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CHI-SQUARE APPROXIMATION OF SQUARED SUMS OF INDEPENDENT RANDOM VARIABLES

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ในงานนี้เรามีวัตถุประสงค์สองประการ คือ ประการแรก เราหาคำตอบของสมการสไตน์สำหรับฟังก์ชันการแจกแจงไคสแควร์ที่มีองศาเสรีเป็นหนึ่ง และสมบัติของคำตอบดังกล่าว

วัตถุประสงค์ที่สอง เราให้ขอบเขตการประมาณค่าฟังก์ชันการแจกแจงของกำลังสองของผลรวมของตัวแปรสุ่มที่อิสระต่อกัน ภายใต้เงื่อนไขว่าโมเมนต์อันดับสามหรือโมเมนต์อันดับสองหาค่าได้ โดยใช้ความสัมพันธ์ระหว่างตัวแปรสุ่มที่มีการแจกแจงไคสแควร์ที่มีองศาเสรีเป็นหนึ่งกับตัวแปรสุ่มที่มีการแจกแจงปกติมาตรฐาน

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In this work, we have 2 objectives. First, we find a solution of the Stein's equation for chi-square distribution with degree of freedom 1 and its properties.

Second, we give bounds on chi-square approximation of the squared sums of independent random variables under the existence of the second or the third moments by using the relation between the chi-square random variable with degree of freedom 1 and the standard normal random variable.

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CONTENTS

CHAPTER I

INTRODUCTION

In 1994, Luk ([6]) gave the Stein's equation,

$$wf''(w) + \frac{1}{2}(1-w)f'(w) = h(w) - \chi_1^2 h \quad \text{for} \quad w \geq 0 \tag{1.1}$$

where $\chi_1^2 h = \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{t}{2}} h(t)dt$ and h is absolutely bounded and the first 3 derivatives of h are bounded. The solution of (1.1) for h is

$$\begin{aligned} f_h(w) = &- \int_0^\infty [\frac{e^{-(\frac{e^{\frac{-t}{2}}}{1-e^{\frac{-t}{2}}})w} we^{\frac{-t}{2}}}{\sqrt{2}(1-e^{\frac{-t}{2}})} \sum_{i=1}^\infty \frac{(\frac{1}{2})^i}{i!\Gamma(i+1)} \int_0^{\frac{w}{1-e^{\frac{-t}{2}}}} e^{(i+\frac{1}{2})u} u^{i-\frac{1}{2}} du \\ &- \frac{1}{\sqrt{2\pi}} \int_0^\infty s^{-\frac{1}{2}} e^{-\frac{s}{2}} h(s)ds]dt, \end{aligned}$$

([6], pp. 13). Later, Reinert([9]) gave a technique to simplify equation (1.1), by letting

$$g(w) = \frac{wf'(w^2)}{2}$$

and (1.1) become,

$$g'(w) - wg(w) = h(w^2) - \chi_1^2 h, \qquad w \geq 0. \tag{1.2}$$

If we choose the test function h in (1.2) to be $I_z : [0, \infty) \to \mathbb{R}$ defined by

$$I_z(w) = \begin{cases} 1 & \text{if} \quad w \leq z, \\ 0 & \text{if} \quad w > z, \end{cases}$$

for a fixed $z \geq 0$, then (1.2) becomes

$$g'(w) - wg(w) = I_z(w^2) - \chi_1^2(z), \qquad w \geq 0 \tag{1.3}$$

where $\chi_1^2(z) = \frac{1}{\sqrt{2\pi}} \int_0^z t^{-\frac{1}{2}} e^{-\frac{t}{2}} dt.$

In Chapter III, we find a solution of a Stein's equation (1.3) and its properties by using the idea of Chen and Shao ([2]).

In the final chapter, we will give bounds on chi-square approximation. Let $X_1, X_2, ..., X_n$ be independent random variables with zero mean and finite variance and $W_n = \sum_{i=1}^{n} X_i$. Assume that $VarW_n = 1$. It is well-known that the distribution of W_n can be approximated by the standard normal distribution Φ, where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$ (see [4], pp.261-268 for example).
Note that $\Phi^2 \overset{d}{=} \chi_1^2$ where χ_1^2 is the chi-square distribution with degree of freedom 1.

Reinert ([9]) used the Stein's method and Taylor expansion to find a uniform bound in chi-square approximation. Her result is Theorem 1.1.

Theorem 1.1. *Let $Y_1, Y_2, ..., Y_n$ be independent random variables with zero mean, variance one and $E|Y_i|^8 < \infty$ for $i = 1, 2, ..., n$. Define $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$. Let $h : \mathbb{R} \to \mathbb{R}$ be absolutely bounded and the first 3 derivatives of h are bounded and $f_h : \mathbb{R} \to \mathbb{R}$ a solution of (1.2). Then a uniform bound in Chi-square approximation is of the form*

$$\left|Eh(S_n^2) - \chi_1^2 h\right| \leq \frac{c(f_h)}{n}$$

where $c(f_h)$ is a constant depending on f_h and on the distribution of the Y_i but it does not depend on n.

In Theorem 1.1, Reinert derived an error estimation for the approximation of the distribution of S_n^2 by a chi-square distribution with degree of freedom 1, which is of the form

$$|Eh(S_n^2) - \chi_1^2 h|.$$

In order to bound

$$|P(W_n^2 \leq z) - \chi_1^2(z)|,$$

we have to choose the function $h = I_z$. But I_z is not continuous. Hence we can not apply Theorem 1.1 to bound $|P(W_n^2 \leq z) - \chi_1^2(z)|$.

In this part, we give uniform and non-uniform bounds on chi-square approximation to the distribution of W_n^2 under the existence of the second and the third moments. To do this, we use a relation between the chi-square random variable with degree of freedom 1 and the standard normal random variable. This is our results.

Theorem 1.2. *(uniform bound) Let $X_1, X_2, ..., X_n$ be independent random variables such that $EX_i = 0$ and $E|X_i|^3 < \infty$ for $i = 1, 2, ..., n$. Assume that $\sum_{i=1}^{n} E{X_i}^2 = 1$. Then*

$$\sup_{z \geq 0} |P(W_n^2 \leq z) - \chi_1^2(z)| \leq 1.583 \sum_{i=1}^{n} E|X_i|^3.$$

Theorem 1.3. *(non-uniform bound) Under the assumption of Theorem 1.2 and $z \geq 0$, we have*

$$|P(W_n^2 \leq z) - \chi_1^2(z)| \leq \frac{63.87}{1 + z^{\frac{3}{2}}} \sum_{i=1}^{n} E|X_i|^3.$$

Theorem 1.4. *(uniform bound) Let $X_1, X_2, ..., X_n$ be independent random variables such that $EX_i = 0$ and $E|X_i|^2 < \infty$ for $i = 1, 2, ..., n$. Assume that $\sum_{i=1}^{n} E{X_i}^2 = 1$. Then*

$$\sup_{z \geq 0} |P(W_n^2 \leq z) - \chi_1^2(z)| \leq 8.2 \left\{ \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1) + \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1) \right\}.$$

Theorem 1.5. *(non-uniform bound) Under the assumption of Theorem 1.4. For* $z \geq 0$*, there exists an absolute constant* C

$$|P(W_n^2 \leq z) - \chi_1^2(z)| \leq C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + \sqrt{z})}{1+z} + \frac{E|X_i|^3 I(|X_i| < 1 + \sqrt{z})}{1 + z^{\frac{3}{2}}} \right\},$$

where

$$C = \begin{cases} 26.22 & if \quad 0 \leq z < 1.69, \\ 57.08 & if \quad 1.69 \leq z < 4, \\ 92.64 & if \quad 4 \leq z < 9, \\ 122.8 & if \quad 9 \leq z < 63.6804, \\ 80.24 & if \quad 63.6804 \leq z < 196, \\ 78.78 & if \quad z \geq 196. \end{cases}$$

We organize our thesis as follows. In Chapter II we give some basic concepts in probability theory. A solution and its properties of the Stein's equation for chi-square distribution with degree of freedom 1 is in Chapter III. Finally, we give uniform and non-uniform bounds on chi-square approximation in Chapter IV.

