CHAPTER III

REGULAR MATRIX SEMIGROUPS

In this chapter we study the regularity of matrix semigroups over some semirings.

It is known that for any field F and for any positive integer n, the matrix semigroup $M_n(F)$ is regular (that is, for any n×n matrix A over F, there exists an n×n matrix B over F such that A = ABA). More generally, it is known that for any ring R and for any positive integer n, the matrix semigroup $M_n(R)$ is regular if and only if R is a regular ring (see [3]). To generalize this result, we characterize regular matrix semigroups $M_n(S)$ with S an additively commutative semiring with 0 and 0 the only additive idempotent. We also characterize regular matrix semigroups $M_n(S)$ with S a semilattice semiring with 0,1.

If S is an additively commutative semiring, the matrix semigroup $M_1(S)$ is isomorphic to the multiplicative structure of the semiring S. Thus we shall study the regularity of the matrix semigroup $M_n(S)$ with S an additively commutative semiring with 0 and $n \geqslant 2$.

The first theorem shows that the regularity of an additively commutative semiring S with 0 is a necessary condition for the matrix semigroup $M_n(S)$ to be regular where n is any positive integer $n \geqslant 2$. However this condition is not a sufficient one.

<u>Theorem 3.1</u>. Let S be an additively commutative semiring with 0, n a positive integer and $n \geqslant 2$. If the matrix semigroup $M_n(S)$ is regular, then S is a regular semiring.

<u>Proof</u>. Let a be an element of S, and let A be the n×n matrix over S defined by

$$A = \begin{bmatrix} a & 0 & 0 & \cdots & a \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $M_n(S)$ is regular, $A = ABA$ for some $B \in M_n(S)$. Let $B = (b_{ij})$. Then

$$AB = \begin{bmatrix} a(b_{11}+b_{n1}) & a(b_{12}+b_{n2}) & a(b_{13}+b_{n3}) & \cdots & a(b_{1n}+b_{nn}) \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ab_{11} & ab_{12} & ab_{13} & \cdots & ab_{1n} \end{bmatrix}$$

$$ABA = \begin{bmatrix} a(b_{11}+b_{n1}+b_{1n}+b_{nn})a & 0 & 0 & \cdots & a(b_{11}+b_{n1})a \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a(b_{11}+b_{1n})a & 0 & 0 & \cdots & ab_{11}a \end{bmatrix}$$

Since $A = ABA$, we have that

$$a = a(b_{11}+b_{n1}+b_{1n}+b_{nn})a,$$

$$a = a(b_{11}+b_{n1})a,$$

$$a = a(b_{11}+b_{1n})a,$$

$$0 = ab_{11}a.$$

Since $ab_{11}a = 0$, the second equality and the third equality give

$a = ab_{n1}a$ and $a = ab_{1n}a$, respectively. Then the first equality gives $a = a+ab_{nn}a+a$. This shows that $a = axa$ and $a = a+y+a$ for some $x,y \in S$. Hence S is a regular semiring. #

<u>Example</u>. Let S be the semiring ({0,1,2,3}, max,min). Then S is an additively commutative semiring with zero 0 and identity 3 and S is a regular semiring. We denote max $\{a,b\}$ and min $\{a,b\}$ by $a+b$ and $a\cdot b$, respectively.

We shall show that the matrix semigroup $M_2(S)$ is not regular. Suppose it is regular. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \in M_2(S)$. Then $A = ABA$ for some $B \in M_2(S)$. Let $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 0\cdot b_1 + 1\cdot b_3 & 0\cdot b_2 + 1\cdot b_4 \\ 2\cdot b_1 + 3\cdot b_3 & 2\cdot b_2 + 3\cdot b_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot b_3 & 1\cdot b_4 \\ 2\cdot b_1 + b_3 & 2\cdot b_2 + b_4 \end{bmatrix}$$

since 0 is the zero and 3 is the identity of the semiring S. Thus

$$ABA = (AB)A = \begin{bmatrix} 1\cdot b_4\cdot 2 & 1\cdot b_3\cdot 1 + 1\cdot b_4\cdot 3 \\ (2\cdot b_2 + b_4)\cdot 2 & (2\cdot b_1 + b_3)\cdot 1 + (2\cdot b_2 + b_4)\cdot 3 \end{bmatrix}.$$

