FUNCTIONAL EQUATION ON PLANAR QUADRILATERALS

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CHAPTER I
INTRODUCTION

1.1 Functional Equation

J. Aczel [2] defined functional equation as follows

> "Functional equations are equations, both sides of which are terms constructed from a finite number of unknown functions and from a finite number of independent variables. This construction is effected by a finite number of known functions of one or several variables and by finitely many substitutions of terms which contain known and unknown functions into other known and unknown functions. The functional equations determine the unknown functions. We speak of functional equations or systems of functional equations, depending on whether we have one or several equations."

We can simply say that functional equations are equations such that unknowns are functions.

Functional equations can be solved by introducing some wise substitutions to yield more information or additional equations as in the following example.

Example 1.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(1-x) + 2f(x) = x + 4 \tag{1.1}$$

for all $x \in \mathbb{R}$.

Solution. Assume that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1).
Replacing x by $1-x$ in (1.1), we have

$$f(x) + 2f(1-x) = 5 - x. \tag{1.2}$$

Solving (1.1) and (1.2), we get $f(x) = x + 1$.
Conversely if a function f is given by $f(x) = x + 1$, then

$$f(1-x) + 2f(x) = (1-x) + 1 + 2(x+1) = x + 4.$$

Therefore the function f defined by $f(x) = x + 1$ for all $x \in \mathbb{R}$ is the unique solution of the functional equation (1.1). $\square$

In some circumstances, functional equations may not be solvable.

Example 1.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$2f(5-x) + xf(x) = 1 \tag{1.3}$$

for all $x \in \mathbb{R}$.

Solution. Assume that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.3).
Substituting $x = 1$ in (1.3), we obtain that

$$2f(4) + f(1) = 1. \tag{1.4}$$

Substituting $x = 4$ in (1.3), we get that

$$2f(1) + 4f(4) = 1. \tag{1.5}$$

We see that $f(1)$ and $f(2)$ cannot simultaneously satisfy (1.4) and (1.5). Thus there is no function $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation (1.3). $\square$

Next, we will give an example of functional equations where the function is defined on $\mathbb{R}^2$.

Example 1.3. Given $a, b \in \mathbb{R}$, find all functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(x, y) + f(x + \lambda a, y + \lambda b) = 0 \tag{1.6}$$

for all $x, y, \lambda \in \mathbb{R}$ and $\lambda \neq 0$.

Solution. Assume that there is a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying (1.6). Replacing x, y by $x + \lambda a, y + \lambda b$, respectively in (1.6), we have

$$f(x + \lambda a, y + \lambda b) + f(x + 2\lambda a, y + 2\lambda b) = 0. \tag{1.7}$$

From (1.6) and (1.7), we obtain that

$$f(x, y) = f(x + 2\lambda a, y + 2\lambda b). \tag{1.8}$$

Replacing λ by $\lambda/2$ in (1.8), we have

$$f(x, y) = f(x + \lambda a, y + \lambda b). \tag{1.9}$$

From (1.6) and (1.9), we conclude that

$$f(x, y) = 0.$$

Conversely, if a function f is given by $f(x, y) = 0$, then it is obvious that

$$f(x, y) + f(x + \lambda a, y + \lambda b) = 0 + 0 = 0.$$

Therefore the function f defined by $f(x, y) = 0$ for all $x, y \in \mathbb{R}$ is the unique solution of the functional equation (1.6). $\square$

1.2 Literature Review

In this section, we will review some research related to this thesis.

In 1968, J. Aczél, H. Haruki, M.A. McKiernan and G. N. Sakovič [1] studied a functional equation

$$f(x + u, y + v) + f(x + u, y - v) + f(x - u, y + v) + f(x - u, y - v) = 4f(x, y). \tag{1.10}$$

This equation states that the value of f at the center of any *rectangle* with sides parallel to the coordinate axes, equals the mean of the values at all vertices.

