CHAPTER V

DECOMPOSITION THEORY OF SKEW RATIO SEMIRINGS AND SKEW RINGS

Necessary and sufficient conditions that groups and rings are decomposable are well-known. In this chapter, we study necessary and sufficient conditions that skew ratio semirings and skew rings are decomposable.

Notation Let $(D,+,\cdot)$ be a skew ratio semiring with 1 as its multiplicative identity and $n \in \mathbf{Z}^+$. Denote $1 + 1 + \ldots + 1$ n times by n. Clearly $nx = xn$ and $n^{-1}x = xn^{-1}$ for all $x \in D$.

Definition 5.1. Let $(D,+,\cdot)$ be a skew ratio semiring and E a multiplicative normal subgroup of D. Then E is said to be a P-set of D iff there exists an $\alpha \in D$ such that

(i) $\alpha x = x\alpha$ for all $x \in E$,

(ii) $(x + y)\alpha \in E$ for all $x,y \in E$

and (iii) $(x + y)\alpha + z = x + (y + z)\alpha$ for all $x,y,z \in E$.

α is called a good element of the P-set E.

Example 5.2.

1) Let D be a skew ratio semiring. Then D and $\{1\}$ are P-sets of D with good elements 1 and 2^{-1}, respectively. D and $\{1\}$ are called the trivial P-sets of D.

2) Let $(C,+,\cdot)$ and $(D,+,\cdot)$ be skew ratio semirings. Define $(x,y) \oplus (z,w) = (x + z, y + w)$ and $(x,y) \odot (z,w) = (x\cdot z, y\cdot w)$ for all $(x,y),(z,w) \in C \times D$. Then $(C \times D, \oplus, \odot)$ is a skew ratio semiring. Let $E = C \times \{1\}$ and $F = \{1\} \times D$. Then E and F and P-sets of $C \times D$ with good elements $(1,2^{-1})$ and $(2^{-1},1)$, respectively.

Theorem 5.3. Let D be a skew ratio semiring and E a P-set of D. Then a good element of E is unique.

Proof. Let α,β be good elements of E. We must show that $\alpha = \beta$. Let $a = 2\alpha$ and $b = 2\beta$. Then $a,b \in E$ since $1 \in E$. Thus $2\alpha\beta = \beta 2\alpha = 2\beta\alpha$, hence $\alpha\beta = \beta\alpha$ which implies that $ab = ba$. Let $x,y,z \in E$. Then $(x + y)\alpha + z = x + (y + z)\alpha$ so $(x + y)a + 2z = 2x + (y + z)a$. .... (*) Similarly, $(x + y)b + 2z = 2x + (y + z)b$ for all $x,y,z \in E$. Consider a. Let $x = y = z = 1$ in (*). We get that $2a + 2 = 2 + 2a$, so

$$a + 1 = 1 + a. \quad \text{.........} (1)$$

Let $x = y = 1$ and $z = a$ in (*). We get that

$$4a = 2 + a + a^2. \quad \text{.........} (2)$$

Let $x = y = 1$ and $z = b$ in (*). We get that

$$2a + 2b = 2 + a + ba. \quad \text{.........} (3)$$

Let $x = z = 1$ and $y = b$ in (*). We get that

$$a + ba + 2 = 2 + ba + a. \quad \text{.........} (4)$$

Let $y = z = 1$ and $x = b$ in (*). We get that

$$ba + a + 2 = 2b + 2a. \quad \text{.........} (5)$$

Similarly, for b we get that

$$b + 1 = 1 + b. \quad \text{.........} (6)$$

$$4b = 2 + b + b^2 \quad \text{.........} (7)$$

$$2b + 2a = 2 + b + ab. \qquad \text{.......... (8)}$$

$$b + ab + 2 = 2 + ab + b. \qquad \text{.......... (9)}$$

$$ab + b + 2 = 2a + 2b. \qquad \text{......... (10)}$$

Thus

$$\begin{aligned} 2a + 2b &= 2 + ba + a && \text{(from (3),(6))} \\ &= ba + a + 2 && \text{(from (4),(6))} \\ &= 2b + 2a. && \text{(from (5))} \end{aligned}$$