Then $1\cdot b_4\cdot 2 = 0$ and $(2\cdot b_1 + b_3)\cdot 1 + (2\cdot b_2 + b_4)\cdot 3 = 3$. Since $\min\{1,b_4,2\} = 1\cdot b_4\cdot 2 = 0$, $b_4 = 0$. Since

$$(2\cdot b_1 + b_3)\cdot 1 = \min\{2\cdot b_1 + b_3, 1\} \leqslant 1$$

and

$$(2\cdot b_2 + b_4)\cdot 3 = 2\cdot b_2 = \min\{2,b_2\} \leqslant 2 ,$$

we have that

$$3 = (2\cdot b_1 + b_3)\cdot 1 + (2\cdot b_2 + b_4)\cdot 3 = \max\{(2\cdot b_1 + b_3)\cdot 1, (2\cdot b_2 + b_4)\cdot 3\} \leqslant 2,$$

a contradiction. #

The next theorem gives a necessary and sufficient condition for an additively commutative semiring S with 0 to have the property that the matrix semigroup $M_n(S)$ is regular for $n \geqslant 3$.

<u>Theorem 3.2</u>. Let S be an additively commutative semiring with 0 and n a positive integer and $n \geqslant 3$. Then the matrix semigroup $M_n(S)$ is regular if and only if S is a regular ring.

<u>Proof</u>. Assume the matrix semigroup $M_n(S)$ is regular. To show that S is a regular ring, let $a \in S$. Let A be an $n\times n$ matrix defined by

$$A = \begin{bmatrix} a & a & 0 & \cdots & 0 \\ 0 & a & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & 0 & 0 & \cdots & a \end{bmatrix},$$

that is, $A_{ij} = \begin{cases} a & \text{if } j = i \text{ or } j = i+1 \text{ or } (j=1 \text{ and } i=n), \\ 0 & \text{otherwise.} \end{cases}$ Since the matrix semigroup $M_n(S)$ is regular, $A = ABA$ for some $B \in M_n(S)$. Let $B = (b_{ij})$. Then

$$AB = \begin{bmatrix} a(b_{11}+b_{21}) & a(b_{12}+b_{22}) & a(b_{13}+b_{23}) & \cdots & a(b_{1n}+b_{2n}) \\ a(b_{21}+b_{31}) & a(b_{22}+b_{32}) & a(b_{23}+b_{33}) & \cdots & a(b_{2n}+b_{3n}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a(b_{11}+b_{n1}) & a(b_{12}+b_{n2}) & a(b_{13}+b_{n3}) & \cdots & a(b_{1n}+b_{nn}) \end{bmatrix}$$

so ABA = (AB)A =

$$\begin{bmatrix} a(b_{11}+b_{21}+b_{1n}+b_{2n})a & a(b_{11}+b_{21}+b_{12}+b_{22})a & \cdots & a(b_{1,n-1}+b_{2,n-1}+b_{1n}+b_{2n})a \\ a(b_{21}+b_{31}+b_{2n}+b_{3n})a & a(b_{21}+b_{31}+b_{22}+b_{32})a & \cdots & a(b_{2,n-1}+b_{3,n-1}+b_{2n}+b_{3n})a \\ \cdots & \cdots & \cdots & \cdots \\ a(b_{11}+b_{n1}+b_{1n}+b_{nn})a & a(b_{11}+b_{n1}+b_{12}+b_{n2})a & \cdots & a(b_{1,n-1}+b_{n,n-1}+b_{1n}+b_{nn})a \end{bmatrix}$$