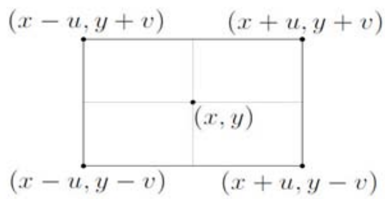

Figure 1.1 : Rectangle

The general solution of (1.10) is given by

$$f(x,y) = A(x,y) + B(x) + C(y) + D \tag{1.11}$$

where $B(x)$ and $C(x)$ are arbitrary additive functions (a function $\phi : \mathbb{R} \to \mathbb{R}$ is additive if and only if $\phi(x+y) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{R}$), $A(x,y)$ is an arbitrary function which is additive in each variable, and D is a constant. They also studied the functional equation

$$f(x+u,y) + f(x-u,y) + f(x,y+v) + f(x,y-v) = 4f(x,y) \tag{1.12}$$

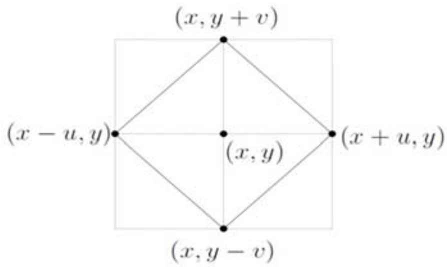

Figure 1.2 : Rhombus

which states that the value of f at the center of any *rhombus* equals the mean of the values at all vertices and obtained the same solutions of the form (1.11). Finally, they investigated the functional equation

$$f(x+v,y+v) + f(x+v,y-v) + f(x-v,y+v) + f(x-v,y-v) = 4f(x,y) \tag{1.13}$$

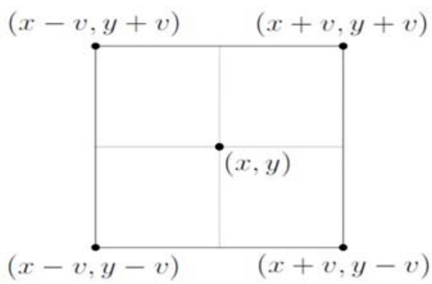

Figure 1.3 : Square

which states that the value of f at the center of any *square* with sides parallel to the coordinate axes equals the mean of the values at all vertices, and found that the general solution is given by

$$
\begin{aligned}
f(x,y) &= A^0 + A^1(x) + B^1(y) + A^{1,1}(x;y) + A^2(y) - A^2(x) - A^3(x) - 3A_3(y,y,x) \\
&\quad + B^3(y) - 3B_3(x,x,y) + A^{1,3}(x;y) - A_{1,3}(x;y,x,x)
\end{aligned}
$$

where $A^3(y) = A_3(y,y,y), B^3(y) = B_3(y,y,y), A^{1,3}(x;y) = A_{1,3}(x;y,y,y)$ with the added symmetry condition $A_{1,3}(x;y,z,z) = A_{1,3}(z;y,x,z)$.
Here, $A_{m,n}(x_1,\ldots,x_m;y_1,\ldots,y_n)$ denotes a symmetric multi-additive function and $A^{m,n}(x;y)$ denotes the diagonalization of $A_{m,n}$.

In 1969, J.A. Baker [7] studied the functional equation

$$f(x+t,y) + f(x-t,y) = f(x,y+t) + f(x,y-t) \tag{1.14}$$

which is analogous to the well-known wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f(x,y) = 0.$$

He found that all continuous solutions are of the form

$$f(x,y) = \alpha(x+y) + \beta(x-y), \tag{1.15}$$

where α and β are arbitrary continuous functions.

In the following year, H. Haruki [5] also solved (1.14) by a different method and obtained results similar to that of J.A. Baker. In the same year, M. Kucharzewski [8] studied (1.14) which satisfies the relation

$$f(x,y) - f(-x,y) - f(x,-y) + f(-x,-y) = \alpha_1(x+y) + \beta_1(x-y),$$

where α_1 and β_1 are arbitrary functions, and got the solutions of the form (1.15) but the function α and β need not be continuous.

In 1972, D.P. Flemming [4] solved (1.14) using a transformation of coordinates (x,y) into (ζ,η) according to the relations

$$\zeta = \frac{k+1}{2}x + \frac{k-1}{2}y$$

and

$$\eta = \frac{k-1}{2}x + \frac{k+1}{2}y$$

and obtained the solutions of the form (1.15) when there is no assumption regarding the continuity of α, β and f. In the same year, M. A. McKiernan [9] gave the solutions of (1.14) in the following more general context which take the form of

$$f(x, y) = \alpha(x + y) + \beta(x - y) + A(x, y),$$

where α, β are arbitrary functions and $A(x, y)$ is an arbitrary skew-symmetric bi-additive function,

$$A(x, y) = -A(y, x)$$

and

$$A(x, y + z) = A(x, y) + A(x, z).$$

In 1973, D. Girod [3] studied more general version of (1.14) and obtained a result similar to that of M. A. McKiernan.

