CHAPTER I

PRELIMINARIES

A nonempty subset I of a semigroup S is called an ideal of S if xa, ax ε I for all x ε S, a ε I.

Let S be a semigroup. An element e of S is called an identity of S if xe = ex = x for all x ε S. An element z of S is called a zero of S if xz = zx = z for all x ε S.

Any semigroup can have at most one identity and at most one zero. If an identity of a semigroup exists, it is usually denoted by 1, and if a semigroup has a zero, then 0 is usually used to denote its zero.

A semigroup S with zero 0 is called a zero semigroup if xy = 0 for all x, y ε S.

A subsemigroup G of a semigroup S is called a subgroup of S if G is also a group. A semigroup S with zero 0 is called a group with zero if $S \smallsetminus \{0\}$ is a subgroup of S.

A semigroup S is called a regular semigroup if for every element a of S, there is an element x of S such that a = axa.

A semigroup S is called an inverse semigroup if for every element a of S, there is a unique element a^{-1} of S such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Let S and T be semigroups and $\varphi: S \to T$ a map. The map φ is called a <u>homomorphism</u> of S into T if

$$(xy)\varphi = (x\varphi)(y\varphi)$$

for all $x, y \in S$.

Let X be a nonempty set. A nonempty finite sequence $a_1, a_2, \ldots, a_n$ usually written by juxtaposition, $a_1 a_2 \ldots a_n$, of elements of X is called a <u>word</u> over the alphabet X. The set $\mathcal{F}_X$ of all words with the operation of juxtaposition

$$(a_1 a_2 \ldots a_m)(b_1 b_2 \ldots b_n) = (a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n)$$

is a semigroup called the <u>free semigroup</u> on the set X.

Let X be a set and B_X the set of all binary relations on X. For any $\rho, \sigma \in B_X$, define their composition $\rho\sigma$ by

$$\rho\sigma = \{(a,b) \in X \times X \mid (a,x) \in \rho \text{ and } (x,b) \in \sigma \text{ for some } x \in X\}.$$

Then B_X is a semigroup under composition of relations, which is called the <u>semigroup of binary relations</u> on X.

A <u>partial transformation</u> of X is a map from a subset of X into X. The empty transformation of X is the partial transformation of X with empty domain and it is denoted by 0. For a partial transformation α of X, the domain and range of α are denoted by $\Delta\alpha$ and $\nabla\alpha$, respectively. Let P_X be the set of all partial transformations of X (including 0). For $\alpha, \beta \in P_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \emptyset$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \emptyset$, let $\alpha\beta = (\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}})(\beta|_{\nabla\alpha \cap \Delta\beta})$ (the composition of the maps $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$) where $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$ denote the restrictions of α and β to

$(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$ and $\nabla\alpha \cap \Delta\beta$, respectively. Then P_X is a regular semigroup having 0 and 1_X as its zero and identity, respectively where 1_X is the identity map on X. The semigroup P_X is called the partial transformation semigroup on X. Observe that $\alpha, \beta \in P_X$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$. In fact, P_X is a subsemigroup of B_X.

By a transformation semigroup on X, we mean a subsemigroup of P_X.

Let I_X be the set of all 1-1 partial transformation of X. Then I_X is an inverse subsemigroup of P_X and it is called the 1-1 partial transformation semigroup or the symmetric inverse semigroup on X.

By a transformation of X, we mean a map of X into itself.

Let T_X be the set of all transformations of X. Then T_X is a regular subsemigroup of P_X with identity 1_X and it is called the full transformation semigroup on X.

Let

G_X = the symmetric group on X,

M_X = the set of all 1-1 transformations of X

and

O_X = the set of all onto transformations of X.

Then M_X and O_X are subsemigroups of T_X containing G_X.

