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EVENTUAL REGULARITY AND ISOMORPHISM THEOREMS OF SOME REGRESSIVE TRANSFORMATION SEMIGROUPS

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เราเรียกการแปลงบางส่วน α บนโพเซตว่าเป็น*การแปลงบางส่วนถดถอย* ถ้า $x\alpha \le x$ สำหรับทุก x ในโดเมนของ α สำหรับโพเซต X ให้ $P_{RE}(X)$, $I_{RE}(X)$ และ $T_{RE}(X)$ แทนกึ่งกรุปการแปลงบางส่วนถดถอยบน X กึ่งกรุปการแปลงบางส่วนหนึ่งต่อหนึ่งถดถอยบน X และ กึ่งกรุปการแปลงเต็มถดถอยบน X ตามลำดับ ผลเกี่ยวกับการเป็นปกติและการเป็นปกติในที่สุด ต่อไปนี้เป็นที่รู้กันแล้ว $P_{RE}(X)$ $[I_{RE}(X)]$ เป็นกึ่งกรุปปกติ ก็ต่อเมื่อ ทุกจุดใน X เป็นจุดเอกเทศ และ $T_{RE}(X)$ เป็นกึ่งกรุปปกติ ก็ต่อเมื่อ $|C| \le 2$ สำหรับทุกเซตย่อยอันดับทุกส่วน C ของ X ถ้า $S(X)$ คือ $P_{RE}(X)$, $T_{RE}(X)$ หรือ $I_{RE}(X)$ แล้ว $S(X)$ เป็นกึ่งกรุปปกติในที่สุด ก็ต่อเมื่อ มีจำนวนเต็มบวก n ซึ่ง $|C| \le n$ สำหรับทุกเซตย่อยอันดับทุกส่วน C ของ X เอ อูมาร์ได้พิสูจน์ทฤษฎีบทสมสัณฐานที่สำคัญดังนี้ สำหรับเซตอันดับทุกส่วน X และ Y ใดๆ $T_{RE}(X) \cong T_{RE}(Y)$ ก็ต่อเมื่อ X และ Y สมสัณฐานอันดับกัน

วัตถุประสงค์ของการวิจัยนี้คือขยายผลจากสิ่งที่รู้แล้วข้างต้น เราพิจารณากึ่งกรุปย่อยของ $P_{RE}(X)$, $I_{RE}(X)$ และ $T_{RE}(X)$ ต่อไปนี้ โดยที่ X' เป็นโพเซตย่อยของ X, $P_{RE}(X,X') = \{\alpha \in P_{RE}(X) \mid \operatorname{ran}\alpha \subseteq X'\}$, $\overline{P}_{RE}(X,X') = \{\alpha \in P_{RE}(X) \mid X'\alpha \subseteq X'\}$ และ $I_{RE}(X,X')$, $\overline{I}_{RE}(X,X')$, $T_{RE}(X,X')$ และ $\overline{T}_{RE}(X,X')$ นิยามในทำนองเดียวกัน เราจะให้ลักษณะว่าเมื่อใดที่กึ่งกรุป $P_{RE}(X,X')$, $I_{RE}(X,X')$, $T_{RE}(X,X')$, $\overline{P}_{RE}(X,X')$, $\overline{I}_{RE}(X,X')$ และ $\overline{T}_{RE}(X,X')$ เหล่านี้เป็นปกติ ซึ่งทำให้ผลเกี่ยวกับการเป็นปกติ ข้างต้นนั้น กลายเป็นกรณีเฉพาะของผลที่ได้นี้ สำหรับการเป็นปกติในที่สุดนั้น เราจะใช้ผลที่ทราบมาแล้ว และบทตั้งบทหนึ่งของผลนี้ มาใช้ในการหาเงื่อนไขที่จำเป็นและเพียงพอในการเป็นปกติในที่สุดของกึ่งกรุปเหล่านี้ ทฤษฎีบทสมสัณฐานที่สำคัญที่ได้จากการวิจัยนี้มีดังนี้ ถ้า $P_{RE}(X,X') \cong P_{RE}(Y,Y')$ แล้ว X' และ Y' สมสัณฐานอันดับกัน และ ถ้า $I_{RE}(X,X') \cong I_{RE}(Y,Y')$ แล้ว X' และ Y' สมสัณฐานอันดับกัน โดยเฉพาะ $P_{RE}(X) \cong P_{RE}(Y)$ ก็ต่อเมื่อ X และ Y สมสัณฐานอันดับกัน และในทำนองเดียวกัน $I_{RE}(X) \cong I_{RE}(Y)$ ก็ต่อเมื่อ X และ Y สมสัณฐานอันดับกัน สำหรับเซตอันดับทุกส่วน X และ Y ถ้า $T_{RE}(X,X') \cong T_{RE}(Y,Y')$ แล้ว X' และ Y' สมสัณฐานอันดับกัน จะเห็นว่าทฤษฎีบทสมสัณฐานทฤษฎีบทหลังที่ได้นี้ขยายทฤษฎีบทสมสัณฐาณของอูมาร์

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต.......................................

สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา.......................................

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A partial transformation α on a poset is *regressive* if $x\alpha \le x$ for all $x \in \operatorname{dom}\alpha$. For a poset X, let $P_{RE}(X)$, $I_{RE}(X)$ and $T_{RE}(X)$ denote respectively the regressive partial transformation semigroup on X, the regressive 1-1 partial transformation semigroup on X and the full regressive transformation semigroup on X. The following results relating to regularity and eventual regularity are known. The semigroup $P_{RE}(X)$ [$I_{RE}(X)$] is regular if and only if X is isolated, and $T_{RE}(X)$ is regular if and only if $|C| \le 2$ for every subchain C of X. If $S(X)$ is $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$, then $S(X)$ is eventually regular if and only if there is a positive integer n such that $|C| \le n$ for every subchain C of X. A. Umar has proved an important isomorphism theorem as follows : For chains X and Y, $T_{RE}(X) \cong T_{RE}(Y)$ if and only if X and Y are order-isomorphic.

Our purpose is to extend the above known results. The following subsemigroups of $P_{RE}(X)$, $I_{RE}(X)$ and $T_{RE}(X)$ are considered where X' is a subposet of X. $P_{RE}(X, X') = \{ \alpha \in P_{RE}(X) \mid \operatorname{ran}\alpha \subseteq X' \}$, $\overline{P}_{RE}(X, X') = \{ \alpha \in P_{RE}(X) \mid X'\alpha \subseteq X' \}$ and $I_{RE}(X, X')$, $\overline{I}_{RE}(X, X')$, $T_{RE}(X, X')$ and $\overline{T}_{RE}(X, X')$ are defined similarly. We characterize when the semigroups $P_{RE}(X, X')$, $I_{RE}(X, X')$, $T_{RE}(X, X')$, $\overline{P}_{RE}(X, X')$, $\overline{I}_{RE}(X, X')$ and $\overline{T}_{RE}(X, X')$ are regular, and then the above known results of regularity become our special cases. For eventual regularity, the known result mentioned above and one of its lemmas are used to obtain necessary and sufficient conditions for all of these semigroups to be eventually regular. Our main isomorphism theorems are as follows : If $P_{RE}(X, X') \cong P_{RE}(Y, Y')$, then X' and Y' are order-isomorphic. If $I_{RE}(X, X') \cong I_{RE}(Y, Y')$, then X' and Y' are order-isomorphic. In particular, $P_{RE}(X) \cong P_{RE}(Y)$ if and only if X and Y are order-isomorphic, and also $I_{RE}(X) \cong I_{RE}(Y)$ if and only if X and Y are order-isomorphic. If X and Y are chains and $T_{RE}(X, X') \cong T_{RE}(Y, Y')$, then X' and Y' are order-isomorphic. It can be seen that the last isomorphism theorem extends Umar's Isomorphism Theorem.

Department **Mathematics** Student's signature......................................

Field of study **Mathematics** Advisor's signature......................................

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CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

For a set X, let $|X|$ denote the cardinality of X. The set of positive integers, the set of integers and the set of real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$, respectively.

An element a of a semigroup S is called an *idempotent* of S if $a^2 = a$. For a semigroup S, let $E(S)$ be the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

An element a of a semigroup S is said to be *regular* if $a = aba$ for some $b \in S$, and we call S a *regular semigroup* if every element of S is regular. The set of all regular elements of a semigroup S will be denoted by $Reg(S)$, that is,

$$Reg(S) = \{a \in S \mid a = aba \text{ for some } b \in S\}.$$

Consequently, $E(S) \subseteq Reg(S)$. By an *eventually regular element* of a semigroup S we mean an element a of S such that $a^k \in Reg(S)$ for some $k \in \mathbb{N}$. If every element of S is eventually regular, we call S an *eventually regular semigroup.* Therefore a regular semigroup is eventually regular.

For an element a of a semigroup S, let $<a>$ denote the subsemigroup of S generated by a, that is,

$$<a> = \{a^n \mid n \in \mathbb{N}\}.$$

We call S a *periodic semigroup* if $<a>$ is finite for every $a \in S$. It is known that for $a \in S$, if $<a>$ is finite, then $a^k \in E(S)$ for some $k \in \mathbb{N}$ ([1], page 3-4). Since

$E(S) \subseteq Reg(S)$ for every semigroup S, it follows that every periodic semigroup is eventually regular. In particular, every finite semigroup is eventually regular.

A *partial transformation* of a set X is a map from a subset of X into X. The *empty transformation* 0 is the partial transformation with empty domain. Let $P(X)$ be the set of all partial transformations of X, that is,

$$P(X) = \{\alpha \colon A \to X \mid A \subseteq X\}.$$

The identity map on a nonempty set A is denoted by 1_A. Then $1_A \in P(X)$ for every nonempty subset A of X. In particular, $1_X \in P(X)$. We denote the domain and the range of $\alpha \in P(X)$ by $\mathrm{dom}\alpha$ and $\mathrm{ran}\alpha$, respectively. Also, for $\alpha \in P(X)$ and $x \in \mathrm{dom}\alpha$, the image of x under α is written by $x\alpha$. The composition $\alpha\beta$ of α, $\beta \in P(X)$ is defined as follows : $\alpha\beta = 0$ if $\mathrm{ran}\alpha \cap \mathrm{dom}\beta = \varnothing$, otherwise, $\alpha\beta$ is the usual composition of the functions $\alpha_{|(\mathrm{ran}\alpha \cap \mathrm{dom}\beta)\alpha^{-1}}$ and $\beta_{|(\mathrm{ran}\alpha \cap \mathrm{dom}\beta)}$. Then under this composition, $P(X)$ is a semigroup having 0 and 1_X as its zero and identity, respectively. Observe that for $\alpha, \beta \in P(X)$,

$$\begin{aligned}
\mathrm{dom}(\alpha\beta) \quad &= \quad (\mathrm{ran}\alpha \cap \mathrm{dom}\beta)\alpha^{-1} \;\subseteq\; \mathrm{dom}\alpha, \\
\mathrm{ran}(\alpha\beta) \quad &= \quad (\mathrm{ran}\alpha \cap \mathrm{dom}\beta)\beta \;\subseteq\; \mathrm{ran}\beta, \\
x \in \mathrm{dom}(\alpha\beta) \quad &\Longleftrightarrow \quad x \in \mathrm{dom}\alpha \text{ and } x\alpha \in \mathrm{dom}\beta.
\end{aligned}$$

The semigroup $P(X)$ is called the *partial transformation semigroup* on X. By a *transformation semigroup* on X we mean a subsemigroup of $P(X)$.

By a *transformation* of X we mean a map of X into itself. Let $T(X)$ be the set of all transformations of X. Then

$$T(X) = \{\alpha \in P(X) \mid \mathrm{dom}\alpha = X\}$$

which is a subsemigroup of $P(X)$ containing 1_X and it is called the *full transformation semigroup* on X. Let $I(X)$ denote the set of all 1-1 partial transformations of X, that is,

$$I(X) = \{\alpha \in P(X) \mid \alpha \text{ is 1-1}\}.$$

Then $I(X)$ is a subsemigroup of $P(X)$ containing 0 and 1_X and it is called the *1-1 partial transformation semigroup* on X or the *symmetric inverse semigroup* on X.

It is well-known that all $P(X)$, $T(X)$ and $I(X)$ are regular ([1], page 4) and for $\alpha \in P(X)$, $\alpha^2 = \alpha$ $(\alpha \in E(P(X)))$ if and only if $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $x\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Thus

$$\begin{aligned} E(T(X)) &= \{\alpha \in T(X) \mid x\alpha = x \text{ for all } x \in \operatorname{ran}\alpha\}, \\ E(I(X)) &= \{1_A \mid \varnothing \neq A \subseteq X\} \cup \{0\}. \end{aligned}$$

For a nonempty subset A of X and $x \in X$, let A_x denote the element of $P(X)$ with domain A and range $\{x\}$. Observe that $A_x \in E(P(X))$ if and only if $x \in A$, in particular, $X_a \in E(T(X))$ for all $a \in X$.

For convenience, we sometimes write an element in $P(X)$ by using a bracket notation. For examples,

$$\begin{pmatrix} a & b & c \\ b & b & d \end{pmatrix} \quad \text{stands for the transformation } \{(a,b),(b,b),(c,d)\}.$$

$$\begin{pmatrix} A & x \\ y & x \end{pmatrix}_{x \in X \setminus A} \quad \text{stands for } \alpha \in T(X) \text{ defined by}$$

$$x\alpha = \begin{cases} y & \text{if } \ x \in A, \\ x & \text{if } \ x \in X \setminus A. \end{cases}$$

In the area of semigroups, the full transformation semigroup $T(X)$ is considered very important. In 1975, J. S. Y. Symons [7] introduced the semigroup

$T(X, X')$, $\varnothing \neq X' \subseteq X$, under composition consisting of all mappings in $T(X)$ whose range are contained in X', that is,

$$T(X, X') = \{\alpha \in T(X) \mid \text{ran}\alpha \subseteq X'\}.$$

Then $X_a \in T(X, X')$ for all $a \in X'$ and $T(X, X')$ is a subsemigroup of $T(X)$. The semigroup $T(X, X')$ can be considered as a generalization of $T(X)$ since $T(X, X) = T(X)$. In fact, in 1966, K. D. Magrill [3] studied the semigroup

$$\overline{T}(X, X') = \{\alpha \in T(X) \mid X'\alpha \subseteq X'\}$$

which is also a generalization of $T(X)$ since $\overline{T}(X, X) = T(X)$. We can see that $1_X \in \overline{T}(X, X')$ but $1_X \notin T(X, X')$ if $X' \subsetneq X$. It is clearly seen that $T(X, X') \subseteq \overline{T}(X, X') \subseteq T(X)$.

For $\alpha \in P(X)$ and $A \subseteq X$, we let $A\alpha$ stand for the set $(A \cap \text{dom}\alpha)\alpha$ $(= \{x\alpha \mid x \in A \cap \text{dom}\alpha\})$.

