CHAPTER IV

MAXIMAL COMMUTATIVE SUBSEMIGROUPS OF MATRIX SEMIGROUPS

Let S be a commutative semiring with 0,1 and n a positive integer. If $n > 1$, then the matrix semigroup $M_n(S)$ is not commutative since $0 \neq 1$. Let $D_n(S)$ and $C_n(S)$ denote the set of all diagonal n×n matrices over S and the set of all circulant n×n matrices over S, respectively, where a circulant n×n matrix over S is a matrix over S in the form

$$\begin{bmatrix} a_1 & a_2 & a_3 \dots a_n \\ a_n & a_1 & a_2 \dots a_{n-1} \\ a_{n-1} & a_n & a_1 \dots a_{n-2} \\ \dots & \dots & \dots \\ a_2 & a_3 & a_4 \dots a_1 \end{bmatrix}.$$

Kim Jin Bai proved in [5] that $D_n(S)$ and $C_n(S)$ are maximal commutative subsemigroups of the matrix semigroup $M_n(S)$ if S is the semiring ([0,1],max,min). It can be seen easily from his proofs that $D_n(S)$ and $C_n(S)$ are maximal commutative subsemigroups of the matrix semigroup $M_n(S)$ for any commutative semiring S with 0,1.

In this chapter we shall introduce three other maximal commutative subsemigroups of the matrix semigroup $M_n(S)$ with S any commutative semiring with 0,1 and n any positive integer. Also, we give one different maximal commutative subsemigroup of the matrix semigroup $M_n(R)$ with R any commutative ring with identity and n any positive integer.

__Theorem 4.1__. Let S be a commutative semiring with 0,1 and n a positive integer. Then the set of all n×n matrices over S in the form

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & b_2 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & b_2 & \cdots & a_2 & 0 \\ b_1 & 0 & \cdots & 0 & a_1 \end{bmatrix}$$

is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$.

__Proof__. Let $\mathcal{U}$ be the set of all n×n matrices over S in the above form. If n = 1 then $\mathcal{U} = M_1(S)$, so we are done. Assume n > 1 and let

$$\Lambda = \begin{cases} \{1,2,\ldots,\frac{n}{2}\} & \text{if } n \text{ is even,} \\ \{1,2,\ldots,\frac{n-1}{2}\} & \text{if } n \text{ is odd.} \end{cases}$$

Then for $A \in M_n(S)$, $A \in \mathcal{U}$ if and only if $A_{ii} = A_{(n-i)+1,(n-i)+1}$, $A_{i,(n-i)+1} = A_{(n-i)+1,i}$ for all $i \in \Lambda$ and $A_{ij} = 0$ for all $i,j \in \{1,2,\ldots,n\}$, $j \neq i$ and $j \neq (n-i)+1$. Clearly, $aI_n \in \mathcal{U}$ for all $a \in S$ where I_n is the identity n×n matrix over S.

To show that $\mathcal{U}$ is a commutative subsemigroup of the matrix semigroup $M_n(S)$, let

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & b_2 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & b_2 & \cdots & a_2 & 0 \\ b_1 & 0 & \cdots & 0 & a_1 \end{bmatrix}, \quad A' = \begin{bmatrix} a'_1 & 0 & \cdots & 0 & b'_1 \\ 0 & a'_2 & \cdots & b'_2 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & b'_2 & \cdots & a'_2 & 0 \\ b'_1 & 0 & \cdots & 0 & a'_1 \end{bmatrix}$$

be elements of $\mathcal{U}$. Then

$$AA' = \begin{bmatrix} a_1a_1'+b_1b_1' & 0 & \cdots & 0 & a_1b_1'+b_1a_1' \\ 0 & a_2a_2'+b_2b_2' & \cdots & a_2b_2'+b_2a_2' & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_2a_2'+a_2b_2' & \cdots & b_2b_2'+a_2a_2' & 0 \\ b_1a_1'+a_1b_1' & 0 & \cdots & 0 & b_1b_1'+a_1a_1' \end{bmatrix},$$

$$A'A = \begin{bmatrix} a_1'a_1+b_1'b_1 & 0 & \cdots & 0 & a_1'b_1+b_1'a_1 \\ 0 & a_2'a_2+b_2'b_2 & \cdots & a_2'b_2+b_2'a_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_2'a_2+a_2'b_2 & \cdots & b_2'b_2+a_2'a_2 & 0 \\ b_1'a_1+a_1'b_1 & 0 & \cdots & 0 & b_1'b_1+a_1'a_1 \end{bmatrix},$$

so $AA' = A'A$ which is an element of $\mathcal{U}$ since S is commutative.