Hence

$$a + b = b + a. \qquad \text{......... (11)}$$

Therefore

$$\begin{aligned} 2b(2a + 1) &= (4b)a + 2b \\ &= 2a + ba + b^2a + 2b && \text{(from (7))} \\ &= 2a + b(a + ba + 2) \\ &= 2a + b(2b + 2a) && \text{(from (6),(5))} \\ &= (2a + 2b) + b^2 + bab && \text{(from (8))} \\ &= 2 + b + ab + b^2 + bab && \text{(from (11),(8))} \\ &= 2 + b + (ba + b^2) + b^2a && (ab = ba) \\ &= (2 + b + b^2) + ba + b^2a && \text{(from (11))} \\ &= 4b + ba + b^2a && \text{(from (7))} \\ &= b(2 + 2 + a + ba) \\ &= b(2 + 2a + 2b) && \text{(from (3))} \\ &= 2b(1 + a + b). \end{aligned}$$

Thus $2a + 1 = 1 + a + b$. Similarly, $2b + 1 = 1 + b + a$. Hence

$$2a + 1 = 2b + 1. \qquad \text{......... (12)}$$

So

$$\begin{aligned} 16(a + 1) &= 4(4a) + 16 \\ &= 8 + 4a + 4a^2 + 16 && \text{(from (2))} \\ &= 8 + (4a^2 + 4a + 1) + 15 && \text{(from (1))} \\ &= 8 + (4b^2 + 4b + 1) + 15 && \text{(from (12))} \\ &= 8 + 4b + 4b^2 + 16 && \text{(from (6))} \\ &= 4(4b) + 16 && \text{(from (7))} \\ &= 16(b + 1). \end{aligned}$$

Thus a + 1 = b + 1 which implies that 1 + a = 1 + b. .... (13)

Therefore

$$
\begin{aligned}
4a &= 2 + a + a^2 && \text{(from (2))} \\
&= 2 + a + ba && \text{(from (13))} \\
&= 2 + b + ab && \text{(from (3),(11),(8))} \\
&= 2 + b + b^2 && \text{(from (13))} \\
&= 4b. && \text{(from (7))}
\end{aligned}
$$

Hence a = b which implies that $\alpha = \beta$ and thus the good element of E is unique.
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Theorem 5.4. Let D be a skew ratio semiring and E a P-set of D with good element α. Then the following are equivalent :

(1) (E,+) is a subsemigroup of (D,+).

(2) $\alpha = 1$.

(3) $\alpha \in E$.

Proof. (1) implies (2). Assume that (1) holds. Then 1 is the good element of E. By the uniqueness, $\alpha = 1$.

Clearly (2) implies (3).

(3) implies (1). Assume that (3) holds. Let $x,y \in E$. Then $(x + y)\alpha \in E$ which implies that $x + y = (x + y)\alpha\alpha^{-1} \in E$. Thus (1) holds.
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Example 5.5. Give $\mathbb{Q}^+$ with the usual addition and multiplication. Give $\mathbb{R}^+$ with the usual multiplication and define $x + y = \min \{x,y\}$ for all $x,y \in \mathbb{R}^+$. Let $D = \mathbb{Q}^+ \times \mathbb{R}^+$ with addition and multiplication defined by $(x,y) + (z,w) = (x + z, y + w)$ and $(x,y) \cdot (z,w) = (xz,yw)$ for all $(x,y),(z,w) \in \mathbb{Q}^+ \times \mathbb{R}^+$. Let $E = \{(x,1) | x \in \mathbb{Q}^+\}$. Then E is a P-set with good element $\alpha = (1,1)$. Here $(1,1) + (1,1) = (2,1) \neq (1,1)$ so the three conditions above do not imply that 1 + 1 = 1. However, 1 + 1 = 1 implies the three conditions above as the following theorem shows.

