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SOME LOCAL SUBSEMIGROUPS OF SEMIGROUPS OF PARTIAL TRANSFORMATIONS AND LINEAR TRANSFORMATIONS

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เรืองวรินท์ อินทรวงษ์ สราญรักษ์สกุล : กึ่งกรุปย่อยเฉพาะที่บางชนิดของกึ่งกรุปการแปลงบางส่วนและการแปลงเชิงเส้น. (SOME LOCAL SUBSEMIGROUPS OF SEMIGROUPS OF PARTIAL TRANSFORMATIONS AND LINEAR TRANSFORMATIONS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : รศ. ดร. อมร วาสนาวิจิตร์, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : ผศ. ดร. ศจี เพียรสกุล, 55 หน้า.

สำหรับกึ่งกรุป S ให้ $E(S)$ เป็นเซตของนิจพลของ S *เซตย่อยเฉพาะที่*ของกึ่งกรุป S คือเซตย่อยของ S ในรูปแบบ eAe โดยที่ $e \in E(S)$ และ A เป็นกึ่งกรุปย่อยของ S เซตย่อยเฉพาะที่ของ S ไม่จำเป็นต้องเป็นกึ่งกรุปย่อยของ S *กึ่งกรุปย่อยเฉพาะที่*ของ S หมายถึง เซตย่อยเฉพาะที่ของ S ซึ่งเป็นกึ่งกรุปย่อยของ S จะสังเกตได้ว่าสำหรับทุก $e \in E(S)$ เซตย่อยเฉพาะที่ eSe เป็นกึ่งกรุปย่อยเฉพาะที่ของ S

เราเรียกกึ่งกรุป S ว่า *กึ่งกรุปปรกติ* เมื่อทุก $a \in S$ มี $x \in S$ ซึ่ง $a = axa$

ให้ X เป็นเซตไม่ว่าง ให้ $P(X)$, $T(X)$, $I(X)$ และ $G(X)$ เป็นกึ่งกรุปการแปลงบางส่วน กึ่งกรุปการแปลงเต็ม กึ่งกรุปการแปลงบางส่วนหนึ่งต่อหนึ่ง (กึ่งกรุปผกผันสมมาตร) และกรุปสมมาตรบน X ตามลำดับ ในการวิจัยนี้ เราแสดงว่าสำหรับ $\alpha \in E(P(X))$ ใดๆ $\alpha T(X)\alpha$ เป็นกึ่งกรุปย่อยเฉพาะที่ และเราให้ลักษณะ $\alpha \in E(P(X))$ เมื่อ X เป็นเซตจำกัดที่ทำให้เซตย่อยเฉพาะที่ $\alpha I(X)\alpha$ และ $\alpha G(X)\alpha$ ของ $P(X)$ เป็นกึ่งกรุปย่อยเฉพาะที่ของ $P(X)$ การให้ลักษณะเหล่านี้แสดงให้เห็นโดยอัตโนมัติว่ากึ่งกรุปย่อยเฉพาะที่เหล่านี้เป็นกึ่งกรุปปรกติ

เราศึกษากึ่งกรุป $L(V)$ ของการแปลงเชิงเส้นทั้งหมดของปริภูมิเวกเตอร์ V ที่มีมิติจำกัดในเรื่องเช่นเดียวกันด้วย เราให้เงื่อนไขที่จำเป็นและเพียงพอของ $\alpha \in E(L(V))$ ที่ทำให้ $\alpha GL(V)\alpha$ เป็นกึ่งกรุปย่อยเฉพาะที่ของ $L(V)$ โดยที่ $GL(V)$ คือกรุปของสมสัณฐานของ V ทั้งหมด ยิ่งไปกว่านั้น เราพิจารณาเซตย่อยเฉพาะที่ $AG_n(F)A$ ของกึ่งกรุปเมทริกซ์ $M_n(F)$ ขนาด $n \times n$ เต็มบนฟิลด์ F ในทำนองเดียวกัน เมื่อ $G_n(F)$ เป็นกรุปของเมทริกซ์ไม่เอกฐานขนาด $n \times n$ บน F ทั้งหมด เราได้ด้วยว่ากึ่งกรุปย่อยเฉพาะที่ของ $L(V)$ และ $M_n(F)$ เหล่านี้เป็นกึ่งกรุปปกติ

ภาควิชา...........คณิตศาสตร์.......... ลายมือชื่อนิสิต..

สาขาวิชา..........คณิตศาสตร์.......... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..........................

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RUANGVARIN INTARAWONG SARARNRAKSKUL : SOME LOCAL SUBSEMIGROUPS OF SEMIGROUPS OF PARTIAL TRANSFORMATIONS AND LINEAR TRANSFORMATIONS. ADVISOR : ASSOC. PROF. AMORN WASANAWICHIT, Ph.D., CO-ADVISOR : ASST. PROF. SAJEE PIANSKOOL, Ph. D., 55 pp.

The set of all idempotents of a semigroup S is denoted by $E(S)$. A *local subset* of a semigroup S is a subset of S of the form eAe where $e \in E(S)$ and A is a subsemigroup of S. A local subset of S need not be a subsemigroup of S. By a *local subsemigroup* of S we mean a local subset of S which is a subsemigroup of S. Notice that for every $e \in E(S)$, the local subset eSe is a local subsemigroup of S.

A semigroup S is *regular* if for every $a \in S$, $a = axa$ for some $x \in S$.

Let X be a nonempty set. Denote by $P(X)$, $T(X)$, $I(X)$ and $G(X)$ the partial transformation semigroup, the full transformation semigroup, the 1-1 partial transformation semigroup (the symmetric inverse semigroup) and the symmetric group on X, respectively. In this research, it is shown that for any $\alpha \in E(P(X))$, $\alpha T(X)\alpha$ is a local subsemigroup of $P(X)$ and we characterize $\alpha \in E(P(X))$ when X is finite for which the local subsets $\alpha I(X)\alpha$ and $\alpha G(X)\alpha$ of $P(X)$ are local subsemigroups of $P(X)$. These characterizations automatically imply that these local subsemigroups of $P(X)$ are regular semigroups.

We also study the semigroup $L(V)$ of all linear transformations of a finite-dimensional vector space V in the same maner. We provide a necessary and sufficient condition for $\alpha \in E(L(V))$ guaranteeing that $\alpha GL(V)\alpha$ is a local subsemigroup of $L(V)$ where $GL(V)$ is the group of all isomorphisms of V. In addition, the local subset $AG_n(F)A$ of the full $n \times n$ matrix semigroup $M_n(F)$ over a field F is considered similarly where $G_n(F)$ is the group of all nonsingular $n \times n$ matrices over F. These local subsemigroups of $L(V)$ and $M_n(F)$ are also regular.

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CONTENTS

CHAPTER I
INTRODUCTION

In [9], [10], [7] and [8] the authors used the word a " local subsemigroup "of a semigroup S to mean a subsemigroup of S of the form eSe where e is an idempotent of S. It is easily seen that if S is a regular semigroup, then so is the local subsemigroup eSe of S. We are motivated by this definition to define " local subsets "and " local subsemigroups "of S in a more general sense as follows : By a *local subset* of a semigroup S we mean a subset of S of the form eAe where A is a subsemigroup of S and e is an idempotent of S. A local subset of a semigroup S need not be a subsemigroup of S. Then it is interesting to find a necessary and sufficient condition for an idempotent e of S which guarantees that eAe becomes a subsemigroup of S for a given subsemigroup A of S. We call a local subset eAe of S a *local subsemigroup* of S if eAe is a subsemigroup of S. It is also interesting to investigate the regularity of certain local subsemigroups of some regular semigroups.

Transformation semigroups are considered very important in the area of semigroups as symmetric groups are crucial in the area of groups. It is well-known that every group can be embedded in a symmetric group. In Semigroup Theory, it is known that every semigroup can be embedded in some full transformation semigroup while every inverse semigroup can be embedded in a symmetric inverse semigroup (1-1 partial transformation semigroup).

In Linear Algebra, linear transformations and matrices play very important roles. Also, semigroups of linear transformations under composition and matrix semigroups are certainly important in Semigroup Theory.

Let X be a nonempty set and let $P(X)$, $T(X)$, $I(X)$ and $G(X)$ be the partial transformation semigroup, the full transformation semigroup, the symmetric inverse semigroup (the 1-1 partial transformation semigroup) and the symmetric group on X, respectively. It is well-known that $P(X)$ and $T(X)$ are regular and $I(X)$ is an inverse semigroup.

If V is a vector space over a field F, let $L(V)$ be the semigroup, under composition, of all linear transformations on V. It is known that $L(V)$ is a regular semigroup. Let $GL(V)$ be the group of all isomorphisms on V. The full $n \times n$ matrix semigroup over F is denoted by $M_n(F)$ and denote by $G_n(F)$ for the group of all nonsingular $n \times n$ matrices over F.

The preliminaries and notation used for this work are given in Chapter II.

In Chapter III, we prove that for every idempotent α of $P(X)$, $\alpha T(X)\alpha$ is a local subsemigroup of $P(X)$. We provide necessary and sufficient conditions for an idempotent α in $P(X)$ when X is finite for which $\alpha I(X)\alpha$ and $\alpha G(X)\alpha$ are local subsemigroups of $P(X)$. These characterizations automatically imply that these local subsemigroups of $P(X)$ are regular semigroups.

We study local subsemigroups of $L(V)$ when V is finite-dimensional and $M_n(F)$ in Chapter IV. Note that if $\dim V = n$, then there is an isomorphism $\theta : L(V) \to M_n(F)$ which preserves ranks. We characterize an idempotent α in $L(V)$ for which $\alpha GL(V)\alpha$ is a local subsemigroup of $L(V)$. By making use of this characterization, we give a necessary and sufficient condition for an idempotent A of $M_n(F)$ so that $AG_n(F)A$ is a local subsemigroup of $M_n(F)$. An explicit form of the local subsemigroup $AG_n(F)A$ is also determined. Moreover, we have that these local sunsemigroups of $L(V)$ and $M_n(F)$ are regular.

CHAPTER II

PRELIMINARIES

The cardinality of a set A will be denoted by $|A|$.

For any mapping α, the image of x in the domain of α will be written as $x\alpha$.

For semigroups S and S', we write $S \cong S'$ if S and S' are isomorphic, that is, there is a bijection $\varphi : S \to S'$ such that $(xy)\varphi = (x\varphi)(y\varphi)$ for all $x, y \in S$.

The element e of a semigroup S is called an *idempotent* if $e^2 = e$. The set of all idempotents of S is denoted by $E(S)$, that is,

$$E(S) = \{x \in S \,|\, x^2 = x\}.$$

If S has an identity and $U(S)$ is the unit group (or the group of units of S), then it is clear that

$$a^{-1}ea \in E(S) \qquad \text{for all } e \in E(S) \text{ and } a \in U(S).$$

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$ and S is called a *regular semigroup* if every element of S is regular. A semigroup S is said to be an *inverse semigroup* if for every $a \in S$, there is a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. It is well-known that a semigroup S is inverse if and only if S is regular and any two idempotents of S commute with each other ([1], p. 28).

If $e \in E(S)$, then $(eSe)(eSe) = e(SeeS)e \subseteq eSe$, so eSe is a subsemigroup of S. In [9], [10], [7] and [8] the authors used the word a " local subsemigroup " of a semigroup S to mean a subsemigroup of S of the form eSe for some $e \in E(S)$. Notice that e is the identity of the semigroup eSe. Moreover, if S is regular, then so is the local subsemigroup eSe of S. To see this, let $a \in S$. Then $eae = (eae)x(eae)$ for some $x \in S$, so $eae = (eae)(exe)(eae)$ and $exe \in eSe$. This definition motivates us to define local subsemigroups of S in a more general sense as follows : By a *local subset* of a semigroup S we mean a subset of S of the form eAe where $e \in E(S)$ and A is a subsemigroup of S. A local subset of S need not be a subsemigroup of S. By a *local subsemigroup* of S we mean a local subset of S which is also a subsemigroup of S. Observe that the local subsemigroup eAe of S has e as its identity if $e \in A$ or S has an identity and A contains the identity of S.

Let X be a nonempty set. A mapping from a subset of X into X is called a *partial transformation* of X. Let 0 be the partial transformation of X with empty domain. The domain and the range (image) of a partial transformation α of X are denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. The identity mapping on a set A is denoted by 1_A. Here $1_\varnothing = 0$. For $\varnothing \neq A \subseteq X$, and $x \in X$, let A_x denote the partial transformation of X whose domain and range are A and $\{x\}$, respectively. For a partial transformation α of X and $\varnothing \neq A \subseteq \operatorname{dom}\alpha$, let $\alpha_{|_A}$ be the restriction of α to A. The *partial transformation semigroup* on X, denoted by $P(X)$, consists of all partial transformations of X and the semigroup operation is composition, that is,

$$\alpha\beta = \begin{cases} 0 & \text{if } \operatorname{ran}\alpha \cap \operatorname{dom}\beta = \varnothing, \\ (\alpha_{|_{(\operatorname{ran}\alpha\cap\operatorname{dom}\beta)\alpha^{-1}}})(\beta_{|_{\operatorname{ran}\alpha\cap\operatorname{dom}\beta}}) & \text{if } \operatorname{ran}\alpha \cap \operatorname{dom}\beta \neq \varnothing. \end{cases}$$

Then 0 and 1_X are the zero and identity of $P(X)$, respectively. Notice that for $\alpha, \beta \in P(X)$,

$$\operatorname{dom}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha,$$

$$\operatorname{ran}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{ran}\beta$$

$$\text{for } x \in X,\ x \in \operatorname{dom}(\alpha\beta) \Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } x\alpha \in \operatorname{dom}\beta,$$

$$\operatorname{dom}\alpha = \dot{\bigcup_{x\in\operatorname{ran}\alpha}} x\alpha^{-1},$$

where $\dot{\bigcup}$ means a disjoint union. For $\alpha \in P(X)$, α may be written by a bracket notation as follows :

$$\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in\operatorname{ran}\alpha}.$$

The following subsets of $P(X)$ are cleary subsemigroups of $P(X)$:

$$T(X) = \{\alpha \in P(X) \mid \operatorname{dom}\alpha = X\},$$
$$I(X) = \{\alpha \in P(X) \mid \alpha \text{ is injective}\},$$
$$G(X) = \{\alpha : X \to X \mid \alpha \text{ is bijective}\}.$$

Notice that $G(X)$ is a subgroup of $P(X)$. The semigroup $T(X)$, $I(X)$ and $G(X)$ are called the *full transformation semigroup*, the 1-1 *partial transformation semigroup* or the *symmetric inverse semigroup* on X and the *symmetric group* on X, respectively. It can be seen that

$$G(X) \subseteq T(X) \subseteq P(X)\ ,\ G(X) \subseteq I(X) \subseteq P(X),$$

$G(X)$ is the unit group or the group of units of all $P(X)$, $T(X)$ and $I(X)$, that is, $G(X)$ is the greatest subgroup of $P(X)$, $T(X)$ and $I(X)$ having 1_X as its identity. It is well-known that $P(X)$ and $T(X)$ are regular semigroups ([3], p. 4 or [5], p. 63) and $I(X)$ is an inverse semigroup ([1], p. 29, [3], p. 4 or [5], p. 149). It can be seen that for $\alpha \in P(X)$,

$$\alpha \in E(P(X)) \Leftrightarrow \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha \text{ and } x\alpha = x \text{ for all } x \in \operatorname{ran}\alpha,$$

that is,

$$E(P(X)) = \{\alpha \in P(X) \mid \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha \text{ and } \alpha_{|\operatorname{ran}\alpha} = 1_{\operatorname{ran}\alpha}\}.$$

Hence

$$E(T(X)) = \{\alpha \in T(X) \mid \alpha_{|\operatorname{ran}\alpha} = 1_{\operatorname{ran}\alpha}\} \quad ([3], \text{p. } 12),$$
$$E(I(X)) = \{1_A \mid A \subseteq X\} \quad ([3], \text{p. } 4),$$
$$E(T(X)) \cup E(I(X)) \subseteq E(P(X)).$$

Consequently, if $\alpha \in E(P(X))$, then for every $x \in \operatorname{ran}\alpha$, $x \in x\alpha^{-1}$. Also, for $a \in X$, $X_a \in E(T(X))$ and for $\varnothing \neq A \subseteq X$ and $x \in X$, $A_x \in E(P(X))$ if and only if $x \in A$.