CHAPTER II

PRELIMINARIES

In this chapter, we give basic concepts in probability which will be used in our work.

2.1 Probability Space and Random Variables

Let Ω be a nonempty set and $\mathcal{F}$ be a σ-algebra of subsets of Ω .
Let $P : \mathcal{F} \to [0, 1]$ be a measure such that $P(\Omega) = 1$. Then $(\Omega, \mathcal{F}, P)$ is called a **probability space** and P, **a probability measure.** The set Ω is the **sample space** and the elements of $\mathcal{F}$ are called **events.** For any event A, the value $P(A)$ is called the **probability of** A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is said to be a **random variable** if for every Borel set B in $\mathbb{R}$,

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

We shall usually use the notation $P(X \in B)$ in stead of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

Let X be a random variable with the distribution function F. X is said to be a **discrete random variable** if the image of X is countable and X is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability density function** of X.

A sequence of events $(E_n)_{n\geq 1}$ is said to be *an increasing sequence* if

$$E_1 \subseteq E_2 \subseteq ... \subseteq E_n \subseteq E_{n+1} \subseteq ...$$

where as it is said to be *an decreasing sequence* if

$$E_1 \supseteq E_2 \supseteq ... \supseteq E_n \supseteq E_{n+1} \supseteq ...$$

Theorem 2.1. *Let $(E_n)_{n\geq 1}$ be a sequence of events. Then*

1. *If (E_n) is increasing, then $\lim_{n\to\infty} P(E_n) = P(\bigcup_{n=1}^{\infty} E_n)$.*
2. *If (E_n) is decreasing, then $\lim_{n\to\infty} P(E_n) = P(\bigcap_{n=1}^{\infty} E_n)$.*

Now we will give some examples of random variables. We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim \mathcal{N}(\mu, \sigma^2)$, if its probability density function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big).$$

Moreover, if $X \sim \mathcal{N}(0,1)$ then X is said to be the **standard normal** random variable.

A random variable X is said to have a **gamma distribution with parameters α and β** (denoted by $X \sim Gam(\alpha, \beta)$),$\alpha > 0$ and $\beta > 0$, if its density function is given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} & \text{if} \quad x > 0, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

In special case, if $X \sim Gam(\frac{n}{2}, 2)$ for some $n \in \mathbb{N}$, then a random variable X is said to be a **chi-square random variable with degree of freedom** n, denoted by $X \sim \chi^2_n$.

Let $X_1, X_2, ..., X_n$ be independent random variables and $X_i \sim \mathcal{N}(0, 1)$ for $i = 1, 2, ..., n$. It is well-know that $X_1^2 + \cdots + X_n^2 \sim \chi^2_n$.

2.2 Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_\alpha$ be sub σ-algebras of $\mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J = \{j_1, j_2, ..., j_k\}$ of Λ and $A_m \in \mathcal{F}_{j_m}$ for $m = 1, ..., k$,

$$P(\bigcap_{m=1}^{k} A_m) = \prod_{m=1}^{k} P(A_m).$$

A set of random variables $\{X_\alpha | \ \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha) | \ \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \sigma(\{X^{-1}(B) \mid B \text{ is a Borel subset of } \mathbb{R}\})$.

We say that $X_1, X_2, ..., X_n$ are **independent** if $\{X_1, X_2, ..., X_n\}$ is **independent**.

Theorem 2.2. *Random variables $X_1, X_2, ..., X_n$ are* **independent** *if and only if for any Borel sets $B_1, B_2, ..., B_n$ we have*

$$P(\bigcap_{i=1}^{n}\{X_i \in B_i\}) = \prod_{i=1}^{n} P(X_i \in B_i).$$

2.3 Expectation and Variance

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$.
If $\int_{\Omega} |X| dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_{\Omega} X dP.$$

Proposition 2.3. *Let X be a random variable and $E|X| < \infty$.*

1. *If X is a discrete random variable, then $E(X) = \sum_{x \in ImX} xP(X = x)$.*
2. *If X is a continuous random variable with density function f, then*

$$E(X) = \int_{\mathbb{R}} xf(x)dx.$$

Let X be a random variable with $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the k-**th moment** of X about the origin and $E[X - E(X)]^k$ is called the k-**th moment** of X about the mean.

We call the second moment of X about the mean, the **variance** of X and denoted by $Var(X)$. Then

$$Var(X) = E[X - E(X)]^2.$$

Note that

1. $Var(X) = E(X^2) - E^2(X)$.

2. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $E(X) = \mu$ and $Var(X) = \sigma^2$.

3. If $X \sim \chi^2_n$, then $E(X) = n$ and $Var(X) = 2n$.

Proposition 2.4. *If $X_1, X_2, ..., X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, ..., n$, then*

1. $E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2)\cdots E(X_n)$,

2. $Var(a_1X_1+a_2X_2+\cdots+a_nX_n) = a_1^2Var(X_1)+a_2^2Var(X_2)+\cdots+a_n^2Var(X_n)$ *for any real numbers $a_1, a_2 \ldots, a_n$.*

CHAPTER III

STEIN'S EQUATION FOR CHI-SQUARE DISTRIBUTION WITH DEGREE OF FREEDOM 1

In this chapter, we give a solution of a Stein's equation for chi-square distribution with degree of freedom 1 and its properties.

In 1972, Stein ([10]) gave a new method in order to approximate the distribution of the sum of dependent random variables to the standard normal distribution Φ, where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt.$$

His method, which is called "Stein's method", was free from Fourier transform and relied instead on the elementary differential equation. Stein's method has been widely applied in the area of normal approximation. The method is as follows: Let Z be the standard normal distributed random variable and let C_{bd} be the set of continuous and piecewise continuously differentiable functions on $\mathbb{R}$ to itself with $E|f'(Z)| < \infty$ for all $f \in C_{bd}$. For $f \in C_{bd}$ and any real valued function h with $E|h(Z)| < \infty$, the Stein's equation for normal distribution is

$$f'(w) - wf(w) = h(w) - Eh(Z).$$

If $h = I_z$, where $z \in \mathbb{R}$ and I_z is an indicator defined by

$$I_z(w) = \begin{cases} 1 & \text{if} \quad w \leq z, \\ 0 & \text{if} \quad w > z, \end{cases}$$

then the Stein's equation becomes

$$f'(w) - wf(w) = I_z(w) - \Phi(z). \tag{3.1}$$

Hence for any random variable W,

$$E(f'(W) - Wf(W)) = P(W \leq z) - \Phi(z).$$

However, Stein's method can be applied to other distributions. For example, Chen ([1]), gave a Stein's equation for Poisson distribution with parameter λ,

$$\lambda f(w+1) - wf(w) = I_z(w) - Poi_\lambda(z), \quad w \in \mathbb{Z}^+ \cup \{0\}$$

where Poi_λ is a Poisson distribution with parameter λ, i.e.,

$$Poi_\lambda(z) = \sum_{k=0}^{z} \frac{e^{-\lambda}\lambda^k}{k!}, \qquad z \in \mathbb{Z}^+ \cup \{0\}.$$

Other examples are binomial distribution ([11]), gamma distribution ([6]) and geometric distribution ([7]).