To show that $\mathcal{U}$ is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$, it suffices to show that if X is an element of $M_n(S)$ such that $AX = XA$ for every $A \in \mathcal{U}$, then $X \in \mathcal{U}$. Let $X = (x_{ij}) \in M_n(S)$ be such that $XA = AX$ for every $A \in \mathcal{U}$. For each $k \in \Lambda$, let $A^{(k)}$ be the n×n matrix over S defined by

$$A_{ij}^{(k)} = \begin{cases} 1 & \text{if } i = j = k \text{ or } i = j = (n-k)+1, \\ 0 & \text{otherwise.} \end{cases}$$ Then $A^{(k)} \in \mathcal{U}$ for every $k \in \Lambda$, so $XA^{(k)} = A^{(k)}X$ for every $k \in \Lambda$. If $k \in \Lambda, j \in \{1,2,\ldots n\}$, $j \neq k$ and $j \neq (n-k)+1$, then

$$A_{kt}^{(k)} = \begin{cases} 1 & \text{if } t = k, \\ 0 & \text{if } t \neq k, \end{cases}$$

$$A_{(n-k)+1,t}^{(k)} = \begin{cases} 1 & \text{if } t = n-k+1, \\ 0 & \text{if } t \neq n-k+1, \end{cases}$$

$$A_{tj}^{(k)} = 0 \quad \text{for every } t \in \{1,2,\ldots,n\}.$$

hence

$$(A^{(k)}X)_{kj} = \sum_{t=1}^{n} A^{(k)}_{kt}X_{tj} = X_{kj} ,$$

$$(XA^{(k)})_{kj} = \sum_{t=1}^{n} X_{kt}A^{(k)}_{tj} = 0 ,$$

$$(A^{(k)}X)_{(n-k)+1,j} = \sum_{t=1}^{n} A^{(k)}_{(n-k)+1,t}X_{tj} = X_{(n-k)+1,j} ,$$

$$(XA^{(k)})_{(n-k)+1,j} = \sum_{t=1}^{n} X_{(n-k)+1,t}A^{(k)}_{tj} = 0 .$$

Therefore $X_{kj} = 0 = X_{(n-k)+1,j}$ for all $k \varepsilon \Lambda$ and for all $j \varepsilon \{1,2,\ldots,n\}$, $j \neq k$ and $j \neq (n-k)+1$ which implies that $X_{ij} = 0$ for all $i,j \varepsilon \{1,2,\ldots,n\}$, $j \neq i$ and $j \neq n-i+1$. Now we have

$$X = \begin{bmatrix} x_{11} & 0 & \cdots & 0 & x_{1n} \\ 0 & x_{22} & \cdots & x_{2,n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_{n-1,2} & \cdots & x_{n-1,n-1} & 0 \\ x_{n1} & 0 & \cdots & 0 & x_{nn} \end{bmatrix}$$

Let B be the n×n matrix over S defined by $B_{ij} = \begin{cases} 1 & \text{if } j = (n-i)+1, \\ 0 & \text{otherwise.} \end{cases}$

Then $B \varepsilon \mathfrak{U}$, so BX = XB. Since

$$BX = \begin{bmatrix} x_{n1} & 0 & \cdots & 0 & x_{nn} \\ 0 & x_{n-1,2} & \cdots & x_{n-1,n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_{22} & \cdots & x_{2,n-1} & 0 \\ x_{11} & 0 & \cdots & 0 & x_{1n} \end{bmatrix}$$

$$\text{and} \qquad XB = \begin{bmatrix} x_{1n} & 0 & \cdots & 0 & x_{11} \\ 0 & x_{2,n-1} & \cdots & x_{22} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & x_{n-1,n-1} & \cdots & x_{n-1,2} & 0 \\ x_{nn} & 0 & \cdots & 0 & x_{n1} \end{bmatrix}$$

we have that $X_{ii} = X_{(n-i)+1,(n-i)+1}$ for all $i \varepsilon \Lambda$, $X_{i,(n-i)+1} = X_{(n-i)+1,i}$ for all $i \varepsilon \Lambda$. Hence $X \varepsilon \mathcal{U}$. #

<u>Theorem 4.2</u>. Let S be a commutative semiring with 0,1 and n a positive integer. Then the set of all n×n matrices over S in the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_1 \end{bmatrix}$$

is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$.

