CHAPTER II

SOME CERTAIN SEMIGROUPS

The purpose of this chapter is to characterize the following semigroups which are absolutely closed :

(1) semigroups having group extensions,

(2) semigroups having extensions which are group with zero,

(3) free semigroups,

(4) zero semigroups.

To characterize semigroups having group extensions which are absolutely closed, the following lemma is required.

<u>Lemma 2.1</u>. Let S be a subsemigroup of a group G. Then S is closed in G if and only if S is a subgroup of G.

<u>Proof</u> : Assume that S is closed in G. Then Dom(S,G) = S. Let $a \in S$. Since

$$\begin{aligned} 1 &= aa^{-1} &&, \; a \in S, \; a^{-1} \in G, \\ &= a^{-1}a^2a^{-1} &&, \; a^2 \in S, \; a = a^{-1}a^2, \\ &= a^{-1}a &&, \; a = a^2a^{-1}, \end{aligned}$$

it follows from Theorem 1.1 that $1 \in \mathrm{Dom}(S,G) = S$.

Also, for each $x \in S$, we have

$$
\begin{aligned}
x^{-1} &= xx^{-2} && , \; x \in S \, , \; x^{-2} \in G \, , \\
&= x^{-1}x^{2}x^{-2} && , \; x^{2} \in S, \; x^{-1} \in G \, , \; x = x^{-1}x^{2} \, , \\
&= x^{-1}1 && , \; 1 \in S, \; 1 = x^{2}x^{-2} \, ,
\end{aligned}
$$

which implies by Theorem 1.1 that $x^{-1} \in \mathrm{Dom}(S,G) = S$.

This proves that S is a subgroup of G.

The converse follows directly from Theorem 1.3. #

<u>Theorem 2.2</u>. Let S be a semigroup having a group extension. Then S is absolutely closed if and only if S is a group.

<u>Proof</u> : Let G be a group extension of S. Then S is a subsemigroup of G.

If S is absolutely closed, then S is closed in G, so by Lemma 2.1, S is a subgroup of G which implies that S is a group.

The converse follows directly from Theorem 1.3. #

Because the set of all complex numbers forms a group under usual addition, the following corollary is clearly obtained from Theorem 2.2.

<u>Corollary 2.3</u>. An additive semigroup of (complex) numbers is absolutely closed if and only if it is a group.

It follows from Corollary 2.3 that each of the semigroups $(\mathbb{N},+)$, $(\mathbb{Q}^{+},+)$ and $(\mathbb{R}^{+},+)$ is not absolutely closed where $\mathbb{N}$, $\mathbb{Q}^{+}$ and $\mathbb{R}^{+}$ denote the set of all positive integers, the set of all positive rational numbers and the set of all positive real numbers, respectively and + denotes the usual addition of complex numbers. Since $(\mathbb{N},+)$ is not absolutely closed, it follows that every infinite cyclic semigroup is not absolutely closed.

Corollary 2.4. Every infinite cyclic semigroup is not absolutely closed.

The next theorem characterizes absolutely closed semigroups having extensions which are groups with zero. To prove this, we need the following lemma :

Lemma 2.5. Let T be a group with zero 0 and S a subsemigroup of T. Then S is closed in T if and only if S is either a subgroup of T or a subgroup with zero of T. (S is called a subgroup with zero of T if $0 \in S$ and $S \setminus \{0\}$ is a subgroup of T.)

Proof : Assume that S is closed in T. Then Dom(S,T) = S. To prove that S is either a subgroup of T or a subgroup with zero of T, it suffices to prove that if $S \neq \{0\}$, then $1 \in S$ and for $x \in S$, $x \neq 0$ implies $x^{-1} \in S$. Assume that $S \neq \{0\}$. Let $a \in S \setminus \{0\}$. From

$$\begin{aligned} 1 &= aa^{-1} &&, a \in S , a^{-1} \in T , \\ &= a^{-1}a^2a^{-1} &&, a^2 \in S, a = a^{-1}a^2 , \\ &= a^{-1}a &&, a = a^2a^{-1} , \end{aligned}$$

we have that $1 \in$ Dom(S,T) (Theorem 1.1), so $1 \in S$. Next, let $x \in S$ and $x \neq 0$. Then

$$x^{-1} = xx^{-2} \quad , x \in S , x^{-2} \in T ,$$

$$= x^{-1}x^{2}x^{-2} \quad , x^{2} \in S , x^{-1} \in T , x = x^{-1}x^{2} ,$$

$$= x^{-1}1 \quad , 1 \in S , 1 = x^{2}x^{-2} .$$

By Theorem 1.1, $x^{-1} \in S$.