An important question is that whether the Stein's equation is unique or not. Chatterjee ([3]) et al. gave two versions of Stein's equation for exponential distribution, $Exp(z) = 1 - e^{-z}$, as follows:

$$f'(w) - f(w) = I_z(w) - Exp(z), \qquad w \geq 0$$

and

$$wf''(w) - (1-w)f'(w) = I_z(w) - Exp(z), \quad w \geq 0.$$

We see that the Stein's equation may be not unique. Normally, a useful equation is the one that its solution and the derivative of its solution are bounded. For the chi-square distribution with degree of freedom 1, Luk ([6]) gave the Stein's equation,

$$wf''(w) + \frac{1}{2}(1-w)f'(w) = h(w) - \chi_1^2 h, \quad w \geq 0, \tag{3.2}$$

where $\chi_1^2 h = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{t}{2}} h(t)dt$ and h is absolutely bounded and the first 3 derivatives of h are bounded.

The solution of (3.2) for h is

$$f_h(w) = -\int_0^\infty [\frac{e^{-(\frac{e^{\frac{-t}{2}}}{1-e^{\frac{-t}{2}}})w} we^{\frac{-t}{2}}}{\sqrt{2}(1-e^{\frac{-t}{2}})} \sum_{i=1}^{\infty} \frac{(\frac{1}{2})^i}{i!\Gamma(i+1)} \int_0^{\frac{w}{1-e^{\frac{-t}{2}}}} e^{(i+\frac{1}{2})u} u^{i-\frac{1}{2}} du$$
$$- \frac{1}{\sqrt{2\pi}} \int_0^\infty s^{-\frac{1}{2}} e^{-\frac{s}{2}} h(s) ds] dt,$$

([6], pp. 13). Later, Reinert([9]) gave a technique to simplify equation (3.2), by letting

$$g(w) = \frac{wf'(w^2)}{2}$$

and showed that (3.2) become

$$g'(w) - wg(w) = h(w^2) - \chi_1^2 h, \quad w \geq 0. \tag{3.3}$$

In this work, we use the idea of (3.3) by choosing the test function $h = I_z$ where $z \geq 0$. Then we have

$$g'(w) - wg(w) = I_z(w^2) - \chi_1^2(z), \quad w, z \geq 0. \tag{3.4}$$

Next, we will find a solution of (3.4) and give its properties by using the ideas of Chen and Shao ([2]).

Proposition 3.1. *A solution $g_z : [0, \infty) \to \mathbb{R}$ of Stein's equation (3.4) is of the form*

$$g_z(w) = \begin{cases} \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))e^{\frac{w^2}{2}}\chi_1^2(w^2) & \text{if} \quad w \leq \sqrt{z}, \\ \frac{\sqrt{2\pi}}{2}\chi_1^2(z)e^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) & \text{if} \quad w > \sqrt{z}. \end{cases} \tag{3.5}$$

Proof. Observe that (3.4) is a linear first order differential equation of the form

$$g'(t) + p(t)g(t) = q(t), \ t \geq 0, \tag{3.6}$$

where $p(t) = -t$ and $q(t) = I_z(t^2) - \chi_1^2(z)$. We use integrating factor

$$\mu = e^{\int -tdt} = e^{-\frac{t^2}{2}}$$

to find a solution. Multiply equation (3.6) by μ. Then we get

$$\frac{d}{dt}[e^{-\frac{t^2}{2}}g(t)] = e^{-\frac{t^2}{2}}[g'(t) - tg(t)] = e^{-\frac{t^2}{2}}[I_z(t^2) - \chi_1^2(z)].$$

So

$$\int_0^w d(e^{-\frac{t^2}{2}}g(t)) = \int_0^w e^{-\frac{t^2}{2}}[I_z(t^2) - \chi_1^2(z)]dt,$$

and then

$$e^{-\frac{w^2}{2}}g(w) = \int_0^w e^{-\frac{t^2}{2}}[I_z(t^2) - \chi_1^2(z)]dt,$$

and we have

$$g(w) = e^{\frac{w^2}{2}}\int_0^w e^{-\frac{t^2}{2}}[I_z(t^2) - \chi_1^2(z)]dt.$$

We claim that $g_z : [0, \infty) \to \mathbb{R}$ which is defined by

$$g_z(w) = e^{\frac{w^2}{2}}\int_0^w e^{-\frac{t^2}{2}}[I_z(t^2) - \chi_1^2(z)]dt, \ \text{ for } \ w \geq 0,$$

is a solution of the Stein's equation (3.4).

Note that

$$\chi_1^2(w^2) = \frac{1}{\sqrt{2\pi}}\int_0^{w^2} t^{-\frac{1}{2}}e^{-\frac{t}{2}}dt = \frac{2}{\sqrt{2\pi}}\int_0^w e^{-\frac{u^2}{2}}du. \tag{3.7}$$

If $w \leq \sqrt{z}$, then (3.7) implies

$$\begin{aligned}
g_z(w) &= e^{\frac{w^2}{2}} \int_0^w e^{-\frac{t^2}{2}} [I_z(t^2) - \chi_1^2(z)] dt \\
&= e^{\frac{w^2}{2}} \int_0^w e^{-\frac{t^2}{2}} (1 - \chi_1^2(z)) dt \\
&= (1 - \chi_1^2(z)) e^{\frac{w^2}{2}} \int_0^w e^{-\frac{t^2}{2}} dt \\
&= \frac{\sqrt{2\pi}}{2} (1 - \chi_1^2(z)) e^{\frac{w^2}{2}} \chi_1^2(w^2).
\end{aligned}$$