Theorem 5.6. Let D be a skew ratio semiring with $1 + 1 = 1$ and E a P-set of D with good element α. Then the following hold :

(1) $(E,+)$ is a subsemigroup of $(D,+)$.

(2) $\alpha = 1$.

(3) $\alpha \in E$.

Proof. Since $(1 + 1)\alpha \in E$, $\alpha \in E$, so (3) holds. By Theorem 5.4, (1) and (2) hold.

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Definition 5.7. A skew ratio semiring D is said to be decomposable iff there exist skew ratio semirings D_1, D_2 such that $|D_1| > 1, |D_2| > 1$ and $D \cong D_1 \times D_2$.

Example 5.8. $(C \times D, \oplus, \odot)$ in Example 5.2(2) is a decomposable skew ratio semiring.

Theorem 5.9. Let $(D,+,\cdot)$ be a skew ratio semiring. Then D is decomposable iff there exist nontrivial P-sets E,F of D with good elements α and β, respectively, such that

1) $E \cap F = \{1\}$,

2) $D = EF$

and 3) $ef + gh = (e + g)\alpha(f + h)\beta$ for all $e,g \in E, f,h \in F$.

Proof. Assume that there exist nontrivial P-sets E,F of D with good elements α and β, respectively, such that 1) - 3) hold. Define $x \oplus y = (x + y)\alpha$ for all $x,y \in E$. To show that $(E,\oplus,\cdot)$ is a skew ratio semiring. We must show that $\oplus$ is associative and $\cdot$ distributes over $\oplus$. Let $x,y,z \in E$. Then

$$
\begin{aligned}
(x \oplus y) \oplus z &= ((x \oplus y) + z)\alpha \\
&= ((x + y)\alpha + z)\alpha \\
&= (x + (y + z)\alpha)\alpha \\
&= (x + (y \oplus z))\alpha \\
&= x \oplus (y \oplus z).
\end{aligned}
$$

Hence $\oplus$ is associative. Since $x(y \oplus z) = x(y + z)\alpha = (xy + xz)\alpha = xy \oplus xz$ and $(y \oplus z)x = (y + z)\alpha x = (y + z)x\alpha = (yx + zx)\alpha = yx \oplus zx$, $\cdot$ distributes over $\oplus$. Hence $(E,\oplus,\cdot)$ is a skew ratio semiring of order greater than 1. Similarly, defining $a \oplus b = (a + b)\beta$ for all $a,b \in F$ we get that $(F,\oplus,\cdot)$ is a skew ratio semiring of order greater than 1. On $E \times F$ define $(x,a) \oplus (y,b) = (x \oplus y, a \oplus b)$ and $(x,a) \cdot (y,b) = (xy,ab)$ for all $(x,a),(y,b) \in E \times F$. Then $(E \times F,\oplus,\cdot)$ is a skew ratio semiring.

Define $i : E \times F \to D$ by $i(e,f) = ef$ for all $(e,f) \in E \times F$. To show that i is a surjection, let $d \in D$. By 2), there exist $e \in E$, $f \in F$ such that $d = ef = i(e,f)$. To show that i is an injection, let $(e,f),(g,h) \in E \times F$ be such that $i(e,f) = i(g,h)$. Then $ef = gh$, so $g^{-1}e = hf^{-1} \in E \cap F = \{1\}$ which implies that $e = g$ and $f = h$. Hence i is a bijection. To show that i is a homomorphism, let $(e,f),(g,h) \in E \times F$. Since $i((e,f)(g,h)) = i(eg,fh) = egfh$, by Proposition 1.63, $egfh = efgh$, so $i((e,f)(g,h)) = i(e,f)i(g,h)$. Since $i((e,f) \oplus (g,h)) = i(e \oplus g, f \oplus h) = (e \oplus g)(f \oplus h) = (e + g)\alpha(f + h)\beta$ and $i(e,f) + i(g,h) = ef + gh$, by 3) $i((e,f) \oplus (g,h)) = i(e,f) + i(g,h)$. Thus i is a homomorphism. Hence $D \cong E \times F$.