1.3 Proposed Work

We will first find the general solution of functional equation on planar triangles. Afterward, we will find the general solution of functional equation on planar quadrilaterals.

CHAPTER II
FUNCTIONAL EQUATION ON PLANAR TRIANGLES

2.1 Geometry of Lines and Triangles on Cartesian Plan $\mathbb{R}^2$

First, we will mentioned some background knowledges essential to this thesis. We start by considering translation and dilation of a nonzero vector $\vec{a}$ shown in Fig 2.1.

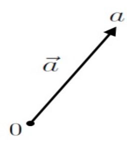

Figure 2.1

If vectors $\vec{0}$ and $\vec{a}$ are translated by another vector $\vec{x}$, then they will become $\vec{x}$ and $\vec{x}+\vec{a}$, respectively, as shown in Fig 2.2.

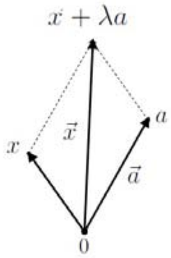

Figure 2.2

If vectors $\vec{0}$ and $\vec{a}$ are dilated by a nonzero real number λ, then they will become $\vec{0}$ and $\lambda\vec{a}$, respectively, as shown in Fig 2.3.

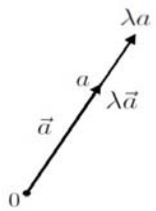

Figure 2.3

Therefore, the vectors $\vec{x}$ and $\vec{x}+\lambda\vec{a}$ can be obtained from vectors $\vec{0}$ and $\vec{a}$, respectively, by first dilating $\vec{a}$ by a factor $\lambda(\lambda \neq 0)$ followed by a translating the result by the vector x.

Next, we will explain a geometry of planar triangles. For given nonzero complex numbers a_1, a_2 such that a_1 is not a multiple of a_2, we can construct a triangle on the Cartesian plan $\mathbb{R}^2$, as shown in Fig 2.4.

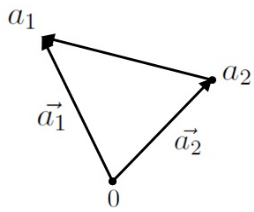

Figure 2.4: Triangle

If vectors $\vec{0}, \vec{a_1}$ and $\vec{a}$ are translated by another vector $\vec{x}$, then they will become $\vec{x}, \vec{x}+\vec{a_1}$ and $\vec{x}+\vec{a_2}$, respectively, as shown in Fig 2.5.

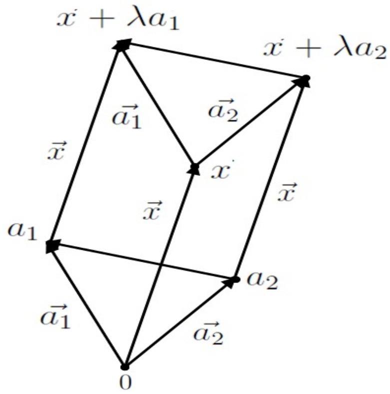

Figure 2.5: translation of triangle

If vectors $\vec{0}, \vec{a_1}$ and $\vec{a_2}$ are dilated by a factor $\lambda(\lambda \neq 0)$, then they will become $\vec{0}, \lambda\vec{a_1}$ and $\lambda\vec{a_2}$, respectively, as shown in Fig 2.6.

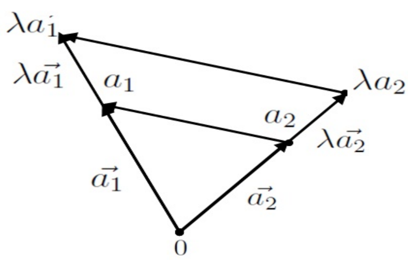

Figure 2.6: dilation of triangle

Therefore, the vectors $\vec{x}, \vec{x}+\lambda\vec{a_1}$ and $\vec{x}+\lambda\vec{a_2}$ can be obtained from $\vec{0}, \vec{a_1}$ and $\vec{a_2}$, respectively, by a dilation by $\lambda(\lambda \neq 0)$ followed by a translation by vector $\vec{x}$.