Since A = ABA, we have

$$a = a(b_{11}+ b_{21}+ b_{1n}+ b_{2n})a \quad \ldots\ldots (1)$$

$$0 = a(b_{1,n-1}+ b_{2,n-1}+ b_{1n}+ b_{2n})a \quad \ldots\ldots (2)$$

$$0 = a(b_{21}+ b_{31}+ b_{2n}+ b_{3n})a \quad \ldots\ldots (3)$$

$$0 = a(b_{11}+ b_{n1}+ b_{12}+ b_{n2})a \quad \ldots\ldots (4)$$

From (1),(2),(3) and (4) we have

$$a = ab_{11}a + ab_{21}a + a(b_{1n}+ b_{2n})a \quad \ldots\ldots (1')$$

$$0 = a(b_{1,n-1}+ b_{2,n-1})a + a(b_{1n}+ b_{2n})a \quad \ldots\ldots (2')$$

$$0 = ab_{21}a + a(b_{31}+ b_{2n}+ b_{3n})a \quad \ldots\ldots (3')$$

$$0 = ab_{11}a + a(b_{n1}+ b_{12}+ b_{n2})a \quad \ldots\ldots (4')$$

respectively. Then (2′) +(3′) +(4′) gives $a+a(b_{1,n-1}+ b_{2,n-1})a + a(b_{31}+ b_{2n}+ b_{3n})a + a(b_{n1}+ b_{12}+ b_{n2})a = 0$. Hence $a+x = 0$ for some $x \in S$. The equality (1) shows that $a = aya$ for some $y \in S$. Therefore we have that S is a regular ring.

The converse follows from [3; Theorem 24 of Part II]. #

Let S be an additively commutative semiring with 0. If S is a regular ring, then the matrix semigroup $M_2(S)$ is regular [3; Theorem 24 of Part II]. If the matrix semigroup $M_2(S)$ is regular, by Theorem 3.1, S is a regular semiring. The following example shows that if the matrix semigroup $M_2(S)$ is regular, S is not necessary to be a ring

<u>Example</u>. Let S be a Boolean algebra of 2 elements 0,1. Then 0+0 = 0, 0+1 = 1+0 = 1+1 = 1 and 0•0 = 0•1 = 1•0 = 0, 1•1 = 1. Then S is not a ring. We shall show that the matrix semigroup $M_2(S)$ is regular. The matrix semigroup $M_2(S)$ has exactly 16 elements and they are

$$\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},$$
$$\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

It is easy to check that the following 11 matrices are all of idempotents of the matrix semigroup $M_2(S)$.

$$\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},$$
$$\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

Consider the remaining 5 matrices :

$$\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix},\begin{bmatrix}1&1\\1&0\end{bmatrix}.$$

Since

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix},$$
$$\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix},$$
$$\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix},$$
$$\begin{bmatrix}0&1\\1&1\end{bmatrix}=\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&1\end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

It follows that each of element of the matrix semigroup $M_2(S)$ is regular. #

Corollary 3.3. Let S be a semilatice semiring with 0 and $|S| > 1$. Then the matrix semigroup $M_n(S)$ is not regular if $n \geqslant 3$

Proof. Since every element of S is an additive idempotent and $|S| > 1$, S is not a ring. By Theorem 3.2, the matrix semigroup $M_n(S)$ is not regular for $n \geqslant 3$. #

In any ring R, 0 is the only additive idempotent of R. However an additively commutative semiring with 0 which 0 is the only additive idempotent is not necessary to be a ring. The semirings $(\mathbb{N} \cup \{0\},+,\cdot)$, $(\mathbb{Q}^+ \cup \{0\},+,\cdot)$ and $(\mathbb{R}^+ \cup \{0\},+,\cdot)$ are examples. Then additively commutative semirings with 0 having 0 as the only additive idempotent are a generalization of rings. Next, we characterize regular matrix semigroup $M_n(S)$ with $n \geqslant 2$ and S an additively commutative semiring with 0 having 0 as the only additive idempotent.

Theorem 3.4. Let S be an additively commutative semiring with 0 and assume that 0 is the only additive idempotent of S. Then for $n \geqslant 2$, the matrix semigroup $M_n(S)$ is regular if and only if S is a regular ring.

Proof. Let $n \geqslant 2$ and assume that the matrix semigroup $M_n(S)$ is regular. By Theorem 3.1, S is a regular semiring. Then (S,+) is a regular semigroup containing exactly one idempotent 0. Hence (S,+) is a group with identity 0, so S is a ring. Therefore S is a regular ring.

The converse follows from [3; Theorem 24 of Part II]. #

Corollary 3.5. If S is one of the following semirings : $(\mathbb{N} \cup \{0\},+,\cdot)$, $(\mathbb{Q}^+ \cup \{0\},+,\cdot)$, $(\mathbb{R}^+ \cup \{0\},+,\cdot)$, then the matrix semigroup $M_n(S)$ is not regular for every $n \geqslant 2$.

