mathbb{R}$. For each $(x_1, x_2) \in \overline{PQ}(L : (1, r))$ there exist $y = (y_1, y_2) \in \overline{PQ}$ such that $\frac{x_2 - y_2}{x_1 - y_1} = r$. In fact $y = (u_1, r(x_1 - u_1) + u_2)$ and $(x - y)//(1, r)$. Hence there exists a polynomial $P_y = f(u_1, r(x_1 - u_1) + x_2)$ such that for all $z \in \mathbb{R}^2$ and $(z - y)//(1, r)$,

$$|f(z) - f(y)| \leq C\|z - y\|^\alpha.$$

Since ψ_{ast} has rapid decay (Lemma 1.11(1)), the integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(u_1, r(x_1 - u_1) + x_2)\psi_{ast}(x_1, x_2)dx_1dx_2 < \infty.$$

So we use Fubini's theorem, letting $p_1 = (x_1 - u_1) - rx_2$ and $p_2 = r(x_1 - u_1) + x_2$. Since $|s + 1/r| \geq \sqrt{a}$, Lemma 1.11(2) implies that ψ_{ast} has vanishing directional moment of any order along $(1, r)^T$. So we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} f(u_1, r(x_1 - u_1) + x_2)\psi_{ast}(x_1, x_2)dx_1dx_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u_1, p_2)\psi_{ast}\left(\frac{rp_2 + p_1}{r^2 + 1} + u_1, \frac{p_2 - rp_1}{-r^2 + 1}\right) \left|\frac{\partial(x_1, x_2)}{\partial(p_1, p_2)}\right| dp_1dp_2 \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u_1, p_2)\psi_{ast}\left(p_1\left(\frac{1}{r^2 + 1}, \frac{r}{r^2 + 1}\right) + \left(\frac{rp_2}{r^2 + 1} + u_1, \frac{p_2 + 2rp_1}{r^2 + 1}\right)\right)\left|\frac{1}{r^2 + 1}\right|\right) dp_2 \\
&= \int_{\mathbb{R}} 0dp_2 = 0.
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
|\langle\psi_{ast}, f\rangle| &= \left|\int_{\mathbb{R}^2} f(x)\psi_{ast}(x)dx\right| \\
&\leq \left|\int_{\mathbb{R}^2} (f(x) - f(u_1, r(x_1 - u_1) + x_2))\,\psi_{ast}(x)dx\right| \\
&+ \left|\int_{\mathbb{R}^2} f(u_1, r(x_1 - u_1) + x_2)\psi_{ast}(x)dx\right| \\
&= \left|\int_{\mathbb{R}^2} (f(x) - f(r(x_1 - u_1) + x_2))\,\psi_{ast}(x)dx\right| + 0
\end{aligned}$$

$$= \left| \int_{\overline{PQ}(L:(1,r))} (f(x) - f(r(x_1 - u_1) + x_2))\psi_{ast}(x)dx \right|$$
$$+ \left| \int_{\mathbb{R}^2 \setminus \overline{PQ}(L:(1,r))} (f(x) - f(r(x_1 - u_1) + x_2))\psi_{ast}(x)dx \right|$$
$$\equiv I_{\text{in}} + I_{\text{out}}.$$

Consider I_{in} : By a change of variable $y = D_{1/a}B_{-s}(x-t)$ in (3), the assumption $|s| \leq \sqrt{a}$ in (4) and the fact that $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$ in (5), we have

$$\begin{aligned}
I_{\text{in}} &\leq \int_{\overline{PQ}(L:(1,r))} |(f(x) - f(u_1, r(x_1 - u_1) + x_2)|\, |\psi_{ast}(x)|\, dx \\
&\leq C \int_{\mathbb{R}^2} \|(x_1 - u_1, r(x_1 - u_1))\|^\alpha \left(\frac{a^{\frac{-3}{4}}}{1 + \|D_{1/a}B_{-s}(x-t)\|^{2N}} \right) dx \\
&= C\|(1,r)\|^\alpha \int_{\mathbb{R}^2} |x_1 - u_1|^\alpha \left(\frac{a^{\frac{-3}{4}}}{1 + \|D_{1/a}B_{-s}(x-t)\|^{2N}} \right) dx && (3) \\
&\leq Ca^{\frac{3}{4}} \int_{\mathbb{R}^2} \left|ay_1 - s\sqrt{a}y_2 + t_1 - u_1\right|^\alpha \left(\frac{1}{1 + \|y\|^{2N}} \right) dy && (4) \\
&\leq Ca^{\frac{3}{4}} \int_{\mathbb{R}^2} (a|y_1| + a|y_2| + |t_1 - u_1|)^\alpha \left(\frac{1}{1 + \|y\|^{2N}} \right) dy \\
&\leq Ca^{\frac{3}{4}} \int_{\mathbb{R}^2} (a\|y\| + a\|y\| + |t_1 - u_1|)^\alpha \left(\frac{1}{1 + \|y\|^{2N}} \right) dy \\
&\leq Ca^{\frac{3}{4}} \int_{\mathbb{R}^2} (2a\|y\| + |t_1 - u_1|)^\alpha \left(\frac{1}{1 + \|y\|^{2N}} \right) dy && (5) \\
&\leq Ca^{\frac{3}{4}} \int_{\mathbb{R}^2} 2^\alpha((2a\|y\|)^\alpha + |t_1 - u_1|^\alpha) \left(\frac{1}{1 + \|y\|^{2N}} \right) dy \\
&= Ca^{\frac{3}{4}} \left[a^\alpha \int_{\mathbb{R}^2} \frac{2^\alpha \|y\|^\alpha}{1 + \|y\|^{2N}} dy + |t_1 - u_1|^\alpha \int_{\mathbb{R}^2} \frac{1}{1 + \|y\|^{2N}} dy \right] \\
&\leq Ca^{\frac{3}{4}} \left[Ca^\alpha + C|t_1 - u_1|^\alpha \right] \\
&\leq Ca^{\frac{3}{4}} \left(a^\alpha + |t_1 - u_1|^\alpha \right).
\end{aligned}$$