In the case that $w > \sqrt{z}$, we use (3.7) to get that

$$\begin{aligned}
g_z(w) &= e^{\frac{w^2}{2}} \int_0^w e^{-\frac{t^2}{2}} [I_z(t^2) - \chi_1^2(z)] dt \\
&= e^{\frac{w^2}{2}} \int_0^{\sqrt{z}} e^{-\frac{t^2}{2}} [I_z(t^2) - \chi_1^2(z)] dt + e^{\frac{w^2}{2}} \int_{\sqrt{z}}^w e^{-\frac{t^2}{2}} [I_z(t^2) - \chi_1^2(z)] dt \\
&= (1 - \chi_1^2(z)) e^{\frac{w^2}{2}} \int_0^{\sqrt{z}} e^{-\frac{t^2}{2}} dt - \chi_1^2(z) e^{\frac{w^2}{2}} \int_{\sqrt{z}}^w e^{-\frac{t^2}{2}} dt \\
&= e^{\frac{w^2}{2}} \int_0^{\sqrt{z}} e^{-\frac{t^2}{2}} dt - \chi_1^2(z) e^{\frac{w^2}{2}} \int_0^w e^{-\frac{t^2}{2}} dt \\
&= \frac{\sqrt{2\pi}}{2} e^{\frac{w^2}{2}} \chi_1^2(z) - \frac{\sqrt{2\pi}}{2} e^{\frac{w^2}{2}} \chi_1^2(z) \chi_1^2(w^2) \\
&= \frac{\sqrt{2\pi}}{2} \chi_1^2(z) e^{\frac{w^2}{2}} (1 - \chi_1^2(w^2)).
\end{aligned}$$

Then

$$g_z(w) = \begin{cases} \frac{\sqrt{2\pi}}{2} (1 - \chi_1^2(z)) e^{\frac{w^2}{2}} \chi_1^2(w^2) & \text{if} \quad w \leq \sqrt{z}, \\ \frac{\sqrt{2\pi}}{2} \chi_1^2(z) e^{\frac{w^2}{2}} (1 - \chi_1^2(w^2)) & \text{if} \quad w > \sqrt{z}. \end{cases}$$

Note that

$$g_z'(w) = \begin{cases} \dfrac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))[we^{\frac{w^2}{2}}\chi_1^2(w^2) + \dfrac{2}{\sqrt{2\pi}}] & \text{if} \quad w < \sqrt{z}, \\ \dfrac{\sqrt{2\pi}}{2}\chi_1^2(z)[we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \dfrac{2}{\sqrt{2\pi}}] & \text{if} \quad w > \sqrt{z}, \end{cases}$$

and g_z' is not differentiable at $w = \sqrt{z}$. To satisfy (3.4), we define

$$\begin{aligned} g_z'(\sqrt{z}) &= \sqrt{z}g_z(\sqrt{z}) + I_z(w^2) - \chi_1^2(z) \\ &= \sqrt{z}\frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))e^{\frac{z}{2}}\chi_1^2(z) + 1 - \chi_1^2(z) \\ &= \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))[\sqrt{z}e^{\frac{z}{2}}\chi_1^2(z) + \frac{2}{\sqrt{2\pi}}]. \end{aligned}$$

This imply

$$g_z'(w) = \begin{cases} \dfrac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))[we^{\frac{w^2}{2}}\chi_1^2(w^2) + \dfrac{2}{\sqrt{2\pi}}] & \text{if} \quad w \leq \sqrt{z}, \\ \dfrac{\sqrt{2\pi}}{2}\chi_1^2(z)[we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \dfrac{2}{\sqrt{2\pi}}] & \text{if} \quad w > \sqrt{z}. \end{cases} \tag{3.8}$$

Hence, for $w \leq \sqrt{z}$,

$$\begin{aligned} g_z'(w) - wg_z(w) =& \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))[we^{\frac{w^2}{2}}\chi_1^2(w^2) + \frac{2}{\sqrt{2\pi}}] \\ & - w\{\frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))e^{\frac{w^2}{2}}\chi_1^2(w^2)\} \\ =& \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))we^{\frac{w^2}{2}}\chi_1^2(w^2) + (1-\chi_1^2(z)) \\ & - \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))we^{\frac{w^2}{2}}\chi_1^2(w^2) \\ =& 1 - \chi_1^2(z) \\ =& I_z(w^2) - \chi_1^2(z), \end{aligned}$$

and for $w > \sqrt{z}$,

$$
\begin{aligned}
g_z'(w) - wg_z(w) =& \frac{\sqrt{2\pi}}{2}\chi_1^2(z)[we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \frac{2}{\sqrt{2\pi}}] \\
& - w[\frac{\sqrt{2\pi}}{2}\chi_1^2(z)e^{\frac{w^2}{2}}(1-\chi_1^2(w^2))] \\
=& \frac{\sqrt{2\pi}}{2}\chi_1^2(z)we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \chi_1^2(z) \\
& - \frac{\sqrt{2\pi}}{2}\chi_1^2(z)we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) \\
=& -\chi_1^2(z) \\
=& I_z(w^2) - \chi_1^2(z).
\end{aligned}
$$

Therefore, g_z is a solution of Stein's equation (3.4). □

For the rest of this chapter, g_z is defined as in Proposition 3.1.

Proposition 3.2. *For each $z \geq 0$, let $h_z : [0, \infty) \to \mathbb{R}$ be defined by*

$$h_z(w) = wg_z(w) \quad \textit{for} \quad w \geq 0.$$

Then h_z is increasing.

Proof. By the definition of g_z and h_z, we have

$$
h_z(w) = \begin{cases} \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))we^{\frac{w^2}{2}}\chi_1^2(w^2) & \text{if} \quad w \leq \sqrt{z}, \\ \frac{\sqrt{2\pi}}{2}\chi_1^2(z)we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) & \text{if} \quad w > \sqrt{z} \end{cases}
$$

and

$$
h_z'(w) = \begin{cases} \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))[(e^{\frac{w^2}{2}} + w^2e^{\frac{w^2}{2}})\chi_1^2(z) + \frac{2w}{\sqrt{2\pi}}] & \text{if} \quad w < \sqrt{z}, \\ \frac{\sqrt{2\pi}}{2}\chi_1^2(z)[(e^{\frac{w^2}{2}} + w^2e^{\frac{w^2}{2}})(1-\chi_1^2(w^2)) - \frac{2w}{\sqrt{2\pi}}] & \text{if} \quad w > \sqrt{z}. \end{cases}
$$

Since h_z is continuous at $w = \sqrt{z}$, to prove h_z is increasing, it suffice to show that $h_z' \geq 0$ on $(0, \sqrt{z})$ and $(\sqrt{z}, \infty)$. It is obvious that $h_z' \geq 0$ on $(0, \sqrt{z})$. Hence, to show $h_z' \geq 0$ on $(\sqrt{z}, \infty)$ we have to prove that

$$(e^{\frac{w^2}{2}} + w^2e^{\frac{w^2}{2}})(1-\chi_1^2(w^2)) - \frac{2w}{\sqrt{2\pi}} \geq 0 \quad \text{for} \quad w > \sqrt{z},$$

i.e.,

$$1-\chi_1^2(w^2)-\frac{2w}{\sqrt{2\pi}(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})}\geq 0 \quad \text{for} \quad w>\sqrt{z}. \tag{3.9}$$

Let $k:[0,\infty)\to\mathbb{R}$ be defined by

$$k(w)=1-\chi_1^2(w^2)-\frac{2w}{\sqrt{2\pi}(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})}, \quad \text{for} \quad w\geq 0.$$