Let $\alpha \in E(P(X))$. Then $\alpha P(X)\alpha$ is a local subsemigroup of $P(X)$. It will be shown in Section 3.1 that the local subset $\alpha T(X)\alpha$ is always a local subsemigroup of $P(X)$. However, the local subsets $\alpha I(X)\alpha$ and $\alpha G(X)\alpha$ of $P(X)$ need not be local subsemigroups of $P(X)$, as shown by the following examples.

Example 2.1. Let $X = \{1, 2, 3, 4\}$ and let $\alpha \in E(P(X))$ be defined by

$$\alpha = \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix}.$$

Then

$$\begin{aligned} \alpha I(X) &= \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix} I(X) \\ &= \{0\} \bigcup \left\{ \begin{pmatrix} 1 \\ a \end{pmatrix} \mid a \in \{1,2,3,4\} \right\} \bigcup \left\{ \begin{pmatrix} \{2,3\} \\ a \end{pmatrix} \mid a \in \{1,2,3,4\} \right\} \\ &\quad \bigcup \left\{ \begin{pmatrix} 1 & \{2,3\} \\ a & b \end{pmatrix} \mid a, b \in \{1,2,3,4\} \text{ and } a \neq b \right\} \end{aligned}$$

and hence

$$\begin{aligned} \alpha I(X)\alpha &= (\alpha I(X)) \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix} \\ &= \left\{ 0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} \{2,3\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{2,3\} \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix}, \right. \\ &\qquad \left. \begin{pmatrix} 1 & \{2,3\} \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} \\ 2 \end{pmatrix} \right\}. \end{aligned}$$

But $\begin{pmatrix} \{1,2,3\} \\ 2 \end{pmatrix} \begin{pmatrix} 1 & \{2,3\} \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \{1,2,3\} \\ 1 \end{pmatrix} \notin \alpha I(X)\alpha$, so $\alpha I(X)\alpha$ is not a local subsemigroup of $P(X)$.

Example 2.2. Let $X = \{1,2,3\}$ and let $\alpha \in E(T(X))$ be defined by

$$\alpha = \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix}.$$

Then

$$\begin{aligned} \alpha G(X) &= \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix} G(X) \\ &= \left\{ \begin{pmatrix} 1 & \{2,3\} \\ a & b \end{pmatrix} \mid a, b \in \{1,2,3\} \text{ and } a \neq b \right\}. \end{aligned}$$

Thus

$$\begin{aligned} \alpha G(X)\alpha &= (\alpha G(X)) \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & \{2,3\} \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} \\ 2 \end{pmatrix} \right\}. \end{aligned}$$

But $\begin{pmatrix} \{1,2,3\} \\ 2 \end{pmatrix} \begin{pmatrix} 1 & \{2,3\} \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \{1,2,3\} \\ 1 \end{pmatrix} \notin \alpha G(X)\alpha$, so $\alpha G(X)\alpha$ is not a local subsemigroup of $P(X)$.

Example 2.3. Let $X = \{1,2,3,4\}$ and define $\alpha \in E(P(X))$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \ (= 1_{\{1,2,3\}}).$$

Then

$$\begin{aligned} \alpha G(X) &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} G(X) \\ &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix} \mid a, b, c \in \{1,2,3,4\} \text{ are distinct} \right\}, \end{aligned}$$

so

$$\alpha G(X)\alpha = (\alpha G(X))\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \mid a, b, c \in \{1, 2, 3, 4\} \text{ are distinct} \right\}.$$

It follows that for every $\beta \in \alpha G(X)\alpha$, $|\operatorname{dom}\beta| \geq 2$. We have that

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \in \alpha G(X)\alpha,$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \in \alpha G(X)\alpha.$$

But $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \alpha G(X)\alpha$, thus $\alpha G(X)\alpha$ is not a local subsemigroup of $P(X)$.

Observe that $0I(X)0 = \{0\} = 0G(X)0$, $1_X I(X)1_X = I(X)$, $1_X G(X)1_X = G(X)$ and for every $a \in X$, $X_a I(X)X_a = \{0, X_a\}$ and $X_a G(X)X_a = \{X_a\}$. These are trivial local subsemigroups of $P(X)$. The following examples show nontrivial local subsets of the form $\alpha I(X)\alpha$ and $\alpha G(X)\alpha$ of $P(X)$ which are local subsemigroups of $P(X)$.

Example 2.4. Let $X = \{1, 2, 3, 4\}$ and define $\alpha \in E(T(X))$ by

$$\alpha = \begin{pmatrix} \{1, 2\} & \{3, 4\} \\ 1 & 3 \end{pmatrix}.$$

Then

$$\alpha I(X) = \begin{pmatrix} \{1, 2\} & \{3, 4\} \\ 1 & 3 \end{pmatrix} I(X)$$

$$= \{0\} \bigcup \left\{ \begin{pmatrix} \{1,2\} \\ a \end{pmatrix} \mid a \in \{1,2,3,4\} \right\} \bigcup \left\{ \begin{pmatrix} \{3,4\} \\ a \end{pmatrix} \mid a \in \{1,2,3,4\} \right\}$$

$$\bigcup \left\{ \begin{pmatrix} \{1,2\} & \{3,4\} \\ a & b \end{pmatrix} \mid a, b \in \{1,2,3,4\} \text{ and } a \neq b \right\}.$$

It follows that

$$\alpha I(X)\alpha = (\alpha I(X)) \begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix}$$

$$= \left\{ 0, \begin{pmatrix} \{1,2\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{1,2\} \\ 3 \end{pmatrix}, \begin{pmatrix} \{3,4\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{3,4\} \\ 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \{1,2\} & \{3,4\} \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} \{1,2,3,4\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{1,2,3,4\} \\ 3 \end{pmatrix} \right\}$$

which is clearly a subsemigroup of $P(X)$. Hence $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$.

Example 2.5. Let $X = \{1,2,3,4\}$ and let $\alpha \in E(T(X))$ be defined by

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix}.$$

We have that

$$\alpha G(X) = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix} G(X)$$

$$= \left\{ \begin{pmatrix} \{1,2\} & \{3,4\} \\ a & b \end{pmatrix} \mid a, b \in \{1,2,3,4\} \text{ and } a \neq b \right\}.$$

Then

$$\alpha G(X)\alpha = (\alpha G(X)) \begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} \{1,2,3,4\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{1,2,3,4\} \\ 3 \end{pmatrix} \right\}$$

which is clearly a subsemigroup of $P(X)$. Hence $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$.

Example 2.6. Let $X = \{1, 2, 3, 4\}$ and let $\alpha \in E(P(X))$ be defined by

$$\alpha = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \ (= 1_{\{1,2\}}).$$

Then

$$\begin{aligned} \alpha G(X) &= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} G(X) \\ &= \left\{ \begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} \mid a, b \in \{1,2,3,4\} \text{ are distinct} \right\} \end{aligned}$$

and hence

$$\begin{aligned} \alpha G(X)\alpha &= (\alpha G(X)) \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \left\{ 0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\} \\ &= I(\{1,2\}). \end{aligned}$$

Therefore $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$.

Next, let V be a vector space over a field F and $L(V)$ the semigroup, under composition, of all linear transformations $\alpha : V \to V$. Recall that for $\alpha \in L(V)$, the kernel of α is

$$\ker \alpha = \{v \in V \mid v\alpha = 0\}.$$

Then for $\alpha \in L(V)$, α is a monomorphism if and only if $\ker\alpha = \{0\}$. We also have that

$$\dim V = \dim(\ker\alpha) + \dim(\operatorname{ran}\alpha)$$

([6], p. 187). Thus if V is finite-dimensional, then for $\alpha \in L(V)$, α is a monomorphism if and only if α is an epimorphism. Let $GL(V)$ be the set of all isomorphisms of V. Then

$$GL(V) = \{\alpha \in L(V) \mid \ker\alpha = \{0\} \text{ and } \operatorname{ran}\alpha = V\}.$$

Then $GL(V)$ is the unit group (or the group of units) of $L(V)$. Therefore we have

$$\begin{aligned}\dim V < \infty \Rightarrow GL(V) &= \{\alpha \in L(V) \mid \ker\alpha = \{0\}\} \\ &= \{\alpha \in L(V) \mid \operatorname{ran}\alpha = V\}.\end{aligned}$$

For $X \subseteq V$, let $\langle X\rangle$ be the subspace of V spanned by X. If $X \subseteq V$, then for $v \in \langle X\rangle$, v may be written as

$$v = \sum_{x\in X} a_x x \qquad \text{where } a_x \in F \text{ for all } x \in X$$

which means a finite sum in the sense that $a_x = 0$ for all but a finite number of x.

If B is a basis of V and $\{w_v \mid v \in B\} \subseteq V$, then there is a unique $\alpha \in L(V)$ such that $v\alpha = w_v$ for all $v \in B$, in this case, α can be written as

$$\alpha = \begin{pmatrix} v \\ w_v \end{pmatrix}_{v\in B}.$$

Hence if $\alpha \in L(V)$ and B is a basis of V, then we may write α as

$$\alpha = \begin{pmatrix} v \\ v\alpha \end{pmatrix}_{v\in B}.$$

If $\alpha \in L(V)$ and X is a nonempty subset of V, then

$$\begin{aligned}\langle X\alpha\rangle &= \{\sum_{x\in X} a_x(x\alpha) \mid a_x \in F \text{ and } a_x = 0 \text{ for all but a finite number of } x\} \\ &= \{(\sum_{x\in X} a_x x)\alpha \mid a_x \in F \text{ and } a_x = 0 \text{ for all but a finite number of } x\}\end{aligned}$$

$$= \{\sum_{x \in X} a_x x \mid a_x \in F \text{ and } a_x = 0 \text{ for all but a finite number of } x\}\, \alpha$$
$$= \langle X \rangle\, \alpha.$$

The following facts of linear transformations will be used.

Proposition 2.7. *Let $\alpha \in L(V)$ and B a basis of V. If $\alpha_{|_B}$ is injective and $B\alpha$ is a basis of V, then $\alpha \in GL(V)$.*

Proof. Since $B\alpha$ spans V, it follows that

$$V = \langle B\alpha \rangle = \langle B \rangle\, \alpha = V\alpha = \operatorname{ran}\alpha.$$

Let $v \in \ker\alpha$. Then $v = \sum_{u \in B} a_u u$, a finite sum, for some $a_u \in F$. Thus

$$0 = v\alpha = \sum_{u \in B} a_u (u\alpha).$$

Since $\alpha_{|_B}$ is 1-1, $u\alpha \neq w\alpha$ if $u \neq w$ in B. But $B\alpha$ is linearly independent over F, it follows from the above equality that $a_u = 0$ for all $u \in B$. This implies that $v = 0$. Thus $\ker\alpha = \{0\}$. Hence $\alpha \in GL(V)$, as desired. $\square$

Proposition 2.8. *Let $\alpha \in L(V)$, B_1 a basis of $\ker\alpha$ and B_2 a basis of $\operatorname{ran}\alpha$. If for every $v \in B_2$, let $v' \in v\alpha^{-1}$, then $B_1 \dot{\cup} \{v' \mid v \in B_2\}$ is a basis of V.*

Proof. To show that $B_1 \dot{\cup} \{v' \mid v \in B_2\}$ is linearly independent over F, let

$$\sum_{u \in B_1} a_u u + \sum_{v \in B_2} b_v v' = 0 \qquad \text{where } a_u \in F \text{ for } u \in B_1 \text{ and } b_v \in F \text{ for } v \in B_2.$$

Then

$$0 = (\sum_{u \in B_1} a_u u + \sum_{v \in B_2} b_v v')\alpha$$
$$= 0 + \sum_{v \in B_2} b_v (v'\alpha) \qquad \text{since } B_1 \subseteq \ker\alpha$$

$$= \sum_{v \in B_2} b_v v \qquad \text{since } v' \in v\alpha^{-1} \text{ for all } v \in B_2.$$

Since B_2 is linearly independent over F, $b_v = 0$ for all $v \in B_2$. Thus

$$\sum_{u \in B_1} a_u u = 0.$$

But since B_1 is linearly independent over F, we have $a_u = 0$ for all $u \in B_1$.

Next, to show that $\langle B_1 \dot{\cup} \{v' \mid v \in B_2\}\rangle = V$, let $w \in V$. Then $w\alpha \in \langle B_2 \rangle$, so

$$w\alpha = \sum_{v \in B_2} c_v v \qquad \text{for some } c_v \in F.$$

But $v'\alpha = v$ for all $v \in B_2$, so

$$w\alpha = \sum_{v \in B_2} c_v (v'\alpha) = \left(\sum_{v \in B_2} c_v v' \right) \alpha.$$

Thus $w - \sum_{v \in B_2} c_v v' \in \ker \alpha = \langle B_1 \rangle$ which implies that

$$w - \sum_{v \in B_2} c_v v' = \sum_{u \in B_1} d_u u \qquad \text{for some } d_u \in F.$$

Hence $w = \sum_{u \in B_1} d_u u + \sum_{v \in B_2} c_v v' \in \langle B_1 \dot{\cup} \{v' \mid v \in B_2\}\rangle$.

Therefore the proposition is proved. □

Proposition 2.9. ([4], p. 211). *If $\alpha \in E(L(V))$, then $V = \ker \alpha \oplus \operatorname{ran} \alpha$.*

Proposition 2.9 yields the following result.

Corollary 2.10. *If $\alpha \in E(L(V))$, B_1 is a basis of $\ker \alpha$ and B_2 is a basis of $\operatorname{ran} \alpha$, then $B_1 \dot{\cup} B_2$ is a basis of V.*

Hence for every $w \in \ker \alpha \smallsetminus \{0\}$, $\{w\} \cup B_2$ is a linearly independent subset of V (since $w \notin \langle B_2 \rangle = \operatorname{ran} \alpha$).

Notice that for $\alpha \in E(L(V))$, $v\alpha = v$ for all $v \in \operatorname{ran}\alpha$. Then Corollary 2.10 can be considered as a consequence of Proposition 2.8. If B_1 is a basis of $\ker\alpha$ and B_2 is a basis of $\operatorname{ran}\alpha$, then α can be defined on the basis $B_1\dot{\cup}B_2$ of V by

$$\alpha = \begin{pmatrix} u & v \\ u & 0 \end{pmatrix}_{\substack{u\in B_2 \\ v\in B_1}}.$$

Proposition 2.11. *Assume that U and W are vector spaces over F and U and W are vector space isomorphic by an isomorphism $\varphi : U \to W$. Define $\bar{\varphi} : L(U) \to L(W)$ by*

$$\alpha\bar{\varphi} = \varphi^{-1}\alpha\varphi \qquad \text{for all } \alpha \in L(U).$$

Then $\bar{\varphi}$ is a semigroup isomorphism.