Consider I_{out} : Since f is bounded and decay properties of ψ_{ast} (Lemma 1.11(2)), we have

$$I_{\text{out}} < \left| \int_{\mathbb{R}^2 \setminus \overline{PQ}(L:(1,r))} (f(x) - f(u_1, r(x_1 - u_1) + x_2)\psi_{ast}(x)dx \right|$$
$$\leq C \int_{|x_1-u_1|\geq L \text{ or } r(x_1-u_1)+u_2>x_2 \text{ or } x_2>r(x_1-u_1)+v_2} \frac{a^{\frac{-3}{4}}}{1 + ||D_{1/a}B_{-s}(x-t)||^{2N}} dx.$$

By a change of variable $y = D_{1/a}B_{-s}(x-t)$, so $x = B_s D_a y + t$ and $dy = a^{-\frac{3}{2}}dx$ so

$$I_{\text{out}} \leq C a^{\frac{3}{4}} \int_{|ay_1+s\sqrt{a}y_2+t_1-u_1|\geq L \text{ or } r(x_1-u_1)+u_2>\sqrt{a}y_2+t_2 \text{ or } \sqrt{a}y_2+t_2>r(x_1-u_1)+v_2} \frac{1}{1+||y||^{2N}} dy.$$

Since $s < \sqrt{a}$,$a < \sqrt{a}$, $|t_1 - u_1| < (1-\delta)L$ and $r(x_1 - u_1) + u_2 - \delta(v_2 - u_2) < t_2 < r(x_1 - u_1) + v_2 - \delta(v_2 - u_2)$

$$I_{\text{out}} \leq C a^{\frac{3}{4}} \int_{|ay_1+ay_2|\geq \delta L \text{ or } \delta(v_2-u_2)\leq ray_1+(r-1)ay_2 \text{ or } (r+1)ay_2-ray_1\geq\delta(v_2-u_2)} \frac{1}{1+||y||^{2N}} dy$$

By another change of variable $z = ay$, so $y = a^{-1}z$ and $dz = ady$. Denote

$A := \{(z_1, z_2) : |z_1 + z_2| \geq \delta L$ or $\delta(v_2 - u_2) \leq rz_1 + (r-1)z_2$ or $(r+1)z_2 - rz_1 \geq \delta(v_2 - u_2)\}$

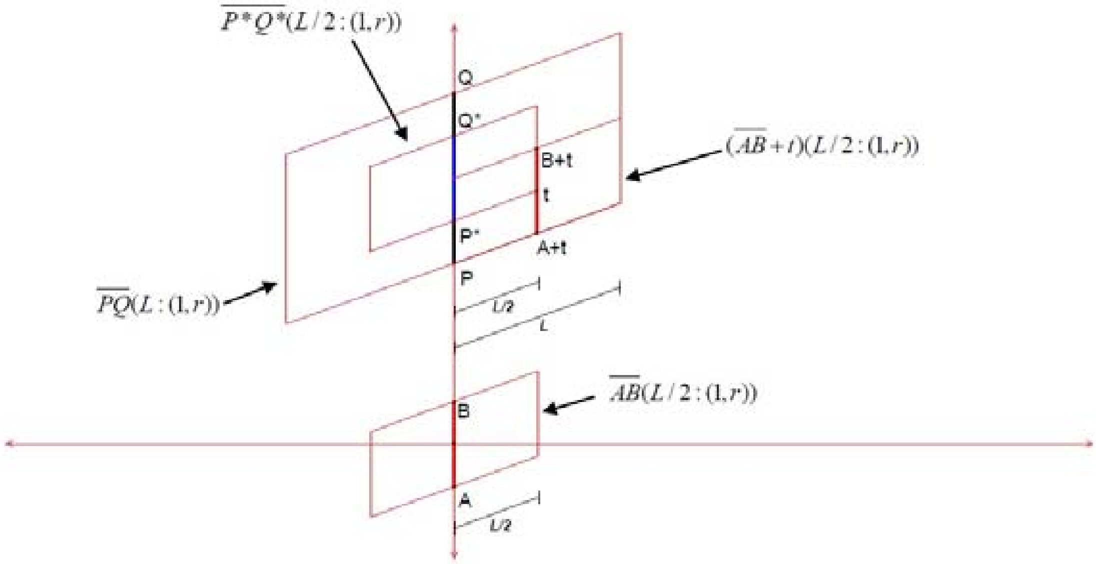
$$\begin{aligned}
I_{\text{out}} &\leq C a^{\frac{3}{4}} \int_A \frac{a^{-1}}{1 + ||\frac{z}{a}||^{2N}} dz \\
&\leq C a^{\frac{3}{4}} \int_A \frac{a^{2N-1}}{a^{2N} + ||z||^{2N}} dz \\
&\leq C a^{2N-1+\frac{3}{4}} \int_A \frac{1}{a^{2N} + ||z||^{2N}} dz \\
&\leq C a^{2N-1+\frac{3}{4}} \int_A \frac{1}{||z||^{2N}} dz \\
&\leq C a^{2N-1+\frac{3}{4}}
\end{aligned}$$