By the fact that

$$\begin{aligned}
k'(w) &= -\frac{2e^{\frac{-w^2}{2}}}{\sqrt{2\pi}}-\frac{2}{\sqrt{2\pi}}[\frac{(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})-w(we^{\frac{w^2}{2}}+2we^{\frac{w^2}{2}}+w^3e^{\frac{w^2}{2}})}{(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2}] \\
&= -\frac{2e^{\frac{-w^2}{2}}}{\sqrt{2\pi}}-\frac{2}{\sqrt{2\pi}}[\frac{(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})-w^2e^{\frac{w^2}{2}}-2w^2e^{\frac{w^2}{2}}-w^4e^{\frac{w^2}{2}})}{(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2}] \\
&= -\frac{2}{\sqrt{2\pi}}\{e^{-\frac{w^2}{2}}+\frac{1}{(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2}(e^{\frac{w^2}{2}}-2w^2e^{\frac{w^2}{2}}-w^4e^{\frac{w^2}{2}})\} \\
&= -\frac{2}{\sqrt{2\pi}(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2}\{e^{-\frac{w^2}{2}}(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2+e^{\frac{w^2}{2}}-2w^2e^{\frac{w^2}{2}}-w^4e^{\frac{w^2}{2}}\} \\
&= -\frac{2}{\sqrt{2\pi}(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2}\{e^{-\frac{w^2}{2}}(e^{w^2}+2w^2e^{w^2}+w^4e^{w^2})+e^{\frac{w^2}{2}}-2w^2e^{\frac{w^2}{2}}-w^4e^{\frac{w^2}{2}}\} \\
&= -\frac{2}{\sqrt{2\pi}(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2}\{e^{\frac{w^2}{2}}+2w^2e^{\frac{w^2}{2}}+w^4e^{\frac{w^2}{2}}+e^{\frac{w^2}{2}}-2w^2e^{\frac{w^2}{2}}-w^4e^{\frac{w^2}{2}}\} \\
&= -\frac{4e^{\frac{w^2}{2}}}{\sqrt{2\pi}(e^{\frac{w^2}{2}}+w^2e^{\frac{w^2}{2}})^2} \\
&\leq 0,
\end{aligned}$$

we have that k is a decreasing function. This implies that

$$k(w)\geq \lim_{w'\to\infty}k(w')=0$$

for all $w\in[0,\infty)$. Hence we have (3.9). □

Proposition 3.3. *The function g_z has the following properties:*

1. $0 \le g_z(w) \le \dfrac{1}{\sqrt{z}}$ *for* $w \ge 0$ *and* $z > 0$,
2. $0 \le g_z(w) \le 1$ *for* $w, z \ge 0$,
3. $|g_z'(w)| \le 1$ *for* $w, z \ge 0$.

Proof. 1. By the definition of g_z, it is easy to see that $g_z(w) \ge 0$. To prove $g_z(w) \le \frac{1}{\sqrt{z}}$, we note that

$$\begin{aligned}
1 - \chi_1^2(z) &= \frac{1}{\sqrt{2\pi}} \int_z^\infty t^{-\frac{1}{2}} e^{-\frac{t}{2}} dt \\
&\le \frac{1}{\sqrt{2\pi}} \int_z^\infty z^{-\frac{1}{2}} e^{-\frac{t}{2}} dt \\
&= \frac{1}{\sqrt{2\pi z}} \int_z^\infty e^{-\frac{t}{2}} dt \\
&= \frac{2}{\sqrt{2\pi z}} e^{-\frac{z}{2}}, \quad \text{for } z > 0. \qquad (3.10)
\end{aligned}$$

If $w \le \sqrt{z}$, then (3.10) implies

$$\begin{aligned}
g_z(w) &= \frac{\sqrt{2\pi}}{2}(1 - \chi_1^2(z)) e^{\frac{w^2}{2}} \chi_1^2(w^2) \\
&\le \frac{\sqrt{2\pi}}{2} \frac{2}{\sqrt{2\pi}\sqrt{z}} e^{-\frac{z}{2}} e^{\frac{w^2}{2}} \chi_1^2(w^2) \\
&= \frac{1}{\sqrt{z}} e^{-\frac{z}{2}} e^{\frac{w^2}{2}} \chi_1^2(w^2) \\
&\le \frac{1}{\sqrt{z}} e^{-\frac{z}{2}} e^{\frac{z}{2}} \chi_1^2(w^2) \\
&= \frac{1}{\sqrt{z}} \chi_1^2(w^2) \\
&\le \frac{1}{\sqrt{z}}.
\end{aligned}$$

Suppose that $w > \sqrt{z}$. By (3.10), we have

$$\begin{aligned} g_z(w) &= \frac{\sqrt{2\pi}}{2}\chi_1^2(z)e^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) \\ &\leq \frac{\sqrt{2\pi}}{2}\chi_1^2(z)e^{\frac{w^2}{2}}\frac{2}{\sqrt{2\pi}w}e^{-\frac{w^2}{2}} \\ &= \frac{1}{w}\chi_1^2(z) \\ &\leq \frac{1}{\sqrt{z}}. \end{aligned}$$

Hence, $0 \leq g_z(w) \leq \dfrac{1}{\sqrt{z}}$ for $z > 0$.

2. By (3.8), it is obvious that $g_z'(w) \geq 0$ for $w < \sqrt{z}$. For $w > \sqrt{z}$, we use (3.10) to show that

$$\begin{aligned} g_z'(w) &= \frac{\sqrt{2\pi}}{2}\chi_1^2(z)we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \chi_1^2(z) \\ &\leq \frac{\sqrt{2\pi}}{2}\chi_1^2(z)we^{\frac{w^2}{2}}\frac{2}{\sqrt{2\pi}w}e^{-\frac{w^2}{2}} - \chi_1^2(z) \\ &= \chi_1^2(z) - \chi_1^2(z) \\ &= 0. \end{aligned}$$

Note that g_z is continuous. Hence, g_z has maximum value at $w = \sqrt{z}$.

If $0 < z \leq 1$, then, by definition of g_z,

$$\begin{aligned} g_z(\sqrt{z}) &= \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))e^{\frac{z}{2}}\chi_1^2(z) \\ &\leq (\frac{\sqrt{2\pi}}{2})(\frac{1}{4})e^{\frac{z}{2}} \\ &\leq (\frac{\sqrt{2\pi}}{2})(\frac{1}{4})e^{\frac{1}{2}} \\ &\leq 0.53. \end{aligned}$$

For $z > 1$, by Proposition 3.3 (1), we have

$$g_z(\sqrt{z}) \leq \frac{1}{\sqrt{z}} \leq 1.$$

If $z = 0$, by the definition of g_z,

$$0 \leq g_0(w) \leq g_0(0) = 0.$$

Hence, $0 \le g_z(w) \le 1$ for $z \ge 0$.

3. Case 1. $w \le \sqrt{z}$.

We will show that $0 \le g_z'(w) \le 1$ by using (3.8) and (3.10).

If $z > 0$, then

$$\begin{aligned}
0 \le g_z'(w) &= \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))[we^{\frac{w^2}{2}}\chi_1^2(w^2) + \frac{2}{\sqrt{2\pi}}] \\
&= \frac{\sqrt{2\pi}}{2}(1-\chi_1^2(z))we^{\frac{w^2}{2}}\chi_1^2(w^2) + (1-\chi_1^2(z)) \\
&\le \frac{\sqrt{2\pi}}{2}\frac{2}{\sqrt{2\pi z}}e^{-\frac{z}{2}}\sqrt{z}e^{\frac{z}{2}}\chi_1^2(w^2) + 1 - \chi_1^2(z) \\
&\le \chi_1^2(z) + 1 - \chi_1^2(z) \\
&= 1.
\end{aligned}$$

For $z = 0$,

$$0 \le g_0'(w) = g_0'(0) = 1.$$

We conclude that for $w \le \sqrt{z}$ and $z \ge 0$, we have

$$0 \le g_z'(w) \le 1. \tag{3.11}$$

Case 2. $w > \sqrt{z}$.