Proof. It is evident that $\varphi^{-1}\alpha\varphi \in L(W)$ for all $\alpha \in L(U)$. Let $\alpha, \beta \in L(U)$. Then

$$\varphi^{-1}(\alpha\beta)\varphi = (\varphi^{-1}\alpha\varphi)(\varphi^{-1}\beta\varphi),$$
$$\varphi^{-1}\alpha\varphi = \varphi^{-1}\beta\varphi \Rightarrow \alpha = \varphi(\varphi^{-1}\alpha\varphi)\varphi^{-1} = \varphi(\varphi^{-1}\beta\varphi)\varphi^{-1} = \beta.$$

If $\lambda \in L(W)$, then $\varphi\lambda\varphi^{-1} \in L(U)$ and $\varphi^{-1}(\varphi\lambda\varphi^{-1})\varphi = \lambda$. This proves that $\bar{\varphi} : L(U) \to L(W)$ is a semigroup isomorphism, as desired. □

Next, let n be a positive integer and F a field. We let $M_n(F)$ be the multiplicative semigroup of all $n \times n$ matrices over F which may be called the *full $n \times n$ matrix semigroup over F*. Let $G_n(F)$ be the set of all nonsingular $n \times n$ matrices over F. Then $G_n(F)$ is the unit group of $M_n(F)$. Let V be a vector space over F of dimension n. Then there is an isomorphism $\theta : L(V) \to M_n(F)$ which preserves ranks, that is, $\dim(\operatorname{ran}\alpha)$ $(= \operatorname{rank}\alpha) = \operatorname{rank}(\alpha\theta)$ for all $\alpha \in L(V)$ ([6], p. 330, 336, 339). This implies that $GL(V)\theta = G_n(F)$ and $1_V\theta = I_n$, the identity $n \times n$ matrix over F.

For $A \in M_n(F)$ and $i, j \in \{1, \ldots, n\}$, let A_{ij} be the entry of A in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column.

For $k \in \{0, 1, \ldots, n\}$, let $D_n^{(k)} \in M_n(F)$ be defined by

$$D_n^{(0)} = 0, \text{ the zero matrix in } M_n(F),$$

$$\text{for } k > 0, \quad (D_n^{(k)})_{ij} = \begin{cases} 1 & \text{if } i = j \in \{1, \ldots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$D_3^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_5^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that for all $k \in \{1, \ldots, n\}$, $D_n^{(k)} \in E(M_n(F))$, $\operatorname{rank}(D_n^{(k)}) = k$ and for every $C \in G_n(F)$,

$$\operatorname{rank}(C^{-1} D_n^{(k)} C) = \operatorname{rank} D_n^{(k)} = k$$

([6], p. 338). Moreover, $C^{-1} D_n^{(k)} C \in E(M_n(F))$ for all $k \in \{1, \ldots, n\}$ and $C \in G_n(F)$. In fact, the following result is known.

Proposition 2.12. ([2], p. 226).

$$E(M_n(F)) = \{C^{-1} D_n^{(k)} C \mid k \in \{0, 1, \ldots, n\} \textit{ and } C \in G_n(F)\}.$$

Recall that for $A \in M_n(F)$, A is row-equivalent to I_n if and only if $A \in G_n(F)$ ([4], p. 23).

CHAPTER III
LOCAL SUBSEMIGROUPS OF PARTIAL TRANSFORMATION SEMIGROUPS

Throughout this chapter, X will be represented a nonempty set. In this chapter, we are concerned with the local subsets $\alpha T(X)\alpha$, $\alpha I(X)\alpha$ and $\alpha G(X)\alpha$ of $P(X)$ where $\alpha \in E(P(X))$. We consider when they become local subsemigroups of $P(X)$ in terms of α. The local subsets $\alpha I(X)\alpha$ and $\alpha G(X)\alpha$ of $P(X)$ are considered when X is finite. In addition, we show that these local subsemigroups are regular semigroups. If $\alpha = 0$, they are all $\{0\}$, a trivial local subsemigroup. Then we consider only $\alpha \in E(P(X)) \smallsetminus \{0\}$.

3.1 The Local Subsemigroups $\alpha T(X)\alpha$ of $P(X)$

The aim of this section is to show that for every $\alpha \in E(P(X))$, the local subset $\alpha T(X)\alpha$ of $P(X)$ is a local subsemigroup of $P(X)$.

First, we provide the following series of lemmas.

Lemma 3.1.1. *If $\alpha \in E(P(X)) \smallsetminus \{0\}$, then for every $\beta \in P(\operatorname{ran}\alpha), \beta\alpha = \beta$.*

Proof. Recall that $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $x\alpha = x$ for all $x \in \operatorname{ran}\alpha$. If $\beta \in P(\operatorname{ran}\alpha)$, then $\alpha_{|\operatorname{ran}\beta} = 1_{\operatorname{ran}\beta}$, so

$$\beta\alpha = \beta(\alpha_{|\operatorname{ran}\beta}) = \beta 1_{\operatorname{ran}\beta} = \beta.$$

□

Lemma 3.1.2. *If $\alpha \in E(P(X)) \smallsetminus \{0\}$, then $\alpha T(\operatorname{ran}\alpha)$ and $\alpha P(\operatorname{ran}\alpha)$ are subsemigroups of $P(X)$ and*

$$\alpha T(\operatorname{ran}\alpha) \cong T(\operatorname{ran}\alpha),\ \ \alpha P(\operatorname{ran}\alpha) \cong P(\operatorname{ran}\alpha).$$

Proof. By Lemma 3.1.1, we have $T(\operatorname{ran}\alpha)\alpha = T(\operatorname{ran}\alpha)$ and $P(\operatorname{ran}\alpha)\alpha = P(\operatorname{ran}\alpha)$. It follows that

$$\begin{aligned}(\alpha T(\operatorname{ran}\alpha))(\alpha T(\operatorname{ran}\alpha)) &= \alpha(T(\operatorname{ran}\alpha)\alpha)T(\operatorname{ran}\alpha)\\ &= \alpha T(\operatorname{ran}\alpha)T(\operatorname{ran}\alpha) = \alpha T(\operatorname{ran}\alpha),\\ (\alpha P(\operatorname{ran}\alpha))(\alpha P(\operatorname{ran}\alpha)) &= \alpha(P(\operatorname{ran}\alpha)\alpha)P(\operatorname{ran}\alpha)\\ &= \alpha P(\operatorname{ran}\alpha)P(\operatorname{ran}\alpha) = \alpha P(\operatorname{ran}\alpha).\end{aligned}$$

Hence $\alpha T(\operatorname{ran}\alpha)$ and $\alpha P(\operatorname{ran}\alpha)$ are subsemigroups of $P(X)$.

Define $\varphi : P(\operatorname{ran}\alpha) \to \alpha P(\operatorname{ran}\alpha)$ by

$$\beta\varphi = \alpha\beta \quad \text{for all } \beta \in P(\operatorname{ran}\alpha).$$

Then φ is onto. Let $\beta, \gamma \in P(\operatorname{ran}\alpha)$. We have that

$$\begin{aligned}(\beta\gamma)\varphi &= \alpha(\beta\gamma)\\ &= \alpha(\beta\alpha)\gamma \qquad \text{from Lemma 3.1.1}\\ &= (\alpha\beta)(\alpha\gamma)\\ &= (\beta\varphi)(\gamma\varphi).\end{aligned}$$

Next, assume that $\beta\varphi = \gamma\varphi$. Then $\alpha\beta = \alpha\gamma$. Let $x \in \operatorname{dom}\beta$. Then $x \in \operatorname{ran}\alpha$, so $x\alpha = x$. Thus

$$x\beta = (x\alpha)\beta = x(\alpha\beta) = x(\alpha\gamma) = (x\alpha)\gamma = x\gamma,$$

so $x \in \operatorname{dom}\gamma$. This shows that $\operatorname{dom}\beta \subseteq \operatorname{dom}\gamma$ and $x\beta = x\gamma$ for all $x \in \operatorname{dom}\beta$. It can be shown similary that $\operatorname{dom}\gamma \subseteq \operatorname{dom}\beta$ and $x\gamma = x\beta$ for all $x \in \operatorname{dom}\gamma$. Hence $\beta = \gamma$. Therefore we deduce that φ is an isomorphism from $P(\operatorname{ran}\alpha)$ onto $\alpha P(\operatorname{ran}\alpha)$. But $(T(\operatorname{ran}\alpha))\varphi = \alpha T(\operatorname{ran}\alpha)$, thus $\varphi_{|_{T(\operatorname{ran}\alpha)}}$ is an isomorphism from $T(\operatorname{ran}\alpha)$ onto $\alpha T(\operatorname{ran}\alpha)$. This proves that $\alpha T(\operatorname{ran}\alpha) \cong T(\operatorname{ran}\alpha)$ and $P(\operatorname{ran}\alpha) \cong P(\operatorname{ran}\alpha)$, as desired. □

Lemma 3.1.3. *If $\alpha \in E(T(X))$, then $\alpha T(X)\alpha = \alpha T(\operatorname{ran}\alpha)$.*

Proof. Since $\operatorname{dom}\alpha = X$, it follows that for every $\beta \in T(\operatorname{ran}\alpha)$, $\operatorname{dom}(\alpha\beta) = X$. Thus $\alpha T(\operatorname{ran}\alpha) \subseteq T(X)$. By Lemma 3.1.1, $T(\operatorname{ran}\alpha)\alpha = T(\operatorname{ran}\alpha)$. Hence

$$\begin{aligned}\alpha T(\operatorname{ran}\alpha) &= \alpha T(\operatorname{ran}\alpha)\alpha \\ &= \alpha(\alpha T(\operatorname{ran}\alpha))\alpha \\ &\subseteq \alpha T(X)\alpha.\end{aligned}$$

If $\beta \in T(X)$, then $\operatorname{dom}(\beta\alpha) = X$ and $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha$, so

$$\alpha\beta\alpha = \alpha((\beta\alpha)_{|\operatorname{ran}\alpha}) \in \alpha T(\operatorname{ran}\alpha).$$

Thus $\alpha T(X)\alpha \subseteq \alpha T(\operatorname{ran}\alpha)$, so the result follows. □

Lemma 3.1.4. *If $\alpha \in E(P(X)) \smallsetminus \{0\}$, then $\alpha P(X)\alpha = \alpha P(\operatorname{ran}\alpha)$.*

Proof. By Lemma 3.1.1, $P(\operatorname{ran}\alpha)\alpha = P(\operatorname{ran}\alpha)$. It follows that

$$\alpha P(\operatorname{ran}\alpha) = \alpha P(\operatorname{ran}\alpha)\alpha \subseteq \alpha P(X)\alpha\,.$$

If $\beta \in P(X)$, then $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha$, so

$$\begin{aligned}\alpha\beta\alpha &= \alpha((\beta\alpha)_{|\operatorname{dom}(\beta\alpha)\cap\operatorname{ran}\alpha}) \\ &\in\ \alpha P(\operatorname{ran}\alpha).\end{aligned}$$

Hence $\alpha P(X)\alpha \subseteq \alpha P(\operatorname{ran}\alpha)$. Therefore $\alpha P(X)\alpha = \alpha P(\operatorname{ran}\alpha)$, as desired. □

Lemma 3.1.5. *If $\alpha \in E(P(X)) \smallsetminus \{0\}$ and $\operatorname{dom}\alpha \subsetneq X$, then $\alpha T(X)\alpha = \alpha P(\operatorname{ran}\alpha)$.*

Proof. Since $T(X) \subseteq P(X)$, $\alpha T(X)\alpha \subseteq \alpha P(X)\alpha$. But $\alpha P(X)\alpha = \alpha P(\operatorname{ran}\alpha)$ by Lemma 3.1.4, so $\alpha T(X)\alpha \subseteq \alpha P(\operatorname{ran}\alpha)$.

For the reverse inclusion, let $\gamma \in P(\operatorname{ran}\alpha)$. Let $a \in X \smallsetminus \operatorname{dom}\alpha$. Define $\overline{\gamma} \in T(X)$ by

$$\overline{\gamma} = \begin{pmatrix} x & y \\ x\gamma & a \end{pmatrix}_{\substack{x\in\operatorname{dom}\gamma,\\ y\in X\smallsetminus\operatorname{dom}\gamma}}.$$

Since $\operatorname{ran}\gamma \subseteq \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $a \notin \operatorname{dom}\alpha$, we have

$$\operatorname{dom}\alpha \cap \operatorname{ran}\overline{\gamma} = \operatorname{dom}\alpha \cap (\operatorname{ran}\gamma \cup \{a\}) = \operatorname{ran}\gamma.$$

Thus

$$\begin{aligned}
\overline{\gamma}\alpha &= \overline{\gamma}(\alpha_{|_{\operatorname{dom}\alpha \cap \operatorname{ran}\overline{\gamma}}}) \\
&= \overline{\gamma}(\alpha_{|_{\operatorname{ran}\gamma}}) \\
&= \gamma\alpha \\
&= \gamma \qquad \text{from Lemma 3.1.1}
\end{aligned}$$

This implies that $\alpha\gamma = \alpha\overline{\gamma}\alpha \in \alpha T(X)\alpha$.

Hence $\alpha T(X)\alpha = \alpha P(\operatorname{ran}\alpha)$. □

Theorem 3.1.6. *For $\alpha \in E(P(X)) \smallsetminus \{0\}$, the local subset $\alpha T(X)\alpha$ of $P(X)$ is a local subsemigroup of $P(X)$ and*

$$\alpha T(X)\alpha \cong \begin{cases} T(\operatorname{ran}\alpha) & \textit{if } \operatorname{dom}\alpha = X, \\ P(\operatorname{ran}\alpha) & \textit{if } \operatorname{dom}\alpha \subsetneq X. \end{cases}$$

Proof. By Lemma 3.1.3 and Lemma 3.1.5, we have

$$\alpha T(X)\alpha = \begin{cases} \alpha T(\operatorname{ran}\alpha) & \text{if } \operatorname{dom}\alpha = X, \\ \alpha P(\operatorname{ran}\alpha) & \text{if } \operatorname{dom}\alpha \subsetneq X. \end{cases}$$

This fact and Lemma 3.1.2 yield the result that $\alpha T(X)\alpha$ is a local subsemigroup of $P(X)$ and

$$\alpha T(X)\alpha \cong \begin{cases} T(\operatorname{ran}\alpha) & \text{if } \operatorname{dom}\alpha = X, \\ P(\operatorname{ran}\alpha) & \text{if } \operatorname{dom}\alpha \subsetneq X. \end{cases}$$

□

Theorem 3.1.7. *For every $\alpha \in E(P(X)) \smallsetminus \{0\}$, then $\alpha T(X)\alpha$ is a regular semigroup.*

Proof. By Theorem 3.1.6, $\alpha T(X)\alpha$ is a local subsemigroup of $P(X)$. But since $T(\operatorname{ran}\alpha)$ and $P(\operatorname{ran}\alpha)$ are regular semigroups, so by Theorem 3.1.6, $\alpha T(X)\alpha$ is a regular semigroup, as desired.

□

3.2 The Local Subsemigroups $\alpha I(X)\alpha$ of $P(X)$

In this section, we give a necessary and sufficient condition for $\alpha \in E(P(X)) \setminus \{0\}$ when X is finite for which the local subset $\alpha I(X)\alpha$ of $P(X)$ is a local subsemigroup of $P(X)$. It is also shown that this local subsemigroup of $P(X)$ is always regular.

Observe that in Example 2.1,

$$|1\alpha^{-1}| = |\{1\}| = 1 < 2 = |\operatorname{ran}\alpha| = |2\alpha^{-1}|$$

and $\alpha I(X)\alpha$ is not a local subsemigroup of $P(X)$ but in Example 2.4,

$$|a\alpha^{-1}| = 2 \geq |\operatorname{ran}\alpha| \qquad \text{for all } a \in \operatorname{ran}\alpha$$

and $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$. This fact is generally true for a local subset $\alpha I(X)\alpha$ of $P(X)$ to be a local subsemigroup of $P(X)$ when X is finite.

We shall show that if X is finite, $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$ if and only if either

(i) $\alpha = 1_A$ for some nonempty subset A of X or

(ii) $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for every $a \in \operatorname{ran}\alpha$,

and this local subsemigroup of $P(X)$ is a regular semigroup.

To obtain the main results, the following series of lemmas is needed.