We just choose N as large that the integral is finite.

To prove supposition 2,3 we assume without loss of generality that $u_1 = 0$. Let $c \in (0, 1/2)$ to be chosen later depending on which statement we want to prove and $R_a = [-a^{-c}, a^{-c}]^2$, then $D_a R_a = [-a^{1-c}, a^{1-c}] \times [-a^{\frac{1}{2}-c}, a^{\frac{1}{2}-c}]$, and hence $B_sD_aR_a \longrightarrow \{0\}$ while $R_a \longrightarrow \mathbb{R}^2$ as $a \longrightarrow 0^+$. We notice that $B_sD_aR_a$ is sheared similarly to essential support of ψ_{ast}, let $v_x = (x_1, \frac{x_1}{-s})$ so $x - v_x = (0, x_2 + \frac{x_1}{s})$. For $w \in \mathbb{R}$, denote $l_w = \{(w, x_2) : x_2 \in \mathbb{R}^2\}$. For each $x \in B_sD_aR_a = B_s\left([-a^{1-c}, a^{1-c}] \times [-a^{1/2-c}, a^{1/2-c}]\right)$, $x \in B_sl_{w_1}$ where $|w_1| \le a^{1-c}$. Therefore $\|x - v_x\| = \left|x_2 + \frac{x_1}{s}\right| = \left|\frac{w_1}{s}\right| \le \frac{a^{1-c}}{s}$.

Let $\overline{AB}$ be such that for each $t \in \overline{P^*Q^*}(L/2 : (1,r))$, $(\overline{AB} + t)(L/2 : (1,r)) \subseteq \overline{P^*Q^*}(L/2 : (1,r))$. Choosing $a_0 < 1$ be such that for all $0 < a < a_0$ and $s \in [-2,2]$, $B_sD_aR_a \subseteq \overline{A^*B^*}(L/2 : (1,r))$. Let $t \in \overline{P^*Q^*}(L/2 : (1,r))$ and denote $h_t(x) = f(x+t)$. Then $h_t \in C^{(1+\epsilon)\beta}(\overline{AB}(L/2 : (1,r)))$. Hence, for each $x \in B_sD_aR_a \subseteq \overline{A^*B^*}(L/2 : (1,r))$ and $|s| \ge \sqrt{a}$, $|x_1| \le |a^{1-c} + sa^{1/2-c}|$ so $\left|\frac{x_1}{-s}\right| \le \frac{a^{1-c}}{s} + a^{1/2-c} \le 2a^{1/2-c}$. Therefore $v_x \in \overline{AB}(L/2 : (1,r))$, there exists a polynomial P_{v_x} of a degree less than $(1+\epsilon)\beta$ such that

$$|h_t(x) - P_{v_x}(x - v_x)| \le C\|(x - v_x)\|^{(1+\epsilon)\beta} \le Ca^{(1-c)(1+\epsilon)\beta}.$$