In this case we will use (3.8) and (3.10) to show that $-1 \le g_z'(w) \le 0$.

We see that

$$\begin{aligned}
g_z'(w) &= \frac{\sqrt{2\pi}}{2}\chi_1^2(z)we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \chi_1^2(z) \\
&\ge -\chi_1^2(z) \\
&\ge -1
\end{aligned}$$

and

$$\begin{aligned} g_z'(w) &= \frac{\sqrt{2\pi}}{2}\chi_1^2(z)we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \chi_1^2(z) \\ &\le \frac{\sqrt{2\pi}}{2}we^{\frac{w^2}{2}}\frac{2}{\sqrt{2\pi}w}e^{-\frac{w^2}{2}}\chi_1^2(z) - \chi_1^2(z) \\ &= \chi_1^2(z) - \chi_1^2(z) \\ &= 0. \end{aligned}$$

Hence, $-1 \le g_z'(w) \le 0$ for $z > 0$. In the case that $z = 0$, we see that

$$g_0'(w) = \frac{\sqrt{2\pi}}{2}\chi_1^2(0)we^{\frac{w^2}{2}}(1-\chi_1^2(w^2)) - \chi_1^2(0) = 0.$$

Thus for $w > \sqrt{z}$ and $z \ge 0$, we have

$$-1 \le g_z'(w) \le 0. \tag{3.12}$$

Therefore, $|g_z'(w)| \le 1$ for all $w, z \ge 0$. □

Proposition 3.4. *Let $z, v, w, s, t \ge 0$. Then*

1. $|g_z'(w) - g_z'(v)| \le 1$,

2. $g_z'(w+s) - g_z'(w+t) \le \begin{cases} 1 & \text{if } w+s \le \sqrt{z} \text{ and } w+t > \sqrt{z}, \\ (|w|+1)(|s|+|t|) & \text{if } s > t, \\ 0 & \text{otherwise}, \end{cases}$

3. $g_z'(w+s) - g_z'(w+t) \ge \begin{cases} -1 & \text{if } w+s > \sqrt{z} \text{ and } w+t \le \sqrt{z}, \\ -(|w|+1)(|s|+|t|) & \text{if } s < t, \\ 0 & \text{otherwise}. \end{cases}$

Proof.

1. Let $z \geq 0$. From (3.11) and (3.12), we have $|g_z'(w) - g_z'(v)| \leq 1$ for $w, v \leq \sqrt{z}$ or $w, v > \sqrt{z}$. Suppose that $w \leq \sqrt{z}$ and $v > \sqrt{z}$. By (3.4), we have

$$g_z'(w) = \begin{cases} wg_z(w) + 1 - \chi_1^2(z) & \text{if} \quad w \leq \sqrt{z}, \\ wg_z(w) - \chi_1^2(z) & \text{if} \quad w > \sqrt{z}. \end{cases} \tag{3.13}$$

From Proposition 3.2, (3.11), (3.12) and (3.13), we have

$$0 \leq g_z'(w) \leq \sqrt{z}g_z(\sqrt{z}) + 1 - \chi_1^2(z) \quad \text{for} \quad w \leq \sqrt{z} \tag{3.14}$$

and

$$\sqrt{z}g_z(\sqrt{z}) - \chi_1^2(z) \leq g_z'(v) \leq 0 \qquad \text{for} \quad v > \sqrt{z}. \tag{3.15}$$

By (3.11), (3.12), (3.14) and (3.15), we get that

$$|g_z'(w) - g_z'(v)| \leq \max\{1, \sqrt{z}g_z(\sqrt{z}) + 1 - \chi_1^2(z) - (\sqrt{z}g_z(\sqrt{z}) - \chi_1^2(z))\} = 1.$$

2. From (3.4), we have

$$g_z'(w+s) - (w+s)g_z(w+s) = I_z((w+s)^2) - \chi_1^2(z)$$

and

$$g_z'(w+t) - (w+t)g_z(w+t) = I_z((w+t)^2) - \chi_1^2(z).$$

Since $h_z(w) = wg_z(w)$ is increasing,

$$g_z'(w+s) - g_z'(w+t) = (w+s)g_z(w+s) - (w+t)g_z(w+t) + I_z((w+s)^2) - I_z((w+t)^2)$$

$$= \begin{cases} (w+s)g_z(w+s) - (w+t)g_z(w+t) + 1 & \text{if} \quad w+s \leq \sqrt{z} \text{ and } \ w+t > \sqrt{z}, \\ (w+s)g_z(w+s) - (w+t)g_z(w+t) - 1 & \text{if} \quad w+s > \sqrt{z} \text{ and } \ w+t \leq \sqrt{z}, \\ (w+s)g_z(w+s) - (w+t)g_z(w+t) & \text{if} \quad w+s \leq \sqrt{z} \text{ and } \ w+t \leq \sqrt{z} \\ & \quad \text{or } \ w+s > \sqrt{z} \text{ and } \ w+t > \sqrt{z} \end{cases}$$

$$\leq \begin{cases} 1 & \text{if } \ w+s \leq \sqrt{z} \text{ and } w+t > \sqrt{z}, \\ w[g_z(w+s) - g_z(w+t)] + sg_z(w+s) - tg_z(w+t) & \text{if } \ s > t, \\ 0 & \text{otherwise.} \end{cases}$$

From Proposition 3.3((i) and (ii)) and by mean-valued theorem, there exists $r \in \mathbb{R}$ such that

$$\begin{aligned} & w[g_z(w+s) - g_z(w+t)] + sg_z(w+s) - tg_z(w+t) \\ & \leq |w[g_z(w+s) - g_z(w+t)]| + |sg_z(w+s) - tg_z(w+t)| \\ & \leq |w|\,|g_z(w+s) - g_z(w+t)| + |s|\,|g_z(w+s)| + |t|\,|g_z(w+t)| \\ & = |w| \left| \frac{g_z(w+s) - g_z(w+t)}{(w+s)-(w+t)}(s-t) \right| + |s|\,|g_z(w+s)| + |t|\,|g_z(w+t)| \\ & = |w|\,|g_z'(r)|\,|s-t| + |s|\,|g_z(w+s)| + |t|\,|g_z(w+t)| \\ & \leq |w|\,(|s| + |t|) + (|s| + |t|) \\ & = (|w|+1)(|s|+|t|). \end{aligned}$$

Hence,

$$g_z'(w+s) - g_z'(w+t) = \begin{cases} 1 & \text{if } w+s \leq \sqrt{z} \text{ and } \ w+t > \sqrt{z}, \\ (|w|+1)(|s|+|t|) & \text{if } \ s > t, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of 3 is similar to 2. □

CHAPTER IV

BOUNDS ON CHI-SQUARE APPROXIMATION

For each $n \in \mathbb{N}$, let $X_1, X_2, ..., X_n$ be independent and not necessarily identically distributed random variables with zero mean, finite variance and $\sum_{i=1}^{n} E{X_i}^2 = 1$. Define

$$W_n = \sum_{i=1}^{n} X_i.$$

Let F_n be the distribution function of W_n and Φ the standard normal distribution function. The central limit theorem in probability theory and statistics guarantee that for each $z \in \mathbb{R}$,

$$F_n(z) \to \Phi(z) \text{ as } n \to \infty,$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt.$

Many researchers try to find the rate of this convergence having two types of bound; uniform bound and non-uniform bound. Consider a bound for the distance between two distribution functions F and G,

$$|F(x) - G(x)| \le K.$$

If K depends on x, then K is considered to be a non-uniform bound. On the other hand, if K does not depend on x, then K is considered to be a uniform bound.