Lemma 3.2.1. *For a nonempty subset A of X, $1_A I(X) 1_A = I(A)$.*

Proof. Let $\beta \in I(X)$. Since $1_A\beta 1_A \in I(X)$, $\operatorname{dom}(1_A\beta 1_A) \subseteq A$ and $\operatorname{ran}(1_A\beta 1_A) \subseteq A$, it follows that $1_A\beta 1_A \in I(A)$. This shows that $1_A I(X) 1_A \subseteq I(A)$. Since 1_A

is the identity of $I(A)$, we have $1_A I(A) 1_A = I(A)$. But since $I(A) \subseteq I(X)$, it follows that

$$I(A) = 1_A I(A) 1_A \subseteq 1_A I(X) 1_A.$$

Hence $1_A I(X) 1_A = I(A)$, as desired. □

Lemma 3.2.2. *Let $\alpha \in E(P(X)) \setminus \{0\}$. If $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for every $a \in \operatorname{ran}\alpha$, then $\alpha I(X)\alpha = \alpha P(\operatorname{ran}\alpha)$.*

Proof. Since $I(X) \subseteq P(X)$, it follows that $\alpha I(X)\alpha \subseteq \alpha P(X)\alpha$. Therefore by Lemma 3.1.4, we have $\alpha I(X)\alpha \subseteq \alpha P(X)\alpha = \alpha P(\operatorname{ran}\alpha)$.

For the reverse inclusion, let $\lambda \in P(\operatorname{ran}\alpha)$. Then $\operatorname{dom}\alpha\lambda \subseteq \operatorname{dom}\alpha$ and $\operatorname{ran}\alpha\lambda \subseteq \operatorname{ran}\lambda \subseteq \operatorname{ran}\alpha$. It follows that

$$\begin{aligned} &\text{for every } c \in \operatorname{ran}\alpha\lambda,\ c(\alpha\lambda)^{-1} = (c\lambda^{-1})\alpha^{-1}, \\ &\operatorname{dom}\alpha\lambda = \dot{\bigcup_{c\in\operatorname{ran}\alpha\lambda}} c(\alpha\lambda)^{-1} = \dot{\bigcup_{c\in\operatorname{ran}\alpha\lambda}} (c\lambda^{-1})\alpha^{-1}. \end{aligned}$$

Therefore

$$\alpha\lambda = \begin{pmatrix} (c\lambda^{-1})\alpha^{-1} \\ c \end{pmatrix}_{c\in\operatorname{ran}\alpha\lambda}.$$

We also have that

$$\text{for every } c \in \operatorname{ran}\alpha\lambda,\ |c\lambda^{-1}| \leq |\operatorname{ran}\alpha| \leq |c\alpha^{-1}|.$$

Then for each $c \in \operatorname{ran}\alpha\lambda$, there is an injective mapping $\varphi_c : c\lambda^{-1} \to c\alpha^{-1}$. This implies that

$$\begin{pmatrix} x \\ x\varphi_c \end{pmatrix}_{\substack{c\in\operatorname{ran}\alpha\lambda \\ x\in c\lambda^{-1}}} \in I(X)$$

and

$$\text{for every } c \in \operatorname{ran}\alpha\lambda,\ ((c\lambda^{-1})\varphi_c)\alpha = \{c\}.$$

Thus $\alpha \begin{pmatrix} x \\ x\varphi_c \end{pmatrix}_{\substack{c\in\operatorname{ran}\alpha\lambda \\ x\in c\lambda^{-1}}} \alpha \in \alpha I(X)\alpha$ and

$$\alpha \begin{pmatrix} x \\ x\varphi_c \end{pmatrix}_{\substack{c\in \operatorname{ran}\alpha\lambda \\ x\in c\lambda^{-1}}} \alpha = \alpha \begin{pmatrix} x \\ c \end{pmatrix}_{\substack{c\in \operatorname{ran}\alpha\lambda \\ x\in c\lambda^{-1}}}$$

$$= \alpha \begin{pmatrix} c\lambda^{-1} \\ c \end{pmatrix}_{c\in \operatorname{ran}\alpha\lambda}$$

$$= \begin{pmatrix} (c\lambda^{-1})\alpha^{-1} \\ c \end{pmatrix}_{c\in \operatorname{ran}\alpha\lambda}$$

$$= \alpha\lambda.$$

Then $\alpha P(\operatorname{ran}\alpha) \subseteq \alpha I(X)\alpha$.

Hence the proof is complete. □

Lemma 3.2.3. *Let $\alpha \in E(P(X)) \smallsetminus \{0\}$ and assume that $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$. If $|a\alpha^{-1}| = 1$ for some $a \in \operatorname{ran}\alpha$, then α is injective, that is, $\alpha = 1_{\operatorname{dom}\alpha}$.*

Proof. Note that $a\alpha = a$ since $a \in \operatorname{ran}\alpha$ and $\alpha \in E(P(X)) \smallsetminus \{0\}$. Let $b \in \operatorname{ran}\alpha$ and suppose that $|b\alpha^{-1}| > 1$. Let $b' \in b\alpha^{-1}$ and $b \neq b'$. Define $\beta, \gamma \in I(X)$ by

$$\beta = \begin{pmatrix} a & b \\ b' & b \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} b \\ a \end{pmatrix}.$$

Then

$$\alpha\beta\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in\operatorname{ran}\alpha} \begin{pmatrix} a & b \\ b' & b \end{pmatrix} \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in\operatorname{ran}\alpha}$$

$$= \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in\operatorname{ran}\alpha} \begin{pmatrix} a & b \\ b & b \end{pmatrix}$$

$$= \begin{pmatrix} a\alpha^{-1} \cup b\alpha^{-1} \\ b \end{pmatrix} \in \alpha I(X)\alpha \quad \text{and}$$

$$\alpha\gamma\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in\operatorname{ran}\alpha} \begin{pmatrix} b \\ a \end{pmatrix} \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in\operatorname{ran}\alpha}$$

$$= \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in\operatorname{ran}\alpha} \begin{pmatrix} b \\ a \end{pmatrix}$$

$$= \begin{pmatrix} b\alpha^{-1} \\ a \end{pmatrix} \in \alpha I(X)\alpha.$$

Thus $(\alpha\beta\alpha)(\alpha\gamma\alpha) = \begin{pmatrix} a\alpha^{-1}\cup b\alpha^{-1} \\ a \end{pmatrix}$. Since $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$, we have $(\alpha\beta\alpha)(\alpha\gamma\alpha) = \alpha\lambda\alpha$ for some $\lambda \in I(X)$. Therefore $a(\alpha\lambda\alpha) = a\lambda\alpha = a$ and $b(\alpha\lambda\alpha) = b\lambda\alpha = a$. Hence $\{a,b\}\lambda \subseteq a\alpha^{-1} = \{a\}$ which is a contradiction since $|\{a,b\}\lambda| = 2$.

This proves that for every $b \in \operatorname{ran}\alpha, b\alpha^{-1} = \{b\}$. Thus for every $x \in \operatorname{dom}\alpha$, $x \in (x\alpha)\alpha^{-1} = \{x\alpha\}$. Therefore $x\alpha = x$ for all $x \in \operatorname{dom}\alpha$, that is, $\alpha = 1_{\operatorname{dom}\alpha}$, as desired. □

Lemma 3.2.4. *Let X be finite and $\alpha \in E(P(X)) \smallsetminus \{0\}$. If $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$ and α is not injective, then $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for all $a \in \operatorname{ran}\alpha$.*

Proof. By Lemma 3.2.3, for every $a \in \operatorname{ran}\alpha$, $|a\alpha^{-1}| > 1$. For every $a \in \operatorname{ran}\alpha$, let $a' \in a\alpha^{-1}$ and $a' \neq a$.

To show that $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for every $a \in \operatorname{ran}\alpha$, we are done if $|\operatorname{ran}\alpha| = 1$. Assume that $|\operatorname{ran}\alpha| = k > 1$ and let $b \in \operatorname{ran}\alpha$. Let

$$\operatorname{ran}\alpha = \{b = a_1, a_2, \ldots, a_k\}.$$

For each $i \in \{2, \ldots, k\}$, let $\beta_i \in I(X)$ be defined by

$$\beta_i = \begin{pmatrix} a_1 & a_i & x \\ a_1 & a_1' & x \end{pmatrix}_{x\in\operatorname{ran}\alpha\smallsetminus\{a_1,a_i\}}.$$

Then for each $i \in \{2, \ldots, k\}$,

$$\alpha\beta_i\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in \operatorname{ran}\alpha} \begin{pmatrix} a_1 & a_i & x \\ a_1 & a_1' & x \end{pmatrix}_{x\in \operatorname{ran}\alpha \smallsetminus \{a_1,a_i\}} \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in \operatorname{ran}\alpha}$$

$$= \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x\in \operatorname{ran}\alpha} \begin{pmatrix} a_1 & a_i & x \\ a_1 & a_1 & x \end{pmatrix}_{x\in \operatorname{ran}\alpha \smallsetminus \{a_1,a_i\}}$$

$$= \begin{pmatrix} a_1\alpha^{-1} \cup a_i\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x\in \operatorname{ran}\alpha \smallsetminus \{a_1,a_i\}}.$$

If $k = 2$, then $\alpha\beta_2\alpha = (\operatorname{dom}\alpha)_{a_1}$. If $k > 2$, then

$$(\alpha\beta_2\alpha)(\alpha\beta_3\alpha)$$

$$= \begin{pmatrix} a_1\alpha^{-1} \cup a_2\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x\in \operatorname{ran}\alpha \smallsetminus \{a_1,a_2\}} \begin{pmatrix} a_1\alpha^{-1} \cup a_3\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x\in \operatorname{ran}\alpha \smallsetminus \{a_1,a_3\}}$$

$$= \begin{pmatrix} a_1\alpha^{-1} \cup a_2\alpha^{-1} \cup a_3\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x\in \operatorname{ran}\alpha \smallsetminus \{a_1,a_2,a_3\}}.$$

The following result is easily obtained by induction :

$$(\alpha\beta_2\alpha)(\alpha\beta_3\alpha)\cdots(\alpha\beta_k\alpha) = \begin{pmatrix} \bigcup_{i=1}^{k} a_i\alpha^{-1} \\ a_1 \end{pmatrix} = (\operatorname{dom}\alpha)_{a_1}.$$

Since $\alpha I(X)\alpha$ is a subsemigroup of $P(X)$, we have that $(\operatorname{dom}\alpha)_{a_1} = \alpha\gamma\alpha$ for some $\gamma \in I(X)$. Consequently,

$$(\operatorname{dom}\alpha)\alpha\gamma\alpha = ((\operatorname{ran}\alpha)\gamma)\alpha = \{a_1\} = \{b\}.$$

It follows that $\operatorname{ran}\alpha \subseteq \operatorname{dom}\gamma$ and $(\operatorname{ran}\alpha)\gamma \subseteq b\alpha^{-1}$. But $\gamma \in I(X)$, so we have

$$|\operatorname{ran}\alpha| = |(\operatorname{ran}\alpha)\gamma| \le |b\alpha^{-1}|.$$

Hence the desired result follows. □

Theorem 3.2.5. *Let X be finite and $\alpha \in E(P(X)) \setminus \{0\}$. Then the local subset $\alpha I(X)\alpha$ of $P(X)$ is a local subsemigroup of $P(X)$ if and only if either*

(i) $\alpha = 1_A$ *for some nonempty subset A of X or*

(ii) $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ *for every $a \in \operatorname{ran}\alpha$.*

Moreover,

$$\alpha I(X)\alpha \begin{cases} = I(A) & \text{if } \alpha \text{ satisfies (i)}, \\ \cong P(\operatorname{ran}\alpha) & \text{if } \alpha \text{ satisfies (ii)}. \end{cases}$$

Proof. Assume that $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$ and suppose that α does not satisfies (i), that is, α is not injective. By Lemma 3.2.4, α satisfies (ii).

Conversely, assume that α satisfies (i) or (ii). If α satisfies (i), then by Lemma 3.2.1, $\alpha I(X)\alpha = 1_A I(X) 1_A = I(A)$ which is a subsemigroup of $P(X)$. Assume that α satisfies (ii). Then Lemma 3.2.2 yields $\alpha I(X)\alpha = \alpha P(\operatorname{ran}\alpha)$. It follows from Lemma 3.1.2 that $\alpha I(X)\alpha$ is a subsemigroup of $P(X)$ and $\alpha I(X)\alpha \cong P(\operatorname{ran}\alpha)$.

Therefore the theorem is proved. □

Theorem 3.2.6. *Let X be finite and $\alpha \in E(P(X)) \setminus \{0\}$. If $\alpha I(X)\alpha$ is a local subsemigroup of $P(X)$, then $\alpha I(X)\alpha$ is a regular semigroup. Moreover, $\alpha I(X)\alpha$ is an inverse semigroup if α is injective, that is, $\alpha = 1_{\operatorname{dom}\alpha} \in I(X)$.*

Proof. Since for any nonempty set Y, $I(Y)$ is an inverse semigroup and $P(Y)$ is a regular semigroup, the result follows directly from Theorem 3.2.5. □

3.3 The Local Subsemigroups $\alpha G(X)\alpha$ of $P(X)$

In this section, we are concerned with the local subset $\alpha G(X)\alpha$ of $P(X)$. We shall give a characterization in terms of α determining when the local subset $\alpha G(X)\alpha$ of $P(X)$ becomes a local subsemigroup of $P(X)$ when X is finite. Also, the local subsemigroup of $P(X)$ is always regular.

We can see from Example 2.2 that $\operatorname{dom}\alpha = X$,

$$|1\alpha^{-1}| = |\{1\}| = 1 < |\operatorname{ran}\alpha| = |2\alpha^{-1}|$$

and $\alpha G(X)\alpha$ is not a local subsemigroup of $P(X)$ while in Example 2.5, $\operatorname{dom}\alpha = X$,

$$|a\alpha^{-1}| \geq |\operatorname{ran}\alpha| \qquad \text{for all } a \in \operatorname{ran}\alpha$$

and $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$. Also, in Example 2.3, $\alpha = 1_{\{1,2,3\}}$, $|\{1,2,3\}| > |X \smallsetminus \{1,2,3\}|$ and $\alpha G(X)\alpha$ is not a local subsemigroup of $P(X)$ but in Example 2.6, $\alpha = 1_{\{1,2\}}$, $|\{1,2\}| \leq |X \smallsetminus \{1,2\}|$ and $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$. It will be shown that these are generally true when X is finite.

We shall prove that if X is finite, then for $\alpha \in E(P(X)) \smallsetminus \{0\}$, $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$ if and only if one of the following statements holds :

(i) $\alpha = 1_X$.

(ii) $\alpha = 1_A$ for some nonempty proper subset A of X with $|A| \leq |X \smallsetminus A|$.

(iii) $\operatorname{dom}\alpha = X$ and $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for all $a \in \operatorname{ran}\alpha$.

(iv) $|\operatorname{ran}\alpha| \leq |X \smallsetminus \operatorname{dom}\alpha|$ and $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for all $a \in \operatorname{ran}\alpha$.

This local subsemigroup of $P(X)$ is also regular.

First, we give the following series of lemmas.

Lemma 3.3.1. *Assume that X is finite and A is a nonempty proper subset of X. Then $1_A G(X) 1_A$ is a local subsemigroup of $P(X)$ if and only if $|A| \leq |X \smallsetminus A|$.*

If this is the case, $1_A G(X) 1_A = I(A)$.