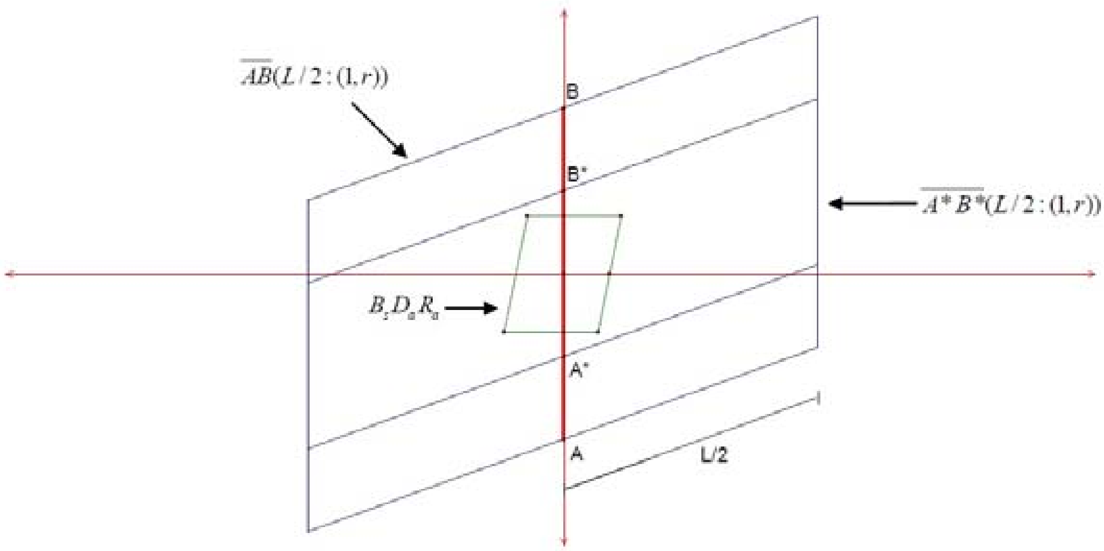


Figure 4.1: A picture of parallelogram involved in the proof of Theorem 4.1(2).

If $|s| \geq \sqrt{a}$, choosing $c = \frac{\epsilon}{2(1+\epsilon)}$, we have $|f(x+t) - P_{v_x}((x - v_x)| \leq Ca^{\beta/2}$. While if $|s| \geq a^\gamma$ then $|f(x+t) - P_{v_x}((x-v_x)| \leq Ca^{(1-c-\gamma)(1+\epsilon)\beta}$. Choosing $c = \frac{\epsilon}{1+\epsilon} - \gamma$, so $|f(x+t) - P_{v_x}((x-v_x)| \leq Ca^\beta$. Consider

$$\begin{aligned}
|\langle \psi_{ast}, f\rangle| &= |\langle \psi_{as0}, h_t\rangle| \\
&\leq \left| \int_{\mathbb{R}^2} h_t(x)\psi_{as0}(x)dx \right| \\
&\leq \left| \int_{\mathbb{R}^2} (h_t(x) - P_{v_x}(x - v_x))\psi_{as0}(x)dx \right| \\
&\leq \left| \int_{\mathbb{R}^2 \setminus B_sD_aR_a} (h_t(x) - P_{v_x}(x - v_x))\psi_{as0}(x)dx \right| \\
&+ \left| \int_{B_sD_aR_a} (f(x+t) - P_{v_x}(x - v_x))\psi_{as0}(x)dx \right|
\end{aligned}$$

This inequality on $B_sD_aR_a$ for $|s| \geq a^\gamma$ yields the estimate,

$$\begin{aligned}
\left| \int_{B_sD_aR_a} (f(x+t) - P_{v_x}(x - v_x))\psi_{as0}(x)dx \right| &\leq C \int_{B_sD_aR_a} \left(\frac{a^{\beta - \frac{3}{4}}}{1 + \|D_{1/a}B_{-s}x\|^{2N}} \right) dx \\
&= Ca^{\beta + \frac{3}{4}} \int_{R_a} \frac{1}{1 + \|y\|^{2N}} dy \\
&\leq Ca^{\beta + \frac{3}{4}},
\end{aligned}$$

If we only assume $|s| \geq \sqrt{a}$, similar we have

$$\left| \int_{B_sD_aR_a} (f(x+t) - P_{v_x}(x - v_x))\psi_{as0}(x)dx \right| \leq Ca^{\frac{\beta}{2} + \frac{3}{4}}.$$

Next, we will bound the integral on $\mathbb{R}^2 \setminus B_sD_aR_a$. By the decay estimate of ψ_{as0} from Lemma 1.11(2) and change of variable $x = B_sD_aR_ay$, if M is an upper bound of $|f(\cdot + t)|$,

$$\begin{aligned}
\left| \int_{\mathbb{R}^2 \setminus B_sD_aR_a} (h_t(x) - P_{v_x}(x - v_x))\psi_{as0}(x)dx \right| &\leq Ca^{\frac{-3}{4}} \int_{\mathbb{R}^2 \setminus B_sD_aR_a} \frac{|h_t(x) - P_{v_x}(x)|}{1 + \|D_{1/a}B_{-s}x + t\|^{2N}} dx \\
&\leq Ca^{\frac{3}{4}} \int_{\mathbb{R}^2 \setminus R_a} \frac{M + P_{y'}(C'\|y\|)}{1 + \|y\|^{2N}} dy \quad \text{(for some } C' > 0)
\end{aligned}$$