2.2 General Solution of Functional Equation on Planar Triangles

In this chapter, given a triangle ABC in the Cartesian plane $\mathbb{R}^2$. We will find the general solution of the functional equation that the sum of the function values, taken at all vertices of a triangle which is obtained by any translation and dilation of a fixed triangle ABC, is equal to zero.
Fig 2.7 illustrates the case where the triangle ABC is nondegenerate.

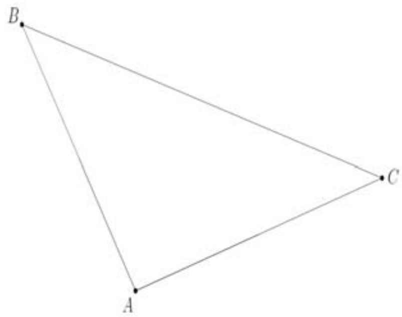

Figure 2.7 : Triangle

Theorem 2.1. Let $a_1, a_2 \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation

$$f(x) + f(x+\lambda a_1) + f(x+\lambda a_2) = 0$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ if and only if $f(x) = 0$ for all $x \in \mathbb{C}$.

Proof. Let $a_1, a_2 \in \mathbb{C}$. Assume that there is a functional $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$f(x) + f(x+\lambda a_1) + f(x+\lambda a_2) = 0 \tag{2.1}$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.
For convenience, we define

$$F_\lambda(x) \equiv f(x) + f(x+\lambda a_1) + f(x+\lambda a_2) \tag{2.2}$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Then $F_\lambda(x) = 0$ for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. As formulated in (2.2) with $A = x, B = x + \lambda a_1$ and $C = x + \lambda a_2$, and with $B' = x + 2\lambda a_1, D = x + \lambda(a_1 + a_2)$ and $C' = x + 2\lambda a_2$.

$$F_\lambda(A) \equiv f(A) + f(B) + f(C), \tag{2.3}$$
$$F_\lambda(B) \equiv f(B) + f(B') + f(D),$$
$$F_\lambda(C) \equiv f(C) + f(D) + f(C'),$$
$$F_{2\lambda}(A) \equiv f(A) + f(B') + f(C'),. \tag{2.4}$$

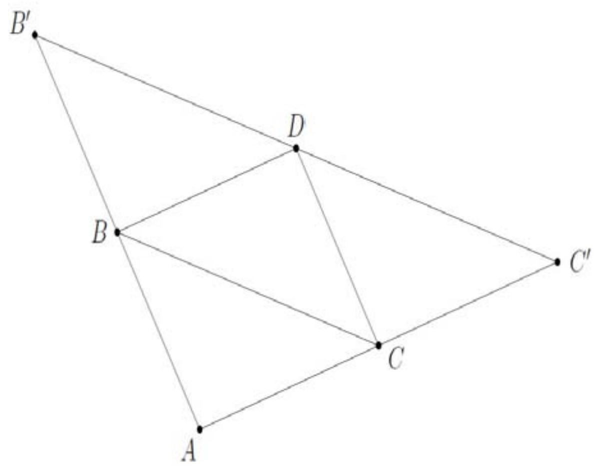

Figure 2.8: Geometrical representation when a_1 is not a multiple of a_2

It is not hard to see that

$$\frac{1}{2}\Big(F_\lambda(A) + F_\lambda(B) + F_\lambda(C) - F_{2\lambda}(A)\Big) = f(C) + f(B) + f(D).$$

That is

$$f(C) + f(B) + f(D) = 0. \tag{2.5}$$

From (2.3) and (2.5), we have

$$f(A) = f(D). \tag{2.6}$$

From (2.6) and (2.4), we get

$$f(B') + f(D) + f(C') = 0. \tag{2.7}$$

Now in (2.7) we have

$$f(x+2\lambda a_1)+f(x+\lambda(a_1+a_2))+f(x+2\lambda a_2)=0. \tag{2.8}$$

Recall that $B'=x+2\lambda a_1, D=x+\lambda(a_1+a_2)$ and $C'=x+2\lambda a_2$, which are collinear points in Fig 2.8.
Thus, we define

$$G_\lambda(x)\equiv f(x+2\lambda a_1)+f(x+\lambda(a_1+a_2))+f(x+2\lambda a_2).$$