Conversely, assume that D is decomposable. Then there exist skew ratio semirings D_1,D_2 of orders greater than 1 and an isomorphism

$i : D_1 \times D_2 \to D$. Let $E = i(D_1 \times \{1\})$ and $F = i(\{1\} \times D_2)$. Since $D_1 \times \{1\}$ and $\{1\} \times D_2$ are multiplicative normal subgroups of $D_1 \times D_2$, E and F are multiplicative normal subgroups of D. Let $\alpha = i(1,2^{-1})$. Then $\alpha \varepsilon D$. Let $x = i(a,1), y = i(b,1)$ and $z = i(c,1) \varepsilon E$. Then

$$\begin{aligned}
\alpha x &= i(1,2^{-1})i(a,1) \\
&= i(a,2^{-1}) \\
&= i(a,1)i(1,2^{-1}) \\
&= x\alpha, \\
(x + y)\alpha &= (i(a,1) + i(b,1))i(1,2^{-1}) \\
&= i(a + b,2)i(1,2^{-1}) \\
&= i(a + b,1) \ \varepsilon E
\end{aligned}$$

and

$$\begin{aligned}
(x + y)\alpha + z &= (i(a,1) + i(b,1))i(1,2^{-1}) + i(c,1) \\
&= i(a + b,1) + i(c,1) \\
&= i(a + b + c,2) \\
&= i(a,1) + i(b + c,1) \\
&= i(a,1) + (i(b,1) + i(c,1))i(1,2^{-1}) \\
&= x + (y + z)\alpha.
\end{aligned}$$

Hence E is a nontrivial P-set of D. Similarly, F is a nontrivial P-set of D with good element $\beta = i(2^{-1},1)$.

Clearly $E \cap F = \{i(1,1)\}$ and $EF \subseteq D$. Let $d \varepsilon D$. Then there exist $e \varepsilon D_1$ and $f \varepsilon D_2$ such that $d = i(e,f)$. Since $i(e,1) \varepsilon E$, $i(1,f) \varepsilon F$ and $i(e,f) = i(e,1)i(1,f)$, $d \varepsilon EF$, so $D \subseteq EF$. Thus $D = EF$. Let $x = i(e,1), y = i(g,1) \varepsilon E$ and $a = i(1,f), b = i(1,h) \varepsilon F$. Then

$$\begin{aligned}
xa + yb &= i(e,1)i(1,f) + i(g,1)i(1,h) \\
&= i(e,f) + i(g,h) \\
&= i(e + g,f + h) \\
&= i(e + g,1)i(1,f + h) \\
&= [i(e,1) + i(g,1)]i(1,2^{-1})[i(1,f) + i(1,h)]i(2^{-1},1)
\end{aligned}$$

$= (x + y)\alpha(a + b)\beta.$

Hence we have the theorem.

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Remark 5.10. Setting $e = f = g = h = 1$ in 3) we get that $\alpha\beta = 2^{-1}$.

Definition 5.11. Let R be a skew ring and $I \subseteq R$. Then I is said to be an ideal of R iff

1) I is an additive normal subgroup of R

and 2) $ri \in I$ and $ir \in I$ for all $i \in I$, $r \in R$.

Example 5.12.

1) Let R be a skew ring. Then R and $\{0\}$ are ideals of R. R and $\{0\}$ are called the trivial ideals of R.

2) Let $(R,+,\cdot)$ and $(T,+,\cdot)$ be skew rings. Define $(x,y) \oplus (z,w) = (x + z, y + w)$ and $(x,y) \odot (z,w) = (x \cdot z, y \cdot w)$ for all $(x,y),(z,w) \in R \times T$. Then $(R \times T, \oplus, \odot)$ is a skew ring. Let $I = R \times \{0\}$ and $J = \{0\} \times T$. Then I and J are ideals of $R \times T$.