$$\leq Ca^{\frac{3}{4}} \int_{\|z\|>1} \frac{M + P_{y'}(C'a^{-c}\|z\|)}{1+\|z\|^{2N}} a^{-c} dz$$
$$\leq Ca^{\frac{3}{4}} \int_{\|z\|>1} \frac{C'(a^{-c}\|z\|)^{\deg P_{y'}}}{1+\|a^{-c}z\|^{2N}} a^{-c} dz$$
$$\leq Ca^{\frac{3}{4}+c(-1-\deg P_{y'})} \int_{\|z\|>1} \frac{C'(\|z\|)^{\deg P_{y'}}}{1+\|a^{-c}z\|^{2N}} dz$$
$$= Ca^{\frac{3}{4}+c(2N-1-\deg P_{y'})} \int_{\|z\|>1} \frac{(\|z\|)^{\deg P_{y'}}}{a^{c(2N)}+\|z\|^{2N}} dz$$
$$\leq Ca^{\frac{3}{4}+c(2N-1-\deg P_{y'})} \int_{\|z\|>1} \|z\|^{\deg P_{y'}-2N} dz$$
$$= Ca^{\frac{3}{4}+c(2N-1-\deg P_{y'})}$$
$$= Ca^{K}$$

where $y' = v_{B_s D_a y}$ and K can be chosen arbitrary large as c is fixed and N is arbitrary. □

The following Lemma extends Lemma 5.1 in Lakhonchai, Sampo and Sumetkijakan(2010).

Lemma 4.2. Let $0 < \alpha \leq 1$, $P(u_1, u_2), Q(v_1, v_2) \in \mathbb{R}^2$, $L > 0$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function.

- If $f \in C^\alpha(\overline{PQ}, \overline{PQ}(L : (1, s_0)); (1, s_0))$ then $f \circ B_{s_0} \in C^\alpha(\overline{P'Q'}, \overline{P'Q'}(L' : B_{-s_0}(1, s_0)); B_{-s_0}(1, s_0))$.
- If $f \in C^\alpha(\overline{PQ}(L : (1, s_0)); (-s_0, 1))$ then $f \circ B_{s_0} \in C^\alpha(\overline{P'Q'}(L' : B_{-s_0}(1, s_0)); (0, 1))$.

Proof. Assume $f \in C^\alpha(\overline{PQ}, \overline{PQ}(L : (1, s_0)); (1, s_0))$, so there exist $C > 0$ such that for each $y \in \overline{PQ}$ there is a polynomial P_y degree less than α ,

$$|f(x) - P_y(x-y)| \leq C\|x-y\|^\alpha$$

for all $x \in \overline{PQ}(L : (1, s_0))$ and $(x - y)^T // (1, s_0)^T$. Since $B_{s_0}\overline{P'Q'} = \overline{PQ}$, for each $y' \in \overline{P'Q'}$ we have $B_{s_0}y' \in \overline{PQ}$. there exists a polynomial $P_{y'}$ such that for all $x \in \overline{P'Q'}(L' : B_{-s_0}(1, s_0))$ and $B_{s_0}x' - B_{s_0}y' = B_{s_0}(x' - y')^T // (1, s_0)^T$ that is $(x' - y')^T // B_{-s_0}(1, s_0)^T$ we obtain

$$\begin{aligned} |f \circ B_{s_0}(x') - P_{B_{s_0}y'}(x' - y')| &\leq C\|B_{s_0}(x' - y')\|^\alpha \\ &\leq C\|B_{s_0}\|_{op}^\alpha (x' - y')\|^\alpha \\ &\leq C\|(x' - y')\|^\alpha, \end{aligned}$$

where $\|B_{s_0}\|_{op} = \left(1 + s^2/2 + (s^2 + s^4/4)^{1/2}\right)^{1/2} \leq \sqrt{3 + \sqrt{8}} = 1 + \sqrt{2}$ (See Kutyniok, Labate(2009)). Next, since $B_{s_0}(0, 1) = (-s_0, 1)$ so we prove similarly. $\square$

The following Theorem generalized Theorem 4.1 to the case with a sheared singularity line segment.

Theorem 4.3. Let $0 < \alpha \leq 1$ and $0 < \delta < 1$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded and let $P(u_1, u_2)$, $Q(v_1, v_2) \in \mathbb{R}^2$ and $L > 0$ be given.

- If $f \in C^\alpha(\overline{PQ}, \overline{PQ}(L : (1, s_0)); (1, s_0))$ then there exist a constant $C < \infty$ such that for all $a \in (0, 1)$, $t \in \overline{P^\delta Q^\delta}((1 - \delta)L : (1, r))$ and $s \in [-2, 2]$ if $|s - s_0| \leq \sqrt{a}$, then

$$|\langle \psi_{ast}, f \rangle| \leq C a^{3/4}(a^\alpha + |t_1 + s_0 t_2 - u_1 - s_0 u_2|^\alpha)$$

- If $f \in C^{(1+\epsilon)\beta}(\overline{PQ}(L : (1, s_0); (-s_0, 1)))$ and $0 < \gamma < \frac{\epsilon}{1+\epsilon} < \frac{1}{2}$ then there exists $C < \infty$ and a fixed coarse scale $a_0 \in (0, 1)$ such that for all $a \in (0, a_0)$,