In this research the semigroups $P(X, X')$, $\overline{P}(X, X')$, $I(X, X')$ and $\overline{I}(X, X')$ are defined analogously, that is,

$$P(X, X') = \{\alpha \in P(X) \mid \text{ran}\alpha \subseteq X'\},\ \overline{P}(X, X') = \{\alpha \in P(X) \mid X'\alpha \subseteq X'\},$$
$$I(X, X') = \{\alpha \in I(X) \mid \text{ran}\alpha \subseteq X'\},\ \overline{I}(X, X') = \{\alpha \in I(X) \mid X'\alpha \subseteq X'\}.$$

Then $P(X, X') \subseteq \overline{P}(X, X') \subseteq P(X)$ and $I(X, X') \subseteq \overline{I}(X, X') \subseteq I(X)$. Since $P(X, X) = \overline{P}(X, X) = P(X)$ and $I(X, X) = \overline{I}(X, X) = I(X)$, both $P(X, X')$ and $\overline{P}(X, X')$ are generalizations of $P(X)$ while $I(X, X')$ and $\overline{I}(X, X')$ are generalizations of $I(X)$.

Next, let X be a poset. By a *subchain* of X we mean a subposet of X which is also a chain. A point $a \in X$ is said to be *isolated* if

$$\text{for any } x \in X,\ x \leq a \text{ or } x \geq a \Longrightarrow x = a,$$

and we call a subposet Y of X *isolated* if every point of Y is isolated in Y.

For $\alpha \in P(X)$, α is said to be *regressive* if

$$x\alpha \le x \quad \text{for all } x \in \text{dom}\alpha.$$

A transformation semigroup on X is said to be *regressive* if all of its elements are regressive. Let

$$P_{RE}(X) = \{\alpha \in P(X) \mid \alpha \text{ is regressive}\},$$
$$T_{RE}(X) = \{\alpha \in T(X) \mid \alpha \text{ is regressive}\},$$
$$I_{RE}(X) = \{\alpha \in I(X) \mid \alpha \text{ is regressive}\}.$$

Then $P_{RE}(X), T_{RE}(X)$ and $I_{RE}(X)$ are respectively subsemigroups of $P(X)$, $T(X)$ and $I(X)$. Observe that 0 and 1_X belong to $P_{RE}(X)$ and $I_{RE}(X)$ and $1_X \in T_{RE}(X)$. By a *regressive transformation semigroup* on X we mean a subsemigroup of $P_{RE}(X)$.

Let X and Y be posets. A bijection $\varphi : X \to Y$ is called an *order-isomorphism* if

$$\text{for } x_1, x_2 \in X,\ x_1 \le x_2 \text{ in } X \ \Leftrightarrow\ x_1\alpha \le x_2\alpha \text{ in } Y.$$

We say that X and Y are *order-isomorphic* if there is an order-isomorphism from X onto Y.

Example 1.1. Let $\alpha : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$x\alpha = x - 1 \quad \text{for all } x \in \mathbb{Z}.$$

Then α is an element of $P_{RE}(\mathbb{Z})$, $T_{RE}(\mathbb{Z})$ and $I_{RE}(\mathbb{Z})$. Also, α is a bijection and

$$x\alpha^n = x - n \quad \text{for all } x \in \mathbb{Z} \text{ and } n \in \mathbb{N}$$

which implies that

$$x(\alpha^n)^{-1} = x + n \quad \text{for all } x \in \mathbb{Z} \text{ and } n \in \mathbb{N}.$$

Hence for every $n \in \mathbb{N}$, $(\alpha^n)^{-1}$ is not regressive, so it belongs to none of $P_{RE}(\mathbb{Z})$, $T_{RE}(\mathbb{Z})$ and $I_{RE}(\mathbb{Z})$. If $\alpha^n = \alpha^n\beta\alpha^n$ for some $n \in \mathbb{N}$ and $\beta \in P_{RE}(\mathbb{Z})$, then $\beta = (\alpha^n)^{-1}$ which is not regressive. This proves that α is not eventually regular.

Some known results of regressive transformation semigroups are as follows: A. Umar [5] has shown that if X is a finite chain, then the subsemigroup $S = \{\alpha \in T_{RE}(X) \mid |\text{ran}\alpha| < |X|\}$ of $T_{RE}(X)$ is generated by $E(S)$, that is, for $\alpha \in S$, $\alpha = \delta_1\delta_2\ldots\delta_k$ for some $\delta_1, \delta_2, \ldots, \delta_k \in E(S)$, and S is not a regular semigroup if $|X| \geq 3$. Y. Kemprasit [2] showed that in any regressive transformation semigroup on a poset, its idempotents and regular elements are identical.

Proposition 1.2. ([2]) *If $S(X)$ is a regressive transformation semigroup on a poset X, then $Reg(S(X)) = E(S(X))$.*

Y. Kemprasit ([2]) also characterized when $P_{RE}(X), T_{RE}(X)$ and $I_{RE}(X)$ are regular semigroups as follows:

Theorem 1.3. ([2]) *For a poset X, if $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$, then $S(X)$ is a regular semigroup if and only if X is isolated.*

Theorem 1.4. ([2]) *For a poset X, $T_{RE}(X)$ is a regular semigroup if and only if for every subchain C of X, $|C| \leq 2$.*

A necessary and sufficient condition for $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ to be eventually regular has been given in [2]. The next proposition was used as a lemma to obtain this characterization. Both will be referred for our work.

Proposition 1.5. ([2]) *If X is a poset and there is no positive integer n such that $|C| \leq n$ for every subchain C of X, then there is a sequence of disjoint finite subchains $C_1, C_2, C_3, \ldots$ of X such that $|C_1| < |C_2| < |C_3| < \ldots$.*

Theorem 1.6. ([2]) *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. Then $S(X)$ is eventually regular if and only if there is a positive integer n such that $|C| \leq n$ for every subchain C of X.*

A significant isomorphism theorem on full regressive transformation semigroups was given by A. Umar [6] in 1996 as follows:

Theorem 1.7. ([6]) *If X and Y are chains, then $T_{RE}(X) \cong T_{RE}(Y)$ if and only if X and Y are order-isomorphic.*

Notice that the converse of Theorem 1.7 is true for any posets X and Y as follows:

Proposition 1.8. *For posets X and Y, if $\varphi : X \to Y$ is an order-isomorphism, then the map $\alpha \mapsto \varphi^{-1}\alpha\varphi$ is an isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$.*

Proof. If $\alpha \in T_{RE}(X)$ and $y \in Y$, then $(y\varphi^{-1})\alpha \leq y\varphi^{-1}$. Since φ is an order-isomorphism, $y\varphi^{-1}\alpha\varphi \leq y\varphi^{-1}\varphi = y$. Thus $\varphi^{-1}\alpha\varphi \in T_{RE}(Y)$. Also, for $\alpha, \beta \in T_{RE}(X)$, $\varphi^{-1}\alpha\beta\varphi = (\varphi^{-1}\alpha\varphi)(\varphi^{-1}\beta\varphi)$, and if $\varphi^{-1}\alpha\varphi = \varphi^{-1}\beta\varphi$, then $\alpha = \beta$. For $\lambda \in T_{RE}(Y)$, we have $\varphi\lambda\varphi^{-1} \in T_{RE}(X)$ and $\varphi^{-1}(\varphi\lambda\varphi^{-1})\varphi = \lambda$. □

T. Saito, K. Aoki and K. Kajitori [4] have given necessary and sufficient conditions for any posets X and Y so that $T_{RE}(X) \cong T_{RE}(Y)$. Umar's Isomorphism Theorem became a special case of their result.

Example 1.9. (1) For each $n \in \mathbb{N}$, $\mathbb{Z}$ is order-isomorphic to $n\mathbb{Z}$ through the map $x \mapsto nx$, by Theorem 1.7, $T_{RE}(\mathbb{Z}) \cong T_{RE}(n\mathbb{Z})$.

(2) We have that $T_{RE}(\mathbb{R}) \cong T_{RE}(\mathbb{R}^+)$ where $\mathbb{R}^+$ is the set of positive real numbers because the map $x \mapsto e^x$ is an order-isomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$.

Due to the semigroup introduced by J. S. V. Symons [7], the semigroup studied by K. D. Magrill [3] and those we define analogously, the following regressive transformation semigroups are defined for a poset X and a subposet X' of X analogously as follows:

$$P_{RE}(X,X') = \{\alpha \in P_{RE}(X) \mid \text{ran}\alpha \subseteq X'\},$$
$$\overline{P}_{RE}(X,X') = \{\alpha \in P_{RE}(X) \mid X'\alpha \subseteq X'\},$$
$$T_{RE}(X,X') = \{\alpha \in T_{RE}(X) \mid \text{ran}\alpha \subseteq X'\},$$
$$\overline{T}_{RE}(X,X') = \{\alpha \in T_{RE}(X) \mid X'\alpha \subseteq X'\},$$
$$I_{RE}(X,X') = \{\alpha \in I_{RE}(X) \mid \text{ran}\alpha \subseteq X'\},$$
$$\overline{I}_{RE}(X,X') = \{\alpha \in I_{RE}(X) \mid X'\alpha \subseteq X'\}.$$

It is clear that

$$P_{RE}(X,X') \subseteq \overline{P}_{RE}(X,X') \subseteq P_{RE}(X), \;\; T_{RE}(X,X') \subseteq \overline{T}_{RE}(X,X') \subseteq T_{RE}(X),$$
$$I_{RE}(X,X') \subseteq \overline{I}_{RE}(X,X') \subseteq I_{RE}(X), \quad P_{RE}(X,X) = \overline{P}_{RE}(X,X) = P_{RE}(X),$$
$$T_{RE}(X,X) = \overline{T}_{RE}(X,X) = T_{RE}(X) \;\text{ and }\; I_{RE}(X,X) = \overline{I}_{RE}(X,X) = I_{RE}(X).$$

Observe that 0 belongs to $P_{RE}(X,X'), \overline{P}_{RE}(X,X')$, $I_{RE}(X,X')$ and $\overline{I}_{RE}(X,X')$ and 1_X belongs to $\overline{P}_{RE}(X,X'), \overline{T}_{RE}(X,X')$ and $\overline{I}_{RE}(X,X')$. Moreover, $T_{RE}(X,X') \neq \varnothing$ (or equivalently, $T_{RE}(X,X')$ is a subsemigroup of $T_{RE}(X)$) if and only if

$$\text{for every } x \in X,\; x' \leq x \text{ for some } x' \in X'. \tag{*}$$

Then whenever we consider $T_{RE}(X,X')$, the condition (*) is always assumed.

Example 1.10. Let $\alpha : \mathbb{Z} \to 2\mathbb{Z}$ be defined by

$$x\alpha = \begin{cases} 0 & \text{if } \; x = 2, \\ x & \text{if } \; x \in 2\mathbb{Z}\backslash\{2\}, \\ x-1 & \text{if } \; x \notin 2\mathbb{Z}. \end{cases}$$

Then $\alpha \in T_{RE}(\mathbb{Z}, 2\mathbb{Z})$. Suppose that $T_{RE}(\mathbb{Z}, 2\mathbb{Z})$ has an identity element, say η. Thus

$$\beta\eta = \eta\beta = \beta \text{ for every } \beta \in T_{RE}(\mathbb{Z}, 2\mathbb{Z}).$$

Since $3\eta \leq 3$ and ran$\eta \subseteq 2\mathbb{Z}$, $3\eta \leq 2$ which implies that $(3\eta)\alpha < 2$. But $3\alpha = 3\eta\alpha < 2$, so it is contrary to the definition of α. Therefore $T_{RE}(\mathbb{Z}, 2\mathbb{Z})$ has no identity. Since $T_{RE}(\mathbb{Z})$ and $T_{RE}(2\mathbb{Z})$ have an identity, we conclude that

$$T_{RE}(\mathbb{Z}) = T_{RE}(\mathbb{Z}, \mathbb{Z}) \ncong T_{RE}(\mathbb{Z}, 2\mathbb{Z}) \ncong T_{RE}(2\mathbb{Z}, 2\mathbb{Z}) = T_{RE}(2\mathbb{Z}).$$

In Chapter II, we deal with the regularity of the six regressive transformation semigroups introduced previously. The aim is to generalize Theorem 1.3 and Theorem 1.4. Our proofs are independent to those given in [2] for Theorem 1.3 and Theorem 1.4. Then these two theorems become consequences of our obtained results.

Eventual regularity of our target regressive transformation semigroups is studied in Chapter III. The purpose is to extend Theorem 1.6. We characterize in this chapter when these regressive transformation semigroups are eventually regular. For these characterizations, Proposition 1.5 and Theorem 1.6 are referred as tools.

Finally, some isomorphism theorems of two regressive transformation semigroups of the same kinds are determined in Chapter IV. The interesting isomorphism theorems obtained in this chapter are as follows: For chains X and Y, a subchain X' of X and a subchain Y' of Y, if $T_{RE}(X, X') \cong T_{RE}(Y, Y')$, then X' and Y' are order-isomorphic. This result generalizes Umar's Isomorphism Theorem. For posets X and Y, X' a subposet of X and Y' a subposet of Y, if $P_{RE}(X, X') \cong P_{RE}(Y, Y')$, then X' and Y' are order-isomorphic, also if $I_{RE}(X, X') \cong I_{RE}(Y, Y')$, then X' and Y' are order-isomorphic. Some nice and remarkable consequences of the later two isomorphism theorems are that for any posets X and Y, $P_{RE}(X) \cong P_{RE}(Y)$ if and only if X and Y are order-isomorphic and $I_{RE}(X) \cong I_{RE}(Y)$ if and only if X and Y are order-isomorphic.

CHAPTER II

REGULAR REGRESSIVE TRANSFORMATION SEMIGROUPS

The purpose of this chapter is to generalize Theorem 1.3 and Theorem 1.4 by considering the regularity of $P_{RE}(X,X')$, $I_{RE}(X,X')$, $T_{RE}(X,X')$, $\overline{P}_{RE}(X,X')$, $\overline{I}_{RE}(X,X')$ and $\overline{T}_{RE}(X,X')$. More interesting results are obtained.

Throughout this chapter, X denotes any poset and X' denotes any subposet of X, otherwise stated.

2.1 Regularity of $P_{RE}(X,X')$, $I_{RE}(X,X')$, $\overline{P}(X,X')$ and $\overline{I}_{RE}(X,X')$

Recall that if $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$, then $S(X)$ is regular if and only if X is isolated (Theorem 1.3). By the definition of regressive partial transformations of X, it is clearly seen that

$$X \text{ is isolated} \Rightarrow P_{RE}(X) = I_{RE}(X) = \{1_A \mid \varnothing \neq A \subseteq X\} \cup \{0\}.$$

Theorem 2.1.1. *Let $S(X,X')$ be $P_{RE}(X,X')$ or $I_{RE}(X,X')$. Then the semigroup $S(X,X')$ is regular if and only if*

(i) X' is isolated and

(ii) for any $x \in X \setminus X'$ and $x' \in X'$, either $x < x'$ or x and x' are uncomparable.

Proof. Suppose first that X' is not isolated. Then there are $a, b \in X'$ such that $a < b$. Let $\alpha = \begin{pmatrix} b \\ a \end{pmatrix}$. Then $\alpha \in S(X,X')$ and $\alpha^2 = 0$, so $\alpha \notin E(S(X,X'))$. By

Proposition 1.2, $\alpha \notin Reg(S(X,X'))$. Next, suppose that there are $c \in X\backslash X'$ and $d \in X'$ such that $c > d$. Thus $\beta = \begin{pmatrix} c \\ d \end{pmatrix} \in S(X,X')$ and $\beta^2 = 0 \neq \beta$. Hence $\beta \notin Reg(S(X,X'))$ by Proposition 1.2. This shows that if $S(X,X')$ is a regular semigroup, then (i) and (ii) hold.