Proof. We prove by contrapositive that if $1_A G(X) 1_A$ is a local subsemigroup of $P(X)$, then $|A| \leq |X \smallsetminus A|$. Assume that $|A| > |X \smallsetminus A|$. First, we claim that

$$|\operatorname{dom}(1_A\beta 1_A)| \geq |A| \smallsetminus |X \smallsetminus A| \quad \text{for every } \beta \in G(X). \tag{1}$$

To prove (1), let $\beta \in G(X)$. Then

$$|\operatorname{dom}(1_A\beta 1_A)| = |\operatorname{dom} 1_A(\beta 1_A)|$$

$$= |(A \cap \mathrm{dom}(\beta 1_A))1_A^{-1}|$$
$$= |(A \cap ((X \cap A)\beta^{-1}))1_A^{-1}|$$
$$= |(A \cap (A\beta^{-1}))1_A^{-1}|$$
$$= |A \cap (A\beta^{-1})|$$
$$= |A \smallsetminus (A \cap (X \smallsetminus A\beta^{-1}))|$$
$$= |A| - |A \cap (X \smallsetminus A\beta^{-1})|$$
$$\geq |A| - |X \smallsetminus A\beta^{-1}|$$
$$= |A| - |X \smallsetminus A| \qquad \text{since } |A\beta^{-1}| = |A|,$$

so (1) is proved. Let

$$k = |A| - |X \smallsetminus A| \text{ and } l = |X \smallsetminus A|.$$

Then $k \geq 1$ since $|A| > |X \smallsetminus A|$, $|A| = k + l$ and so

$$|X| = |A| + |X \smallsetminus A| = k + 2l.$$

Since $A \subsetneq X$, $l = |X \smallsetminus A| \geq 1$, so $|A| = k + l > k$. Next, we claim that

for all distinct $a_1, a_2, \ldots, a_k \in A$ and distinct $b_1, b_2, \ldots, b_k \in A$,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix} \in 1_A G(X) 1_A. \tag{2}$$

To prove (2), let $a_1, \ldots, a_k \in A$ be distinct and let $b_1, \ldots, b_k \in A$ be distinct. Then

$$|A \smallsetminus \{a_1, \ldots, a_k\}| = |A \smallsetminus \{b_1, \ldots, b_k\}| = l = |X \smallsetminus A|,$$

so there are bijections $\varphi : A \smallsetminus \{a_1, \ldots, a_k\} \to X \smallsetminus A$ and $\psi : X \smallsetminus A \to A \smallsetminus \{b_1, \ldots, b_k\}$. Then

$$\beta = \begin{pmatrix} a_1 & \cdots & a_k & x & y \\ b_1 & \cdots & b_k & x\varphi & y\psi \end{pmatrix}_{\substack{x \in A \smallsetminus \{a_1, \ldots, a_k\} \\ y \in X \smallsetminus A}} \in G(X).$$

Consequently,

$$1_A \beta 1_A = 1_A \beta_{|_A} 1_A$$

$$= \beta_{|_A} 1_A$$

$$= \begin{pmatrix} a_1 & \cdots & a_k & x \\ b_1 & \cdots & b_k & x\varphi \end{pmatrix}_{x \in A \smallsetminus \{a_1,\ldots,a_k\}} 1_A$$

$$= \begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix} \qquad \text{since } \operatorname{ran}\varphi \subseteq X \smallsetminus A,$$

so $\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix} \in 1_A G(X) 1_A$. Hence (2) is proved.

Since $|A| > k$, there are distinct elements $a_1, \ldots, a_k, a_{k+1} \in A$. Then by (2),

$$\gamma = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_2 & a_3 & \cdots & a_{k+1} \end{pmatrix} \text{ and } \lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_k & a_1 & \cdots & a_{k-1} \end{pmatrix}$$

are elements in $1_A G(X) 1_A$. But since

$$\gamma\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} \\ a_1 & a_2 & \cdots & a_{k-1} \end{pmatrix}$$

and $|\operatorname{dom}(\gamma\lambda)| = k - 1 < k = |A| - |X \smallsetminus A|$. It follows from (1) that $\gamma\lambda \notin 1_A G(X) 1_A$. This shows that $1_A G(X) 1_A$ is not a local subsemigroup of $P(X)$.

For the converse, assume that $|A| \leq |X \smallsetminus A|$. We will show that $1_A G(X) 1_A = I(A)$. By Lemma 3.2.1, $1_A I(X) 1_A = I(A)$. But since $G(X) \subseteq I(X)$, we have

$$1_A G(X) 1_A \subseteq 1_A I(X) 1_A = I(A).$$

For the reverse inclusion, let $\beta \in I(A)$. Then $\operatorname{dom}\beta \subseteq A$, $\operatorname{ran}\beta \subseteq A$ and $|\operatorname{dom}\beta| = |\operatorname{ran}\beta|$. Since $|A \smallsetminus \operatorname{dom}\beta| \leq |A| \leq |X \smallsetminus A|$, there is an injective mapping $\varphi : A \smallsetminus \operatorname{dom}\beta \to X \smallsetminus A$. Now, we have

$$\beta : \operatorname{dom}\beta \subseteq A \to \operatorname{ran}\beta \subseteq A \text{ is bijective,}$$

$$\varphi : A \smallsetminus \operatorname{dom}\beta \to (A \smallsetminus \operatorname{dom}\beta)\varphi \subseteq X \smallsetminus A \text{ is bijective.}$$

Since X is finite, it follows that

$$|X \smallsetminus A| = |(A \smallsetminus \operatorname{ran}\beta) \,\dot\cup\, ((X \smallsetminus A) \smallsetminus (A \smallsetminus \operatorname{dom}\beta)\varphi)|.$$

Then there is a bijection $\psi : X \setminus A \to (A \setminus \operatorname{ran}\beta)\,\dot{\cup}\,((X \setminus A) \setminus (A \setminus \operatorname{dom}\beta)\varphi)$. Define $\lambda : X \to X$ by

$$\lambda = \begin{pmatrix} x & y & z \\ x\beta & y\varphi & z\psi \end{pmatrix}_{\substack{x \in \operatorname{dom}\beta \\ y \in A \setminus \operatorname{dom}\beta \\ z \in X \setminus A}}.$$

Then $\lambda \in G(X)$ and

$$\begin{aligned} 1_A G(X) 1_A \ni 1_A \lambda 1_A &= 1_A \lambda_{|_A} 1_A \\ &= \lambda_{|_A} 1_A \\ &= \begin{pmatrix} x & y \\ x\beta & y\varphi \end{pmatrix}_{\substack{x \in \operatorname{dom}\beta \\ y \in A \setminus \operatorname{dom}\beta}} 1_A \\ &= \begin{pmatrix} x \\ x\beta \end{pmatrix}_{x \in \operatorname{dom}\beta} \qquad \text{since } \operatorname{ran}\varphi \subseteq X \setminus A \\ &= \beta. \end{aligned}$$

This shows that $1_A G(X) 1_A = I(A)$.

The lemma is thereby established. $\square$

Lemma 3.3.2. *Assume that X is finite. If $\alpha \in E(T(X))$ is such that $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for every $a \in \operatorname{ran}\alpha$, then $\alpha G(X) \alpha = \alpha T(\operatorname{ran}\alpha)$.*

Proof. If $\beta \in G(X)$, then $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha$, so we have $\alpha\beta\alpha = \alpha((\beta\alpha)_{|_{\operatorname{ran}\alpha}}) \in \alpha T(\operatorname{ran}\alpha)$. Hence $\alpha G(X)\alpha \subseteq \alpha T(\operatorname{ran}\alpha)$.

For the reverse inclusion, let $\lambda \in T(\operatorname{ran}\alpha)$. Then

$$\operatorname{ran}\alpha = \operatorname{dom}\lambda = \dot{\bigcup_{c \in \operatorname{ran}\lambda}} c\lambda^{-1}.$$

We also have that

$$\text{for every } c \in \operatorname{ran}\lambda,\ |c\lambda^{-1}| \leq |\operatorname{ran}\alpha| \leq |c\alpha^{-1}|.$$

Then for each $c \in \operatorname{ran}\lambda$, there is an injective mapping $\varphi_c : c\lambda^{-1} \to c\alpha^{-1}$. It follows that

$$\text{for every } c \in \operatorname{ran}\lambda,\ ((c\lambda^{-1})\varphi_c)\alpha = \{c\}.$$

Define $\beta : \dot{\bigcup\limits_{c\in \operatorname{ran}\lambda}} c\lambda^{-1} \longrightarrow \dot{\bigcup\limits_{c\in \operatorname{ran}\lambda}} c\alpha^{-1}$ by

$$\beta = \begin{pmatrix} x \\ x\varphi_c \end{pmatrix}_{\substack{c\in \operatorname{ran}\lambda \\ x\in c\lambda^{-1}}}.$$

Then $\operatorname{dom}\beta = \operatorname{dom}\lambda = \operatorname{ran}\alpha$. Since each φ_c is injective, β is injective, and thus $|\operatorname{dom}\beta| = |\operatorname{ran}\beta|$. Since X is finite, we have $|X \smallsetminus \operatorname{dom}\beta| = |X \smallsetminus \operatorname{ran}\beta|$. Let $\psi : X \smallsetminus \operatorname{dom}\beta \to X \smallsetminus \operatorname{ran}\beta$ be a bijection. Define $\bar{\beta} : X \to X$ by

$$\bar{\beta} = \begin{pmatrix} x & y \\ x\beta & y\psi \end{pmatrix}_{\substack{x\in \operatorname{dom}\beta \\ y\in X\smallsetminus \operatorname{dom}\beta}}.$$

It follows that $\bar{\beta} \in G(X)$. Also, we have

$$\begin{aligned}
\alpha\bar{\beta}\alpha &= \alpha(\bar{\beta}_{|\operatorname{ran}\alpha})\alpha \\
&= \alpha\beta\alpha \\
&= \alpha \begin{pmatrix} x \\ x\varphi_c \end{pmatrix}_{\substack{c\in \operatorname{ran}\lambda \\ x\in c\lambda^{-1}}} \alpha \\
&= \alpha \begin{pmatrix} x \\ c \end{pmatrix}_{\substack{c\in \operatorname{ran}\lambda \\ x\in c\lambda^{-1}}} \\
&= \alpha \begin{pmatrix} c\lambda^{-1} \\ c \end{pmatrix}_{c\in \operatorname{ran}\lambda} \\
&= \alpha\lambda.
\end{aligned}$$

Hence the lemma is proved. □

Lemma 3.3.3. *Assume that X is finite. If $\alpha \in E(P(X))$ is such that $|\operatorname{ran}\alpha| \leq |X \smallsetminus \operatorname{dom}\alpha|$ and $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for all $a \in \operatorname{ran}\alpha$, then $\alpha G(X)\alpha = \alpha P(\operatorname{ran}\alpha)$.*

Proof. By Lemma 3.1.4, $\alpha P(X)\alpha = \alpha P(\operatorname{ran}\alpha)$. Since $G(X) \subseteq P(X)$, it follows that $\alpha G(X)\alpha \subseteq \alpha P(\operatorname{ran}\alpha)$

For the reverse inclusion, let $\lambda \in P(\operatorname{ran}\alpha)$.

Case 1 : $\lambda = 0$. Since $|\operatorname{ran}\alpha| \leq |X \smallsetminus \operatorname{dom}\alpha|$, there exists an injective mapping $\beta : \operatorname{ran}\alpha \to X \smallsetminus \operatorname{dom}\alpha$. Since X is finite, $|X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus \operatorname{dom}\beta| = |X \smallsetminus \operatorname{ran}\beta|$, so there is a bijection $\varphi : X \smallsetminus \operatorname{dom}\beta \to X \smallsetminus \operatorname{ran}\beta$. Define

$$\bar{\beta} = \begin{pmatrix} x & y \\ x\beta & y\varphi \end{pmatrix}_{\substack{x \in \operatorname{dom}\beta \\ y \in X \smallsetminus \operatorname{dom}\beta}} .$$

Then $\bar{\beta} \in G(X)$ and

$$\begin{aligned} \alpha G(X)\alpha \ni \alpha\bar{\beta}\alpha &= \alpha(\bar{\beta}_{|\operatorname{ran}\alpha})\alpha \\ &= \alpha\beta\alpha \\ &= \alpha 0 \qquad \text{since } \operatorname{ran}\beta \subseteq X \smallsetminus \operatorname{dom}\alpha \\ &= 0 \\ &= \alpha\lambda. \end{aligned}$$

Case 2 : $\lambda \neq 0$. Then

$$\operatorname{ran}\alpha \supseteq \operatorname{dom}\lambda = \dot{\bigcup_{c \in \operatorname{ran}\lambda}} c\lambda^{-1}$$

and

$$\text{for every } c \in \operatorname{ran}\lambda,\ |c\lambda^{-1}| \leq |\operatorname{ran}\alpha| \leq |c\alpha^{-1}|.$$

Then for each $c \in \operatorname{ran}\lambda$, there is an injective mapping $\varphi_c : c\lambda^{-1} \to c\alpha^{-1}$. This implies that

$$\text{for every } \ c \in \operatorname{ran}\alpha,\ ((c\lambda^{-1})\varphi_c)\alpha = \{c\}.$$

Define $\beta : \dot{\bigcup}_{c \in \operatorname{ran}\lambda} c\lambda^{-1} \to \dot{\bigcup}_{c \in \operatorname{ran}\lambda} c\alpha^{-1}$ by

$$\beta = \begin{pmatrix} x \\ x\varphi_c \end{pmatrix}_{\substack{c \in \operatorname{ran}\lambda \\ x \in c\lambda^{-1}}} .$$

Then β is injective, $\operatorname{dom}\beta = \operatorname{dom}\lambda \subseteq \operatorname{ran}\alpha$ and $\operatorname{ran}\beta \subseteq \operatorname{dom}\alpha$. Since $|\operatorname{ran}\alpha \smallsetminus \operatorname{dom}\beta| \leq |\operatorname{ran}\alpha| \leq |X \smallsetminus \operatorname{dom}\alpha|$, so there is an injective mapping $\varphi : \operatorname{ran}\alpha \smallsetminus \operatorname{dom}\beta \to X \smallsetminus \operatorname{dom}\alpha$. Since X is finite, we have

$$|X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus (\operatorname{ran}\beta \,\dot\cup\, \operatorname{ran}\varphi)|.$$

Let $\psi : X \smallsetminus \operatorname{ran}\alpha \to X \smallsetminus (\operatorname{ran}\beta \,\dot\cup\, \operatorname{ran}\varphi)$ be bijective and define

$$\bar\beta = \begin{pmatrix} x & y & z \\ x\varphi_c & y\varphi & z\psi \end{pmatrix}_{\substack{c\in\operatorname{ran}\lambda,\, x\in c\lambda^{-1} \\ y\in\operatorname{ran}\alpha\smallsetminus\operatorname{dom}\beta \\ z\in X\smallsetminus\operatorname{ran}\alpha}}.$$

Then $\bar\beta \in G(X)$ and

$$\begin{aligned}
\alpha G(X)\alpha \ni \alpha\bar\beta\alpha &= \alpha(\bar\beta_{|\operatorname{ran}\alpha})\alpha \\
&= \alpha \begin{pmatrix} x & y \\ x\varphi_c & y\varphi \end{pmatrix}_{\substack{c\in\operatorname{ran}\lambda,\, x\in c\lambda^{-1} \\ y\in\operatorname{ran}\alpha\smallsetminus\operatorname{dom}\beta}} \alpha \\
&= \alpha \begin{pmatrix} x \\ x\varphi_c \end{pmatrix}_{\substack{c\in\operatorname{ran}\lambda \\ x\in c\lambda^{-1}}} \alpha \qquad \text{since } \operatorname{ran}\varphi \subseteq X \smallsetminus \operatorname{dom}\alpha \\
&= \alpha \begin{pmatrix} c\lambda^{-1} \\ c \end{pmatrix}_{c\in\operatorname{ran}\lambda} \qquad \text{since } ((c\lambda^{-1})\varphi_c)\alpha = \{c\} \\
&= \alpha\lambda.
\end{aligned}$$

Therefore the lemma is proved. □

Lemma 3.3.4. *Assume that* $\alpha \in E(P(X)) \smallsetminus \{0\}$ *and* $\alpha G(X)\alpha$ *is a local subsemigroup of* $P(X)$. *If* $|a\alpha^{-1}| = 1$ *for some* $a \in \operatorname{ran}\alpha$, *then* α *is injective, that is,* $\alpha = 1_{\operatorname{dom}\alpha}$.