From (2.8), we have $G_\lambda(x)=0$ for all $x\in\mathbb{C}$ and $\lambda\in\mathbb{R}\setminus\{0\}$.
We can verify that

$$0=G_{\lambda/3}(x+\frac{\lambda}{3}a_1)-G_{\lambda/3}(x+\frac{\lambda}{3}a_2)=f(x+\lambda a_1)-f(x+\lambda a_2).$$

That is

$$f(x+\lambda a_1)=f(x+\lambda a_2) \tag{2.9}$$

From (2.1) and (2.9),

$$f(x)+2f(x+\lambda a_1)=0. \tag{2.10}$$

Replacing x by $x+\lambda a_1$ in (2.10), we get

$$f(x+\lambda a_1)+2f(x+2\lambda a_1)=0. \tag{2.11}$$

From (2.10) and (2.11) and by replacing λ with $\lambda/2$, we have

$$f(x)=4f(x+\lambda a_1). \tag{2.12}$$

From (2.10) and (2.12), we obtain

$$f(x)=0 \tag{2.13}$$

for all $x\in\mathbb{C}$.
Conversely, we can clearly see that if $f(x)=0$ for all $x\in\mathbb{C}$ then (2.1) hold for all $x\in\mathbb{C}$ and $\lambda\in\mathbb{R}\setminus\{0\}$. □

CHAPTER III
FUNCTIONAL EQUATION ON PLANAR QUADRILATERALS

In this chapter, we will find the general solution of the functional equation stating that the sum of the function taken at all vertices of a quadrilateral, obtained by any translation and dilation of a fixed quadrilateral $ABCD$, is equal to zero. In order to better understand the method, let us start with the case of parallelograms.

3.1 The Case of Parallelograms

Given $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ such that a_1 is not a multiple of a_2, we can construct a parallelogram on the Cartesian plan $\mathbb{R}^2$ as shown in Fig 3.1.1.

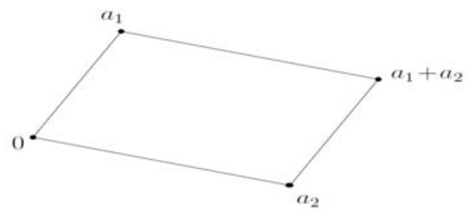

Figure 3.1.1 : Parallelogram

Then the vectors $\vec{x}, \vec{x}+\lambda\vec{a_1}, \vec{x}+\lambda\vec{a_2}$ and $\vec{x}+\lambda(\vec{a_1}+\vec{a_2})$ can be obtained from $\vec{0}, \vec{a_1}, \vec{a_2}$ and $\vec{a_1}+\vec{a_2}$, respectively, by a dilation $\vec{a}$ by a factor $\lambda(\lambda \neq 0)$ and followed by a translation by vector $\vec{x}$ as shown in Fig 3.1.2

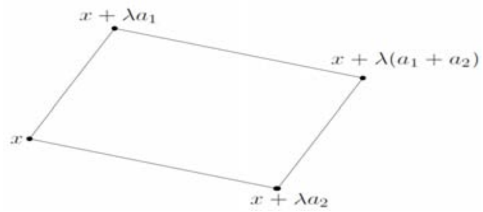

Figure 3.1.2

We will first prove the following lemma.

Lemma 3.1. Let $a \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation

$$f(x) + f(x + \lambda a) = 0 \tag{3.1}$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ if and only if $f(x) = 0$ for all $x \in \mathbb{C}$.

Proof. Let $a \in \mathbb{C}$. Assume that there is a functional $f : \mathbb{C} \to \mathbb{C}$ satisfying (3.1). Replacing x by $x + \lambda a$ in (3.1), we obtain

$$f(x + \lambda a) + f(x + 2\lambda a) = 0. \tag{3.2}$$

By (3.1) and (3.2), we get

$$f(x) = f(x + 2\lambda a). \tag{3.3}$$

Replacing λ by $\lambda/2$ in (3.3), we have

$$f(x) = f(x + \lambda a). \tag{3.4}$$

From (3.1) and (3.4), we get $f(x) = 0$ for all $x \in \mathbb{C}$.
Conversely, we can clearly see that if $f(x) = 0$ for all $x \in \mathbb{C}$ then (3.1) hold for all $x \in \mathbb{C}$ and all $\lambda \in \mathbb{R} \setminus \{0\}$. $\square$

Note: It should be noted that Lemma 3.1 is equivalent to Example 1.3.