The converse follows directly from Theorem 1.3. #

Theorem 2.6. Let S be a semigroup having an extension which is a group with zero. Then S is absolutely closed if and only if S is either a group or a group with zero.

Proof : Let T be a group with zero having S as a subsemigroup.

If S is absolutely closed, then S is closed in T, so by Lemma 2.5 , S is either a subgroup of T or a subgroup with zero of T. Hence S is a group or a group with zero.

The converse holds by Theorem 1.3. #

Since the set of all complex numbers under usual multiplication forms a group with zero, by Theorem 2.6, we have the following corollary :

Corollary 2.7. A multiplicative semigroup of (complex) numbers is absolutely closed if and only if it is either a group or a group with zero.

Corollary 2.7 implies that each of the semigroups $(\mathbb{N},\cdot)$, $([0,1],\cdot)$ and $(\{z \in \mathbb{C} \mid |z| \leq 1\},\cdot)$ is not absolutely closed where $\mathbb{C}$

denotes the set of all complex numbers and $\cdot$ denotes the usual multiplication of complex numbers.

We show in the next theorem by Corollary 2.4 and Theorem 1.4 that every free semigroup on a nonempty set is not absolutely closed.

Theorem 2.8. For any nonempty set X, the free semigroup on X, $\mathcal{F}_X$ is not absolutely closed.

Proof : If $|X| = 1$, then $\mathcal{F}_X$ is an infinite cyclic semigroup, hence by Corollary 2.4, $\mathcal{F}_X$ is not absolutely closed.

Suppose that $|X| > 1$. Let a and b be two distinct elements in X. Then $a, b \in \mathcal{F}_X$. It follows from the definition of free semigroup that $\mathcal{F}_X a \cap \mathcal{F}_X b = b\mathcal{F}_X \cap a\mathcal{F}_X = \emptyset$ since $a \neq b$. Hence by Theorem 1.4, $\mathcal{F}_X$ is not absolutely closed. #

Finally, we shall characterize zero semigroups which are absolutely closed. First, we prove the following lemma :

Lemma 2.9. Let U be a semigroup having a zero 0 and S a semigroup containing U as a subsemigroup. Assume that $d \in Dom(U,S)$ and

$$
\begin{aligned}
d &= u_0 y_1 , \quad u_0 \in U , \ y_1 \in S , \\
&= x_1 u_1 y_1 , \ u_1 \in U , \ x_1 \in S , \ u_0 = x_1 u_1 , \\
&= x_1 u_2 y_2 , \ u_2 \in U , \ y_2 \in S , \ u_1 y_1 = u_2 y_2 , \\
&= x_2 u_3 y_2 , \ u_3 \in U , \ x_2 \in S , \ x_1 u_2 = x_2 u_3 , \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots \\
&= x_m u_{2m-1} y_m , \ u_{2m-1} \in U, \ x_m \in S, \ x_{m-1} u_{2m-2} = x_m u_{2m-1} , \\
&= x_m u_{2m} , \ u_{2m} \in U , \ u_{2m-1} y_m = u_{2m} ,
\end{aligned}
$$

that is,

$$d = u_0 y_1 \ , \ u_0 = x_1 u_1 \ ,$$

$$x_i u_{2i} = x_{i+1} u_{2i+1}, \ u_{2i-1} y_i = u_{2i} y_{i+1} \ (i=1,\ldots,m-1),$$

$$u_{2m-1} y_m = u_{2m},$$

for some $u_0, u_1, \ldots, u_{2m} \in U$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in S$.
Then $d = 0$ if one of the following conditions holds :

(1) $d = x_1 0$.

