INTRODUCTION

The Zigzag Theorem of Isbell is a well-known and powerful theorem in semigroup theory. Studying dense subsemigroups of semigroups is one of the many applications of this theorem. Dense subsemigroups of semigroups of certain types have been studied by Hall in [1] and by Higgins in [2]. An example of a finite dense subsemigroup of an infinite semigroup was given by Hall in [1]. Isbell has constructed a finite semigroup having a proper dense subsemigroup in [5]. It is clear that every left zero semigroup and every right zero semigroup has no proper dense subsemigroup. Moreover, every finite group has no proper dense subsemigroup. In [2], Higgins has characterized the three standard transformation semigroups $\mathcal{T}_X, \mathcal{PT}_X$ and $\mathcal{I}_X$ which have proper dense subsemigroups in terms of the cardinality of X as follows:

(*) " For a set X, if $\mathcal{S} = \mathcal{T}_X$, $\mathcal{PT}_X$ or $\mathcal{I}_X$, then $\mathcal{S}$ has a proper dense subsemigroup if and only if X is infinite."

It is accepted that the following generalized transformation semigroups are natural and extensive generalizations of $\mathcal{T}_X$, $\mathcal{PT}_X$ and $\mathcal{I}_X$, respectively:

(1) $(\mathcal{T}(X,Y),\theta)$,

(2) $(\mathcal{PT}(X,Y),\theta)$ and

(3) $(\mathcal{I}(X,Y),\theta)$

where $\mathcal{T}(X,Y)$ = the set of all mappings from X into Y,

$\mathcal{PT}(X,Y)$ = the set of all mappings from subsets of X into Y and

$\mathcal{I}(X,Y)$ = the set of all 1–1 mappings from subsets of X into Y

and if $\mathcal{S}(X,Y)$ is any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$, then $\theta \in \mathcal{S}(Y,X)$ and $(\mathcal{S}(X,Y),\theta)$ denotes the semigroup $(\mathcal{S}(X,Y),*)$ with $\alpha * \beta = \alpha\theta\beta$ for all $\alpha,\beta \in \mathcal{S}(X,Y)$. Magill has studied semigroups of type (1) in a series of papers [6], [7] and [8]. Also, semigroups of type (1) have also been studied by Symon in [10]. Sullivan has proved in [9] that every subsemigroup of a semigroup of type (2) is isomorphic to a subsemigroup of a semigroup of type (1).

The purpose of this research is to generalize result (*) given by Higgins. We shall characterize generalized transformation semigroups of types (1), (2) and (3) given above which have proper dense subsemigroups in terms of X, Y and the range of θ.

The notation and quoted results used for this work are given in Chapter I. In Chapter II, we introduce certain properties of generalized transformation semigroups which are required for Chapter III. Chapter III gives the main result of this research which is the following:

(**) " For any sets X,Y, if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\theta \in \mathcal{S}(Y,X)$, then the semigroup $(\mathcal{S}(X,Y),\theta)$ has a proper dense subsemigroup if and only if X and Y are both infinite and $|\nabla\theta| = \min\{|X|,|Y|\}$ where $\nabla\theta$ is the range of θ."

In the beginning of Chapter III, we give a new proof of Higgins' theorem (*). In proving our main result (**), we use this new proof as a guide. However, our proof of (**) is much more complicated than our new proof of Higgins' theorem (*). Moreover, our proofs show that infinitely many proper dense subsemigroups exist in both (*) and (**).