In the following theorem, we will solve a useful functional equation pertain to a parallelogram.

Theorem 3.2. Let $a_1, a_2 \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation

$$f(x) + f(x + \lambda a_1) + f(x + \lambda a_2) + f(x + \lambda(a_1 + a_2)) = 0 \tag{3.5}$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ if and only if $f(x) = 0$ for all $x \in \mathbb{C}$.

Proof. Let $a_1, a_2 \in \mathbb{C}$. Assume that there is a function $f : \mathbb{C} \to \mathbb{C}$ satisfying (3.5). For convenience, we define

$$F_\lambda(x) \equiv f(x) + f(x + \lambda a_1) + f(x + \lambda a_2) + f(x + \lambda(a_1 + a_2))$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.
It is straightforward to verify that

$$\frac{1}{2}\Big(F_\lambda(x) + F_\lambda(x + \lambda a_1) + F_\lambda(x + \lambda(a_1 + a_2)) + F_\lambda(x + \lambda a_2) - F_{2\lambda}(x)\Big)$$
$$-F_\lambda(x) - F_\lambda(x + \lambda(a_1, a_2)) = -f(x) - f(x + 2\lambda(a_1 + a_2)).$$

That is

$$f(x) + f(x + 2\lambda(a_1 + a_2)) = 0 \tag{3.6}$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.
Applying Lemma 3.1 to (3.6), we obtain that $f(x) = 0$ for all $x \in \mathbb{C}$.
Conversely, if $f(x) = 0$ for all $x \in \mathbb{C}$ then (3.5) hold for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.
□

The above theorem motivates us to consider a functional equation peertain to a general quadrilateral in the Cartesian plane $\mathbb{R}^2$ in the next section.

3.2 The Case of General Quadrilaterals

We will now turn to the case of general quadrilaterals. Let us start with a lemma on a slightly different functional equation on a parallelogram.

Lemma 3.3. Let $a_1, a_2 \in \mathbb{C}$. If a function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation

$$f(x) + f(x + \lambda a_1) = f(x + \lambda a_2) + f(x + \lambda(a_1 + a_2))$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ then $f(x) = f(x + \lambda a_2)$ for all $x \in \mathbb{C}$.

Proof. Assume the hypothesis of the lemma.
We define

$$F_\lambda(x) \equiv f(x) + f(x + \lambda a_1) - f(x + \lambda a_2) - f(x + \lambda(a_1 + a_2)) \tag{3.7}$$

for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.
As formulated in (3.7) with $A = x, B = x+\lambda a_1, C = x+\lambda a_2$ and $D = x+\lambda(a_1+a_2)$

we have,

$$F_\lambda(A) \equiv f(A) + f(B) - f(D) - f(C),$$
$$F_\lambda(B) \equiv f(B) + f(B') - f(C) - f(P),$$
$$F_\lambda(C) \equiv f(C) + f(P) - f(Q) - f(C'),$$
$$F_\lambda(D) \equiv f(D) + f(C) - f(D') - f(Q),$$
$$F_{2\lambda}(A) \equiv f(A) + f(B') - f(D') - f(C')$$

where $B' = x + 2\lambda a_1, P = x + \lambda(2a_1 + a_2), D' = x + 2\lambda a_2, Q = x + \lambda(2a_2 + a_1)$ and $C' = x + 2\lambda(a_1 + a_2)$.