If S is a semilattice semiring with 0,1, Corollary 3.3 shows that the matrix semigroup $M_n(S)$ is not regular for $n \geqslant 3$. The matrix semigroup $M_2(S)$ with S a semilattice semiring with 0,1 is now considered. The example after Theorem 3.1 shows that if S is a semilattice semiring with 0,1, the matrix semigroup $M_2(S)$ is not necessarily regular. A necessary and sufficient condition for a semilattice semiring S with 0,1 such that the matrix semigroup $M_2(S)$ is regular is given by the following theorem.

Theorem 3.6. Let S be a semilattice semiring with 0,1 and n a positive integer and $n \geqslant 2$. Then the matrix semigroup $M_n(S)$ is regular if and only if $n = 2$ and S is a Boolean algebra.

Proof. Assume that the matrix semigroup $M_n(S)$ is regular. Since $|S| > 1$ by Corollary 3.3, $n \not\geqslant 3$. But $n \geqslant 2$, so $n = 2$.

Claim that $a+1 = 1$ for every $a \in S$. Let $a \in S$. Since the matrix semigroup $M_n(S)$ is regular and $\begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} \in M_2(S)$, there exist elements x,y,z,w ε S such that

$$\begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} .$$

Then

$$\begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} ax+z & ay+w \\ x+z & y+w \end{bmatrix} \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ax+az+ay+w & ax+z+ay+w \\ ax+az+y+w & x+y+z+w \end{bmatrix}$$

since $a^2 = a$, so

$$a = ax + ay + az + w \quad \ldots\ldots \quad (1)$$

$$1 = ax + ay + z + w \quad \ldots\ldots \quad (2)$$

$$1 = ax + az + y + w \quad \ldots\ldots \quad (3)$$

$$1 = x + y + z + w \quad \ldots\ldots \quad (4)$$

Then (1)+(2)+(3)+(4) gives

$$a+1 = ax + ay + az + x + y + z + w$$

since s+s = s for every s ε S. Hence

$$\begin{aligned} a+1 &= (ax+ax) + ay + az + x + (y+y) + (z+z) + (w+w+w) \\ &= (ax+ay+z+w)+(ax+az+y+w)+(x+y+z+w) \\ &= 1+1+1 \quad \text{(from (2),(3),(4))} \\ &= 1 \end{aligned}$$

Next, we want to show that S is a Boolean algebra, that is, we want to show that

(i) $a+bc = (a+b)(a+c)$ for all a,b,c ε S,

(ii) for every a ε S, there exists an element a' ε S such that $a+a' = 1$ and $aa' = 0$.

To prove (i), let $a,b,c \in S$. Since $x+1 = 1$ for every $x \in S$,

$$\begin{aligned}(a+b)(a+c) &= a^2 + ac + ba + bc \\ &= a + ab + ac + bc \\ &= a(1+b+c) + bc \\ &= a1 + bc \\ &= a + bc.\end{aligned}$$

To prove (ii), let $a \in S$. Since the matrix semigroup $M_2(S)$ is regular and $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \in M_2(S)$, there exist elements $x,y,z,w \in S$ such that

$$\begin{aligned}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} &= \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \\ &= \begin{bmatrix} ax+z & ay+w \\ az & aw \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \\ &= \begin{bmatrix} ax+az & ax+z+ay+aw \\ az & az+aw \end{bmatrix}.\end{aligned}$$

Then

$$a = ax + az \quad \ldots\ldots (1')$$

$$1 = ax + z + ay + aw \quad \ldots\ldots (2')$$

$$0 = az \quad \ldots\ldots (3')$$

$$a = az + aw \quad \ldots\ldots (4')$$

From $(1')$ and $(3')$, we have that $ax = a$. Then

$$\begin{aligned}1 &= ax + ay + aw + z \quad \text{(from } (2')\text{)} \\ &= a + ay + aw + z \\ &= a(1+y+w) + z \\ &= a1 + z \quad \text{(since } s+1 = 1 \text{ for every } s \in S\text{)} \\ &= a + z.\end{aligned}$$

Then a+z = 1, and az = 0 (from (3′)).

Hence S is a Boolean algebra.