For $\alpha \in T_X$, $x \in X$, α is said to be 1-1 at x if $(x\alpha)\alpha^{-1} = \{x\}$. For $\alpha \in T_X$, α is said to be almost 1-1 if the set $\{x \in X \mid \alpha$ is not 1-1 at $x\}$ is finite. Let AM_X be the set of all almost 1-1 transformations of X. Clearly, $M_X \subseteq AM_X$. Claim that AM_X is a subsemigroup of T_X. To prove this, let $\alpha, \beta \in AM_X$. For convenience, for $\gamma \in T_X$, let $A_\gamma = \{x \in X \mid \gamma$ is not 1-1 at $x\}$. Hence for $\gamma \in T_X$, $\gamma \in AM_X$ if and only if A_γ is finite. Let $x \in X \setminus (A_\alpha \cup (A_\beta)\alpha^{-1})$. Then $x \in X \setminus A_\alpha$ and $x\alpha \in X \setminus A_\beta$. Let $y \in (x(\alpha\beta))(\alpha\beta)^{-1}$. Then $(y\alpha)\beta = y(\alpha\beta) = x(\alpha\beta) = (x\alpha)\beta$. Since $x\alpha \in X \setminus A_\beta$, $y\alpha = x\alpha$. This implies that $y = x$ since $x \in X \setminus A_\alpha$. Hence $(x(\alpha\beta))(\alpha\beta)^{-1} = \{x\}$. This proves that if $x \in X \setminus (A_\alpha \cup (A_\beta)\alpha^{-1})$, then $x \in X \setminus A_{\alpha\beta}$. Hence $A_{\alpha\beta} \subseteq A_\alpha \cup (A_\beta)\alpha^{-1}$. Since α and β are almost 1-1, A_α and A_β are finite. The set $(A_\beta)\alpha^{-1}$ is finite since A_β is a finite subset of X and α is almost 1-1. It follows that $A_{\alpha\beta}$ is finite. Therefore $\alpha\beta \in AM_X$.

For $\alpha \in T_X$, α is said to be <u>almost onto</u> if $X \setminus \nabla\alpha$ is finite. Let AO_X be the set of all almost onto transformations of X. Clearly, $O_X \subseteq AO_X$. For $\alpha, \beta \in T_X$, we have that

$$\begin{aligned} X \setminus \nabla\alpha\beta &= (X \setminus \nabla\beta) \cup (\nabla\beta \setminus \nabla\alpha\beta) \\ &= (X \setminus \nabla\beta) \cup (X\beta \setminus (\nabla\alpha)\beta) \\ &\subseteq (X \setminus \nabla\beta) \cup (X \setminus \nabla\alpha)\beta. \end{aligned}$$

Thus, if $\alpha, \beta \in AO_X$, then $X \setminus \nabla\alpha$ and $X \setminus \nabla\beta$ are finite, and it follows that $X \setminus \nabla\alpha\beta$ is finite. This proves that AO_X is a subsemigroup of T_X containing O_X.

The <u>shift</u> of a partial transformation α of X, $S(\alpha)$, is defined to be the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$. A partial transformation α of X is said to be <u>almost identical</u> if the shift of α is finite. Let

U_X = the set of all almost identical partial transformations of X,

V_X = the set of all almost identical transformations of X

and

W_X = the set of all almost identical 1-1 partial transformations of X.

If $\alpha, \beta \in P_X$, then $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$. Hence U_X, V_X and W_X are subsemigroups of P_X, T_X and I_X, respectively. Moreover, W_X is an inverse semigroup.

Let

CP_X = the set of all constant partial transformations of X (including 0)

and

CT_X = the set of all constant transformations of X.

Then CP_X and CT_X are subsemigroups of P_X and T_X, respectively.

A subsemigroup U of a semigroup S is said to be <u>closed</u>* in S if for any element $x \in S \setminus U$, there are a semigroup T and homomorphisms $\varphi, \psi : S \to T$ such that $\varphi|_U = \psi|_U$ and $x\varphi \neq x\psi$.