<u>Proof</u>. Let $\mathcal{V}$ be the set of all n×n matrices over S in the above form, that is, for $A \varepsilon M_n(S)$, $A \varepsilon \mathcal{V}$ if and only if $A_{ij} = 0$ if $i > j$ and $A_{i,i+k-1} = A_{1,1+k-1} = A_{1k}$ for all $i,k \varepsilon \{1,2,\ldots,n\}$ with $i+k-1 < n$. Then $aI_n \varepsilon \mathcal{V}$ for every $a \varepsilon S$.

To show that $\mathcal{V}$ is a commutative subsemigroup of the matrix semigroup $M_n(S)$, let

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_1 \end{bmatrix} , \quad B = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ 0 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & 0 & b_1 & \cdots & b_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_1 \end{bmatrix}$$

be elements of $\mathcal{V}$. Then

$$AB = \begin{bmatrix} a_1b_1 & a_1b_2+ a_2b_1 & a_1b_3+ a_2b_2+ a_3b_1 & \cdots & a_1b_n+\cdots+ a_nb_1 \\ 0 & a_1b_1 & a_1b_2+ a_2b_1 & \cdots & a_1b_{n-1}+\cdots+ a_{n-1}b_1 \\ 0 & 0 & a_1b_1 & \cdots & a_1b_{n-2}+\cdots+ a_{n-2}b_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_1b_1 \end{bmatrix} ,$$

$$BA = \begin{bmatrix} b_1a_1 & b_1a_1+ b_2a_1 & b_1a_3+ b_2a_2+ b_3a_1 & \cdots & b_1a_n+\cdots+ b_na_1 \\ 0 & b_1a_1 & b_1a_2+ b_2a_1 & \cdots & b_1a_{n-1}+\cdots+b_{n-1}a_1 \\ 0 & 0 & b_1a_1 & \cdots & b_1a_{n-2}+\cdots+b_{n-2}a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_1a_1 \end{bmatrix}$$

so $AB \in \mathcal{V}$ and $AB = BA$ since S is commutative.

To show $\mathcal{V}$ is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$, let $X \in M_n(S)$ be such that $XA = AX$ for every $A \in \mathcal{V}$. For each $k \in \{1,2,\ldots,n\}$, let $D^{(k)} \in \mathcal{V}$ be defined by

$$D^{(k)}_{ij} = \begin{cases} 1 & \text{if } j = n-k+i , \\ 0 & \text{otherwise} , \end{cases}$$

that is,

$$D^{(k)} = \begin{array}{cl} \begin{array}{ccccccc} & & & n\text{-}k\text{+}1 \ \underline{\text{th}} \text{ col.} & & & \\ & & & \downarrow & & & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \leftarrow k \ \underline{\text{th}} \text{ row }, \\ \\ \\ \end{array} \end{array}$$

Then $XD^{(k)} = D^{(k)}X$.

To show that $X_{ij} = 0$ for all $i,j \in \{1,2,\ldots,n\}$, $i > j$, let $i,j \in \{1,2,\ldots,n\}$, $i > j$. Then $XD^{(j)} = D^{(j)}X$, so $(XD^{(j)})_{in} = (D^{(j)}X)_{in}$. Since $D^{(j)}_{tn} = \begin{cases} 1 & \text{if } t = j, \\ 0 & \text{if } t \neq j, \end{cases}$ we have that

$$(XD^{(j)})_{in} = \sum_{t=1}^{n} X_{it}D^{(j)}_{tn} = X_{ij} \ .$$

Since $i > j$, $D^{(j)}_{it} = 0$ for every $t \in \{1,2,\ldots,n\}$, so we have

$$(D^{(j)}X)_{in} = \sum_{t=1}^{n} D^{(j)}_{it}X_{tn} = 0.$$

Hence $X_{ij} = 0$.

Next, be shall show that $X_{i,i+k-1} = X_{1k}$ for all $i,k \in \{1,2,\ldots,n\}$, $i+k-1 < n$. Let $i,k \in \{1,2,\ldots,n\}$, $i+k-1 < n$. Then $D^{(n-i+1)}X = XD^{(n-i+1)}$, so $(D^{(n-i+1)}X)_{1,i+k-1} = (XD^{(n-i+1)})_{1,i+k-1}$. Since

$$D^{(n-i+1)}_{1t} = \begin{cases} 1 & \text{if } t = n-(n-i+1)+1 = i, \\ 0 & \text{if } t \neq i, \end{cases}$$

$$(D^{(n-i+1)}X)_{1,i+k-1} = \sum_{t=1}^{n} D^{(n-i+1)}_{1t} X_{t,i+k-1} = X_{i,i+k-1} \ .$$

If $i+k-1 = n-(n-i+1)+t$ then $t = k$. Thus $D^{(n-i+1)}_{t,i+k-1} = \begin{cases} 1 & \text{if } t = k, \\ 0 & \text{if } t \neq k, \end{cases}$ so

$$(XD^{(n-i+1)})_{1,i+k-1} = \sum_{t=1}^{n} X_{1t}D^{(n-i+1)}_{t,i+k-1} = X_{1k}.$$

Hence $X_{i,i+k-1} = X_{1k}$, as required.