To prove the converse, assume that S is a Boolean algebra and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \varepsilon M_2(S)$. Then there exists an element k ε S such that (ad+bc) + k = 1 and (ad+bc)k = 0. Since S is a semilattice semiring and adk + bck = (ad+bc)k = 0, we have that adk = bck = 0. Let $B = \begin{bmatrix} d+k & b+k \\ c+k & a+k \end{bmatrix}$. Then $B \ \varepsilon \ M_2(S)$ and

$$ABA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d+k & b+k \\ c+k & a+k \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a(d+k)+b(c+k) & a(b+k)+b(a+k) \\ c(d+k)+d(c+k) & c(b+k)+d(a+k) \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} (a(d+k)+b(c+k))a+(a(b+k)+b(a+k))c & (a(d+k)+b(c+k))b+(a(b+k)+b(a+k))d \\ (c(d+k)+d(c+k))a+(c(b+k)+d(a+k))c & (c(d+k)+d(c+k))b+(c(b+k)+d(a+k))d \end{bmatrix}.$$

Therefore $(ABA)_{11} = (a(d+k)+b(c+k))a+(a(b+k)+b(a+k))c$

$= a(d+k+bc+bk) + abc + ack + abc + bck$

$= a(d+k+bc+bk+ck) + bck$

$= a(d+k+bc+bk+ck)$ (since bck = 0)

$= a(ad+bc+k+bk+ck)$

$= a(1+bk+ck)$ (since ad+bc+k = 1)

$= a$ (since x+1 = 1 for every x ε S)

$= A_{11}$,

$(ABA)_{12} = (a(d+k)+b(c+k))b+(a(b+k)+b(a+k))d$

$= b(ad+ak+c+k) + abd + bdk + abd + adk$

$= b(ad+ak+c+k+dk) + adk$

$= b(ad+bc+k+ak+dk)$ (since adk = 0)

$= b(1+ak+dk)$ (since $ad+bc+k = 1$)

$= b$ (since $x+1 = 1$ for every $x \in S$)

$= A_{12}$,

$(ABA)_{21} = (c(d+k)+d(c+k)a+(c(b+k)+d(a+k))c$

$= c(c(b+k)+d(a+k)) + adc + ack + acd + adk$

$= c(c(b+k)+d(a+k)+ad+ak+ad) + adk$

$= c(ad+bc+k+dk+ak)$ (since $adk = 0$)

$= c(1+dk+ak)$ (since $ad+bc+k = 1$)

$= c$ (since $x+1 = 1$ for every $x \in S$)

$= A_{21}$,

$(ABA)_{22} = (c(d+k)+d(c+k))b+(c(b+k)+d(a+k))d$

$= d(c(b+k)+d(a+k)) + bcd + bdk + bck$

$= d(c(b+k)+d(a+k)+bc+bk) + bck$

$= d(ad+bc+k+bk+ck)$ (since $bck = 0$)

$= d(1+bk+ck)$ (since $ad+bc+k = 1$)

$= d$ (since $x+1 = 1$ for every $x \in S$)

$= A_{22}$.

Hence A is regular. #

<u>Corollary 3.7</u>. Let S be a set of real numbers with the minimum element and the maximum element and $|S| > 2$. Then matrix semigroup $M_n((S,\max,\min))$ is not regular for every $n \geqslant 2$.

In particular, the matrix semigroup $M_n(([0,1],\max,\min))$ is not regular for every $n \geqslant 2$.

<u>Proof</u>. Let m and M be the minimum element and the maximum elements of S, respectively. Then m and M are the zero and the identity of the semiring (S,max,min), respectively.

To show the semilattice semiring (S,max,min) is not a Boolean algebra, suppose it is a Boolean algebra. Since $|S| > 2$, there exists an element $a \in S$ such that $m < a < M$. Then there exists an element $\acute{a} \in S$ such that $a+\acute{a} = M$ and $a\acute{a} = m$. Since $a < M$ and $M = a+\acute{a} = \max\{a,\acute{a}\}$, it follows that $\acute{a} = M$. Then $m = a\acute{a} = \min\{a,\acute{a}\} = \min\{a,M\} = a$, a contradiction. Hence the semilattice semiring (S,max,min) is not a Boolean algebra. By Theorem 3.6, the matrix semigroup $M_n((S,\max,\min))$ is not regular for every $n \geqslant 2$. #