For the converse, assume that (i) and (ii) hold. Let $\alpha \in S(X,X')$ and $x \in \text{dom}\alpha$. Then $x\alpha \le x$ and $x\alpha \in X'$. Because of (ii), $x \in X'$, so by (i), $x\alpha = x$. This proves that $\alpha = 1_{\text{dom}\alpha}$, the identity map on domα. Hence $\alpha = \alpha^2 \in Reg(S(X,X'))$.

Therefore the theorem is proved. $\square$

Theorem 2.1.2. *Let $\overline{S}(X,X')$ be $\overline{P}_{RE}(X,X')$ or $\overline{I}_{RE}(X,X')$. Then the semigroup $\overline{S}(X,X')$ is regular if and only if*

(i) X' is isolated,

(ii) $X\backslash X'$ is isolated and

(iii) for any $x \in X\backslash X'$ and $x' \in X'$, either $x < x'$ or x and x' are uncomparable.

Proof. Recall that $P_{RE}(X,X') \subseteq \overline{P}_{RE}(X,X')$ and $I_{RE}(X,X') \subseteq \overline{I}_{RE}(X,X')$. By Proposition 1.2 and Theorem 2.1.1, to prove the necessity part, it suffices to show that if $X\backslash X'$ is not isolated, then there is a nonregular element in $\overline{S}(X,X')$. Assume that there are a, b in $X\backslash X'$ such that $a < b$. Then $\gamma = \begin{pmatrix} b \\ a \end{pmatrix} \in \overline{S}(X,X')$ and $\gamma^2 = 0 \neq \gamma$. Hence $\gamma \notin Reg(\overline{S}(X,X'))$ by Proposition 1.2. Therefore if $\overline{S}(X,X')$ is regular, then (i)-(iii) hold.

Conversely, assume that (i), (ii) and (iii) hold. Let $\alpha \in \overline{S}(X,X')$ and $x \in \text{dom}\alpha$. Then $x\alpha \le x$.

Case 1 : $x \in X'$. Since $X'\alpha \subseteq X'$, $x\alpha \in X'$. Because $x\alpha \le x$, it follows from (i) that $x\alpha = x$.

Case 2 : $x \in X\backslash X'$. Since $x\alpha \le x$, it follows from (iii) that $x\alpha \in X\backslash X'$. But

$X \backslash X'$ is isolated by (ii), thus $x\alpha = x$.

This proves that $\alpha = 1_{\mathrm{dom}\alpha}$, so α is regular.

Hence the proof is complete. $\square$

Theorem 1.3 is directly obtained from Theorem 2.1.1 or Theorem 2.1.2 when $X' = X$.

Corollary 2.1.3. *If $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$, then $S(X)$ is a regular semigroup if and only if X is isolated.*

In general, a subsemigroup of a regular semigroup need not be regular. An obvious example is that $(\mathbb{R}, +)$ is a regular semigroup (a group) and $\mathbb{N}$ is a subsemigroup of $(\mathbb{R}, +)$ which is not regular. However, $P_{RE}(X, X')$ and $I_{RE}(X, X')$ are respectively subsemigroups of $\overline{P}_{RE}(X, X')$ and $\overline{I}_{RE}(X, X')$ and by Theorem 2.1.1 and Theorem 2.1.2, the regularity of $\overline{P}_{RE}(X, X')$ $[\overline{I}_{RE}(X, X')]$ implies the regularity of its subsemigroup $P_{RE}(X, X')$ $[I_{RE}(X, X')]$. In fact, it follows directly from Proposition 1.2 that any subsemigroup of a regular regressive partial transformation semigroup on X is also regular.

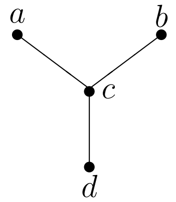
Corollary 2.1.4. *The following statements hold.*

(i) If $\overline{P}_{RE}(X, X')$ is a regular semigroup, then so is $P_{RE}(X, X')$.

(ii) If $\overline{I}_{RE}(X, X')$ is a regular semigroup, then so is $I_{RE}(X, X')$.

Example 2.1.5. Let X and Y be posets, X' a subposet of X and Y' a subposet of Y defined by the Hasse diagrams as follows:

X :



X' :

By Theorem 2.1.1 and Theorem 2.1.2, $P_{RE}(X, X')$ and $I_{RE}(X, X')$ are regular but neither $\overline{P}_{RE}(X, X')$ nor $\overline{I}_{RE}(X, X')$ are regular. Also, from these two theorems, we have that all the semigroups, $P_{RE}(Y, Y')$, $I_{RE}(Y, Y')$, $\overline{P}_{RE}(Y, Y')$ and $\overline{I}_{RE}(Y, Y')$ are regular. Note that by Corollary 2.1.3, none of $P_{RE}(X)$, $I_{RE}(X)$, $P_{RE}(Y)$ and $I_{RE}(Y)$ is regular while all $P_{RE}(X')$, $I_{RE}(X')$, $P_{RE}(Y')$ and $I_{RE}(Y')$ are regular.

2.2 Regularity of $T_{RE}(X, X')$ and $\overline{T}_{RE}(X, X')$

In this section, we intend to generalize Theorem 1.4 stated that $T_{RE}(X)$ is regular if and only if $|C| \leq 2$ for every subchain C of X.

Theorem 2.2.1. *The semigroup $T_{RE}(X, X')$ is regular if and only if for every subchain C of X,*

(i) $|C \cap X'| \leq 2$ and

(ii) if $C \cap X' \neq \phi$ and $C \cap X'$ has an upper bound not in $C \cap X'$, then $|C \cap X'| = 1$.

Proof. Assume that every subchain C of X satisfies (i) and (ii). By Proposition 1.2, it suffices to show that every element of $T_{RE}(X, X')$ is an idempotent. Let $\alpha \in T_{RE}(X, X')$ and $x \in X$. Then $x \geq x\alpha \geq x\alpha^2$ and $x\alpha, x\alpha^2 \in X'$.

Case 1 : $x \in X'$. Then $x, x\alpha, x\alpha^2 \in X'$ and $x \geq x\alpha \geq x\alpha^2$. It follows from (i) that $x = x\alpha$ or $x\alpha = x\alpha^2$. Hence $x\alpha = x\alpha^2$.

Case 2 : $x \in X \backslash X'$. Consider the chain $C = \{x\alpha, x\alpha^2\} \subseteq X'$. Then $x \in X \backslash X'$ as is an upper bound of C. By (ii), $|C| = 1$, and thus $x\alpha = x\alpha^2$.

We therefore conclude that $x\alpha = x\alpha^2$ for all $x \in X$. Hence α is an idempotent.

Conversely, suppose that there exists a chain C of X such that (1) $|C \cap X'| \geq 3$ or (2) $|C \cap X'| \geq 2$ and $C \cap X'$ has an upper bound in $X \backslash (C \cap X')$. In any cases, we have a subchain $a < b < c$ of X with $a, b \in X'$. Recall that X' satisfies the condition (*). Then for each $x \in X$, there exists $x^{'} \in X'$ such that $x^{'} \leq x$. Define $\alpha : X \rightarrow X$ by

$$\alpha = \begin{pmatrix} b & c & x \\ a & b & x^{'} \end{pmatrix}_{x \in X \backslash \{b,c\}}.$$

Then $\alpha \in T_{RE}(X, X')$. Since $b \in \text{ran}\alpha$ and $b\alpha = a \neq b$, we have that α is not an idempotent. By Proposition 1.2, α is not a regular element of $T_{RE}(X, X')$.

Hence if $T_{RE}(X, X')$ is regular, then (i) and (ii) hold. $\square$

Theorem 2.2.2. *The semigroup $\overline{T}_{RE}(X, X')$ is regular if and only if for every subchain C of X,*

(i) $|C \cap X'| \leq 2$,

(ii) $|C \cap (X \backslash X')| \leq 2$,

(iii) if $C \cap X' \neq \phi$ and $C \cap X'$ has an upper bound not in $C \cap X'$, then $|C \cap X'| = 1$ and

(iv) if $C \cap (X \backslash X') \neq \phi$ and $C \cap (X \backslash X')$ has a lower bound not in $C \cap (X \backslash X')$, then $|C \cap (X \backslash X')| = 1$.

Proof. Assume that every chain C of X satisfies (i)-(iv). Let $\alpha \in \overline{T}_{RE}(X, X')$ and $x \in X$. Then $x \geq x\alpha \geq x\alpha^2$.

Case 1 : $x \in X'$. Since $X'\alpha \subseteq X'$, we have that all $x, x\alpha$ and $x\alpha^2$ belong to X'. It therefore follows from (i) that $x = x\alpha$ or $x\alpha = x\alpha^2$, so $x\alpha = x\alpha^2$.

Case 2 : $x \notin X'$ and $x\alpha \in X'$. Then $x\alpha^2 \in X'$ since $X'\alpha \subseteq X'$. We then deduce from (iii) that $x\alpha = x\alpha^2$.

Case 3 : $x \notin X'$, $x\alpha \notin X'$ and $x\alpha^2 \in X'$. Then we have from (iv) that $x = x\alpha$, and hence $x\alpha = x\alpha^2$.

Case 4 : $x \notin X'$, $x\alpha \notin X'$ and $x\alpha^2 \notin X'$. It then follows from (ii) that $x = x\alpha$ or $x\alpha = x\alpha^2$ which implies that $x\alpha = x\alpha^2$.

This shows that $\alpha^2 = \alpha$, so α is a regular element of $\overline{T}_{RE}(X, X')$.

For the converse, suppose that there exists a subchain C satisfying at least one of the following conditions.

(1) $|C \cap X'| \geq 3$,

(2) $|C \cap (X \backslash X')| \geq 3$,

(3) $|C \cap X'| \geq 2$ and $C \cap X'$ has an upper bound not in itself,

(4) $|C \cap (X \backslash X')| \geq 2$ and $C \cap (X \backslash X')$ has a lower bound not in itself.

Case 1 : $|C \cap X'| \geq 3$. Then there are $a, b, c \in C \cap X'$ such that $a < b < c$. Define $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} b & c & x \\ a & b & x \end{pmatrix}_{x \in X \backslash \{b,c\}}.$$

Then $\alpha \in T_{RE}(X)$ and $X'\alpha = \{a, b\} \cup (X' \backslash \{b, c\}) \subseteq X'$, so $\alpha \in \overline{T}_{RE}(X, X')$. But $b \in \operatorname{ran}\alpha$ and $b\alpha = a \neq b$, so $\alpha^2 \neq \alpha$.

Case 2 : $|C \cap (X \backslash X')| \geq 3$. Then $e < f < g$ for some $e, f, g \in C \cap (X \backslash X')$. Let

$$\beta = \begin{pmatrix} f & g & x \\ e & f & x \end{pmatrix}_{x \in X \backslash \{f,g\}}.$$

Then $\beta \in T_{RE}(X)$ and $x\beta = x$ for all $x \in X'$, so $\beta \in \overline{T}_{RE}(X, X')$. Since $f \in \operatorname{ran}\beta$ and $f\beta = e \neq f$, $\beta^2 \neq \beta$.

Case 3 : $|C \cap X'| \geq 2$ and $C \cap X'$ has an upper bound $u \in X \backslash (C \cap X')$. Then $k > h$ for some $k, h \in C \cap X'$, and thus $u > k > h$. Let

$$\gamma = \begin{pmatrix} k & u & x \\ h & k & x \end{pmatrix}_{x \in X\backslash\{k,u\}}.$$

Then $\gamma \in T_{RE}(X)$. If $u \in X'$, then $X'\gamma = \{h,k\} \cup (X'\backslash\{k,u\}) = X'\backslash\{u\} \subseteq X'$. If $u \in X\backslash X'$, then $X'\gamma = \{h\} \cup (X'\backslash\{k\}) = X'\backslash\{k\} \subseteq X'$. Therefore $\gamma \in \overline{T}_{RE}(X, X')$. Since $k \in \text{ran}\gamma$ and $k\gamma = h \neq k$, $\gamma^2 \neq \gamma$.

Case 4 : $|C \cap (X\backslash X')| \geq 2$ and $C \cap (X\backslash X')$ has a lower bound $l \in X\backslash(C \cap (X\backslash X'))$. Then $p > q$ for some $p, q \in C \cap (X\backslash X')$, and so $p > q > l$. Let

$$\lambda = \begin{pmatrix} q & p & x \\ l & q & x \end{pmatrix}_{x \in X\backslash\{p,q\}}.$$

Then $\lambda \in T_{RE}(X)$. Since $X' \subseteq X\backslash\{p,q\}$, $X'\lambda = X'$, thus $\lambda \in \overline{T}_{RE}(X, X')$. But $q \in \text{ran}\lambda$ and $q\lambda = l \neq q$, so $\lambda^2 \neq \lambda$.

We therefore deduce from Proposition 1.2 that $\overline{T}_{RE}(X, X')$ is not a regular semigroup. □

Remark 2.2.3. It can be easily seen from Theorem 2.2.2 that if the semigroup $\overline{T}_{RE}(X, X')$ is regular, then the following statements hold.

(i) Every subchain of X has length at most 4.

(ii) If $C = \{a, b, c, d\}$ is a subchain of X such that $a < b < c < d$, then either $C \cap X' = \{c, d\}$ or $C \cap X' = \{b, d\}$.

We can see easily that Theorem 1.4 is a consequence of Theorem 2.2.1 and Theorem 2.2.2.

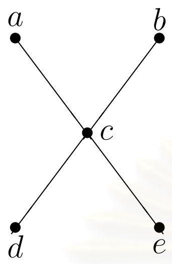
Corollary 2.2.4. *The semigroup $T_{RE}(X)$ is regular if and only if for every subchain C of X, $|C| \leq 2$.*

Also, from Theorem 2.2.1 and Theorem 2.2.2 or from Proposition 1.2, we have

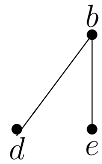
Corollary 2.2.5. *If $\overline{T}_{RE}(X, X')$ is a regular semigroup, then so is $T_{RE}(X, X')$.*

Example 2.2.6. Let X and Y be posets, X' a subposet of X and Y' a subposet of Y defined by the following Hasse diagrams.

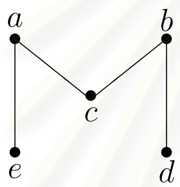
X :



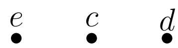
X' :



Y :



Y' :



By Theorem 2.2.1, $T_{RE}(X, X')$ is regular, and by Theorem 2.2.2, $\overline{T}_{RE}(X, X')$ is not regular and both $T_{RE}(Y, Y')$ and $\overline{T}_{RE}(Y, Y')$ are regular.

CHAPTER III

EVENTUALLY REGULAR REGRESSIVE TRANSFORMATION SEMIGROUPS

The purpose of this chapter is to charaterize when our target regressive transformation semigroups are eventually regular. These characterizations will generalize Theorem 1.6.

Throughout this chapter unless mentioned, X denotes any poset and X' denotes a subposet of X.

3.1 Eventual Regularity of $P_{RE}(X,X')$, $I_{RE}(X,X')$ and $T_{RE}(X,X')$

We first give a necessary and sufficient condition of $P_{RE}(X,X')$, $I_{RE}(X,X')$ and $T_{RE}(X,X')$ to be eventually regular. This condition depends only on X'.