Therefore, $X \varepsilon \mathcal{V}$.

Hence $\mathcal{V}$ is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$. #

Lemma 4.3. Let S be a commutative semiring and n a positive integer. If $\mathcal{M}$ is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$, then the set $\mathcal{M}^T = \{A^T/A \varepsilon \mathcal{M}\}$ is also a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$.

Proof. If $\mathcal{T}$ is a commutative subsemigroup of the matrix semigroup $M_n(S)$, then for $A,B \varepsilon \mathcal{T}$, $AB = BA \varepsilon \mathcal{T}$, so $A^TB^T = (BA)^T = (AB)^T = B^TA^T \varepsilon \mathcal{T}^T$, hence $\mathcal{T}^T$ is a commutative subsemigroup of the matrix semigroup $M_n(S)$ where $\mathcal{T}^T = \{A^T/A \varepsilon \mathcal{T}\}$.

Let $\mathcal{M}$ be a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$. Then $\mathcal{M}^T$ is a commutative subsemigroup of the matrix semigroup $M_n(S)$. To show $\mathcal{M}^T$ is maximal, let $\mathcal{T}$ be a commutative subsemigroup of the matrix semigroup $M_n(S)$ and $\mathcal{M}^T \subseteq \mathcal{T}$. Then $\mathcal{M} = (\mathcal{M}^T)^T \subseteq \mathcal{T}^T$, so $\mathcal{M} = \mathcal{T}^T$ since $\mathcal{T}^T$ is a commutative subsemigroup of $M_n(S)$ and $\mathcal{M}$ is a maximal commutative subsemigroup of $M_n(S)$. Therefore, $\mathcal{M}^T = (\mathcal{T}^T)^T = \mathcal{T}$. #

Theorem 4.4. Let S be a commutative semiring with 0,1 and n is positive integer. Then the set of all n×n matrices over S in the form

$$\begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_{n-1} & a_{n-3} & \cdots & a_1 \end{bmatrix}$$

is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$.

Proof. Let $\mathcal{W}$ be the set of all n×n matrices over S in the above form. Then $\mathcal{W}^T$ is the set of all n×n matrices over S in the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_2 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_1 \end{bmatrix}$$

By Theorem 4.2, $\mathcal{W}^T$ ia a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$. Then by Lemma 4.3, $\mathcal{W} = (\mathcal{W}^T)^T$ is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$. #

Theorem 4.5 Let R be a commutative ring with 1 and n a positive integer. Then the set of all n×n matrices over R in the form

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & b_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -b_2 & \cdots & a_2 & 0 \\ -b_1 & 0 & \cdots & 0 & a_1 \end{bmatrix}$$

with $(\frac{n+1}{2}, \frac{n+1}{2})$- entry arbitrary for n being odd, is a maximal commutative subsemigroup of the matrix semigroup $M_n(S)$.

<u>Proof</u>. Let $\mathcal{X}$ be the set of all n×n matrices over R in the above form, that is, for $A \in M_n(R)$, $A \in \mathcal{X}$ if and only if $A_{ii} = A_{(n-i)+1,(n-i)+1}$, $A_{i,(n-i)+1} = -A_{(n-i)+1,i}$ for all $i \in \Lambda$ and $A_{ij} = 0$ for all $i,j \in \{1,2,\ldots,n\}$, $j \neq i$ and $j \neq (n-i)+1$ where

$$\Lambda = \begin{cases} \{1,2,\ldots,\frac{n}{2}\} & \text{if n is even ,} \\ \{1,2,\ldots,\frac{n-1}{2}\} & \text{if n is odd .} \end{cases}$$

Then $aI_n \in \mathcal{X}$ for every $a \in R$ where I_n is the identity n×n matrix over R.