A semigroup S is said to be <u>absolutely closed</u> if S is closed in every semigroup which contains S as a subsemigroup.

Let S be a semigroup and U a subsemigroup of S. For any element d of S, d is said to be <u>dominated</u> by U or U <u>dominates</u> d if for any semigroup T and for any homomorphisms $\varphi, \psi : S \to T$, $\varphi|_U = \psi|_U$ implies $d\varphi = d\psi$. The set of all elements of S which are dominated by U is called the <u>dominion</u> of U in S and it is denoted by Dom(U,S).

The following statements clearly hold :

---

* In Topology, it is known that for a metric space X and for $C \subseteq X$, C is closed in X if and only if for any $x \in X \setminus C$, there are a metric space Y and continuous mappings $f,g : X \to Y$ such that $f|_C = g|_C$ and $f(x) \neq g(x)$.

(i) Dom(U,S) is a subsemigroup of S containing U.

(ii) U is closed in S if and only if Dom(U,S) = U.

(iii) If U and V are subsemigroups of S such that $U \subseteq V$, then $\mathrm{Dom}(U,V) \subseteq \mathrm{Dom}(U,S)$, and hence U is closed in S implies that U is closed in V.

Let U be a subsemigroup of a semigroup S. A <u>zigzag</u> of length $m (m \in \mathbb{N})$ in U over S with value $d \in S$ is a system of equalities

$$(*) \quad \begin{cases} d = u_0 y_1, \ u_0 = x_1 u_1, \\ \qquad x_i u_{2i} = x_{i+1} u_{2i+1}, \ u_{2i-1} y_i = u_{2i} y_{i+1} \quad (i = 1, 2, \ldots, m-1), \\ \qquad\qquad u_{2m-1} y_m = u_{2m}, \end{cases}$$

with $u_0, u_1, \ldots, u_{2m} \in U$, $x_1, \ldots, x_m$, $y_1, \ldots, y_m \in S$.

<u>Remark</u>. If (*) holds, then $d = x_m u_{2m}$.

A form of (*) can be given as follows :

$$\begin{aligned} d &= u_0 y_1 \\ &= x_1 u_1 y_1, && u_0 = x_1 u_1, \\ &= x_1 u_2 y_2, && u_1 y_1 = u_2 y_2, \\ &= x_2 u_3 y_2, && x_1 u_2 = x_2 u_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= x_m u_{2m-1} y_m, && x_{m-1} u_{2m-2} = x_m u_{2m-1}, \\ &= x_m u_{2m}, && u_{2m-1} y_m = u_{2m}. \end{aligned}$$

The following results will be used in this thesis :

<u>Theorem 1.1</u> (Isbell's Zigzag Theorem, [1]). Let U be a subsemigroup of a semigroup S. Then $d \in \mathrm{Dom}(U,S)$ if and only if $d \in U$ or there is a zigzag in U over S with value d.

It follows from the Theorem 1.1 that every ideal of a semigroup S is closed in S.

Let U be a subsemigroup of a semigroup S. Assume that U has an identity 1. If $d \in U$, then

$$\begin{aligned} d &= 1d \\ &= 11d, \quad 1 = 11, \\ &= 1d, \quad 1d = d, \end{aligned}$$

which implies that there is a zigzag in U over S with value d. Hence by Theorem 1.1, we have

Corollary 1.2. Let U be a subsemigroup of a semigroup S. Assume that U has an identity. Then $d \in Dom(U,S)$ if and only if there is a zigzag in U over S with value d.

Theorem 1.3 ([2]). Every inverse semigroup is absolutely closed.

It follows from Theorem 1.3 that every group is absolutely closed.

Theorem 1.4 ([2]). If a semigroup S contains elements a_1, a_2, a_3 such that $a_1S \cap a_2S = Sa_2 \cap Sa_3 = \emptyset$, then S is not absolutely closed.