(2) $d = 0y_m$.

(3) $d = 0y_1$.

(4) $d = x_m 0$.

(5) $d = x_m 0 y_m$.

(6) $d = x_i 0 y_i$ for some $i \in \{1,\ldots,m-1\}$.

(7) $d = x_i 0 y_{i+1}$ for some $i \in \{1,\ldots,m-1\}$.

In particular, if $u_i = 0$ for some $i \in \{0,1,\ldots,2m\}$, then $d = 0$.

Proof : First, we note that $0u_i = 0 = u_i 0$ for all $i \in \{0,1,\ldots,2m\}$.

(1) Assume that $d = x_1 0$. Then $d = x_1 u_1 0$. But $x_1 u_1 = u_0$, so $d = u_0 0 = 0$.

(2) It can be proved similarly to (1) that if $d = 0y_m$, then $d = 0$.

(3) Since $u_{2i-1} y_i = u_{2i} y_{i+1}$ for all $i \in \{1,\ldots,m-1\}$, we have that for each $i \in \{1,\ldots,m-1\}$, $0y_i = 0u_{2i-1} y_i = 0u_{2i} y_{i+1} = 0y_{i+1}$. Hence it follows that if $d = 0y_1$, then $d = 0y_m$ which implies by (2) that $d = 0$.

(4) It can be proved similarly to (3) by using (1) that if $d = x_m 0$, then $d = 0$.

(5) Assume that $d = x_m 0 y_m$. Then $d = x_m 0 u_{2m-1} y_m$. Since $u_{2m-1} y_m = u_{2m}$, $d = x_m 0 u_{2m} = x_m 0$. By (4), we get that $d = 0$.

(6) Since $u_{2i-1} y_i = u_{2i} y_{i+1}$ for all $i \in \{1,\dots,m-1\}$, we have that $0y_i = 0u_{2i-1} y_i = 0u_{2i} y_{i+1} = 0y_{i+1}$ for all $i \in \{1,\dots,m-1\}$. Then

$$x_i 0 y_i = x_i 0 y_{i+1} \qquad \dots\dots\dots\dots\dots\dots (*)$$

for $i = 1,\dots,m-1$. Since $x_i u_{2i} = x_{i+1} u_{2i+1}$ for all $i \in \{1,\dots,m-1\}$, it follows that $x_i 0 = x_i u_{2i} 0 = x_{i+1} u_{2i+1} 0 = x_{i+1} 0$ for all $i \in \{1,\dots,m-1\}$. Then

$$x_i 0 y_{i+1} = x_{i+1} 0 y_{i+1} \qquad \dots\dots\dots\dots\dots\dots (**)$$

for $i = 1,\dots,m-1$. From (*) and (**), we have that

$$x_i 0 y_i = x_{i+1} 0 y_{i+1}$$

for $i = 1,\dots,m-1$. Hence it follows that if $d = x_i 0 y_i$ for some $i \in \{1,\dots,m-1\}$, then $d = x_m 0 y_m$ which implies that $d = 0$ by (5).

(7) If there exists $i \in \{1,\dots,m-1\}$ such that $d = x_i 0 y_{i+1}$, then $d = x_i 0 u_{2i} y_{i+1} = x_i 0 u_{2i-1} y_i$ since $u_{2i-1} y_i = u_{2i} y_{i+1}$, hence $d = x_i 0 y_i$, so $d = 0$ by (6). #

Theorem 2.10. Let S be a zero semigroup. Then S is absolutely closed if and only if $|S| \leq 2$.

Proof : Let 0 be the zero of S.