Theorem 3.1.1. *Let $S(X,X')$ be $P_{RE}(X,X')$, $I_{RE}(X,X')$ or $T_{RE}(X,X')$. Then $S(X,X')$ is eventually regular if and only if there exists a positive integer n such that $|C| \leq n$ for every subchain C of X'.*

Proof. To prove necessity, assume that $S(X,X')$ is eventually regular. Based on Theorem 1.6, it suffices to show that $S(X',X')$ is eventually regular where

$$S(X',X') = \begin{cases} P_{RE}(X',X') & \text{if} \quad S(X,X') = P_{RE}(X,X'), \\ I_{RE}(X',X') & \text{if} \quad S(X,X') = I_{RE}(X,X'), \\ T_{RE}(X',X') & \text{if} \quad S(X,X') = T_{RE}(X,X'). \end{cases}$$

Let $\alpha \in S(X', X')$.

Case 1 : $S(X, X')$ is $P_{RE}(X, X')$ or $I_{RE}(X, X')$. Then $\alpha \in S(X, X')$. Since $S(X, X')$ is eventually regular, $\alpha^k \in Reg(S(X, X'))$ for some $k \in \mathbb{N}$. By Proposition 1.2, $\alpha^k \in E(S(X, X'))$. But $\alpha^k \in S(X', X')$, so $\alpha^k \in E(S(X', X'))$.

Case 2 : $S(X, X')$ is $T_{RE}(X, X')$. By (*), for every $x \in X$, there is an $x' \in X'$ such that $x' \leq x$. Define $\beta : X \to X'$ by

$$x\beta = \begin{cases} x\alpha & \text{if} \quad x \in X', \\ \\ x' & \text{if} \quad x \in X \backslash X'. \end{cases}$$

Since $\alpha \in T_{RE}(X')$, β is clearly an element of $T_{RE}(X, X')$ and $\beta_{|_{X'}} = \alpha$. But $T_{RE}(X, X')$ is eventually regular, thus $\beta^k \in Reg(T_{RE}(X, X'))$ for some $k \in \mathbb{N}$, and hence $\beta^k \in E(T_{RE}(X, X'))$ by Proposition 1.2. But $\alpha = \beta_{|_{X'}} \in T_{RE}(X')$, so $\alpha^k \in E(T_{RE}(X'))$.

It therefore follows from Theorem 1.6 that there exists an $n \in \mathbb{N}$ such that $|C| \leq n$ for every subchain C of X'.

To prove sufficiency, assume that there is an $n \in \mathbb{N}$ such that $|C| \leq n$ for every chain C of X'. To show that $S(X, X')$ is eventually regular, let $\alpha \in S(X, X')$ and $x \in \mathrm{dom}\alpha^{n+1}$. Then

$$x \geq x\alpha \geq x\alpha^2 \geq \ldots \geq x\alpha^n \geq x\alpha^{n+1}.$$

Since $\mathrm{ran}\alpha \subseteq X'$, $x\alpha \geq x\alpha^2 \geq \ldots \geq x\alpha^n \geq x\alpha^{n+1}$ is a subchain of X'. We have by assumption that $x\alpha^i = x\alpha^{i+1}$ for some $i \in \{1, 2, \ldots, n\}$. Since $x \in \mathrm{dom}\alpha^{n+1}$, $x\alpha^i \in \mathrm{dom}\alpha^{n+1-i}$, so we have $x\alpha^{n+1} = (x\alpha^i)\alpha^{n+1-i} = (x\alpha^{i+1})\alpha^{n+1-i} = x\alpha^{n+2}$. This proves that $\mathrm{dom}\alpha^{n+1} \subseteq \mathrm{dom}\alpha^{n+2}$ and $x\alpha^{n+1} = x\alpha^{n+2}$ for every $x \in \mathrm{dom}\alpha^{n+1}$. But $\mathrm{dom}\alpha^{n+2} \subseteq \mathrm{dom}\alpha^{n+1}$, so we have $\alpha^{n+1} = \alpha^{n+2}$. Consequently, $\alpha^{n+1} \in E(S(X, X'))$.

Hence the theorem is proved. $\square$

The following corollary is obtained directly from Theorem 1.6 and Theorem 3.1.1.

Corollary 3.1.2. *The following statements hold.*

(i) $P_{RE}(X, X')$ *is eventually regular if and only if* $P_{RE}(X')$ *is eventually regular.*

(ii) $I_{RE}(X, X')$ *is eventually regular if and only if* $I_{RE}(X')$ *is eventually regular.*

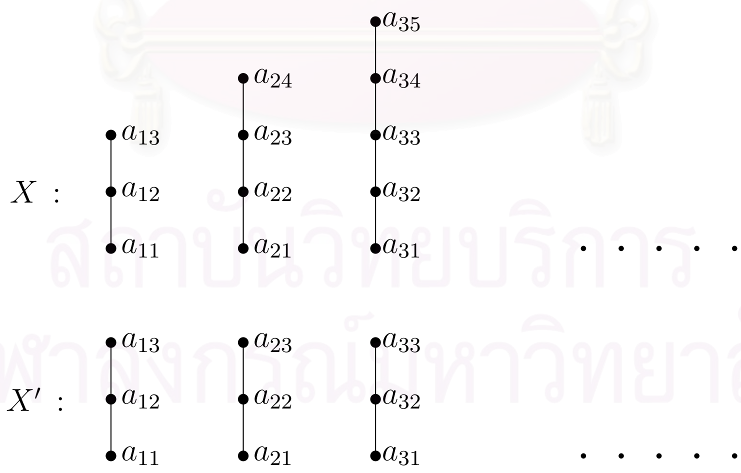
(iii) $T_{RE}(X, X')$ *is eventually regular if and only if* $T_{RE}(X')$ *is eventually regular.*

Some easy consequences of Theorem 3.1.1 are as follows:

Corollary 3.1.3. *If* X' *is a finite subposet of* X*, then all the semigroups* $P_{RE}(X, X')$, $T_{RE}(X, X')$ *and* $I_{RE}(X, X')$ *are eventually regular.*

Corollary 3.1.4. *If* X' *is an infinite subchain of* X*, then none of the semigroups* $P_{RE}(X, X')$, $T_{RE}(X, X')$ *and* $I_{RE}(X, X')$ *is eventually regular.*

Example 3.1.5. Let X be a poset and X' a subposet of X defined by the following Hasse diagrams.



Notice that X and X' satisfy the property (*). We deduce from Theorem 3.1.1 that all $P_{RE}(X, X')$, $I_{RE}(X, X')$ and $T_{RE}(X, X')$ are eventually regular. We give a remark that from Theorem 1.6, $T_{RE}(X)$ is not eventually regular but $T_{RE}(X')$ is eventually regular.

3.2 Eventual Regularity of $\overline{P}_{RE}(X, X')$, $\overline{I}_{RE}(X, X')$ and $\overline{T}_{RE}(X, X')$

In this section, we give a characterization determining when $\overline{P}_{RE}(X, X')$, $\overline{I}_{RE}(X, X')$ and $\overline{T}_{RE}(X, X')$ are eventually regular. The next theorem shows that this characterization depends only on X but not on X', and it is the same as that given for being eventual regularity of $P_{RE}(X)$, $I_{RE}(X)$ and $T_{RE}(X)$. To obtain this result, the following obvious fact is also needed and the proof is omitted.

Lemma 3.2.1. *Let S be a semigroup with $Reg(S) = E(S)$ and T a subsemigroup of S. Then for $a \in T$, if a is an eventually regular element of S, then a is an eventually regular element of T. Hence if S is eventually regular, then so is T.*

Theorem 3.2.2. *Let $\overline{S}(X, X')$ be $\overline{P}_{RE}(X, X')$, $\overline{T}_{RE}(X, X')$ or $\overline{I}_{RE}(X, X')$. Then $\overline{S}(X, X')$ is eventually regular if and only if there is a positive integer n such that $|C| \leq n$ for every subchain C of X.*

Proof. To prove sufficiency, assume that there is a positive integer n such that $|C| \leq n$ for every subchain C of X. Then by Theorem 1.6, all $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are eventually regular. But since $\overline{P}_{RE}(X, X')$, $\overline{T}_{RE}(X, X')$ and $\overline{I}_{RE}(X, X')$ are respectively subsemigroups of $P_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$, we have by Proposition 1.2 and Lemma 3.2.1 that all the semigroups $\overline{P}_{RE}(X, X')$, $\overline{T}_{RE}(X, X')$ and $\overline{I}_{RE}(X, X')$ are eventually regular.

To prove necessary by contrapositive, suppose that there is no $n \in \mathbb{N}$ such that $|C| \leq n$ for every subchain C of X. By Proposition 1.5, there exists a sequence of disjoint finite subchains $C_1, C_2, C_3, \ldots$ of X such that $|C_1| < |C_2| < |C_3| < \ldots$. Therefore we deduce that there is a sequence $k_1, k_2, k_3, \ldots$ of $\mathbb{N}$ such that

$$k_1 < k_2 < k_3 < \ldots \text{ and}$$

$$|C_{k_1} \cap X'| < |C_{k_2} \cap X'| < |C_{k_3} \cap X'| < \ldots \text{ or}$$
$$|C_{k_1} \cap (X \backslash X')| < |C_{k_2} \cap (X \backslash X')| < |C_{k_3} \cap (X \backslash X')| < \ldots$$

Let $D_i = C_{k_i}$ for every $i \in \mathbb{N}$. Then $|D_1 \cap X'| < |D_2 \cap X'| < |D_3 \cap X'| < \ldots$ or $|D_1 \cap (X \backslash X')| < |D_2 \cap (X \backslash X')| < |D_3 \cap (X \backslash X')| < \ldots$.

Case 1 : $|D_1 \cap X'| < |D_2 \cap X'| < |D_3 \cap X'| < \ldots$. We may assume that $|D_1 \cap X'| > 1$. For each $i \in \mathbb{N}$, let

$$D_i \cap X' = \left\{x'_{i_1}, x'_{i_2}, \ldots, x'_{i_{l_i}}\right\} \text{ where } x'_{i_1} < x'_{i_2} < \ldots < x'_{i_{l_i}}.$$

Then $1 < l_1 < l_2 < \ldots$. Define $\alpha : \bigcup_{i=1}^{\infty}((D_i \cap X') \backslash \{x'_{i_1} \mid i \in \mathbb{N}\}) \to X$ by

$$x'_{i_j}\alpha = x'_{i_{j-1}} \quad \text{for all } i \in \mathbb{N} \text{ and } j \in \{2, 3, \ldots, l_i\}. \tag{1}$$

Thus $\alpha \in I_{RE}(X')$ and if $m \in \mathbb{N}$, then $l_k > 2m$ for some $k \in \mathbb{N}$. By (1), $x'_{k_{l_k}} \in \mathrm{dom}\alpha^{2m} \subseteq \mathrm{dom}\alpha^m$ and

$$x'_{k_{l_k}}\alpha^{2m} = x'_{k_{l_k - 2m}} < x'_{k_{l_k - m}} = x'_{k_{l_k}}\alpha^m.$$

This shows that

$$\text{for every } m \in \mathbb{N}, \text{ there is an element } a \in \mathrm{dom}\alpha^{2m} \text{ such that } a\alpha^m \neq a\alpha^{2m}. \tag{2}$$

Hence $\alpha^m \neq \alpha^{2m}$ for every $m \in \mathbb{N}$, that is, $\alpha^m \notin E(I_{RE}(X'))$ for every $m \in \mathbb{N}$. By Proposition 1.2, α is not an eventually regular element of $I_{RE}(X')$. But $I_{RE}(X')$ is a subsemigroup of $\overline{P}_{RE}(X, X')$ and $\overline{I}_{RE}(X, X')$, so by Lemma 3.2.1, α is not eventually regular in $\overline{P}_{RE}(X, X')$ and $\overline{I}_{RE}(X, X')$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x\alpha & \text{if } \ x \in \mathrm{dom}\alpha, \\ x & \text{otherwise.} \end{cases}$$

Then $\beta \in \overline{T}_{RE}(X, X')$ since $X'\beta = ((X' \cap \mathrm{dom}\alpha) \cup (X' \backslash \mathrm{dom}\alpha))\beta = (X' \cap \mathrm{dom}\alpha)\alpha \cup (X' \backslash \mathrm{dom}\alpha) \subseteq X'$. To show that $\beta^m \neq \beta^{2m}$ for every $m \in \mathbb{N}$, let $m \in \mathbb{N}$ be fixed. By (2), there is an element $a \in \mathrm{dom}\alpha^{2m}$ such that $a\alpha^m \neq a\alpha^{2m}$. Then $a, a\alpha, \ldots, a\alpha^{2m-1} \in \mathrm{dom}\alpha$. It follows from the definition of β that

$$a\beta = a\alpha, a\alpha\beta = a\alpha^2, \ldots, a\alpha^{2m-1}\beta = a\alpha^{2m}.$$

Consequently, $a\beta^m = a\alpha^m$ and $a\beta^{2m} = a\alpha^{2m}$ which imply that $a\beta^m \neq a\beta^{2m}$, so $\beta^m \neq \beta^{2m}$. Therefore $\beta^m \notin E(\overline{T}_{RE}(X, X'))$ for every $m \in \mathbb{N}$. Hence we deduce from Proposition 1.2 that β is not an eventually regular element of $\overline{T}_{RE}(X, X')$.

Case 2 : $|D_1 \cap (X\backslash X')| < |D_2 \cap (X\backslash X')| < |D_3 \cap (X\backslash X')| < \ldots$. By considering $X\backslash X'$ as X' in Case 1, we also have a map $\lambda \in I_{RE}(X\backslash X')$ satisfying the property that

$$\text{for every } m \in \mathbb{N}, \text{ there is an element } a \in \mathrm{dom}\lambda^{2m} \text{ such that } a\lambda^m \neq a\lambda^{2m}. \tag{3}$$

This implies by Proposition 1.2 that λ is not an eventually regular element of $I_{RE}(X\backslash X')$. But $I_{RE}(X\backslash X')$ is clearly a subsemigroup of $\overline{I}_{RE}(X, X')$ and $\overline{P}_{RE}(X, X')$. By Lemma 3.2.1, λ is not eventually regular in $\overline{I}_{RE}(X, X')$ and $\overline{P}_{RE}(X, X')$. Define $\mu : X \to X$ by

$$x\mu = \begin{cases} x\lambda & \text{if} \quad x \in \mathrm{dom}\lambda, \\ x & \text{if} \quad x \in X\backslash\mathrm{dom}\lambda. \end{cases}$$

Since $X' \subseteq X\backslash\mathrm{dom}\lambda$, $x\mu = x$ for all $x \in X'$, so $\mu \in \overline{T}_{RE}(X, X')$. From (3) and the definition of μ, we can prove similary as in Case 1 that $\mu^m \neq \mu^{2m}$ for every $m \in \mathbb{N}$. Thus by Proposition 1.2, μ is not eventually regular in $\overline{T}_{RE}(X, X')$.