To show that $\mathcal{X}$ is a commutative subsemigroup of the matrix semigroup $M_n(S)$, let

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & b_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -b_2 & \cdots & a_2 & 0 \\ -b_1 & 0 & \cdots & 0 & a_1 \end{bmatrix}, \quad A' = \begin{bmatrix} a'_1 & 0 & \cdots & 0 & b'_1 \\ 0 & a'_2 & \cdots & b'_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -b'_2 & \cdots & a'_2 & 0 \\ -b'_1 & 0 & \cdots & 0 & a'_1 \end{bmatrix}$$

be elements of $\mathcal{X}$. Then

$$AA' = \begin{bmatrix} a_1a'_1 - b_1b'_1 & 0 & \cdots & 0 & a_1b'_1 + b_1a'_1 \\ 0 & a_2a'_2 - b_2b'_2 & \cdots & a_2b'_2 + b_2a'_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -b_2a'_2 - a_2b'_2 & \cdots & -b_2b'_2 + a_2a'_2 & 0 \\ -b_1a'_1 - a_1b'_2 & 0 & \cdots & 0 & -b_1b'_1 + a_1a'_1 \end{bmatrix},$$

$$A'A = \begin{bmatrix} a'_1a_1 - b'_1b_1 & 0 & & 0 & a'_1b_1 + b'_1a_1 \\ 0 & a'_2a_2 - b'_2b_2 & \cdots & a'_2b_2 + b'_2a_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -b'_2a_2 - a'_2b_2 & \cdots & -b'_2b_2 + a'_2a_2 & 0 \\ -b'_1a_1 - a'_1b_1 & 0 & \cdots & 0 & -b'_1b_1 + a'_1a_1 \end{bmatrix}$$

so $AA' = A'A \in \mathcal{X}$ since R is commutative.

To show that $\mathcal{X}$ is a maximal commutative subsemigroup of the matrix semigroup $M_n(R)$, let $X = (x_{ij}) \in M_n(R)$ be such that $AX = XA$ for every $A \in \mathcal{X}$. For $k \in \Lambda$, define $A^{(k)}$ as in the proof of Theorem 4.1, that is, $A^{(k)}_{ij} = \begin{cases} 1 & \text{if } i = j = k \text{ or } i = j = (n-k)+1, \\ 0 & \text{otherwise.} \end{cases}$ Then $A^{(k)} \in \mathcal{X}$ for every $k \in \Lambda$, so $A^{(k)}X = XA^{(k)}$ for every $k \in \Lambda$. As in the proof of Theorem 4.1, we have that $x_{ij} = 0$ for all $i,j \in \{1,2,\ldots,n\}$, $j \neq i$ or $j \neq (n-i)+1$, thus

$$X = \begin{bmatrix} x_{11} & 0 & \cdots & 0 & x_{1n} \\ 0 & x_{22} & \cdots & x_{2,n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_{n-1,2} & \cdots & x_{n-1,n-1} & 0 \\ x_n & 0 & \cdots & 0 & x_{nn} \end{bmatrix}.$$

Let B be the n×n matrix over R defined by $B_{ij} = \begin{cases} 1 & \text{if } i\in\Lambda \text{ and } j=(n-i)+1, \\ -1 & \text{if } j\in\Lambda \text{ and } i=(n-j)+1, \\ 0 & \text{otherwise.} \end{cases}$

Then $B \in \mathcal{X}$, so $BX = XB$. If $i \in \Lambda$, then

$$B_{t,(n-i)+1} = \begin{cases} 1 & \text{if } t = i\ , \\ 0 & \text{if } t \neq i\ , \end{cases}$$

$$B_{it} = \begin{cases} 1 & \text{if } t = (n-i)+1 , \\ 0 & \text{if } t \neq (n-i)+1 , \end{cases}$$

$$B_{ti} = \begin{cases} -1 & \text{if } t = (n-i)+1 , \\ 0 & \text{if } t \neq (n-i)+1 , \end{cases}$$

so $$(XB)_{i,(n-i)+1} = \sum_{t=1}^{n} X_{it}B_{t,(n-i)+1} = X_{ii} ,$$

$$(BX)_{i,(n-i)+1} = \sum_{t=1}^{n} B_{it}X_{t,(n-i)+1} = X_{(n-i)+1,(n-i)+1} ,$$

$$(BX)_{ii} = \sum_{t=1}^{n} B_{it}X_{ti} = X_{(n-i)+1,i} ,$$

$$(XB)_{ii} = \sum_{t=1}^{n} X_{it}B_{ti} = -X_{i,(n-i)+1} ,$$

which imply $X_{ii} = X_{(n-i)+1,(n-i)+1}$, $X_{i,(n-i)+1} = -X_{(n-i)+1,i}$. Hence $X \varepsilon \mathcal{X}$.

Therefore, $\mathcal{X}$ is a maximal commutative subsemigroup of the matrix semigroup $M_n(R)$. #