Definition 5.13. A skew ring R is said to be decomposable iff there exist skew rings R_1, R_2 such that $|R_1| > 1, |R_2| > 1$ and $R \cong R_1 \times R_2$.

Example 5.14. $(R \times T, \oplus, \odot)$ in Example 5.12(2) is a decomposable skew ring.

Theorem 5.15. Let R be a skew ring. Then R is decomposable iff there exist nontrivial ideals I, J of R such that

1) $I \cap J = \{0\}$

and 2) $R = I + J$.

Proof. Assume that there exist nontrivial ideals I,J of R such that 1) and 2) hold. Clearly I and J are skew rings of orders greater than 1. On $I \times J$ define $(i,j) + (p,q) = (i + p, j + q)$ and $(i,j) \cdot (p,q) = (i \cdot p, j \cdot q)$ for all $(i,j),(p,q) \in I \times J$. Then $(I \times J,+,\cdot)$ is a skew ring. Define $f : I \times J \to R$ by $f(i,j) = i + j$ for all $(i,j) \in I \times J$. To show that f is a surjection, let $r \in R$. By 2), there exist $i \in I, j \in J$ such that $r = i + j = f(i,j)$. To show that f is an injection, let $(i,j),(p,q) \in I \times J$ be such that $f(i,j) = f(p,q)$. Then $i + j = p + q$, so $-p + i = q - j \in I \cap J = \{0\}$ which implies that $i = p$ and $j = q$. Therefore f is a bijection. Claim that $ij = ji = 0$ for all $i \in I, j \in J$. To prove this, let $i \in I$ and $j \in J$. Since I and J are ideals, $ij, ji \in I \cap J = \{0\}$, so $ij = ji = 0$. Hence we have the claim. To show that f is a homomorphism, let $(i,j),(p,q) \in I \times J$. Then

$$
\begin{aligned}
f((i,j)(p,q)) &= f(ip,jq) \\
&= ip + jq \\
&= ip + 0 + 0 + jq \\
&= ip + iq + jp + jq \\
&= (i + j)(p + q) \\
&= f(i,j)f(p,q).
\end{aligned}
$$

Now $f((i,j) + (p,q)) = f(i + p, j + q) = i + p + j + q$. By Proposition 1.63, $i + p + j + q = i + j + p + q$. Hence $f((i,j) + (p,q)) = f(i,j) + f(p,q)$. Thus f is a homomorphism. Hence $R \cong I \times J$.

Conversely, assume that R is decomposable. Then there exist skew rings R_1, R_2 of orders greater than 1 and an isomorphism $f : R_1 \times R_2 \to R$. Let $I = f(R_1 \times \{0\})$ and $J = f(\{0\} \times R_2)$. Clearly I and J are additive subgroups of R. Let $i = f(x,0) \in I$ and $r = f(y,z) \in R$.

Then

$$\begin{aligned} r + i - r &= f(y,z) + f(x,0) - f(y,z) \\ &= f(y + x - y, z + 0 - z) \\ &= f(y + x - y, 0) \ \varepsilon\ I, \\ ir &= f(x,0)f(y,z) \\ &= f(xy,0) \ \varepsilon\ I \end{aligned}$$

and

$$\begin{aligned} ri &= f(y,z)f(x,0) \\ &= f(yx,0) \ \varepsilon\ I. \end{aligned}$$

Thus I is a nontrivial ideal of R. Similarly, J is a nontrivial ideal of R. Clearly $I \cap J = \{f(0,0)\}$ and $I + J \subseteq R$. Let $r \ \varepsilon\ R$. Then there exist $i \ \varepsilon\ R_1$ and $j \ \varepsilon\ R_2$ such that $r = f(i,j)$. Since $f(i,0) \ \varepsilon\ I$, $f(0,j) \ \varepsilon\ J$ and $f(i,j) = f(i,0) + f(0,j)$, $r \ \varepsilon\ I + J$. So $R \subseteq I + J$. Thus $R = I \times J$. Hence we have the theorem. #