Therefore the theorem is completely proved. □

From Theorem 1.6 and Theorem 3.2.2, we have

Corollary 3.2.3. *The following statements hold.*

(i) $\overline{P}_{RE}(X, X')$ *is eventually regular if and only if* $P_{RE}(X)$ *is eventually regular.*

(ii) $\overline{I}_{RE}(X, X')$ *is eventually regular if and only if* $I_{RE}(X)$ *is eventually regular.*

(iii) $\overline{T}_{RE}(X, X')$ *is eventually regular if and only if* $T_{RE}(X)$ *is eventually regular.*

Also, the next result follows directly from Theorem 3.2.2.

Corollary 3.2.4. *If* X *is an infinite chain, then none of the semigroups* $\overline{P}_{RE}(X, X')$, $\overline{I}_{RE}(X, X')$ *and* $\overline{T}_{RE}(X, X')$ *is eventually regular.*

Example 3.2.5. Let X and X' be defined as in Example 3.1.5. By Theorem 3.2.2, $\overline{P}_{RE}(X, X')$, $\overline{I}_{RE}(X, X')$ and $\overline{T}_{RE}(X, X')$ are not eventually regular. However, all of $P_{RE}(X, X')$, $I_{RE}(X, X')$ and $T_{RE}(X, X')$ are eventually regular.

CHAPTER IV

ISOMORPHISM THEOREMS OF REGRESSIVE TRANSFORMATION SEMIGROUPS

We first intend to generalize Umar's Theorem (Theorem 1.7) stated that for chains X and Y, $T_{RE}(X) \cong T_{RE}(Y)$ if and only if X and Y are order-isomorphic. In fact, some other interesting isomorphism theorems are also provided in this chapter.

4.1 Elementary Results

Some required elementary results are provided in this section. These results will be referred later.

Proposition 4.1.1. *Let X be a poset, X' a subposet of X and let $S(X,X')$ be $P_{RE}(X,X')$ or $I_{RE}(X,X')$. Then the following statements are equivalent.*

(i) $S(X,X')$ has an identity.

(ii) For all $a \in X \backslash X'$ and $b \in X'$, either $a < b$ or a and b are uncomparable.

(iii) $S(X,X') = S(X')$, that is,

$$P_{RE}(X,X') = P_{RE}(X') \quad \textit{if} \quad S(X,X') = P_{RE}(X,X') \textit{ and}$$

$$I_{RE}(X,X') = I_{RE}(X') \quad \textit{if} \quad S(X,X') = I_{RE}(X,X').$$

Proof. (i)$\Rightarrow$(ii). To prove by contrapositive, assume that there are $a \in X \backslash X'$ and $b \in X'$ such that $a > b$. Then $\begin{pmatrix} a \\ b \end{pmatrix} \in S(X,X')$. If $\alpha \in S(X,X')$, then ran$\alpha \subseteq X'$, so $a \notin$ ranα which implies that $\alpha \begin{pmatrix} a \\ b \end{pmatrix} = 0 \neq \begin{pmatrix} a \\ b \end{pmatrix}$.

This shows that $S(X, X')$ has no identity.

(ii)$\Rightarrow$(iii). Suppose that (ii) holds. Clearly, $P_{RE}(X') \subseteq P_{RE}(X, X')$ and $I_{RE}(X') \subseteq I_{RE}(X, X')$. Let α be an element of $P_{RE}(X, X')$ and $x \in \mathrm{dom}\alpha$. Then $x\alpha \leq x$ and $x\alpha \in X'$. By (ii), x must be an element of X'. Hence

$$\alpha \in \begin{cases} P_{RE}(X') & \text{if} \quad \alpha \in P_{RE}(X, X'), \\ I_{RE}(X') & \text{if} \quad \alpha \in I_{RE}(X, X'). \end{cases}$$

Therefore (iii) is proved.

(iii)$\Rightarrow$(i). Obvious. $\square$

Proposition 4.1.2. *Let X be a chain and X' a proper subchain of X. If the semigroup $T_{RE}(X, X')$ has an identity, then the following statements hold.*

(i) $minX$ exists.

(ii) For all $a \in X \backslash X'$ and $b \in X' \backslash \{minX\}$, $a < b$.

Proof. Let η be the identity of $T_{RE}(X, X')$. By the property (*), for every $x \in X$, there exists an element $x' \in X'$ such that $x' \leq x$.

Suppose that X has no minimum element. From the above reason, X' has no minimum element. Let $a \in X \backslash X'$. Then $a > a\eta \in X'$, so $a > a\eta > b$ for some $b \in X'$. Define $\alpha : X \to X'$ by

$$x\alpha = \begin{cases} a\eta & \text{if} \quad x = a, \\ b & \text{if} \quad x = a\eta, \\ x' & \text{otherwise}. \end{cases}$$

Then $\alpha \in T_{RE}(X, X')$, so $\alpha\eta = \eta\alpha = \alpha$. Hence $b = (a\eta)\alpha = a\alpha = a\eta$, a contradiction. This shows that (i) holds, that is, $\min X$ exists. By (*), $\min X' = \min X$. Suppose that there are $a \in X \backslash X'$ and $b \in X' \backslash \{\min X\}$ such that $a > b$. Then

$a > b > \min X$. Define $\beta : X \to X'$ by

$$x\beta = \begin{cases} b & \text{if} \quad x = a, \\ \min X & \text{if} \quad x \in X', \\ x' & \text{otherwise.} \end{cases}$$

Then $\beta \in T_{RE}(X, X')$ and thus $\beta\eta = \eta\beta = \beta$. Since $a\eta \in X'$, $a\eta\beta = \min X$. Hence $b = a\beta = a\eta\beta = \min X$, a contradiction. Therefore (ii) is proved. $\square$

The following result is similar to Proposition 1.8. The proof is analogous to that of Proposition 1.8 and we shall omit it

Proposition 4.1.3. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. If there is an order-isomorphism $\varphi : X \to Y$ such that $X'\varphi = Y'$, then $\alpha \mapsto \varphi^{-1}\alpha\varphi$ is an isomorphism of $P_{RE}(X, X')$ onto $P_{RE}(Y, Y')$, of $I_{RE}(X, X')$ onto $I_{RE}(Y, Y')$ and of $T_{RE}(X, X')$ onto $T_{RE}(Y, Y')$.*

Example 4.1.4. Let $n \in \mathbb{N}$. Then $\varphi : \mathbb{Z} \to n\mathbb{Z}$ defined by $x\varphi = nx$ for all $x \in \mathbb{Z}$ is an order-isomorphism and $(m\mathbb{Z})\varphi = mn\mathbb{Z}$ for all $m \in \mathbb{N}$. It follows from Proposition 4.1.3 that

$$P_{RE}(\mathbb{Z}, m\mathbb{Z}) \cong P_{RE}(n\mathbb{Z}, mn\mathbb{Z}), \quad I_{RE}(\mathbb{Z}, m\mathbb{Z}) \cong I_{RE}(n\mathbb{Z}, mn\mathbb{Z}),$$
$$T_{RE}(\mathbb{Z}, n\mathbb{Z}) \cong T_{RE}(n\mathbb{Z}, mn\mathbb{Z})$$

for all $m, n \in \mathbb{N}$.

The converse of Proposition 4.1.3 is not necessary true even when X and Y are chains. To see this, let X and Y be finite chains such that $|X| \neq |Y|$. Then $|T_{RE}(X, \{\min X\})| = 1 = |T_{RE}(Y, \{\min Y\})|$. Hence $T_{RE}(X, \{\min X\})$ and $T_{RE}(Y, \{\min Y\})$ are isomorphic but X and Y are not order-isomorphic. A non-trival example can be seen in the last part of Section 4.2.

4.2 Isomorphism Theorems of $T_{RE}(X, X')$

We shall prove in this section that for chains X and Y, a subchain X' of X and a subchain Y' of Y, if $T_{RE}(X, X') \cong T_{RE}(Y, Y')$, then X' and Y' are order-isomorphic. This result and Proposition 4.1.3 generalize Umar's Isomorphism Theorem (Theorem 1.7). Our idea of the proof is based on the proof of Theorem 1.7 given by A. Umar [6].

An *order-ideal* of a poset X is a nonempty subset A of X having the following property:

$$\text{for } x \in X,\ x \leq a \text{ for some } a \in A \text{ implies } x \in A.$$

Also, for a subposet X' of X, an *order-ideal* of X' is a nonempty subset B of X' having the following property:

$$\text{for } x \in X',\ x \leq b \text{ for some } b \in B \text{ implies } x \in B.$$

Lemma 4.2.1. *Let X' be a subposet of a poset X. If $\alpha \in E(T_{RE}(X, X'))$ is such that ranα is an order-ideal of X', then $\alpha E(T_{RE}(X, X')) \subseteq E(T_{RE}(X, X'))$.*

Proof. Let $\beta \in E(T_{RE}(X, X'))$ and let $x \in X$. Then $x\alpha\beta \leq x\alpha$. Since $x\alpha\beta \in X'$, $x\alpha \in \text{ran}\alpha$ and $\text{ran}\alpha$ is an order-ideal of X', it follows that $x\alpha\beta \in \text{ran}\alpha$. But $\alpha \in E(T_{RE}(X, X'))$, so $x\alpha\beta\alpha = x\alpha\beta$. Since $\beta^2 = \beta$, we have $x\alpha\beta = (x\alpha\beta)\beta = (x\alpha\beta\alpha)\beta = x(\alpha\beta)^2$. We then deduce that $\alpha\beta \in E(T_{RE}(X, X'))$. $\square$

Lemma 4.2.2. *Let X and Y be chains, X' a subchain of X, Y' a subchain of Y and $\varphi : T_{RE}(X, X') \to T_{RE}(Y, Y')$ an isomorphism. Then for $\alpha \in E(T_{RE}(X, X'))$, ran$\alpha$ is an order-ideal of X' if and only if ran$(\alpha\varphi)$ is an order-ideal of Y'.*

Proof. Let $\alpha \in E(T_{RE}(X, X'))$. Assume that $\text{ran}(\alpha\varphi)$ is not an order-ideal of Y'. Then there are $y_1, y_2 \in Y'$ such that $y_1 < y_2$, $y_2 \in \text{ran}(\alpha\varphi)$ and $y_1 \notin \text{ran}(\alpha\varphi)$, so

$$y_1(\alpha\varphi) < y_1. \tag{1}$$

Define $\beta : Y \to Y'$ by

$$y\beta = \begin{cases} y_1 & \text{if} \quad y \geq y_1, \\ y & \text{if} \quad y \in Y' \text{ and } y < y_1, \\ y(\alpha\varphi) & \text{if} \quad y \in Y \backslash Y' \text{ and } y < y_1. \end{cases}$$

Then $\beta \in T_{RE}(Y, Y')$. Since $y_2 > y_1$ and $y_2 \in \operatorname{ran}(\alpha\varphi)$, we have

$$y_1 = y_2\beta \in \operatorname{ran}((\alpha\varphi)\beta). \tag{2}$$

If $y \in Y$ is such that $y \geq y_1$, then $y\beta^2 = y_1\beta = y_1 = y\beta$. If $y \in Y'$ and $y < y_1$, then $y\beta^2 = y = y\beta$. Next, let $y \in Y\backslash Y'$ be such that $y < y_1$. Then $y\beta = y(\alpha\varphi) \leq y < y_1$. Since $y(\alpha\varphi) \in Y'$ and $y < y_1$, we have $y\beta^2 = (y(\alpha\varphi))\beta = y(\alpha\varphi) = y\beta$. This shows that $\beta \in E(T_{RE}(Y, Y'))$. Then $\beta = \gamma\varphi$ for some $\gamma \in E(T_{RE}(X, X'))$. But $y_1(\alpha\varphi)\beta \leq y_1(\alpha\varphi) < y_1$ by (1) and $y_1 \in \operatorname{ran}((\alpha\varphi)\beta)$ by (2), so we have $(\alpha\gamma)\varphi = (\alpha\varphi)(\gamma\varphi) = (\alpha\varphi)\beta \notin E(T_{RE}(Y, Y'))$. Hence $\alpha\gamma \notin E(T_{RE}(X, X'))$. By Lemma 4.2.1, this proves that $\operatorname{ran}\alpha$ is not an order-ideal of X'.

Since $\varphi^{-1} : T_{RE}(Y, Y') \to T_{RE}(X, X')$ is an isomorphism, the converse follows from the above proof. $\square$

Observe that the range of the map β defined in the proof of Lemma 4.2.2 is also an order-ideal of Y' whose maximum element is y_1. To be more precise, $\operatorname{ran}\beta = \{y \in Y' \mid y \leq y_1\}$. It can be easily seen that for any $a \in X$, $\{x \in X \mid x \leq a\}$ is an order-ideal of X whose maximum element is a. For ease in writing, it will be denoted by $(\leftarrow a]_X$. Therefore, for any subposet X' of X and $a \in X'$, $(\leftarrow a]_{X'} = (\leftarrow a]_X \cap X'$ is the order-ideal of X' whose maximum element is a.

The following lemmas are required. The first one is obvious.

Lemma 4.2.3. *Let X be a poset and $K = \{(\leftarrow a]_X \mid a \in X\}$. Partially order K by inclusion. Then the map $a \mapsto (\leftarrow a]_X$ is an order-isomorphism of X onto K.*

Lemma 4.2.4. *Let X' be a subchain of a chain X. Then for every $a \in X'$, there exists a map $\alpha \in E(T_{RE}(X, X'))$ such that ran$\alpha = (\leftarrow a]_{X'}$.*

Proof. By (*), for every $x \in X$, there is an element $x' \in X'$ such that $x' \leq x$. Let $a \in X'$ and define $\alpha : X \to X'$ by

$$x\alpha = \begin{cases} a & \text{if} \quad x \geq a, \\ x & \text{if} \quad x \in X' \text{ and } x < a, \\ x' & \text{if} \quad x \in X \backslash X' \text{ and } x < a. \end{cases}$$

Then $\alpha \in E(T_{RE}(X, X'))$ and ran$\alpha = (\leftarrow a]_{X'}$. □

Lemma 4.2.5. *Let X' be a subposet of a poset X. Then for each $\alpha \in T_{RE}(X, X')$, there is an element $\alpha^* \in E(T_{RE}(X, X'))$ such that ranα^*=ranα and $\alpha\alpha^* = \alpha$.*

Proof. Let $\alpha \in T_{RE}(X, X')$. Define $\alpha^* : X \to X'$ by

$$x\alpha^* = \begin{cases} x & \text{if} \quad x \in \text{ran}\alpha, \\ x\alpha & \text{if} \quad x \in X \backslash ran\alpha. \end{cases}$$

Thus $\alpha^* \in T_{RE}(X, X')$. Let $x \in X$. Then $x\alpha \in$ ranα, so $x\alpha\alpha^* = x\alpha$. If $x \in$ ranα, then $x(\alpha^*)^2 = x = x\alpha^*$. If $x \in X\backslash$ranα, then $x(\alpha^*)^2 = (x\alpha)\alpha^* = x\alpha = x\alpha^*$. This shows that $\alpha\alpha^* = \alpha$ and $(\alpha^*)^2 = \alpha^*$. It is clear by the definition of α^* that ranα^*=ranα. □

Lemma 4.2.6. *Let X and Y be chains, X' a subchain of X, Y' a subchain of Y, $R_1 = \{ran\alpha \mid \alpha \in T_{RE}(X, X')\}$ and $R_2 = \{ran\alpha \mid \alpha \in T_{RE}(Y, Y')\}$. Partially order R_1 and R_2 by inclusion. Let $\varphi : T_{RE}(X, X') \to T_{RE}(Y, Y')$ be an isomorphism and $\overline{\varphi} : R_1 \to R_2$ defined by $(ran\alpha)\overline{\varphi} = ran(\alpha\varphi)$ for all $\alpha \in T_{RE}(X, X')$. Then the following statements hold.*

(i) $\overline{\varphi}$ *is an order-isomorphism of R_1 onto R_2.*

(ii) $(\{(\leftarrow a]_{X'} \mid a \in X'\})\overline{\varphi} = \{(\leftarrow b]_{Y'} \mid b \in Y'\}$.