Assume that $|S| \leq 2$. If $|S| = 1$, then S is a group, hence S is absolutely closed. Assume that $|S| = 2$. Let $S = \{0,a\}$. Let T be a semigroup containing S as a subsemigroup. Suppose that $\mathrm{Dom}(S,T) \neq S$. Let $d \in \mathrm{Dom}(S,T) \setminus S$. Then, by Theorem 1.1, we have

$$d = u_0y_1,\ u_0 = x_1u_1,$$
$$x_iu_{2i} = x_{i+1}u_{2i+1},\ u_{2i-1}y_i = u_{2i}y_{i+1}\ (i=1,\ldots,m-1),$$
$$u_{2m-1}y_m = u_{2m},$$

for some $u_0,u_1,\ldots,u_{2m} \in S$, $x_1,\ldots,x_m$, $y_1,\ldots,y_m \in T$. By Lemma 2.9, we get that for each $i \in \{0,1,\ldots,2m\}$, $u_i \neq 0$, hence $u_i = a$ for all $i \in \{0,1,\ldots,2m\}$. Since for each $i \in \{1,\ldots,m-1\}$, $u_{2i-1}y_i = u_{2i}y_{i+1}$, for each $i \in \{1,\ldots,m-1\}$, $ay_i = ay_{i+1}$ which implies that $ay_1 = ay_m$. Since $d = u_0y_1$ and $u_0 = a$, we have $d = ay_1$, and hence $d = ay_m$. Since $u_{2m} = u_{2m-1}y_m$ and $u_{2m-1} = a = u_{2m}$, $a = u_{2m} = u_{2m-1}y_m = ay_m$. Thus $d = ay_m = a$. It is a contradiction since $d \notin S$. Therefore $\mathrm{Dom}(S,T) = S$.

This proves that S is absolutely closed.

For the converse, assume that $|S| > 2$. Let a_1 and a_2 be two distinct nonzero elements of S. Let a_3, a_4, $a_5,\ldots$ and b_1, b_2, $b_3,\ldots$ be distinct symbols not representing any elements of S and let

$$T = S \cup \{a_3,a_4,a_5,\ldots\} \cup \{b_1,b_2,b_3,\ldots\}.$$

Define the operation $*$ on T as follows :

$$x * y = 0 \quad \text{for all } x, y \in S \cup \{a_3,a_4,a_5,\ldots\},$$
$$x * y = y * x = 0 \quad \text{if } x \in S \setminus \{a_1,a_2\} \text{ and } y \in \{b_1,b_2,b_3,\ldots\},$$
$$b_i * b_j = b_{i+j} \quad \text{for all } i, j \in \mathbb{N} \text{ and}$$
$$a_i * b_j = b_j * a_i = a_{i+j} \text{ for all } i, j \in \mathbb{N}.$$

Then $(S \cup \{a_3,a_4,a_5,\ldots\},*)$ is a zero semigroup with zero 0, $(\{b_1,b_2,b_3,\ldots\},*)$ is an infinite cyclic semigroup (having b_1 as its generator) and $0 * x = x * 0 = 0$ for all $x \in T$.

For $x, y, z \in T$, it is easy to see that if at least one of x, y, z is in $S \setminus \{a_1,a_2\}$ or at least two of x, y, z are in

$\{a_1, a_2, a_3, \ldots\}$, then $(x * y) * z = 0 = x * (y * z)$. For $i, j, k \in \mathbb{N}$,

$$(a_i * b_j) * b_k = a_{i+j} * b_k = a_{i+j+k} = a_i * b_{j+k} = a_i * (b_j * b_k),$$

$$(b_i * a_j) * b_k = a_{i+j} * b_k = a_{i+j+k} = b_i * a_{j+k} = b_i * (a_j * b_k),$$

$$(b_i * b_j) * a_k = b_{i+j} * a_k = a_{i+j+k} = b_i * a_{j+k} = b_i * (b_j * a_k).$$

Hence $*$ is associative on T. Therefore $(T,*)$ is a semigroup having S as a subsemigroup.

Claim that S is not closed in T. We prove the claim by showing that $a_3 \in Dom(S,T)$ (which implies that $Dom(S,T) \neq S$). By defining T and $*$, we have that

$$\begin{aligned} a_3 &= a_2 * b_1 &&, a_2 \in S,\ b_1 \in T, \\ &= b_1 * a_1 * b_1 &&, a_1 \in S,\ a_2 = b_1 * a_2, \\ &= b_1 * a_2 &&, a_2 = a_1 * b_1 . \end{aligned}$$

By Theorem 1.1, $a_3 \in Dom(S,T)$. #