$t \in \overline{P^*Q^*}(L/2 : (1, s_0))$ and $s \in [-2, 2]$,

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} Ca^{\frac{\beta}{2}+\frac{3}{4}} & \text{if } |s - s_0| \geq \sqrt{a} \\ Ca^{\beta+\frac{3}{4}} & \text{if } |s - s_0| \geq a^{\gamma}. \end{cases}$$

Proof. Since $B_{s_0}\overline{P'Q'}(L' : B_{-s_0}(1, s_0)) = \overline{PQ}(L : (1, s_0))$. It is easy to see that $|\langle \psi_{ast}, f \rangle| = |\langle \psi_{a(s-s_0)B_{-s_0}t}, f \circ B_{s_0} \rangle|$ and using Lemma 4.2, we have $f \circ B_{s_0} \in C^{\alpha}(\overline{P'Q'}, \overline{P'Q'}(L' : B_{-s_0}(1, s_0)); B_{-s_0}(1, s_0))$.Since $t \in \overline{P^{\delta}Q^{\delta}}((1-\delta)L : (1, s_0))$, so $B_{-s_0}t \in \overline{(P^{\delta})'(Q^{\delta})'}(((1-\delta)L)' : B_{-s_0}(1, s_0))$. Consider $B_{-s_0}(1, s_0) = (1+s_0^2, s_0)$ we see that $|(s - s_0) + \frac{1+s_0^2}{s_0}| = |s + \frac{1}{s_0}|$. Next, we will show that for each $s \in [-2, 2]$ if $|s - s_0| \leq \sqrt{a}$ then $|s + \frac{1}{s_0}| \geq \sqrt{a}$. In case $s_0 = 0$, trivially. For $s_0 > 0$, since $(s_0 - 1)^2 \geq 0$ we have $s_0 + \frac{1}{s_0} \geq 2$. Since $0 < a \leq 1$, $s_0 + \frac{1}{s_0} \geq 2\sqrt{a}$. So $-\frac{1}{s_0} + \sqrt{a} \leq s_0 - \sqrt{a}$. While if $s_0 < 0$, Let $m = -s_0 > 0$ by above case we got $\frac{1}{m} + \sqrt{a} \leq -t - \sqrt{a}$ then $\frac{1}{s_0} + \sqrt{a} \geq -s_0 + \sqrt{a}$. So clearly on the real line we have that $|s + \frac{1}{s_0}| \geq \sqrt{a}$. So by Theorem 6 we have,

$$|\langle \psi_{ast}, f \rangle| = |\langle \psi_{a(s-s_0)B_{-s_0}t}, f \circ B_{s_0} \rangle| \leq Ca^{3/4}(a^{\alpha} + |t_1 + s_0t_2 - u_1 - s_0u_2|^{\alpha}).$$

Supposition 2 we prove similarly. □

As a result of theorem, we model a linear singularity situation.

Corollary 4.4. Let $0 < \alpha \leq 1$, $P(u_1, u_2)$, $Q(v_1, v_2) \in \mathbb{R}^2$, $s_0 \in [-2, 2]$ and $0 < \gamma < \frac{\epsilon}{1+\epsilon} < \frac{1}{2}$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded in $C^{\alpha}(\overline{PQ}, \overline{PQ}(L : (1, s_0)); (1, s_0))$, $C^N(\overline{PQ}(L : (1, s_0)) \setminus \overline{PQ}; (1, s_0))$ and $C^{(1+\epsilon)N}(\overline{PQ}(L : (-s_0, 1)) \setminus \overline{PQ}; (-s_0, 1))$ for some $L > 0$. Then there is a constant $C < \infty$ and $a_0 < 1$ such that for all

$a \in (0, a_0)$, $t \in \overline{P^*Q^*}(L/2 : (1, s_0))$ and $s \in [-2, 2]$,

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} Ca^{\alpha+\frac{3}{4}} & \text{if } s = s_0 \text{ and } t \in \overline{P^*Q^*} \\ Ca^{N+\frac{3}{4}} & \text{if } s = s_0 \text{ and } t \notin \overline{P^*Q^*} \\ Ca^{N+\frac{3}{4}} & \text{if } |s - s_0| \geq a^{\gamma}. \end{cases}$$

Consequently, for $s = s_0$ and $t \in \overline{P^*Q^*}$, $|\langle \psi_{ast}, f \rangle| = O(a^{\alpha+3/4})$ as $a \to 0^+$ for all other cases $|\langle \psi_{ast}, f \rangle| = O(a^{N+3/4})$ as $a \to 0^+$.