Proof. Let α^* be defined as in Lemma 4.2.5 for $\alpha \in T_{RE}(X, X')$ or $\alpha \in T_{RE}(Y, Y')$.

(i) Let $\alpha \in T_{RE}(X, X')$ be arbitrary fixed. By Lemma 4.2.5, $\alpha^* \in E(T_{RE}(X, X'))$,

$$\text{ran}\alpha^* = \text{ran}\alpha \text{ and } \alpha\alpha^* = \alpha,$$

and $(\alpha\varphi)^* \in E(T_{RE}(Y, Y'))$,

$$\text{ran}(\alpha\varphi)^* = \text{ran}(\alpha\varphi) \text{ and } (\alpha\varphi)(\alpha\varphi)^* = \alpha\varphi.$$

Since $(\alpha\varphi)(\alpha^*\varphi) = \alpha\varphi$, it follows that

$$\text{ran}(\alpha\varphi)^* = \text{ran}(\alpha\varphi) \subseteq \text{ran}(\alpha^*\varphi)$$

and since $(\alpha\varphi)^*\varphi^{-1} \in E(T_{RE}(X, X'))$, $\text{ran}\alpha^* = \text{ran}\alpha = \text{ran}(\alpha\varphi\varphi^{-1}) \subseteq \text{ran}((\alpha\varphi)^*\varphi^{-1})$. This implies that $\alpha^*((\alpha\varphi)^*\varphi^{-1}) = \alpha^*$. Thus $(\alpha^*\varphi)(\alpha\varphi)^* = \alpha^*\varphi$, and so

$$\text{ran}(\alpha^*\varphi) \subseteq \text{ran}(\alpha\varphi)^*.$$

This proves that

$$\text{for every } \alpha \in T_{RE}(X, X'),\ \text{ran}(\alpha^*\varphi) = \text{ran}(\alpha\varphi)^*. \tag{1}$$

Next, to show that $\overline{\varphi}$ is an order-isomorphism of R_1 onto R_2, let $\alpha, \beta \in T_{RE}(X, X')$. Then

$$\begin{aligned}
\text{ran}\alpha \subseteq \text{ran}\beta \quad &\Leftrightarrow \quad \text{ran}\alpha^* \subseteq \text{ran}\beta^* \\
&\Leftrightarrow \quad \alpha^*\beta^* = \alpha^* \qquad\qquad \text{since } \alpha^*, \beta^* \in E(T_{RE}(X, X')) \\
&\Leftrightarrow \quad (\alpha^*\varphi)(\beta^*\varphi) = \alpha^*\varphi \\
&\Leftrightarrow \quad \text{ran}(\alpha^*\varphi) \subseteq \text{ran}(\beta^*\varphi) \\
&\Leftrightarrow \quad \text{ran}(\alpha\varphi)^* \subseteq \text{ran}(\beta\varphi)^* \quad \text{from (1)} \\
&\Leftrightarrow \quad \text{ran}(\alpha\varphi) \subseteq \text{ran}(\beta\varphi) \qquad\qquad (2)
\end{aligned}$$

and hence

$$\text{ran}\alpha = \text{ran}\beta \Leftrightarrow \text{ran}(\alpha\varphi) = \text{ran}(\beta\varphi). \tag{3}$$

We therefore conclude from (3) that $\overline{\varphi}$ is well-defined and one-to-one and from (2) that $\overline{\varphi}$ is order-preserving. Clearly, $\overline{\varphi}$ is onto since $\varphi : T_{RE}(X, X') \to T_{RE}(Y, Y')$ is onto.

(ii) Let $a \in X'$. By Lemma 4.2.4, there is a map $\alpha \in E(T_{RE}(X, X'))$ such that $\text{ran}\alpha = (\leftarrow a]_{X'}$. Since $(\leftarrow a]_{X'}$ is an order-ideal of X', by Lemma 4.2.2, $\text{ran}(\alpha\varphi)$ is an order-ideal of Y'. To show that $\text{ran}(\alpha\varphi) = (\leftarrow e]_{Y'}$ for some $e \in Y'$, let $b \in \text{ran}(\alpha\varphi)$. Then $(\leftarrow b]_{Y'} \subseteq \text{ran}(\alpha\varphi)$. If $(\leftarrow b]_{Y'} = \text{ran}(\alpha\varphi)$, then we are done. Assume that $(\leftarrow b]_{Y'} \subsetneq \text{ran}(\alpha\varphi)$. By Lemma 4.2.4, there is a map $\beta \in E(T_{RE}(Y, Y'))$ such that $\text{ran}\beta = (\leftarrow b]_{Y'}$. Let $\gamma \in E(T_{RE}(X, X'))$ be such that $\gamma\varphi = \beta$. Hence $\text{ran}(\gamma\varphi) \subsetneq \text{ran}(\alpha\varphi)$. We therefore have from (i) that $\text{ran}\gamma \subsetneq \text{ran}\alpha$. Also, by Lemma 4.2.2, $\text{ran}\gamma$ is an order-ideal of X'. Let $c \in \text{ran}\gamma$. Then $(\leftarrow c]_{X'} \subseteq \text{ran}\gamma \subsetneq (\leftarrow a]_{X'}$, so $c < a$. By the property (*), for every $x \in X$, there is an element $x' \in X'$ such that $x' \leq x$. Define $\lambda : X \to X'$ by

$$x\lambda = \begin{cases} c & \text{if} \quad x \geq a, \\ x & \text{if} \quad x \in X' \text{ and } x < a, \\ x' & \text{if} \quad x \in X \backslash X' \text{ and } x < a. \end{cases}$$

Clearly, $\lambda \in E(T_{RE}(X, X'))$ and $\text{ran}\lambda = (\leftarrow a]_{X'} \backslash \{a\} \subsetneq (\leftarrow a]_{X'} = \text{ran}\alpha$. By (i), $\text{ran}(\lambda\varphi) \subsetneq \text{ran}(\alpha\varphi)$. Let $d \in \text{ran}(\alpha\varphi) \backslash \text{ran}(\lambda\varphi)$. Then $\text{ran}(\lambda\varphi) \subsetneq (\leftarrow d]_{Y'} \subseteq \text{ran}(\alpha\varphi)$, and by Lemma 4.2.4, $\text{ran}\eta = (\leftarrow d]_{Y'}$ for some $\eta \in E(T_{RE}(Y, Y'))$. Let $\mu \in E(T_{RE}(X, X'))$ be such that $\mu\varphi = \eta$. Thus $\text{ran}(\lambda\varphi) \subsetneq \text{ran}(\mu\varphi) \subseteq \text{ran}(\alpha\varphi)$ which implies by (i) that $(\leftarrow a]_{X'} \backslash \{a\} = \text{ran}\lambda \subsetneq \text{ran}\mu \subseteq \text{ran}\alpha = (\leftarrow a]_{X'}$. Consequently, $\text{ran}\mu = \text{ran}\alpha$, and from (i), $\text{ran}(\mu\varphi) = \text{ran}(\alpha\varphi)$. Hence $(\leftarrow a]_{X'}\overline{\varphi} = \text{ran}(\alpha\varphi) = \text{ran}(\eta) = (\leftarrow d]_{Y'}$. It means that for any $\alpha \in E(T_{RE}(X, X'))$ such that $\text{ran}\alpha$ is an order-ideal of X' with $\max(\text{ran}\alpha)$ exists, then $\text{ran}(\alpha\varphi)$ is also an order-ideal of Y' with $\max(\text{ran}(\alpha\varphi))$ exists.

By considering φ^{-1} instead of φ, from the above proof, we have that for every $d \in Y'$, there are $\eta \in E(T_{RE}(Y, Y'))$ and $a \in X'$ such that ran$\eta = (\leftarrow d]_{Y'}$ and ran$(\eta\varphi^{-1}) = (\leftarrow a]_{X'}$, and hence $(\leftarrow a]_{X'}\overline{\varphi}$ =(ran$(\eta\varphi^{-1}))\overline{\varphi}$=ran$((\eta\varphi^{-1})\varphi)$=ran$\eta$ = $(\leftarrow d]_{Y'}$.

Therefore the lemma is proved. □

Theorem 4.2.7. *Let X and Y be chains, X' a subchain of X and Y' a subchain of Y. If $T_{RE}(X, X') \cong T_{RE}(Y, Y')$, then X' and Y' are order-isomorphic.*

Proof. From Lemma 4.2.6 (ii), the chains $\{(\leftarrow a]_{X'} \mid a \in X'\}$ and $\{(\leftarrow b]_{Y'} \mid b \in Y'\}$ under inclusion are order-isomorphic. But by Lemma 4.2.3, $\{(\leftarrow a]_{X'} \mid a \in X'\}$ is order-isomorphic to X' and $\{(\leftarrow b]_{Y'} \mid b \in Y'\}$ is order-isomorphic to Y'. Hence X' and Y' are order-isomorphic. □

Since $T_{RE}(X) = T_{RE}(X, X)$ for every chain X, we have that Umar's Isomorphism Theorem is a consequence of Theorem 4.2.7 and Proposition 4.1.3.

Corollary 4.2.8. *For chains X and Y, $T_{RE}(X) \cong T_{RE}(Y)$ if and only if X and Y are order-isomorphic.*

Unlike Umar's Isomorphism Theorem, the necessary condition in Theorem 4.2.7 is not sufficient. An example is given below

Example 4.2.9. Let $X = \{1, 2, 3\}$ be a chain under the natural order, $X_1 = \{1, 2\}$ and $X_2 = \{1, 3\}$. Then X_1 and X_2 are order-isomorphic subchains of X but

$$T_{RE}(X, X_1) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \right\} \text{ and}$$

$$T_{RE}(X, X_2) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \right\}$$

are not isomorphic.

Example 4.2.9 shows that being order-isomorphic of subchains X_1 and X_2 of a finite chain X is not sufficient for the corresponding regressive full transformation semigroups to be isomorphic. In fact, the next theorem shows that they must be equal. The following lemma is required.

Lemma 4.2.10. *Let X be a poset with a minimum element and X_1 and X_2 subposets of X. If $\varphi : T_{RE}(X, X_1) \to T_{RE}(X, X_2)$ is an isomorphism, then the following statements hold.*

(i) For $\alpha \in E(T_{RE}(X, X_1))$ and $n \in \mathbb{N}$, $|ran\alpha| = n \Leftrightarrow |ran(\alpha\varphi)| = n$.

(ii) For $n \in \mathbb{N}$,

$$|\{\alpha \in E(T_{RE}(X, X_1)) \mid |ran\alpha| = n\}| = |\{\alpha \in E(T_{RE}(X, X_2)) \mid |ran\alpha| = n\}|.$$

Proof. (i) By the property (*), $\min X \in X_1$ and $\min X \in X_2$. Note that $X_{\min X}\alpha = X_{\min X}$ for all $\alpha \in T_{RE}(X, X_1) \cup T_{RE}(X, X_2)$. It is easily seen that if $\alpha \in T_{RE}(X, X_1) \cup T_{RE}(X, X_2)$ is such that $|\mathrm{ran}\alpha| = 1$, then $\alpha = X_{\min X} \in E(T_{RE}(X, X_1)) \cap E(T_{RE}(X, X_2))$. Let $\beta \in T_{RE}(X, X_1)$ be such that $\beta\varphi = X_{\min X}$. Since $X_{\min X}\beta = X_{\min X}$, we have

$$X_{\min X}\varphi = (X_{\min X}\beta)\varphi = ((X_{\min X})\varphi)X_{\min X} = X_{\min X}.$$

This shows that (i) holds for $n = 1$.

Assume that $k > 1$ and for $n \in \mathbb{N}$ with $n < k$, $|\mathrm{ran}\alpha| = n \Leftrightarrow |\mathrm{ran}(\alpha\varphi)| = n$ for all $\alpha \in E(T_{RE}(X, X_1))$. Let $\beta \in E(T_{RE}(X, X_1))$ be such that $|\mathrm{ran}\beta| = k$. Then $\beta\varphi \in E(T_{RE}(X, X_2))$ and by assumption, $|\mathrm{ran}(\beta\varphi)| \geq k$. Let $a_1, a_2, \ldots, a_k$ be distinct elements in $\mathrm{ran}(\beta\varphi)$ with $a_k = \min X$. Since $\beta\varphi \in E(T_{RE}(X, X_2))$, it follows that

$$X = \left(\bigcup_{i=1}^{k} a_i(\beta\varphi)^{-1}\right) \cup \left(\bigcup_{\substack{x \in \mathrm{ran}(\beta\varphi) \\ x \notin \{a_1, a_2, \ldots, a_k\}}} x(\beta\varphi)^{-1}\right) \tag{1}$$

which is a disjoint union. Since $x(\beta\varphi) = x$ for all $x \in \mathrm{ran}(\beta\varphi)$, we have

$$x \in x(\beta\varphi)^{-1} \qquad \text{for all } x \in \operatorname{ran}(\beta\varphi). \tag{2}$$

Also,

$$\text{for all } x \in \operatorname{ran}(\beta\varphi),\ x \leq y \text{ for all } y \in x(\beta\varphi)^{-1} \tag{3}$$

since $\beta\varphi$ is regressive. Define $\gamma : X \to X_2$ by

$$x\gamma = \begin{cases} a_i & \text{if} \quad x \in a_i(\beta\varphi)^{-1} \quad \text{for } i = 1, 2, \ldots, k, \\ \min X & \text{if} \quad x \in \bigcup\limits_{\substack{y \in \operatorname{ran}(\beta\varphi) \\ y \notin \{a_1, a_2, \ldots, a_k\}}} y(\beta\varphi)^{-1}. \end{cases}$$

From (1), γ is well-defined and from (3), γ is regressive. By the definition of γ, $\operatorname{ran}\gamma = \{a_1, a_2, \ldots, a_k = \min X\} \subseteq X_2$. By (2), $a_i\gamma = a_i$ for all $i \in \{1, 2, \ldots, k\}$. Thus $\gamma \in E(T_{RE}(X, X_2))$ and $|\operatorname{ran}\gamma| = k$. Since $\operatorname{ran}\gamma = \{a_1, a_2, \ldots, a_k\} \subseteq \operatorname{ran}(\beta\varphi)$ and $a_i(\beta\varphi) = a_i$ for all i, it follows that $\gamma(\beta\varphi) = \gamma$. Thus $(\gamma\varphi^{-1})\beta = \gamma\varphi^{-1}$ which implies that $\operatorname{ran}(\gamma\varphi^{-1}) \subseteq \operatorname{ran}\beta$. Since $|\operatorname{ran}\gamma| = k$, by assumption $|\operatorname{ran}(\gamma\varphi^{-1})| \geq k$. But $|\operatorname{ran}\beta| = k$ and $\operatorname{ran}(\gamma\varphi^{-1}) \subseteq \operatorname{ran}\beta$, so

$$\operatorname{ran}(\gamma\varphi^{-1}) = \operatorname{ran}\beta. \tag{4}$$

If $i \in \{1, \ldots, k\}$ and $x \in a_i(\beta\varphi)^{-1}$, then

$$x(\beta\varphi)\gamma = a_i\gamma = a_i = x\gamma.$$

If $x \in y(\beta\varphi)^{-1}$ for some $y \in \operatorname{ran}(\beta\varphi)$ with $y \notin \{a_1, a_2, \ldots, a_k\}$, then $x, y \in y(\beta\varphi)^{-1}$ by (2), then by the definition of γ,

$$x(\beta\varphi)\gamma = y\gamma = \min X = x\gamma.$$

It follows from (1) that $(\beta\varphi)\gamma = \gamma$, and hence

$$\beta(\gamma\varphi^{-1}) = \gamma\varphi^{-1}. \tag{5}$$

Therefore for every $x \in X$,

$$\begin{aligned} x(\gamma\varphi^{-1}) &= x\beta(\gamma\varphi^{-1}) && \text{from (5)} \\ &= x\beta && \text{from (4) and since } \gamma\varphi^{-1} \in E(T_{RE}(X, X_1)). \end{aligned}$$

We deduce that $\gamma\varphi^{-1} = \beta$ and thus $\beta\varphi = \gamma$. Therefore $|\mathrm{ran}(\beta\varphi)| = |\mathrm{ran}\gamma| = k$.