Proof. Note that $a\alpha = a$ since $a \in \operatorname{ran}\alpha$ and $\alpha \in E(P(X)) \smallsetminus \{0\}$. Let $b \in \operatorname{ran}\alpha$ and suppose that $|b\alpha^{-1}| > 1$. Let $b' \in b\alpha^{-1}$ and $b \neq b'$. Since $|X \smallsetminus \{a, b\}| =$

$|X \smallsetminus \{b, b'\}|$, there is a bijection $\varphi : X \smallsetminus \{a, b\} \to X \smallsetminus \{b, b'\}$. Define $\beta, \gamma \in G(X)$ by

$$\beta = \begin{pmatrix} a & b & x \\ b' & b & x\varphi \end{pmatrix}_{x \in X \smallsetminus \{a,b\}} \quad \text{and} \quad \gamma = \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in X \smallsetminus \{a,b\}}.$$

Then $\{a\} \cup b\alpha^{-1} \subseteq \operatorname{dom}(\alpha\beta\alpha)$, $(\{a\} \cup b\alpha^{-1})(\alpha\beta\alpha) = \{b\}$ and

$$\begin{aligned} \alpha\gamma\alpha &= \alpha \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in X \smallsetminus \{a,b\}} \alpha \\ &= \begin{pmatrix} a & b\alpha^{-1} & x\alpha^{-1} \\ b & a & x \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a,b\}} \alpha \\ &= \begin{pmatrix} a & b\alpha^{-1} & x\alpha^{-1} \\ b & a & x \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a,b\}}. \end{aligned}$$

Thus $\{a\} \cup b\alpha^{-1} \subseteq \operatorname{dom}((\alpha\beta\alpha)(\alpha\gamma\alpha))$ and $(\{a\} \cup b\alpha^{-1})((\alpha\beta\alpha)(\alpha\gamma\alpha)) = \{a\}$. Since $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$, we have $(\alpha\beta\alpha)(\alpha\gamma\alpha) = \alpha\lambda\alpha$ for some $\lambda \in G(X)$. This implies that

$$\{a\} = (\{a\} \cup b\alpha^{-1})\alpha\lambda\alpha = (\{a, b\}\lambda)\alpha.$$

Hence $\{a, b\}\lambda \subseteq a\alpha^{-1} = \{a\}$ which is a contradiction since $|\{a, b\}\lambda| = 2$.

This proves that for every $b \in \operatorname{ran}\alpha$, $b\alpha^{-1} = \{b\}$. Thus for every $x \in \operatorname{dom}\alpha$, $x \in (x\alpha)\alpha^{-1} = \{x\alpha\}$. Therefore $x\alpha = x$ for all $x \in \operatorname{dom}\alpha$, that is, $\alpha = 1_{\operatorname{dom}\alpha}$, as desired. □

Lemma 3.3.5. *Let X be finite and assume that $\alpha \in E(P(X)) \smallsetminus \{0\}$ and α is not injective. If $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$, then $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for every $a \in \operatorname{ran}\alpha$.*

Proof. By Lemma 3.3.4, for every $a \in \operatorname{ran}\alpha$, $|a\alpha^{-1}| > 1$. For every $a \in \operatorname{ran}\alpha$, let $a' \in a\alpha^{-1}$ and $a' \neq a$.

To show that $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for every $a \in \operatorname{ran}\alpha$, we are done if $|\operatorname{ran}\alpha| = 1$. Assume that $|\operatorname{ran}\alpha| = k > 1$ and let $b \in \operatorname{ran}\alpha$. Let

$$\operatorname{ran}\alpha = \{a_1 = b, a_2, \ldots, a_k\}.$$

Since for each $i \in \{2, \ldots, k\}$, $|\{a_1, \ldots, a_{i-1}, a_1', a_{i+1}, \ldots, a_k\}| = k = \operatorname{ran}\alpha$, there is a bijection $\varphi_i : X \smallsetminus \operatorname{ran}\alpha \to X \smallsetminus \{a_1, \ldots, a_{i-1}, a_1', a_{i+1} \ldots, a_k\}$. For each $i \in \{2, \ldots, k\}$, let $\beta_i \in G(X)$ be defined by

$$\beta_i = \begin{pmatrix} a_1 & \ldots & a_{i-1} & a_i & a_{i+1} & \ldots & a_k & x \\ a_1 & \ldots & a_{i-1} & a_1' & a_{i+1} & \ldots & a_k & x\varphi_i \end{pmatrix}_{x \in X \smallsetminus \operatorname{ran}\alpha}.$$

Then for each $i \in \{2, \ldots, k\}$,

$$\begin{aligned}
&\alpha\beta_i\alpha \\
&= \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran}\alpha} \begin{pmatrix} a_1 & \ldots & a_{i-1} & a_i & a_{i+1} & \ldots & a_k & x \\ a_1 & \ldots & a_{i-1} & a_1' & a_{i+1} & \ldots & a_k & x\varphi_i \end{pmatrix}_{x \in X \smallsetminus \operatorname{ran}\alpha} \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran}\alpha} \\
&= \begin{pmatrix} a_1\alpha^{-1} & a_i\alpha^{-1} & x\alpha^{-1} \\ a_1 & a_1' & x \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a_1, a_i\}} \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran}\alpha} \\
&= \begin{pmatrix} a_1\alpha^{-1} \cup a_i\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a_1, a_i\}}.
\end{aligned}$$

If $k = 2$, then $\alpha\beta_2\alpha = (\operatorname{dom}\alpha)_{a_1}$. If $k > 2$, then

$$\begin{aligned}
&(\alpha\beta_2\alpha)(\alpha\beta_3\alpha) \\
&= \begin{pmatrix} a_1\alpha^{-1} \cup a_2\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a_1, a_2\}} \begin{pmatrix} a_1\alpha^{-1} \cup a_3\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a_1, a_3\}}
\end{aligned}$$

$$= \begin{pmatrix} a_1\alpha^{-1} \cup a_2\alpha^{-1} \cup a_3\alpha^{-1} & x\alpha^{-1} \\ a_1 & x \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a_1,a_2,a_3\}}.$$

By induction, we have

$$(\alpha\beta_2\alpha)(\alpha\beta_3\alpha)\cdots(\alpha\beta_k\alpha) = \begin{pmatrix} \bigcup_{i=1}^{k} a_i\alpha^{-1} \\ a_1 \end{pmatrix} = (\operatorname{dom}\alpha)_{a_1}.$$

Since $\alpha G(X)\alpha$ is a subsemigroup of $P(X)$, we have that $(\operatorname{dom}\alpha)_{a_1} = \alpha\gamma\alpha$ for some $\gamma \in G(X)$. Consequently,

$$(\operatorname{dom}\alpha)\alpha\gamma\alpha = ((\operatorname{ran}\alpha)\gamma)\alpha = \{a_1\} = \{b\}.$$

It follows that $\operatorname{ran}\alpha \subseteq \operatorname{dom}\gamma$ and $(\operatorname{ran}\alpha)\gamma \subseteq b\alpha^{-1}$. But $\gamma \in G(X)$, so we have $|\operatorname{ran}\alpha| = |(\operatorname{ran}\alpha)\gamma| \leq |b\alpha^{-1}|$. Hence the desired result follows. $\square$

Lemma 3.3.6. *Assume that X is finite and $\alpha \in E(P(X)) \smallsetminus \{0\}$ is such that $\operatorname{dom}\alpha \subsetneq X$ and α is not injective. If $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$, then $|\operatorname{ran}\alpha| \leq |X \smallsetminus \operatorname{dom}\alpha|$.*

Proof. If $|\operatorname{ran}\alpha| = 1$, then we are done. Assume that $|\operatorname{ran}\alpha| > 1$. Let a, b be distinct elements of $\operatorname{ran}\alpha$ and let $c \in X \smallsetminus \operatorname{dom}\alpha$. By Lemma 3.3.5, $|\operatorname{ran}\alpha| \leq |a\alpha^{-1}|$ and $|\operatorname{ran}\alpha| \leq |b\alpha^{-1}|$. Thus

$$|\operatorname{ran}\alpha \smallsetminus \{a\}| < |b\alpha^{-1}| \quad \text{and} \quad |\operatorname{ran}\alpha \smallsetminus \{b\}| < |a\alpha^{-1}|.$$

Then there are injective mappings $\varphi : \operatorname{ran}\alpha \smallsetminus \{a\} \to b\alpha^{-1} \subseteq \operatorname{dom}\alpha$ and $\psi : \operatorname{ran}\alpha \smallsetminus \{b\} \to a\alpha^{-1} \subseteq \operatorname{dom}\alpha$. Define

$$\beta = \begin{pmatrix} x & a \\ x\varphi & c \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a\}} \quad \text{and} \quad \gamma = \begin{pmatrix} x & b \\ x\psi & c \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{b\}}.$$

Then β and γ are injective and $\operatorname{dom}\beta = \operatorname{dom}\gamma = \operatorname{ran}\alpha$. Since X is finite, there are bijective mappings $\varphi' : X \smallsetminus \operatorname{dom}\beta \to X \smallsetminus \operatorname{ran}\beta$ and $\psi' : X \smallsetminus \operatorname{dom}\gamma \to X \smallsetminus \operatorname{ran}\gamma$. Then

$$\bar{\beta} = \begin{pmatrix} x & a & y \\ x\varphi & c & y\varphi' \end{pmatrix}_{\substack{x \in \operatorname{dom}\beta \smallsetminus \{a\} \\ y \in X \smallsetminus \operatorname{dom}\beta}} \quad \text{and} \quad \bar{\gamma} = \begin{pmatrix} x & b & y \\ x\psi & c & y\psi' \end{pmatrix}_{\substack{x \in \operatorname{dom}\gamma \smallsetminus \{b\} \\ y \in X \smallsetminus \operatorname{dom}\gamma}}$$

are elements of $G(X)$. Also, we have

$$\begin{aligned}
\alpha\bar{\beta}\alpha &= \alpha(\bar{\beta}_{|\operatorname{ran}\alpha})\alpha \\
&= \alpha\beta\alpha \\
&= \alpha \begin{pmatrix} x \\ x\varphi \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a\}} \alpha \qquad \text{since } c \notin \operatorname{dom}\alpha \\
&= \alpha \begin{pmatrix} x \\ b \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{a\}} \qquad \text{since } \operatorname{ran}\varphi \subseteq b\alpha^{-1} \\
&= \begin{pmatrix} \dot{\bigcup}_{x \in \operatorname{ran}\alpha \smallsetminus \{a\}} x\alpha^{-1} \end{pmatrix}_b ,
\end{aligned}$$

$$\begin{aligned}
\alpha\bar{\gamma}\alpha &= \alpha(\bar{\gamma}_{|\operatorname{ran}\alpha})\alpha \\
&= \alpha\gamma\alpha \\
&= \alpha \begin{pmatrix} x \\ x\psi \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{b\}} \alpha \qquad \text{since } c \notin \operatorname{dom}\alpha \\
&= \alpha \begin{pmatrix} x \\ a \end{pmatrix}_{x \in \operatorname{ran}\alpha \smallsetminus \{b\}} \qquad \text{since } \operatorname{ran}\psi \subseteq a\alpha^{-1} \\
&= \begin{pmatrix} \dot{\bigcup}_{x \in \operatorname{ran}\alpha \smallsetminus \{b\}} x\alpha^{-1} \end{pmatrix}_a .
\end{aligned}$$

Consequently, $(\alpha\bar{\beta}\alpha)(\alpha\bar{\gamma}\alpha) = \begin{pmatrix} \dot{\bigcup}_{x \in \operatorname{ran}\alpha \smallsetminus \{a\}} x\alpha^{-1} \end{pmatrix}_b \begin{pmatrix} \dot{\bigcup}_{x \in \operatorname{ran}\alpha \smallsetminus \{b\}} x\alpha^{-1} \end{pmatrix}_a = 0$ since $b \in b\alpha^{-1}$. Thus $0 \in \alpha G(X)\alpha$.

To show that $|\operatorname{ran}\alpha| \leq |X \setminus \operatorname{dom}\alpha|$, suppose on the contrary that $|\operatorname{ran}\alpha| > |X \setminus \operatorname{dom}\alpha|$. Let $\lambda \in G(X)$. Then

$$|\operatorname{ran}(\alpha\lambda)| = |\operatorname{ran}\alpha| > |X \setminus \operatorname{dom}\alpha|$$

which implies that $\operatorname{ran}(\alpha\lambda) \cap \operatorname{dom}\alpha \neq \varnothing$. It follows that

$$\operatorname{dom}(\alpha\lambda\alpha) = (\operatorname{ran}(\alpha\lambda) \cap \operatorname{dom}\alpha)(\alpha\lambda)^{-1} \neq \varnothing,$$

thus $\alpha\lambda\alpha \neq 0$. Since λ is arbitrary in $G(X)$, it follows that $0 \notin \alpha G(X)\alpha$ which is a contradiction. Hence we have $|\operatorname{ran}\alpha| \leq |X \setminus \operatorname{dom}\alpha|$, as desired. □

Theorem 3.3.7. *Let X be finite and $\alpha \in E(P(X)) \setminus \{0\}$. Then the local subset $\alpha G(X)\alpha$ of $P(X)$ is a local subsemigroup of $P(X)$ if and only if one of the following statements holds.*

(i) $\alpha = 1_X$.

(ii) $\alpha = 1_A$ *for some nonempty proper subset A of X with* $|A| \leq |X \setminus A|$.

(iii) $\operatorname{dom}\alpha = X$ *and* $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ *for all* $a \in \operatorname{ran}\alpha$.

(iv) $|\operatorname{ran}\alpha| \leq |X \setminus \operatorname{dom}\alpha|$ *and* $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ *for all* $a \in \operatorname{ran}\alpha$.

Moreover,

$$\alpha G(X)\alpha \begin{cases} = G(X) & \text{if } \alpha \text{ satisfies (i)},\\ = I(A) & \text{if } \alpha \text{ satisfies (ii)},\\ \cong T(\operatorname{ran}\alpha) & \text{if } \alpha \text{ satisfies (iii)},\\ \cong P(\operatorname{ran}\alpha) & \text{if } \alpha \text{ satisfies (iv)}.\end{cases}$$

Proof. Assume that $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$.

Case 1 : α is injective and $\operatorname{dom}\alpha = X$. Then $\alpha = 1_X$.

Case 2 : α is injective and $\operatorname{dom}\alpha \subsetneq X$. Then $\alpha = 1_A$ for some nonempty proper subset A of X. By Lemma 3.3.1, $|A| \leq |X \setminus A|$, so (ii) holds.

Case 3 : α is not injective and $\operatorname{dom}\alpha = X$. Then by Lemma 3.3.5, α satisfies (iii).

Case 4 : α is not injective and $\operatorname{dom}\alpha \subsetneq X$. By Lemma 3.3.5, $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for all $a \in \operatorname{ran}\alpha$. From Lemma 3.3.6, $|\operatorname{ran}\alpha| \leq |X \smallsetminus \operatorname{dom}\alpha|$. Hence (iv) holds.

Conversely, assume that α satisfies one of (i) − (iv). If α satisfies (i), then $\alpha G(X)\alpha = 1_X G(X) 1_X = G(X)$. If α satisfies (ii), then by Lemma 3.3.1, $\alpha G(X)\alpha = 1_A G(X) 1_A = I(A)$.