Proof. Let $t \in \overline{PQ}(L : (1, s_0))$. If $s = s_0$ and t is on the line $\overline{P^*Q^*}$ then $|s - s_0| = 0 \leq \sqrt{a}$ and $s_0(t_2 - u_2) + (t_1 - u_1) = 0$. So Theorem 4.3 gives $|\langle \psi_{ast}, f \rangle| \leq Ca^{\alpha+\frac{3}{4}}$ for all $0 < a \leq 1$.

Let $s = s_0$ and $t \notin \overline{P^*Q^*}$. In light of Theorem 4.3 on the line $\overline{P^*Q^*}$, the assumption that $f \in C^N(\overline{PQ}(L : (1, s_0)) \setminus \overline{PQ}; (1, s_0))$ implies that $|\langle \psi_{ast}, f \rangle| \leq Ca^{N+\frac{3}{4}}$ for all $0 < a \leq 1$.

If $|s - s_0| \geq a^{\gamma}$ then, by Theorem 4.3, $|\langle \psi_{ast}, f \rangle| \leq Ca^{N+\frac{3}{4}}$ for all $0 < a \leq a_0$. $\square$

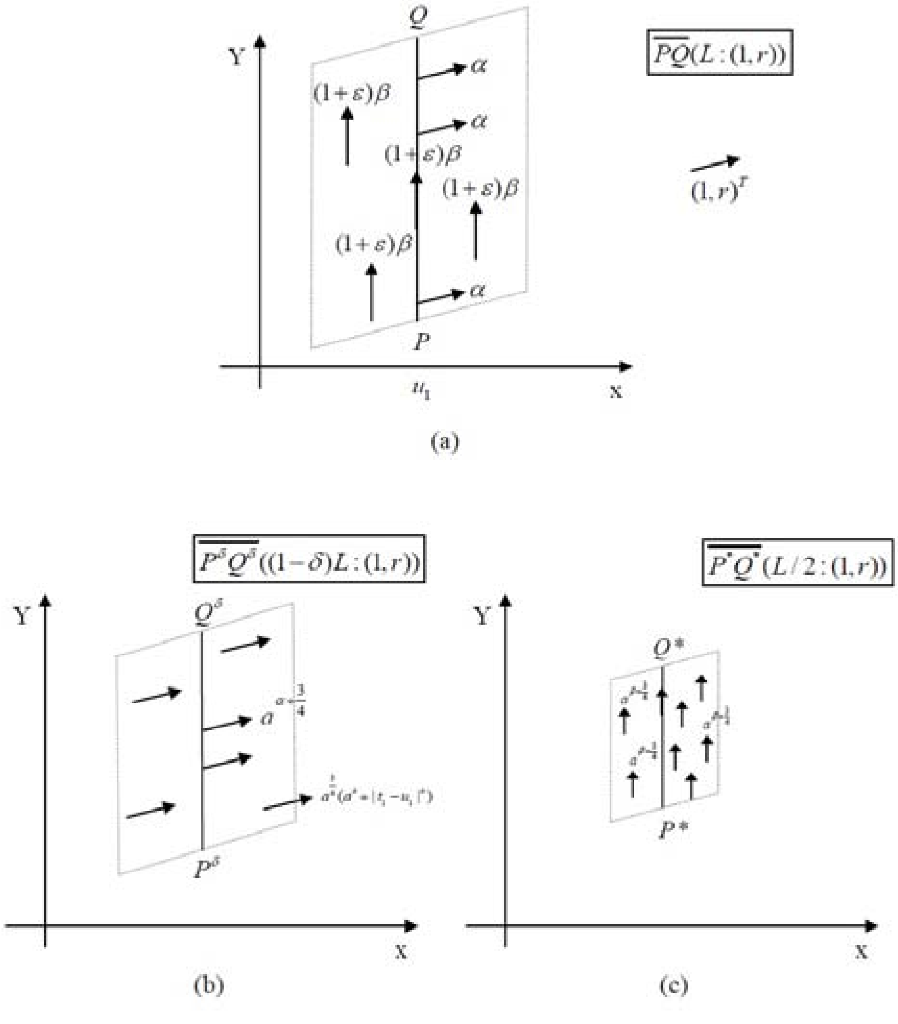


Figure 4.2: An illustration of a vertical line segment singularity in Theorem 4.1. (a): directional Hölder regularity of function. (b): decay rate of the continuous shearlet transform in a direction $(1, r)^T$. (c): decay rate of the continuous shearlet transform in a direction $(0, 1)^T$.