If $\beta \in E(T_{RE}(X, X_1))$ is such that $|\mathrm{ran}(\beta\varphi)| = k$, it can be shown analogously that $|\mathrm{ran}((\beta\varphi)\varphi^{-1})| = k$, so $|\mathrm{ran}\beta| = k$.

Therefore (i) is proved.

(ii) Let $n \in \mathbb{N}$. Since $\varphi : T_{RE}(X, X_1) \to T_{RE}(X, X_2)$ is an isomorphism, by (i), $\varphi_n : \{\alpha \in E(T_{RE}(X, X_1)) \mid |\mathrm{ran}\alpha| = n\} \to \{\alpha \in E(T_{RE}(X, X_2)) \mid |\mathrm{ran}\alpha| = n\}$ defined by $\alpha\varphi_n = \alpha\varphi$ for all $\alpha \in E(T_{RE}(X, X_1))$ is a bijection.

Hence (ii) is proved. $\square$

Theorem 4.2.11. *Let X be a finite chain and X_1 and X_2 subchains of X. Then $T_{RE}(X, X_1) \cong T_{RE}(X, X_2)$ if and only if $X_1 = X_2$.*

Proof. The sufficiency part is immediate. To prove the necessity part, assume that $T_{RE}(X, X_1)$ and $T_{RE}(X, X_2)$ are isomorphic. By Theorem 4.2.7, $|X_1| = |X_2|$. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $x_1 < x_2 < \ldots < x_n$. By the property (*), $x_1 \in X_1 \cap X_2$. Then $x_1\alpha = x_1$ for every $\alpha \in T_{RE}(X, X_1) \cup T_{RE}(X, X_2)$. To show that $X_1 = X_2$, suppose instead that $X_1 \neq X_2$. Since X_1 and X_2 are finite and $|X_1| = |X_2|$, it follows that $X_1 \backslash X_2 \neq \phi$ and $X_2 \backslash X_1 \neq \phi$. Since $(X_1 \backslash X_2) \cap (X_2 \backslash X_1) = \phi$, either $\min(X_1 \backslash X_2) < \min(X_2 \backslash X_1)$ or $\min(X_2 \backslash X_1) < \min(X_1 \backslash X_2)$. Let

$$x_k = \min\{\min(X_1 \backslash X_2), \min(X_2 \backslash X_1)\}.$$

Then $k < n$. Since $x_1 \in X_1 \cap X_2$, $1 < k < n$. Without loss of generality, assume that $x_k = \min(X_1 \backslash X_2)$. For $x \in X_1$ with $x < x_k$, if $x \notin X_2$, then $x \in X_1 \backslash X_2$ which is contrary to that $x < x_k = \min(X_1 \backslash X_2)$. For $x \in X_2$ with $x < x_k$, if $x \notin X_1$, then $x \in X_2 \backslash X_1$, so $x \geq \min(X_2 \backslash X_1) > x_k$, a contradiction. Hence $\{x \in X_1 \mid x < x_k\} = \{x \in X_2 \mid x < x_k\}$. Let $A = \{x \in X_1 \mid x < x_k\}$. Then $x_1 \in A$. Since $A \subseteq X_1 \cap X_2$, it follows that the sets $\{\alpha \in E(T_{RE}(X, X_1)) \mid \mathrm{ran}\alpha \subseteq A \text{ and } |\mathrm{ran}\alpha| \leq 2\}$ and $\{\alpha \in E(T_{RE}(X, X_2)) \mid \mathrm{ran}\alpha \subseteq A \text{ and } |\mathrm{ran}\alpha| \leq 2\}$ are identical. Let m be its cardinality, that is,

$$m = |\{\alpha \in E(T_{RE}(X, X_1)) \mid \text{ran}\alpha \subseteq A \text{ and } |\text{ran}\alpha| \leq 2\}|$$
$$= |\{\alpha \in E(T_{RE}(X, X_2)) \mid \text{ran}\alpha \subseteq A \text{ and } |\text{ran}\alpha| \leq 2\}|. \qquad (1)$$

We can see that for $t \in \{2, \ldots, n\}$ and $\alpha \in E(T_{RE}(X))$,

$$\text{ran}\alpha = \{x_1, x_t\} \Leftrightarrow \{x_1, \ldots, x_{t-1}\}\alpha = \{x_1\},\ x_t\alpha = x_t$$
$$\text{and } \{x_{t+1}, \ldots, x_n\}\alpha \subseteq \{x_1, x_t\}.$$

Consequently,

$$\text{For } t \in \{2, \ldots, n\},\ |\{\alpha \in E(T_{RE}(X)) \mid \text{ran}\alpha = \{x_1, x_t\}\}| = 2^{n-t}. \qquad (2)$$

Hence

$$\begin{aligned}
&|\{\alpha \in E(T_{RE}(X, X_1)) \mid |\text{ran}\alpha| \leq 2\}| \\
&\quad \geq |\{\alpha \in E(T_{RE}(X, X_1)) \mid \text{ran}\alpha \subseteq A \text{ and } |\text{ran}\alpha| \leq 2\}| \\
&\quad\quad + |\{\alpha \in E(T_{RE}(X, X_1)) \mid \text{ran}\alpha = \{x_1, x_k\}\}| \\
&\quad = m + 2^{n-k}. \qquad \text{from (1) and (2)} \qquad (3)
\end{aligned}$$

Since $x_k \notin X_2$, $X_2 = A \cup (\{x_{k+1}, \ldots, x_n\} \cap X_2)$, and hence

$$\begin{aligned}
&|\{\alpha \in E(T_{RE}(X, X_2)) \mid |\text{ran}\alpha| \leq 2\}| \\
&\quad = |\{\alpha \in E(T_{RE}(X, X_2)) \mid \text{ran}\alpha \subseteq A \text{ and } |\text{ran}\alpha| \leq 2\}| + \\
&\quad\quad |\{\alpha \in E(T_{RE}(X, X_2)) \mid \text{ran}\alpha = \{x_1, x\} \text{ for some } x \in \{x_{k+1}, \ldots, x_n\} \cap X_2\}| \\
&\quad \leq m + 2^{n-(k+1)} + 2^{n-(k+2)} + \ldots + 2 + 1 \qquad \text{from (1) and (2)} \\
&\quad = m + 2^{n-k}\left(\tfrac{1}{2} + \tfrac{1}{2^2} + \ldots + \tfrac{1}{2^{n-k}}\right) \\
&\quad < m + 2^{n-k}. \qquad (4)
\end{aligned}$$

From (3) and (4), we have

$$|\{\alpha \in E(T_{RE}(X, X_1)) \mid |\text{ran}\alpha| \leq 2\}| > |\{\alpha \in E(T_{RE}(X, X_2)) \mid |\text{ran}\alpha| \leq 2\}|. \quad (5)$$

Since X is finite and $T_{RE}(X, X_1)$ and $T_{RE}(X, X_2)$ are isomorphic, by Lemma 4.2.10 (ii),

$$|\{\alpha \in E(T_{RE}(X, X_1)) \mid |\text{ran}\alpha| \leq 2\}| = |\{\alpha \in E(T_{RE}(X, X_2)) \mid |\text{ran}\alpha| \leq 2\}|. \quad (6)$$

Therefore (5) and (6) yield a contradiction.

Hence the theorem is completely proved. □

The following example shows that Theorem 4.2.11 need not hold if X is an infinite chain.

Example 4.2.12. Consider the chain $\mathbb{Z}$. We have that $\mathbb{Z}^- \neq \mathbb{Z}^- \cup \{0\}$ and $T_{RE}(\mathbb{Z}, \mathbb{Z}^-) \cong T_{RE}(\mathbb{Z}, \mathbb{Z}^- \cup \{0\})$ by Proposition 4.1.3 since $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined by $x\varphi = x + 1$ is an order-isomorphism and $\mathbb{Z}^-\varphi = \mathbb{Z}^- \cup \{0\}$.

The next theorems characterizes when $T_{RE}(X, X')$ is isomorphic to $T_{RE}(X)$ and when $T_{RE}(X, X')$ is isomorphic to $T_{RE}(X')$ when X' is a subchain of a chain X. Since both $T_{RE}(X)$ and $T_{RE}(X')$ have an identity, Proposition 4.1.2 is also a tool for these characterizations.

Lemma 4.2.13. *Let X be a chain and X' a subchain of X. If $minX$ exists and $a < b$ for all $a \in X \backslash X'$ and $b \in X' \backslash \{minX\}$, then for all $\alpha \in T_{RE}(X, X')$ and $a \in X \backslash X'$, $a\alpha = minX$.*

Proof. Let $\alpha \in T_{RE}(X, X')$ and $a \in X \backslash X'$. Then $a > a\alpha \in X'$, so by assumption, $a\alpha$=min$X$. □

Theorem 4.2.14. *Let X be a chain and X' a proper subchain of X. Then $T_{RE}(X, X') \cong T_{RE}(X)$ if and only if the following statements hold.*

(i) X' and X are order-isomorphic.

(ii) $minX$ exists and $a < b$ for all $a \in X \backslash X'$ and $b \in X' \backslash \{minX\}$.

Proof. Assume that $T_{RE}(X,X')$ and $T_{RE}(X)$ are isomorphic. Then $T_{RE}(X,X')$ and $T_{RE}(X)$ have an identity. By Theorem 4.2.7, X' and X are order-isomorphic. By Proposition 4.1.2, min$X$ exists and $a<b$ for all $a\in X\backslash X'$ and $b\in X'\backslash\{\min X\}$. Recall that min$X$=min$X'$.

For the converse, assume that (i) and (ii) hold. Let $\varphi : X \to X'$ be an order-isomorphism. Then $(\min X)\varphi = \min X$. For $\alpha \in T_{RE}(X)$, define $\alpha' : X \to X'$ by

$$x\alpha' = \begin{cases} x(\varphi^{-1}\alpha\varphi) & \text{if} \quad x \in X', \\ \min X & \text{if} \quad x \in X\backslash X'. \end{cases}$$

We can see from the proof of Proposition 1.8 that $\alpha' \in T_{RE}(X,X')$. Let $\overline{\varphi}$: $T_{RE}(X) \to T_{RE}(X,X')$ be defined by

$$\alpha\overline{\varphi} = \alpha' \qquad \text{for all } \alpha \in T_{RE}(X).$$

Let $\alpha,\beta \in T_{RE}(X)$ and $x \in X$.

Case 1: $x \in X\backslash X'$. Then $x(\alpha\beta)' = \min X$ and $x\alpha'\beta' = (\min X)\beta' = \min X$.

Case 2: $x \in X'$. Then $x\varphi^{-1}\alpha\varphi \in X'$, and thus $x(\alpha\beta)' = x(\varphi^{-1}\alpha\beta\varphi) = x(\varphi^{-1}\alpha\varphi)(\varphi^{-1}\beta\varphi) = x\alpha'\beta'$.

Therefore $\overline{\varphi}$ is a homomorphism.

To show that $\overline{\varphi}$ is one-to-one, let $\alpha,\beta \in T_{RE}(X)$ be such that $\alpha' = \beta'$. Then $x(\varphi^{-1}\alpha\varphi) = x(\varphi^{-1}\beta\varphi)$ for all $x \in X'$ which implies that

$$x\varphi^{-1}\alpha = x\varphi^{-1}\beta \qquad \text{for all } x \in X'.$$

Since $X'\varphi^{-1} = X$, it then follows that $x\alpha = x\beta$ for all $x \in X$, we conclude that $\alpha = \beta$.

Finally, to show that ran$\overline{\varphi} = T_{RE}(X,X')$, let $\beta \in T_{RE}(X,X')$. Then $\varphi\beta\varphi^{-1} \in$

$T_{RE}(X)$. Since $(\min X)\beta = \min X$, by Lemma 4.2.13, $x\beta = \min X$ for all $x \in (X\backslash X') \cup \{\min X\}$. Hence

$$x(\varphi\beta\varphi^{-1})' = \begin{cases} x(\varphi^{-1}(\varphi\beta\varphi^{-1})\varphi) = x\beta & \text{if} \quad x \in X', \\ \min X = x\beta & \text{if} \quad x \in X\backslash X'. \end{cases}$$

Therefore the theorem is completety proved. $\square$

Theorem 4.2.15. *Let X be a chain and X' a proper subchain of X. Then $T_{RE}(X, X') \cong T_{RE}(X')$ if and only if $minX$ exists and $a < b$ for all $a \in X\backslash X'$ and $b \in X'\backslash\{minX\}$.*

Proof. The necessary part follows directly from Proposition 4.1.2.

Conversely, assume that $\min X$ exists and $a < b$ for all $a \in X\backslash X'$ and $b \in X'\backslash\{\min X\}$. Define $\varphi : T_{RE}(X, X') \to T_{RE}(X')$ by

$$\alpha\varphi = \alpha_{|_{X'}}, \text{ the restriction of } \alpha \text{ to } X', \text{ for all } \alpha \in T_{RE}(X, X').$$

Let $\alpha, \beta \in T_{RE}(X, X')$. If $x \in X'$, then $x\alpha \in X'$, so $x(\alpha\beta)_{|_{X'}} = x\alpha\beta = x(\alpha_{|_{X'}}\beta_{|_{X'}})$. Thus φ is a homomorphism. To show that φ is one-to-one, assume that $\alpha_{|_{X'}} = \beta_{|_{X'}}$. Then $x\alpha = x\beta$ for all $x \in X'$. If $x \in X\backslash X'$, then by assumption and Lemma 4.2.13, $x\alpha = \min X = x\beta$. Therefore $\alpha = \beta$. Finally, let $\lambda \in T_{RE}(X')$. Define $\mu : X \to X'$ by

$$x\mu = \begin{cases} x\lambda & \text{if} \quad x \in X', \\ \min X & \text{if} \quad x \in X\backslash X'. \end{cases}$$

Then $\mu \in T_{RE}(X, X')$ and $\mu_{|_{X'}} = \lambda$. Hence φ is an isomorphism of $T_{RE}(X, X')$ onto $T_{RE}(X')$.