Assume that (iii) holds. From Lemma 3.3.2, we have $\alpha G(X)\alpha = \alpha T(\operatorname{ran}\alpha)$. By Lemma 3.1.2, $\alpha T(\operatorname{ran}\alpha)$ is a subsemigroup of $P(X)$ which is isomorphic to $T(\operatorname{ran}\alpha)$. Hence $\alpha G(X)\alpha \cong T(\operatorname{ran}\alpha)$.

Finally, assume that (iv) holds. Then by Lemma 3.3.3, $\alpha G(X)\alpha = \alpha P(\operatorname{ran}\alpha)$. By Lemma 3.1.2, $\alpha P(\operatorname{ran}\alpha)$ is a subsemigroup of $P(X)$ which is isomorphic to $P(\operatorname{ran}\alpha)$. It follows that $\alpha G(X)\alpha \cong P(\operatorname{ran}\alpha)$.

This proves the theorem. □

As a consequence of Theorem 3.3.7, we have

Corollary 3.3.8. *Let X be finite and $\alpha \in E(T(X))$. Then the local subset $\alpha G(X)\alpha$ of $T(X)$ is a local subsemigroup of $T(X)$ if and only if either*

(i) $\alpha = 1_X$ *or*

(ii) $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ *for every* $a \in \operatorname{ran}\alpha$.

Moreover,

$$\alpha G(X)\alpha \begin{cases} = G(X) & \text{if } \alpha \text{ satisfies (i)}, \\ \cong T(\operatorname{ran}\alpha) & \text{if } \alpha \text{ satisfies (ii)}. \end{cases}$$

Theorem 3.3.9. *Let X be finite and $\alpha \in E(P(X)) \smallsetminus \{0\}$. If $\alpha G(X)\alpha$ is a local subsemigroup of $P(X)$, then $\alpha G(X)\alpha$ is a regular semigroup. In addition, if α is injective, that is, $\alpha = 1_{\operatorname{dom}\alpha}$, then $\alpha G(X)\alpha$ is an inverse semigroup.*

Proof. Since for any nonempty set Y, $G(Y)$ is a group, $I(Y)$ is an inverse semigroup and both $T(Y)$ and $P(Y)$ are regular semigroups, the theorem is directly obtained from Theorem 3.3.7. □

CHAPTER IV
LOCAL SUBSEMIGROUPS OF SEMIGROUPS OF LINEAR TRANSFORMATIONS

Throughout this chapter, let F be a field, V a vector space over F and n a positive integer. This chapter deals with the local subset $\alpha GL(V)\alpha$ of the semigroup $L(V)$ and the local subset $AG_n(F)A$ of the semigroup $M_n(F)$. We determine when $\alpha GL(V)\alpha$ becomes a local subsemigroup of $L(V)$ in terms of an idempotent α of $L(V)$ when V is finite-dimensional. Also, we characterize an idempotent A of $M_n(F)$ for which $AG_n(F)A$ is a local subsemigroup of $M_n(F)$. From these characterizations, we have that these local subsemigroups of $L(V)$ and $M_n(F)$ are regular semigroups.

4.1 The Local Subsemigroups $\alpha GL(V)\alpha$ of $L(V)$

The aim of this section is to show that if V is finite-dimensional and $\alpha \in E(L(V))$, then $\alpha GL(V)\alpha$ is a local subsemigroup of $L(V)$ if and only if either

(i) $\alpha = 1_V$ or

(ii) $2\dim(\operatorname{ran}\alpha) \leq \dim V$.

This local subsemigroup of $L(V)$ is regular.

First, we provide the following series of lemmas.

Lemma 4.1.1. *If $\alpha \in E(L(V))$, then for every $\beta \in L(\operatorname{ran}\alpha)$, $\beta\alpha = \beta$.*

Proof. See the proof of Lemma 3.1.1. □

Lemma 4.1.2. *If $\alpha \in E(L(V))$, then $\alpha L(\operatorname{ran}\alpha)$ is a subsemigroup of $L(V)$ and*

$$\alpha L(\operatorname{ran}\alpha) \cong L(\operatorname{ran}\alpha).$$

Proof. See the proof of Lemma 3.1.2. Note that for $\beta \in L(\operatorname{ran}\alpha)$, $\operatorname{dom}\beta = \operatorname{ran}\alpha$, so $\operatorname{dom}(\alpha\beta) = \operatorname{dom}\alpha = V$. Hence $\alpha L(\operatorname{ran}\alpha) \subseteq L(V)$. $\square$

Lemma 4.1.3. *If $\alpha \in E(L(V))$, then $\alpha L(V)\alpha = \alpha L(\operatorname{ran}\alpha)$.*

Proof. See the proof of Lemma 3.1.3. $\square$

Lemma 4.1.4. *Assume that V is finite-dimensional and $\alpha \in E(L(V))$. If $\dim(\ker\alpha) \geq \dim(\operatorname{ran}\alpha)$, then $\alpha GL(V)\alpha = \alpha L(\operatorname{ran}\alpha)$.*

Proof. Since $GL(V) \subseteq L(V)$, by Lemma 4.1.3, we have

$$\alpha GL(V)\alpha \subseteq \alpha L(V)\alpha = \alpha L(\operatorname{ran}\alpha).$$

To show that $\alpha L(\operatorname{ran}\alpha) \subseteq \alpha GL(V)\alpha$, let $\lambda \in L(\operatorname{ran}\alpha)$. Then $\alpha\lambda \in L(V)$ and

$$\operatorname{ran}(\alpha\lambda) \subseteq \operatorname{ran}\lambda \subseteq \operatorname{ran}\alpha = \operatorname{dom}\lambda.$$

But $\dim(\ker\alpha) \geq \dim(\operatorname{ran}\alpha)$, so we have

$$\dim(\ker\lambda) \leq \dim(\operatorname{ran}\alpha) \leq \dim(\ker\alpha). \tag{1}$$

Let B_1 be a basis of $\ker\lambda$ and B_2 be a basis of $\operatorname{ran}\lambda$. For each $v \in B_2$, let $v' \in v\lambda^{-1}$. By Proposition 2.8, $B_1 \dot\cup \{v' \mid v \in B_2\}$ is a basis of $\operatorname{ran}\alpha$ ($= \operatorname{dom}\lambda$). Let B_3 be a basis of $\operatorname{ran}\alpha$ containing B_2. Then

$$\dim(\operatorname{ran}\alpha) = |B_1 \dot\cup \{v' \mid v \in B_2\}| = |B_3| = |(B_3 \smallsetminus B_2) \dot\cup B_2|.$$

Since $\dim V < \infty$ and $|\{v' \mid v \in B_2\}| = |B_2|$, it follows that $|B_1| = |B_3 \smallsetminus B_2|$. Let B_4 be a basis of $\ker\alpha$. By Corollary 2.10,

$$B_1 \dot\cup \{v' \mid v \in B_2\} \dot\cup B_4 \quad \text{is a basis of } V \text{ and} \tag{2}$$

$$B_3 \dot\cup B_4 \text{ is a basis of } V. \tag{3}$$

From (1), we have $|B_1| \le |B_4|$. Let $\varphi : B_1 \to B_4$ be injective. Then we have

$$|B_3 \smallsetminus B_2| = |B_1| = |B_1\varphi|$$

which implies that

$$|B_4| = |(B_4 \smallsetminus B_1\varphi) \dot\cup B_1\varphi| = |(B_4 \smallsetminus B_1\varphi) \dot\cup (B_3 \smallsetminus B_2)|$$

since $B_3 \cap B_4 = \varnothing$ (see (3)). Let $\psi : B_4 \to (B_4 \smallsetminus B_1\varphi) \dot\cup (B_3 \smallsetminus B_2)$ be a bijection. Define $\beta \in L(V)$ on the basis $B_1 \dot\cup \{v' \mid v \in B_2\} \dot\cup B_4$ of V (see (2)) by

$$\beta = \begin{pmatrix} u & v' & w \\ u\varphi & v & w\psi \end{pmatrix}_{\substack{u \in B_1,\, v \in B_2, \\ w \in B_4}}. \tag{4}$$

Since φ, $v' \mapsto v$ ($v \in B_2$), ψ are injective, $B_1\varphi \subseteq B_4$, $B_2 \subseteq B_3$, $B_3 \cap B_4 = \varnothing$ and $B_4\psi = (B_4 \smallsetminus B_1\varphi) \dot\cup (B_3 \smallsetminus B_2)$, it follows that β restricted to the basis $B_1 \dot\cup \{v' \mid v \in B_2\} \dot\cup B_4$ of V is injective. Also,

$$\begin{aligned}(B_1 \dot\cup \{v' \mid v \in B_2\} \dot\cup B_4)\beta &= B_1\varphi \dot\cup B_2 \dot\cup (B_4 \smallsetminus B_1\varphi) \dot\cup (B_3 \smallsetminus B_2) \\ &= B_3 \dot\cup B_4\end{aligned}$$

which is a basis of V by (3). By Proposition 2.7, we deduce that $\beta \in GL(V)$. We claim that $\alpha\beta\alpha = \alpha\lambda$. By (2), it suffices to show that

$$v\alpha\beta\alpha = v\alpha\lambda \quad \text{for all } v \in B_1 \dot\cup \{v' \mid v \in B_2\} \dot\cup B_4.$$

Recall that $v\alpha = v$ for all $v \in \operatorname{ran}\alpha$. We have that

$$\begin{aligned}\text{for } u \in B_1,\ u\alpha\beta\alpha &= u\beta\alpha && \text{since } B_1 \subseteq \operatorname{ran}\alpha \\ &= (u\varphi)\alpha && \text{from (4)} \\ &= 0 && \text{since } B_1\varphi \subseteq B_4 \subseteq \ker\alpha, \\ u\alpha\lambda &= u\lambda \\ &= 0 && \text{since } B_1 \subseteq \ker\lambda,\end{aligned}$$

$$\begin{aligned}
\text{for } u \in B_2,\ u'\alpha\beta\alpha &= u'\beta\alpha && \text{since } u' \in u\lambda^{-1} \subseteq \operatorname{ran}\alpha \\
&= u\alpha && \text{from (4)} \\
&= u, && \text{since } B_2 \subseteq \operatorname{ran}\lambda \subseteq \operatorname{ran}\alpha \\
u'\alpha\lambda &= u'\lambda && \\
&= u && \text{since } u' \in u\lambda^{-1}, \\
\text{for } u \in B_4,\ u\alpha\beta\alpha = 0 &= u\alpha\lambda && \text{since } B_4 \subseteq \ker\alpha.
\end{aligned}$$

This implies that $\alpha\beta\alpha = \alpha\lambda$.

This proves that $\alpha GL(V)\alpha = \alpha L(\operatorname{ran}\alpha)$, as desired. □

Lemma 4.1.5. *Assume that V is finite-demensional, $\alpha \in E(L(V))$ and $\alpha \neq 1_V$. If $\alpha GL(V)\alpha$ is a local subsemigroup of $L(V)$, then* $\dim(\ker\alpha) \geq \dim(\operatorname{ran}\alpha)$.

Proof. Since $\alpha \neq 1_V$, $u\alpha = v$ for some distinct $u, v \in V$. Then $u\alpha = u\alpha^2 = v\alpha$, so α is not a monomorphism. Thus $\ker\alpha \neq \{0\}$. Let $w \in \ker\alpha \setminus \{0\}$.

To show that $\dim(\ker\alpha) \geq \dim(\operatorname{ran}\alpha)$, we are done if $\dim(\operatorname{ran}\alpha) = 0$. Assume that $\dim(\operatorname{ran}\alpha) = k > 0$. Let $\{u_1, \ldots, u_k\}$ be a basis of $\operatorname{ran}\alpha$. By Corollary 2.10, we have that for each $i \in \{1, \ldots, k\}$, $u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_k$ are linearly independent. Let B_0 be a basis of $\ker\alpha$. By Corollary 2.10, $B_0 \dot\cup \{u_1, \ldots, u_k\}$ is a basis of V. For each $i \in \{1, \ldots, k\}$, let B_i be a basis of V containing $\{u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_k\}$. Since $\dim V < \infty$,

$$|B_0| = \dim V - k = |B_i \setminus \{u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_k\}|.$$

For each $i \in \{1, \ldots, k\}$, let $\varphi_i : B_0 \to B_i \setminus \{u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_k\}$ be a bijection. For each $i \in \{1, \ldots, k\}$, define $\beta_i \in L(V)$ on the basis $B_0 \dot\cup \{u_1, \ldots, u_k\}$ of V by

$$\beta_i = \begin{pmatrix} u_1 & \cdots & u_{i-1} & u_i & u_{i+1} & \cdots & u_k & v \\ u_1 & \cdots & u_{i-1} & w & u_{i+1} & \cdots & u_k & v\varphi_i \end{pmatrix}_{v \in B_0}.$$

By Proposition 2.7, $\beta_i \in GL(V)$ for all $i \in \{1, \ldots, k\}$. Note that $u_i\alpha = u_i$ for all $i \in \{1, \ldots, k\}$ and $v\alpha = 0$ for all $v \in B_0$. Since $w \in \ker \alpha$, we have that for $i \in \{1, \ldots, k\}$,

$$\alpha\beta_i\alpha$$

$$= \begin{pmatrix} u_1 & \cdots & u_k & v \\ u_1 & \cdots & u_k & 0 \end{pmatrix}_{v \in B_0} \begin{pmatrix} u_1 & \cdots & u_{i-1} & u_i & u_{i+1} & \cdots & u_k & v \\ u_1 & \cdots & u_{i-1} & w & u_{i+1} & \cdots & u_k & v\varphi_i \end{pmatrix}_{v \in B_0} \begin{pmatrix} u_1 & \cdots & u_k & v \\ u_1 & \cdots & u_k & 0 \end{pmatrix}_{v \in B_0}$$

$$= \begin{pmatrix} u_1 & \cdots & u_{i-1} & u_i & u_{i+1} & \cdots & u_k & v \\ u_1 & \cdots & u_{i-1} & 0 & u_{i+1} & \cdots & u_k & 0 \end{pmatrix}_{v \in B_0}.$$

Then

$$(\alpha\beta_1\alpha)(\alpha\beta_2\alpha) = \begin{pmatrix} u_1 & u_2 & \cdots & u_k & v \\ 0 & u_2 & \cdots & u_k & 0 \end{pmatrix}_{v \in B_0} \begin{pmatrix} u_1 & u_2 & u_3 & \cdots & u_k & v \\ u_1 & 0 & u_3 & \cdots & u_k & 0 \end{pmatrix}_{v \in B_0}$$

$$= \begin{pmatrix} u_1 & u_2 & u_3 & \cdots & u_k & v \\ 0 & 0 & u_3 & \cdots & u_k & 0 \end{pmatrix}_{v \in B_0}.$$

The following result holds by induction :

$$(\alpha\beta_1\alpha)(\alpha\beta_2\alpha)\ldots(\alpha\beta_k\alpha) = \begin{pmatrix} u_1 & \cdots & u_k & v \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{v \in B_0}.$$

Since $\alpha GL(V)\alpha$ is a subsemigroup of $L(V)$, it follows that the zero map 0 on V belongs to $\alpha GL(V)\alpha$. Thus $\alpha\gamma\alpha = 0$ for some $\gamma \in GL(V)$. Consequently,

$$(\operatorname{ran}\alpha)\gamma = (V\alpha)\gamma \subseteq \ker \alpha.$$

Since $\gamma \in GL(V)$, $\dim(\ker \alpha) \geq \dim(\operatorname{ran}\alpha)\gamma = \dim(\operatorname{ran}\alpha)$.

The proof is thereby completed. □

Theorem 4.1.6. *Let V be finite-dimensional and $\alpha \in E(L(V))$. Then the local subset $\alpha GL(V)\alpha$ of $L(V)$ is a local subsemigroup of $L(V)$ if and only if either*

(i) $\alpha = 1_V$ *or*

(ii) $\dim(\ker\alpha) \geq \dim(\operatorname{ran}\alpha)$.