In 1986, Siganov ([12]) gave a uniform bound under the assumption that the third moment is finite. His result is as follows.

Theorem 4.1. *(Siganov,1986). Let* $X_1, X_2, ..., X_n$ *be independent random variables such that* $EX_i = 0$ *and* $E|X_i|^3 < \infty$ *for* $i = 1, 2, ..., n$. *Assume that* $\sum_{i=1}^{n} EX_i^{\ 2} = 1$. *Then*

$$\sup_{z\in\mathbb{R}} |P(W_n \le z) - \Phi(z)| \le 0.7915 \sum_{i=1}^{n} E|X_i|^3.$$

In 1977, Paditz ([8]) gave a non-uniform bound under the assumptions as in Theorem 4.1. His result is as follows.

Theorem 4.2. *(Paditz,1977). Under the assumption of Theorem 4.1, we have*

$$|P(W_n \le z) - \Phi(z)| \le \frac{31.935}{1+|z|^3} \sum_{i=1}^{n} E|X_i|^3$$

for $z \in \mathbb{R}$.

In 2001, Chen and Shao ([2]) gave the versions of a uniform bound and a non-uniform bound without assuming the existence of the third moments. Their results are as follow.

Theorem 4.3. *(uniform bound) Let* $X_1, X_2, ..., X_n$ *be independent random variables such that* $EX_i = 0$ *and* $E|X_i|^2 < \infty$ *for* $i = 1, 2, ..., n$. *Assume that* $\sum_{i=1}^{n} EX_i^{\ 2} = 1$. *Then*

$$\sup_{z\in\mathbb{R}} |P(W_n \le z) - \Phi(z)| \le 4.1 \left\{ \sum_{i=1}^{n} EX_i^2 I(|X_i| \ge 1) + \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1) \right\}.$$

Theorem 4.4. *(non-uniform bound) Under the assumptions of Theorem 4.3, there exists a constant* C *such that*

$$|P(W_n \le z) - \Phi(z)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|z|)}{(1+|z|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|z|)}{(1+|z|)^3} \right\}$$

for $z \in \mathbb{R}$.

In 2007, Neammanee and Thongtha ([5]) calculated the constant by using Paditz-Siganov theorem. Their result is as follow.

Theorem 4.5. *Under the assumption of Theorem 4.3, for $z \in \mathbb{R}$, we have*

$$|P(W_n \leq z) - \Phi(z)| \leq C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + |z|)}{1 + |z|^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |z|)}{1 + |z|^3} \right\}$$

where

$$C = \begin{cases} 13.11 & if \quad 0 \leq |z| < 1.3, \\ 28.54 & if \quad 1.3 \leq |z| < 2, \\ 46.32 & if \quad 2 \leq |z| < 3, \\ 61.40 & if \quad 3 \leq |z| < 7.98, \\ 40.12 & if \quad 7.98 \leq |z| < 14, \\ 39.39 & if \quad |z| \geq 14. \end{cases}$$

In this chapter, we give uniform and non-uniform bounds on chi-square approximation to the distribution of W_n^2 under the existence of the second or the third moments. To do this, we use a relation between the chi-square random variable with degree of freedom 1 and the standard normal random variable. The relation is as follows: Let $Z_1, Z_2, ..., Z_n$ be independent standard normal random variables. It is well-known that

$$Z_1^2 + \cdots + Z_n^2 \sim \chi_n^2.$$

For the first part, we assume $E|X_i|^3 < \infty$ for $i = 1, 2, ..., n$. We apply Theorem 4.1 and Theorem 4.2 to give uniform and non-uniform bounds on chi-square approximation to the distribution of W_n^2 under the existence of the third moment, respectively. The followings are our results.

Theorem 4.6. *(uniform bound) Let* $X_1, X_2, ..., X_n$ *be independent random variables such that* $EX_i = 0$ *and* $E|X_i|^3 < \infty$ *for* $i = 1, 2, ..., n$. *Assume that* $\sum_{i=1}^{n} E{X_i}^2 = 1$. *Then*

$$\sup_{z\geq 0} |P(W_n^2 \leq z) - \chi_1^2(z)| \leq 1.583 \sum_{i=1}^{n} E|X_i|^3.$$

Theorem 4.7. *(non-uniform bound) Under the assumption of Theorem 4.6 and* $z \geq 0$, *we have*

$$|P(W_n^2 \leq z) - \chi_1^2(z)| \leq \frac{63.87}{1 + z^{\frac{3}{2}}} \sum_{i=1}^{n} E|X_i|^3.$$

For the second part, we assume that $E|X_i|^2 < \infty$ for $i = 1, 2, ..., n$. We apply Theorem 4.3 and Theorem 4.5 to give uniform and non-uniform bounds on chi-square approximation to the distribution of W_n^2 under the existence of the second moments respectively. Our results are stated as follows.

Theorem 4.8. *(uniform bound)Let* $X_1, X_2, ..., X_n$ *be independent random variables such that* $EX_i = 0$ *and* $E|X_i|^2 < \infty$ *for* $i = 1, 2, ..., n$. *Assume that* $\sum_{i=1}^{n} E{X_i}^2 = 1$. *Then*

$$\sup_{z\geq 0} |P(W_n^2 \leq z) - \chi_1^2(z)| \leq 8.2\{\sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1) + \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1)\}.$$

Theorem 4.9. *(non-uniform bound) Under the assumption of Theorem 4.8, for* $z \geq 0$, *there exists an absolute constant* C *such that*

$$|P(W_n^2 \leq z) - \chi_1^2(z)| \leq C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + \sqrt{z})}{1 + z} + \frac{E|X_i|^3 I(|X_i| < 1 + \sqrt{z})}{1 + z^{\frac{3}{2}}} \right\},$$

where

$$C = \begin{cases} 26.22 & if \;\; 0 \le z < 1.69, \\ 57.08 & if \;\; 1.69 \le z < 4, \\ 92.64 & if \;\; 4 \le z < 9, \\ 122.8 & if \;\; 9 \le z < 63.6804, \\ 80.24 & if \;\; 63.6804 \le z < 196, \\ 78.78 & if \;\; z \ge 196. \end{cases}$$

In this proof, we give the proof of Theorem 4.6-Theorem 4.9. To prove the theorems, we need following Lemma.