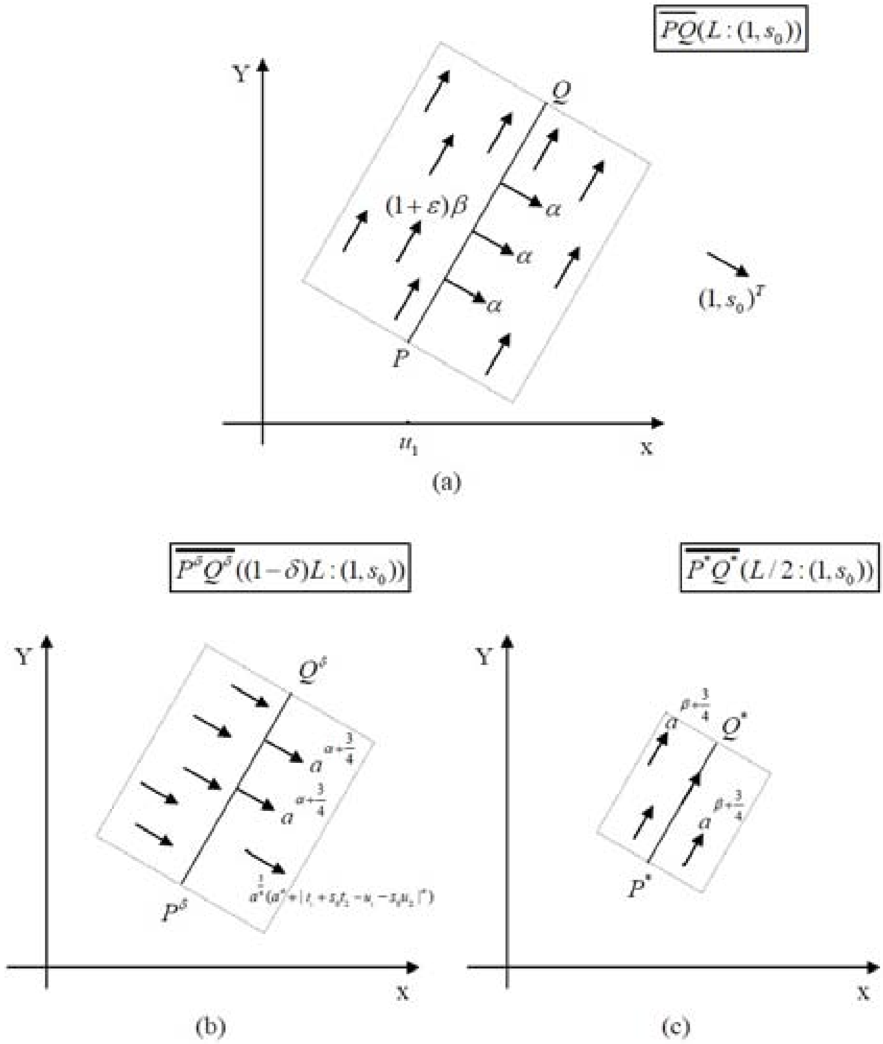


Figure 4.3: An illustration of a sheared line segment singularity in Theorem 4.3. (a): directional Hölder regularity of function. (b): decay rate of the continuous shearlet transform in a direction $(1, s_0)^T$. (c): decay rate of the continuous shearlet transform in a direction $(-s_0, 1)^T$.

REFERENCES

[1] Candes E.J. and Donoho D.L., Continuous curvelet transform I. Resolution of the wave front set, *Applied and Computational Harmonic Analysis*, 19 no.2 (2005) : 162-197.

[2] Candes E.J. and Donoho D.L., Continuous curvelet transform II. Discretization and frames, *Applied and Computational Harmonic Analysis*, 19 no.2 (2005) : 198-222.

[3] Daubechies, I., Ten Lecture on Wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 61, SIAM, Philadelphia, 1992.

[4] Grohs, P., Continuous Shearlet Frames and Resolution of the Wavefront Set. *Monatshefte fr Mathematik*, 164/4 (2011) : 393-426.

[5] Grohs, P., Shearlets and microlocal analysis. *European Mathematical Society*, 7 (2010) : 2573-2611.

[6] Holschneider, M. and Tchamitchian, Ph., Pointwise analysis of Riemann's "nondifferentiable" function, *Invent. Math.*, 105(1) (1991) : 157-175.

[7] Jaffard, S., Pointwise smoothness, two-microlocalization and wavelet coefficients, *Publicacions Matematiques*, 35 (1991) : 155-168.

[8] Kutyniok, G. and Labate, D., Resolution of Wavefront Set using Continuous Shearlets, Tran. Amer.Math Soc., 361 (2009) : 2719-2754.

[9] Lakhonchai, P., Sampo, J., Sumetkijakan, S., Shearlet transform and directional regularities. *International Journal of Wavelets, Multiresolution and Information Processing*, 8 (2010) : 743-771.

[10] Sampo, J., Sumetkijakan, S., Estimations of Hölder regularities and direction of singularity by Hart Smith and curvelet transforms. *J. Fourier Anal. Appl.*, 15 (2009) : 58-79.

[11] Sumetkijakan, S., a linear singularities by the continuous Shearlet transform. *Sampling Theory in Signal and Image Processing*, 10(2011) : 77-87.

VITA

Name	Mr. Jukrapong Tiammee
Date of Birth	17 November 1986
Place of Birth	Tak, Thailand
Education	B.Sc. (Mathematics)(First Class Honours), Chiangmai University, 2008
Scholarship	Science Achievement Scholarship of Thailand (SAST)