Therefore the theorem is proved. $\square$

Example 4.2.16. We can easily see that the map $\varphi : [0, \infty) \to \{0\} \cup (1, \infty)$

defined by

$$x\varphi = \begin{cases} 0 & \text{if} \quad x = 0, \\ x+1 & \text{if} \quad x > 0 \end{cases}$$

is an order-isomorphism. Also, the subchain $\{0\} \cup (1, \infty)$ of $[0, \infty)$ satisfies the necessity parts of Theorem 4.2.14 and Theorem 4.2.15. We therefore have from Theorem 4.2.14 and Theorem 4.2.15 that

$$T_{RE}([0,\infty)) \cong T_{RE}([0,\infty), \{0\} \cup (1,\infty)) \cong T_{RE}(\{0\} \cup (1,\infty)).$$

In fact, that $T_{RE}([0,\infty)) \cong T_{RE}(\{0\} \cup (1,\infty))$ can be considered as a consequence of Umar's Isomorphism Theorem. It is easy to check that $\{0\} \cup [1,\infty)$ and $[0,\infty)$ are not order-isomorphic. However, the subchain $\{0\} \cup [1,\infty)$ of the chain $[0,\infty)$ satisfies the necessity part of Theorem 4.2.15. Consequently,

$$T_{RE}([0,\infty)) \ncong T_{RE}([0,\infty), \{0\} \cup [1,\infty)) \cong T_{RE}(\{0\} \cup [1,\infty)).$$

4.3 Isomorphism Theorems of $P_{RE}(X, X')$ and $I_{RE}(X, X')$

The aim of this section is to give necessary conditions for that $P_{RE}(X, X') \cong P_{RE}(Y, Y')$ and $I_{RE}(X, X') \cong I_{RE}(Y, Y')$ where X and Y are posets, X' is a subposet of X and Y' is a subposet of Y. Consequently, we characterize when $P_{RE}(X) \cong P_{RE}(Y)$ and when $I_{RE}(X) \cong I_{RE}(Y)$.

The following two lemmas are required.

Lemma 4.3.1. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. Then the following statements hold.*

(i) If $\varphi : I_{RE}(X, X') \to I_{RE}(Y, Y')$ is an isomorphism, then for every $a \in X'$, there exists an $\overline{a} \in Y'$ such that

$$\begin{pmatrix} a \\ a \end{pmatrix} \varphi = \begin{pmatrix} \overline{a} \\ \overline{a} \end{pmatrix}.$$

(ii) If $\varphi : P_{RE}(X,X') \to P_{RE}(Y,Y')$ is an isomorphism, then

for every $a \in X'$, there exists an $\overline{a} \in Y'$ and $A \subseteq Y \backslash Y'$ such that

$$\binom{a}{a}\varphi = \binom{A \cup \{\overline{a}\}}{\overline{a}}.$$

Proof. (i) Let $a \in X'$. Then $\binom{a}{a}\varphi \in E(I_{RE}(Y,Y'))\backslash\{0\}$. Let $\overline{a} \in \operatorname{ran}\left(\binom{a}{a}\varphi\right)$. Thus $\overline{a} \in \operatorname{dom}\left(\binom{a}{a}\varphi\right)$ and $\overline{a}\left(\binom{a}{a}\varphi\right) = \overline{a}$. Consequently,

$$\binom{\overline{a}}{\overline{a}}\left(\binom{a}{a}\varphi\right) = \binom{\overline{a}}{\overline{a}}$$

which implies that

$$\left(\binom{\overline{a}}{\overline{a}}\varphi^{-1}\right)\binom{a}{a} = \binom{\overline{a}}{\overline{a}}\varphi^{-1}.$$

Thus $\varnothing \neq \operatorname{ran}\left(\binom{\overline{a}}{\overline{a}}\varphi^{-1}\right) \subseteq \{a\}$ and so $\operatorname{ran}\left(\binom{\overline{a}}{\overline{a}}\varphi^{-1}\right) = \{a\}$. But $\binom{\overline{a}}{\overline{a}}\varphi^{-1} \in E(I_{RE}(X,X'))$, thus $\binom{\overline{a}}{\overline{a}}\varphi^{-1} = \binom{a}{a}$. Hence $\binom{a}{a}\varphi = \binom{\overline{a}}{\overline{a}}$.

Therefore (i) is proved.

(ii) Let $a \in X'$ and $\overline{a} \in \operatorname{ran}\left(\binom{a}{a}\varphi\right)$. As can be seen from the proof in (i) that $\binom{\overline{a}}{\overline{a}}\varphi^{-1} = B_a$ for some nonempty subset B of X containing a, so $\binom{\overline{a}}{\overline{a}} = B_a\varphi$. Hence

$$\left(\binom{a}{a}\varphi\right)\binom{\overline{a}}{\overline{a}} = \left(\binom{a}{a}\varphi\right)(B_a\varphi) = \left(\binom{a}{a}B_a\right)\varphi = \binom{a}{a}\varphi$$

which implies that $\operatorname{ran}\left(\binom{a}{a}\varphi\right) \subseteq \{\overline{a}\}$, so $\operatorname{ran}\left(\binom{a}{a}\varphi\right) = \{\overline{a}\}$. Since $\binom{a}{a}\varphi \in E(P_{RE}(Y,Y'))$ and $\operatorname{ran}\left(\binom{a}{a}\varphi\right) = \{\overline{a}\}$, it follows that

$$\binom{a}{a}\varphi = C_{\overline{a}} \text{ for some } C \subseteq Y \text{ and } \overline{a} \in C. \tag{1}$$

But $\varphi^{-1} : P_{RE}(Y,Y') \to P_{RE}(X,X')$, so from the above proof, we deduce that

$$\text{for every } y \in Y',\ \binom{y}{y}\varphi^{-1} = K_z \text{ for some } K \subseteq X \text{ and } z \in K. \tag{2}$$

If $d \in C \cap Y'$, then from (2), $\binom{d}{d}\varphi^{-1} = D_e$ for some $D \subseteq X$ and $e \in D$. Since

$$\binom{d}{\overline{a}} = \binom{d}{d}C_{\overline{a}} = \binom{d}{d}\left(\binom{a}{a}\varphi\right),$$

we have

$$\begin{aligned} 0 \neq \binom{d}{\overline{a}}\varphi^{-1} &= \left(\binom{d}{d}\varphi^{-1}\right)\binom{a}{a} \\ &= D_e\binom{a}{a} \end{aligned}$$

which implies that $e = a$. Hence $\binom{d}{\overline{a}}\varphi^{-1} = D_a$ and $a \in D$. Hence $D_a \in E(P_{RE}(X, X'))$ and $D_a\varphi = \binom{d}{\overline{a}} \in E(P_{RE}(Y, Y'))$. Consequently, $d = \overline{a}$. This shows that $C \cap Y' = \{\overline{a}\}$. We therefore deduce from (1) that

$$\binom{a}{a}\varphi = \binom{A \cup \{\overline{a}\}}{\overline{a}} \text{ for some } A \subseteq Y \backslash Y'.$$

Hence (ii) is proved. $\square$

Lemma 4.3.2. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. Then the following statements hold.*

(i) If $\varphi : I_{RE}(X, X') \to I_{RE}(Y, Y')$ is an isomorphism, then $\theta : X' \to Y'$ defined by $a\theta = \overline{a}$ in (i) of Lemma 4.3.1 for all $a \in X'$ is an order-isomorphism.

(ii) If $\varphi : P_{RE}(X, X') \to P_{RE}(Y, Y')$ is an isomorphism, then $\theta : X' \to Y'$ defined by $a\theta = \overline{a}$ in (ii) of Lemma 4.3.1 for all $a \in X'$ is an order-isomorphism.

Proof. (i) Since φ is 1-1, θ is clearly 1-1. Let $b \in Y'$. By Lemma 4.3.1 (i), $\binom{b}{b}\varphi^{-1} = \binom{c}{c}$ for some $c \in X'$. Thus $\binom{c}{c}\varphi = \binom{b}{b}$. But $\binom{c}{c}\varphi = \binom{\overline{c}}{\overline{c}}$, so $c\theta = \overline{c} = b$. Hence θ in (i) is bijective.

Next, let $a, b \in X'$ be such that $a < b$. Then $\binom{b}{a} \in I(X, X')$ and

$$\binom{b}{b}\binom{b}{a}\binom{a}{a} = \binom{b}{a}.$$

Thus

$$\left(\binom{b}{b}\varphi\right)\left(\binom{b}{a}\varphi\right)\left(\binom{a}{a}\varphi\right) = \binom{b}{a}\varphi,$$

and so

$$\binom{\overline{b}}{\overline{b}}\left(\binom{b}{a}\varphi\right)\binom{\overline{a}}{\overline{a}} = \binom{b}{a}\varphi.$$

Consequently, $\binom{b}{a}\varphi = \binom{\overline{b}}{\overline{a}} \in I_{RE}(Y,Y')$, so $\overline{a} < \overline{b}$ since $\binom{b}{a} \notin E(I_{RE}(X,X'))$.

(ii) Let $a_1, a_2 \in X'$ be such that $\overline{a}_1 = \overline{a}_2$. Then

$\binom{a_1}{a_1}\varphi = \binom{A_1 \cup \{\overline{a}_1\}}{\overline{a}_1}$ and $\binom{a_2}{a_2}\varphi = \binom{A_2 \cup \{\overline{a}_2\}}{\overline{a}_2}$ for some $A_1, A_2 \subseteq Y\backslash Y'$.

Thus

$$\begin{aligned}\left(\binom{a_1}{a_1}\binom{a_2}{a_2}\right)\varphi &= \left(\binom{a_1}{a_1}\varphi\right)\left(\binom{a_2}{a_2}\varphi\right)\\ &= \binom{A_1 \cup \{\overline{a}_1\}}{\overline{a}_1}\binom{A_2 \cup \{\overline{a}_2\}}{\overline{a}_2}\\ &= \binom{A_1 \cup \{\overline{a}_1\}}{\overline{a}_2} \neq 0 \quad \text{since } \overline{a}_1 = \overline{a}_2,\end{aligned}$$

so $\binom{a_1}{a_1}\binom{a_2}{a_2} \neq 0$ which implies that $a_1 = a_2$. This proves that θ is 1-1. Next, let $b \in Y'$. By Lemma 4.3.1 (ii), $\binom{b}{b}\varphi^{-1} = \binom{B \cup \{c\}}{c}$ for some $B \subseteq X\backslash X'$ and $c \in X'$. Then

$$\binom{c}{c}\left(\binom{b}{b}\varphi^{-1}\right) = \binom{c}{c}\binom{B \cup \{c\}}{c} = \binom{c}{c},$$

and thus

$$\left(\binom{c}{c}\varphi\right)\binom{b}{b} = \binom{c}{c}\varphi.$$

By Lemma 4.3.1 (ii), $\binom{c}{c}\varphi = \binom{C \cup \{\overline{c}\}}{\overline{c}}$ for some $C \subseteq Y\backslash Y'$. Now we have $\binom{C \cup \{\overline{c}\}}{\overline{c}}\binom{b}{b} = \binom{C \cup \{\overline{c}\}}{\overline{c}}$. This implies that $b = \overline{c}$. Hence θ is bijective.

Finally, let $a, b \in X$ be such that $a < b$. Then

$$\binom{b}{b}\binom{b}{a}\binom{a}{a} = \binom{b}{a}.$$

It follows from Lemma 4.3.1 (ii) that there are $A, B \subseteq Y\backslash Y'$ such that

$$\binom{B \cup \{\bar{b}\}}{\bar{b}} \left(\binom{b}{a} \varphi \right) \binom{A \cup \{\overline{a}\}}{\overline{a}} = \binom{b}{a} \varphi \neq 0.$$

We therefore conclude that

$$\binom{b}{a} \varphi = \binom{B \cup \{\bar{b}\}}{\overline{a}}.$$

But $\binom{B \cup \{\bar{b}\}}{\overline{a}} \in P_{RE}(Y, Y')$ and $\binom{b}{a} \notin E(P_{RE}(X, X'))$, so we have $\overline{a} < \bar{b}$.

Hence this lemma is proved. □

From Lemma 4.3.2, we have

Theorem 4.3.3. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. Then:*

(i) If $P_{RE}(X, X') \cong P_{RE}(Y, Y')$, then X' and Y' are order-isomorphic.

(ii) If $I_{RE}(X, X') \cong I_{RE}(Y, Y')$, then X' and Y' are order-isomorphic.

The following interesting consequence follows directly from Theorem 4.3.3 and Proposition 4.1.3.

Corollary 4.3.4. *Let X and Y be posets. Then the following statements hold.*

(i) $P_{RE}(X) \cong P_{RE}(Y)$ if and only if X and Y are order-isomorphic.

(ii) $I_{RE}(X) \cong I_{RE}(Y)$ if and only if X and Y are order-isomorphic.

Theorem 4.3.5. *Let X and Y be posets, X' a subposet X. Then*

(i) $P_{RE}(X, X') \cong P_{RE}(Y)$ if and only if

(1.1) X' and Y are order-isomorphic and

(1.2) for every $a \in X \backslash X'$ and $b \in X'$, either $a < b$ or a and b are uncomparable.

(ii) $I_{RE}(X, X') \cong I_{RE}(Y)$ if and only if

(2.1) X' and Y are order-isomorphic and

(2.2) for every $a \in X \backslash X'$ and $b \in X'$, either $a < b$ or a and b are uncomparable.

Proof. (i) Assume that $P_{RE}(X,X') \cong P_{RE}(Y)$. Then $P_{RE}(X,X')$ has an identity, so (1.2) holds by Proposition 4.1.1. Also, (1.1) follows from Theorem 4.3.3 (i)

Conversely, assume that (1.1) and (1.2) hold. By (1.2) and Proposition 4.1.1, $P_{RE}(X,X') = P_{RE}(X')$. From (1.1) and Corollary 4.3.4 (i), $P_{RE}(X') \cong P_{RE}(Y)$. Hence $P_{RE}(X,X') \cong P_{RE}(Y)$.

(ii) It can be proved similarly by Proposition 4.1.1, Theorem 4.3.3 (ii) and Corollary 4.3.4 (ii) $\square$

Example 4.3.6. We have that $\mathbb{Z}$ and $2\mathbb{Z}$ are order-isomorphic, $\mathbb{N}$ and $2\mathbb{N}$ are order-isomorphic but $2\mathbb{Z}$ and $2\mathbb{N}$ are not order-isomorphic. Therefore we deduce from Theorem 4.3.3, Corollary 4.3.4 and Theorem 4.3.5 that

$$P_{RE}(2\mathbb{Z}) \cong P_{RE}(\mathbb{Z}) \ncong P_{RE}(\mathbb{Z},2\mathbb{Z}) \ncong P_{RE}(2\mathbb{Z},2\mathbb{N}) \cong P_{RE}(\mathbb{N}),$$
$$I_{RE}(2\mathbb{Z}) \cong I_{RE}(\mathbb{Z}) \ncong I_{RE}(\mathbb{Z},2\mathbb{Z}) \ncong I_{RE}(2\mathbb{Z},2\mathbb{N}) \cong I_{RE}(\mathbb{N}).$$

This example also shows that the converses of both Theorem 4.3.3 (i) and Theorem 4.3.3 (ii) are not generally true.

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