Moreover,

$$\alpha GL(V)\alpha \begin{cases} = GL(V) & \textit{if } \alpha \textit{ satisfies (i)}, \\ \cong L(\operatorname{ran}\alpha) & \textit{if } \alpha \textit{ satisfies (ii)}. \end{cases}$$

Proof. Assume that $\alpha GL(V)\alpha$ is a local subsemigroup of $L(V)$. If $\alpha \neq 1_V$, then by Lemma 4.1.5, $\dim(\ker\alpha) \geq \dim(\operatorname{ran}\alpha)$.

Conversely, assume that α satisfies (i) or (ii). If α satisfies (i), then $\alpha GL(V)\alpha = 1_V GL(V) 1_V = GL(V)$. If α satisfies (ii), then by Lemma 4.1.4, $\alpha GL(V)\alpha = \alpha L(\operatorname{ran}\alpha)$ and by Lemma 4.1.2, it is a subsemigroup of $L(V)$ which is isomorphic to $L(\operatorname{ran}\alpha)$.

Therefore the theorem is proved. $\square$

Since for $\alpha \in L(V)$, $\dim V = \dim(\ker\alpha) + \dim(\operatorname{ran}\alpha)$, it follows that if $\dim V < \infty$, then

$$\begin{aligned} \dim(\operatorname{ran}\alpha) \leq \dim(\ker\alpha) &\Leftrightarrow \dim(\operatorname{ran}\alpha) \leq \dim V - \dim(\operatorname{ran}\alpha) \\ &\Leftrightarrow 2\dim(\operatorname{ran}\alpha) \leq \dim V. \end{aligned}$$

Hence Theorem 4.1.6 can be restated as follows:

Theorem 4.1.7. *Let V be finite-dimensional and $\alpha \in E(L(V))$. Then the local subset $\alpha GL(V)\alpha$ of $L(V)$ is a local subsemigroup of $L(V)$ if and only if either*

(i) $\alpha = 1_V$ *or*

(ii) $2\dim(\operatorname{ran}\alpha) \leq \dim V$.

Moreover,

$$\alpha GL(V)\alpha \begin{cases} = GL(V) & \textit{if } \alpha \textit{ satisfies (i)}, \\ \cong L(\operatorname{ran}\alpha) & \textit{if } \alpha \textit{ satisfies (ii)}. \end{cases}$$

As a consequence of Theorem 4.1.7, we have

Corollary 4.1.8. *If* $\dim V \leq 2$*, then for every* $\alpha \in E(L(V))$*,* $\alpha GL(V)\alpha$ *is a local subsemigroup of* $L(V)$*.*

Proof. We have $\alpha_{|\operatorname{ran}\alpha} = 1_{\operatorname{ran}\alpha}$ and $\dim(\operatorname{ran}\alpha) = 0, 1$ or 2. If $\dim(\operatorname{ran}\alpha) = 2$, then $\operatorname{ran}\alpha = V$, so $\alpha = 1_V$. If $\dim(\operatorname{ran}\alpha) = 0$ or 1, then $2\dim(\operatorname{ran}\alpha) \leq 2 = \dim V$. By Theorem 4.1.7, $\alpha GL(V)\alpha$ is a local subsemigroup of $L(V)$. □

Theorem 4.1.9. *Let* V *be finite-dimensional and* $\alpha \in E(L(V))$*. If* $\alpha GL(V)\alpha$ *is a local subsemigroup of* $L(V)$*, then* $\alpha GL(V)\alpha$ *is a regular semigroup.*

Proof. Since for a subspace U of V, $GL(U)$ is a group and $L(U)$ is a regular semigroup, the result follows from Theorem 4.1.7. □

Example 4.1.10. Let F be a field. Consider the vector space F^5 over F with the usual addition and scalar multiplication. Define $\alpha, \beta : F^5 \to F^5$ by

$$(x, y, z, w, t)\alpha = (x, 0, z, 0, t)$$
$$(x, y, z, w, t)\beta = (x, y, 0, 0, 0)$$
$$\text{for all } x, y, z, w, t \in F.$$

Then $\alpha, \beta \in E(L(F^5))$, $\dim(\operatorname{ran}\alpha) = 3$ and $\dim(\operatorname{ran}\beta) = 2$. Since $2\dim(\operatorname{ran}\alpha) = 6 > 5 = \dim F^5$ and $2\dim(\operatorname{ran}\beta) = 4 < 5 = \dim F^5$, by Theorem 4.1.7, $\alpha GL(F^5)\alpha$ is not a local subsemigroup of $L(F^5)$ but $\beta GL(F^5)\beta$ is a local subsemigroup of $GL(F^5)$ and

$$\beta GL(F^5)\beta \cong L(\{(x, y, 0, 0, 0) \mid x, y \in F\}) \cong L(F^2)$$

since $\{(x, y, 0, 0, 0) \mid x, y \in F\}$ and F^2 are vector space isomorphic (see Proposition 2.11).

4.2 The Local Subsemigroups $AG_n(F)A$ of $M_n(F)$

By making use of Theorem 4.1.7 and a relationship between $L(V)$ and $M_n(F)$ if $\dim V = n$, we shall show that for $A \in E(M_n(F))$, $AG_n(F)A$ is a local subsemigroup of $M_n(F)$ if and only if either

(i) $A = I_n$, the identity $n \times n$ matrix over F or

(ii) $2 \operatorname{rank} A \leq n$.

In addition, we show that if $A \neq 0$, then the local subsemigroup $AG_n(F)A$ is isomorphic to $M_k(F)$ where $k = \operatorname{rank} A$ which implies that it is a regular semigroup.

Recall that if $\dim V = n$, then there is a semigroup isomorphism $\theta : L(V) \to M_n(F)$ such that for all $\alpha \in L(V)$, $\operatorname{rank}(\alpha\theta) = \dim(\operatorname{ran} \alpha)$. Note that $(E(L(V)))\theta = E(M_n(F))$.

Theorem 4.2.1. *For $A \in E(M_n(F))$, the local subset $AG_n(F)A$ of $M_n(F)$ is a local subsemigroup of $M_n(F)$ if and only if either*

(i) *$A = I_n$ or*

(ii) *$2 \operatorname{rank} A \leq n$.*

In addition,

$$AG_n(F)A \begin{cases} = G_n(F) & \text{if } A \text{ satisfies (i)}, \\ = \{0\} & \text{if } A = 0, \\ \cong M_k(F) & \text{if } A \neq 0,\ A \text{ satisfies (ii) and } \operatorname{rank} A = k. \end{cases}$$

Proof. Let V be a vector space over F of dimension n and $\theta : L(V) \to M_n(F)$ a semigroup isomorphism such that

$$\operatorname{rank}(\alpha\theta) = \dim(\operatorname{ran} \alpha) \qquad \text{for all } \alpha \in L(V).$$

First, assume that $AG_n(F)A$ is a local subsemigroup of $M_n(F)$. Then

$$(AG_n(F)A)\theta^{-1} = (A\theta^{-1})GL(V)(A\theta^{-1})$$

is a local subsemigroup of $L(V)$. By Theorem 4.1.7, either

(i) $A\theta^{-1} = 1_V$ or

(ii) $2\dim(\mathrm{ran}(A\theta^{-1})) \leq \dim V = n$.

But since $1_V\theta = I_n$ and

$$\dim(\mathrm{ran}(A\theta^{-1})) = \mathrm{rank}((A\theta^{-1})\theta) = \mathrm{rank}\, A,$$

it follows that either

(i) $A = I_n$ or

(ii) $2\,\mathrm{rank}\, A \leq n$.

Conversely, assume that A satisfies (i) or (ii). If A satisfies (i), then $AG_n(F)A = G_n(F)$. Assume that A satisfies (ii). If $A = 0$, then $AG_nA = \{0\}$. Assume that $A \neq 0$. Let $\mathrm{rank}\, A = k$. We have $A\theta^{-1} \in L(V)$ and $\dim V = n \geq 2\,\mathrm{rank}\, A = 2\,\mathrm{rank}((A\theta^{-1})\theta) = 2\dim(\mathrm{ran}(A\theta^{-1}))$. By Theorem 4.1.7, $(A\theta^{-1})GL(V)(A\theta^{-1})$ is a local subsemigroup of $L(V)$ and

$$(A\theta^{-1})GL(V)(A\theta^{-1}) \cong L(\mathrm{ran}(A\theta^{-1})).$$

But $((A\theta^{-1})GL(V)(A\theta^{-1}))\theta = AG_n(F)A$, so

$$AG_n(F)A \cong L(\mathrm{ran}(A\theta^{-1})).$$

Since $\dim(\mathrm{ran}(A\theta^{-1})) = \mathrm{rank}((A\theta^{-1})\theta) = \mathrm{rank}\, A = k$, it follows that

$$L(\mathrm{ran}(A\theta^{-1})) \cong M_k(F).$$

This proves the theorem. □

Corollary 4.2.2. *For every $A \in E(M_2(F))$, $AG_2(F)A$ is a local subsemigroup of $M_2(F)$.*

Proof. If $\mathrm{rank}\, A \leq 1$, then $2\,\mathrm{rank}\, A \leq 2$, so by Theorem 4.2.1, $AG_2(F)A$ is a local subsemigroup of $M_2(F)$.

Assume that $\operatorname{rank} A = 2$. Then $A \in E(G_2(F))$ which implies that $A = I_2$. Thus $AG_2(F)A = I_2G_2(F)I_2 = G_2(F)$. □

Theorem 4.2.3. *Let $A \in E(M_n(F))$. If $AG_n(F)A$ is a local subsemigroup of $M_n(F)$, then $AG_n(F)A$ is a regular semigroup.*

Proof. Since for every positive integer m, $G_m(F)$ is a group and $M_m(F)$ is a regular semigroup, the result follows from Theorem 4.2.1. □

Example 4.2.4. Let F be a field and $x, y, z \in F$. Define

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $A^2 = A$, $B^2 = B$, $\operatorname{rank} A = 2$ and $\operatorname{rank} B = 3$. Then $2 \operatorname{rank} A = 4$ and $2 \operatorname{rank} B = 6 > 4$. By Theorem 4.2.1, $AG_4(F)A$ is a local subsemigroup of $M_4(F)$ but $BG_4(F)B$ is not. Moreover, $AG_4(F)A \cong M_2(F)$.

Finally, by making use of Proposition 2.12, we provide some explicit form of the local subsemigroup $AG_n(F)A$ of $M_n(F)$.

Lemma 4.2.5. *If k is a positive integer such that $2k \leq n$, then $D_n^{(k)}G_n(F)D_n^{(k)}$ is the set of all $n \times n$ matrices over F of the form*

$$\begin{bmatrix} x_{11} & \cdots & x_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{1}$$

Proof. We can see that for any matrix $A \in M_n(F)$,

$$D_n^{(k)} A D_n^{(k)} = \begin{bmatrix} A_{11} & \cdots & A_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It follows that $D_n^{(k)} G_n(F) D_n^{(k)}$ is a subset of the set of all $n \times n$ matrices over F of the form (1).

Conversely, let $B \in M_n(F)$ be of the form (1), that is,

$$B = \begin{bmatrix} B_{11} & \cdots & B_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ B_{k1} & \cdots & B_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $2k \leq n$, it implies that

$$B_{1n} = B_{2,n-1} = \cdots = B_{k,n-(k-1)} = B_{k+1,n-k} = \cdots = B_{n1} = 0.$$

Define $\bar{B} \in M_n(F)$ by

$$\bar{B}_{ij} = \begin{cases} B_{i,j} & \text{if } i, j \in \{1, \ldots, k\}, \\ 1 & \text{if } i + j = n + 1, \\ 0 & \text{otherwise,} \end{cases}$$

that is,

$$\bar{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots & B_{1k} & 0 & \cdots & 0 & 0 & 1 \\ B_{21} & B_{22} & B_{23} & \cdots & B_{2k} & 0 & \cdots & 0 & 1 & 0 \\ B_{31} & B_{32} & B_{33} & \cdots & B_{3k} & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ B_{k1} & B_{k2} & B_{k3} & \cdots & B_{kk} & * & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & * & \bullet & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

where

$$* := 1 \text{ and } \bullet := 0 \quad \text{if} \quad 2k = n,$$
$$* := 0 \text{ and } \bullet := 1 \quad \text{if} \quad 2k = n-1,$$
$$* := 0 \text{ and } \bullet := 0 \quad \text{if} \quad 2k < n-1.$$

It is clearly seen that $\bar{B}$ is row-equivalent to I_n. It follows that $\bar{B} \in G_n(F)$. Moreover, $D_n^{(k)}\bar{B}D_n^{(k)} = B$, so $B \in D_n^{(k)}G_n(F)D_n^{(k)}$.

Hence the proof is complete. □

Theorem 4.2.6. *Let $A = C^{-1}D_n^{(k)}C \in E(M_n(F))$ where $C \in G_n(F)$* (see Proposition 2.12)*. Then the local subset $AG_n(F)A$ of $M_n(F)$ is a local subsemigroup of $M_n(F)$ if and only if either $k = n$ or $2k \le n$. Moreover,*

$k = n \Rightarrow AG_n(F)A = G_n(F)$,

$k = 0 \Rightarrow AG_n(F)A = \{0\}$,

$$0 < 2k \le n \Rightarrow AG_n(F)A = C^{-1}\left\{ \begin{bmatrix} x_{11} & \cdots & x_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \;\middle|\; x_{i,j} \in F \text{ for } i,j \in \{1,\ldots,k\} \right\} C.$$

Proof. Since $A = C^{-1}D_n^{(k)}C$, $\text{rank}(A) = k$.

Assume that $AG_n(F)A$ is a local subsemigroup of $M_n(F)$. By Theorem 4.2.1, either $A = I_n$ or $2k \le n$. If $A = I_n$, then $k = n$, so $AG_n(F)A = G_n(F)$. If $k = 0$, then $AG_nA = \{0\}$.

Assume that $k > 0$ and $2k \le n$. By Lemma 4.2.5,

$$D_n^{(k)}G_n(F)D_n^{(k)} = \left\{ \begin{bmatrix} x_{11} & \cdots & x_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \,\middle|\, x_{i,j} \in F \text{ for } i,j \in \{1,\dots,k\} \right\}.$$

Hence

$$\begin{aligned} AG_n(F)A &= (C^{-1}D_n^{(k)}C)G_n(F)(C^{-1}D_n^{(k)}C) \\ &= C^{-1}D_n^{(k)}(CG_n(F)C^{-1})D_n^{(k)}C \\ &= C^{-1}D_n^{(k)}G_n(F)D_n^{(k)}C \qquad \text{since } C \in G_n(F) \\ &= C^{-1}(D_n^{(k)}G_n(F)D_n^{(k)})C \\ &= C^{-1}\left\{ \begin{bmatrix} x_{11} & \cdots & x_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \,\middle|\, x_{i,j} \in F \text{ for } i,j \in \{1,\dots,k\} \right\} C. \end{aligned}$$

For the converse, assume that either $k = n$ or $2k \le n$. If $k = n$, then $A = I_n$, and if $2k \le n$, then $2\,\text{rank}(A) \le n$. It follows from Theorem 4.2.1 that $AG_n(F)A$ is a local subsemigroup of $M_n(F)$.

The proof is thereby completed. □

Remark 4.2.7. Let A be as in the assumption of Theorem 4.2.6 and $0 < 2k \le n$. By Theorem 4.2.1, we have that $AG_n(F)A \cong M_k(F)$. This can be seen from

Theorem 4.2.6 since

$$\theta : M_k(F) \to C^{-1}\left\{\begin{bmatrix} x_{11} & \cdots & x_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \mid x_{i,j} \in F \text{ for } i,j \in \{1,\ldots,k\}\right\} C$$

defined by

$$A\theta = C^{-1}\begin{bmatrix} A_{11} & \cdots & A_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} C$$

is an isomorphism. See the proof of Proposition 2.11.

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