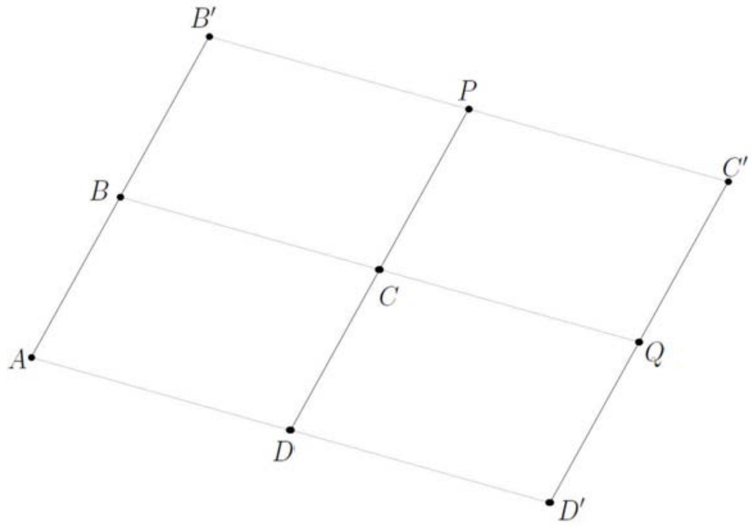

Figure 3.2.2: Geometrical representation when a_1 is not a multiple of a_2

We can simply verify that

$$\frac{1}{2}\Big(F_\lambda(A) + F_\lambda(D) - F_\lambda(B) - F_\lambda(C) + F_{2\lambda}(A)\Big) = f(A) - f(D').$$

That is

$$f(x) = f(x + 2\lambda a_2) \tag{3.8}$$

Replacing λ by $\lambda/2$ in (3.8), we get

$$f(x) = f(x + \lambda a_2) \tag{3.9}$$

for all $x \in \mathbb{C}, \lambda \in \mathbb{R} \setminus \{0\}$. $\square$

The following theorem is the main result in this thesis.
Figure 3.2.3 illustrates the case when the quadrilateral $AEFB$ is nondegenerate.

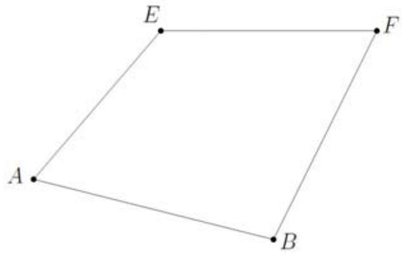

Figure 3.2.3 : nondegenerate quadrilateral

Theorem 3.4. Let $a_1, a_2, a_3 \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation

$$f(x) + f(x + \lambda a_1) + f(x + \lambda a_2) + f(x + \lambda a_3) = 0 \tag{3.10}$$

for all $x \in \mathbb{C}, \lambda \in \mathbb{R} \setminus \{0\}$ if and only if $f(x) = 0$ for all $x \in \mathbb{C}$.

Proof. Let $a_1, a_2, a_3 \in \mathbb{C}$. Assume there is a functional equation $f : \mathbb{C} \to \mathbb{C}$ satisfying (3.10).
For convenience, we will define

$$F_\lambda(x) \equiv f(x) + f(x + \lambda a_1) + f(x + \lambda a_2) + f(x + \lambda a_3) \tag{3.11}$$

for all $x \in \mathbb{C}, \lambda \in \mathbb{R}$ and $\lambda \neq 0$.
From (3.11), we get $F_\lambda(x) = 0$ for all $x \in \mathbb{C}, \lambda \in \mathbb{R} \setminus \{0\}$.
As formulated in (3.11) with $A = x, E = x + \lambda a_1, F = x + \lambda a_2$ and $B = x + \lambda a_3$

we have,

$$F_\lambda(A) \equiv f(A) + f(E) + f(F) + f(B), \tag{3.12}$$

$$F_\lambda(B) \equiv f(B) + f(G) + f(H) + f(C), \tag{3.13}$$

$$F_\lambda(E) \equiv f(E) + f(K) + f(L) + f(G), \tag{3.14}$$

$$F_\lambda(F) \equiv f(F) + f(L) + f(N) + f(H), \tag{3.15}$$

$$F_\lambda(G) \equiv f(G) + f(M) + f(O) + f(I), \tag{3.16}$$

$$F_\lambda(H) \equiv f(H) + f(O) + f(P) + f(J), \tag{3.17}$$

$$F_\lambda(L) \equiv f(L) + f(R) + f(S) + f(O), \tag{3.18}$$

$$F_{2\lambda}(A) \equiv f(A) + f(K) + f(N) + f(C), \tag{3.19}$$

$$F_{2\lambda}(E) \equiv f(E) + f(Q) + f(S) + f(I), \tag{3.20}$$

$$F_{2\lambda}(F) \equiv f(F) + f(R) + f(T) + f(J), \tag{3.21}$$

$$F_{2\lambda}(B) \equiv f(B) + f(M) + f(P) + f(D), \tag{3.22}$$

$$F_{3\lambda}(A) \equiv f(A) + f(Q) + f(T) + f(D), \tag{3.23}$$

where

$$K = x + 2\lambda a_1, Q = x + 3\lambda a_1, N = x + 2\lambda a_2, T = x + 3\lambda a_2,$$