Lemma 4.10. *For* $z \ge 0$,

$$\{w \mid W_n(w) < -\sqrt{z}\} = \bigcup_{m=1}^{\infty} \{w \mid W_n(w) \le -\sqrt{z} - \frac{1}{m}\}. \tag{4.1}$$

Proof. For the exclusion, it is obvious.

For the inclusion, let $w \in \{w \mid W_n(w) < -\sqrt{z}\}$.

Then $-\sqrt{z} - W_n(w) > 0$. By Archimedian principle there exists $m_0 \in \mathbb{N}$ such that

$$-\sqrt{z} - W_n(w) > \frac{1}{m_0}.$$

Hence, $W_n(w) \le -\sqrt{z} - \frac{1}{m_0}$. □

Proof of Theorem 4.6.

Proof. Let Z be a standard normal random variable. From (4.1) and Theorem 2.1(1),

$$P(W_n < -\sqrt{z}) = P(\bigcup_{m=1}^{\infty} \{W_n \le -\sqrt{z} - \frac{1}{m}\}) = \lim_{m \to \infty} P(W_n \le -\sqrt{z} - \frac{1}{m})$$

and the fact that $\Phi(-\sqrt{z}) = \lim_{m\to\infty} \Phi(-\sqrt{z} - \frac{1}{m})$,

$$
\begin{aligned}
&|P(W_n^2 \le z) - \chi_1^2(z)| \\
&= |P(-\sqrt{z} \le W_n \le \sqrt{z}) - \chi_1^2(z)| \\
&= |P(-\sqrt{z} \le W_n \le \sqrt{z}) - P(Z^2 \le z)| \\
&= |P(-\sqrt{z} \le W_n \le \sqrt{z}) - P(-\sqrt{z} \le Z \le \sqrt{z})| \\
&= |P(W_n \le \sqrt{z}) - P(W_n < -\sqrt{z}) - [P(Z \le \sqrt{z}) - P(Z < -\sqrt{z})]| \\
&\le |P(W_n \le \sqrt{z}) - P(Z \le \sqrt{z})| + |P(W_n < -\sqrt{z}) - P(Z < -\sqrt{z})| \\
&= |P(W_n \le \sqrt{z}) - \Phi(\sqrt{z})| + |P(W_n < -\sqrt{z}) - \Phi(-\sqrt{z})| \\
&= |P(W_n \le \sqrt{z}) - \Phi(\sqrt{z})| + |P(\bigcup_{m=1}^{\infty}\{W_n \le -\sqrt{z} - \frac{1}{m}\} - \Phi(-\sqrt{z} - \frac{1}{m})| \\
&= |P(W_n \le \sqrt{z}) - \Phi(\sqrt{z})| + |\lim_{m\to\infty} P(W_n \le -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})| \\
&= |P(W_n \le \sqrt{z}) - \Phi(\sqrt{z})| + \lim_{m\to\infty} |P(W_n \le -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})|. \quad (4.2)
\end{aligned}
$$

From (4.2) and Theorem 4.1, we have

$$
\begin{aligned}
&\sup_{z\ge 0} |P(W_n^2 \le z) - \chi_1^2(z)| \\
&\le \sup_{z\ge 0} \left\{ |P(W_n \le \sqrt{z}) - \Phi(\sqrt{z})| + \lim_{m\to\infty} |P(W_n \le -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})| \right\} \\
&\le \sup_{z\ge 0} |P(W_n \le \sqrt{z}) - \Phi(\sqrt{z})| + \sup_{z\ge 0} \lim_{m\to\infty} |P(W_n \le -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})| \\
&\le 0.7915 \sum_{i=1}^{n} E|X_i|^3 + 0.7915 \sum_{i=1}^{n} E|X_i|^3 \\
&= 1.583 \sum_{i=1}^{n} E|X_i|^3.
\end{aligned}
$$

Hence,

$$
\sup_{z\ge 0} |P(W_n^2 \le z) - \chi_1^2(z)| \le 1.583 \sum_{i=1}^{n} E|X_i|^3.
$$

□

Proof of Theorem 4.7.

Proof. From (4.2) and Theorem 4.2, we have for each $z \geq 0$

$$
\begin{aligned}
&|P(W_n^2 \leq z) - \chi_1^2(z)| \\
&\leq |P(W_n \leq \sqrt{z}) - \Phi(\sqrt{z})| + \lim_{m\to\infty} |P(W_n \leq -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})| \\
&\leq \frac{31.935}{1+|\sqrt{z}|^3} \sum_{i=1}^{n} E|X_i|^3 + \lim_{m\to\infty} \frac{31.935}{1+|-\sqrt{z}-\frac{1}{m}|^3} \sum_{i=1}^{n} E|X_i|^3 \\
&\leq \frac{31.935}{1+|\sqrt{z}|^3} \sum_{i=1}^{n} E|X_i|^3 + \frac{31.935}{1+|-\sqrt{z}|^3} \sum_{i=1}^{n} E|X_i|^3 \\
&= \frac{31.935}{1+z^{\frac{3}{2}}} \sum_{i=1}^{n} E|X_i|^3 + \frac{31.935}{1+z^{\frac{3}{2}}} \sum_{i=1}^{n} E|X_i|^3 \\
&= \frac{63.87}{1+z^{\frac{3}{2}}} \sum_{i=1}^{n} E|X_i|^3.
\end{aligned}
$$

Hence, $|P(W_n^2 \leq z) - \chi_1^2(z)| \leq \dfrac{63.87}{1+z^{\frac{3}{2}}} \displaystyle\sum_{i=1}^{n} E|X_i|^3.$ □

Proof of Theorem 4.8.

Proof. From (4.2) and Theorem 4.3, we have for each $z \geq 0$

$$
\begin{aligned}
&\sup_{z\geq 0} |P(W_n^2 \leq z) - \chi_1^2(z)| \\
&\leq \sup_{z\geq 0} \left\{ |P(W_n \leq \sqrt{z}) - \Phi(\sqrt{z})| + \lim_{m\to\infty} |P(W_n \leq -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})| \right\} \\
&\leq \sup_{z\geq 0} |P(W_n \leq \sqrt{z}) - \Phi(\sqrt{z})| + \sup_{z\geq 0} \lim_{m\to\infty} |P(W_n \leq -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})| \\
&= \sup_{z\geq 0} |P(W_n \leq \sqrt{z}) - \Phi(\sqrt{z})| + \lim_{m\to\infty} \sup_{z\geq 0} |P(W_n \leq -\sqrt{z} - \frac{1}{m}) - \Phi(-\sqrt{z} - \frac{1}{m})| \\
&\leq 4.1 \left\{ \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1) + \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1) \right\} \\
&\quad + 4.1 \left\{ \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1) + \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1) \right\} \\
&= 8.2 \left\{ \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1) + \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1) \right\}.
\end{aligned}
$$

Hence,

$$\sup_{z\geq 0}|P(W_n^2\leq z)-\chi_1^2(z)|\leq 8.2\left\{\sum_{i=1}^{n}EX_i^2I(|X_i|\geq 1)+\sum_{i=1}^{n}E|X_i|^3I(|X_i|<1)\right\}.$$

□

Proof of Theorem 4.9.

Proof. We can prove the theorem by using (4.2), Theorem 4.5 and the same technique of Theorem 4.8. □

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