$$C = x + 2\lambda a_3, D = x + 3\lambda a_3, H = x + \lambda(a_2 + a_3), J = x + \lambda(a_2 + 2a_3),$$

$$G = x + \lambda(a_1 + a_3), I = x + \lambda(a_1 + 2a_3), L = x + \lambda(a_1 + a_2),$$

$$R = x + \lambda(2a_1 + a_2), M = x + \lambda(2a_1 + a_3), S = x + \lambda(a_1 + 2a_2),$$

$$O = x + \lambda(a_1 + a_2 + a_3), P = x + \lambda(2a_2 + a_3).$$

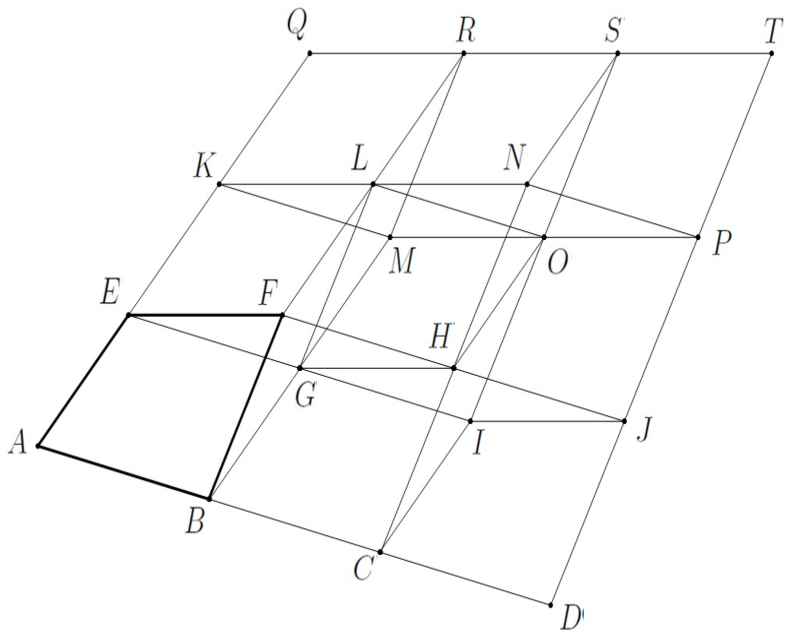

Figure 3.2.4: Geometrical representation when a_1, a_2, a_3 are not collinear points

Setting $S_1 = F_\lambda(B) + F_\lambda(E) + F_\lambda(F), S_2 = F_\lambda(G) + F_\lambda(H) + F_\lambda(L)$ and $S_3 = F_{2\lambda}(A) + F_{2\lambda}(E) + F_{2\lambda}(F) + F_{2\lambda}(B)$.
By the definition of $F_\lambda(x)$ in (3.11), we verify that

$$0 = \frac{1}{3}(S_1 + S_2 - S_3) = f(G) + f(L) + f(O) + f(H).$$

That is

$$f(G) + f(L) + f(O) + f(H) = 0. \tag{3.24}$$

Combine (3.24) with (3.13),

$$f(L) + f(O) = f(B) + f(C). \tag{3.25}$$

Applying Lemma 3.3 with (3.25), we obtain that

$$f(C) = f(O). \tag{3.26}$$

By combing (3.24) with (3.14) and then applying Lemma 3.3, we obtain that

$$f(K) = f(O). \tag{3.27}$$

Now we have

$$f(K) = f(C).$$

That is

$$f(x + 2\lambda a_1) = f(x + 2\lambda a_3). \tag{3.28}$$

Replacing λ by $\lambda/2$, we get

$$f(E) = f(B). \tag{3.29}$$

Similarly, we can prove that

$$f(A) = f(F). \tag{3.30}$$

Recall that $f(A) + f(E) + f(F) + f(B) = 0$, and by using (3.29) and (3.30), we get

$$f(A) + f(E) = 0. \tag{3.31}$$

From (3.31) and Lemma 3.1, we obtain that $f(x) = 0$ for all $x \in \mathbb{C}$.
Conversely, clearly that if $f(x) = 0$ for all $x \in \mathbb{C}$ then (3.10) hold for all $x \in \mathbb{C}$ and $\lambda \in \mathbb{R} \smallsetminus \{0\}$. $\square$

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