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CONNECTIONS AND MEANS FOR POSITIVE OPERATORS ON A HILBERT SPACE

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การเชื่อมโยงคือการดำเนินการทวิภาคสำหรับตัวดำเนินการเชิงบวกบนปริภูมิฮิลเบิร์ตซึ่งสอดคล้องกับความเป็นทางเดียว อสมการหม้อแปลงไฟฟ้าและภาวะต่อเนื่องจากข้างบน ค่าเฉลี่ยคือการเชื่อมโยงที่ถูกทำให้เป็นบรรทัดฐาน

ในงานวิจัยนี้ เราแสดงว่าข้อสมมติภาวะต่อเนื่องในบทนิยามของการเชื่อมโยงสามารถผ่อนคลายได้ เราให้ลักษณะสมบัติเชิงสัจพจน์อีกหลายอย่างของการเชื่อมโยงและค่าเฉลี่ยซึ่งเกี่ยวข้องกับความเว้าและสมบัติการอยู่ระหว่าง การเชื่อมโยงของตัวดำเนินการทำให้เกิดการเชื่อมโยงของสเกลาร์ในข้อเท็จจริง   มีฟังก์ชันสมสัณฐานอันดับเชิงสัมพรรคระหว่างการเชื่อมโยงและการเชื่อมโยงของสเกลาร์ที่ถูกชักนำ เราให้การพรรณนาที่ชัดแจ้งของการเชื่อมโยงทั่วไปโดยการแยกการเชื่อมโยง เราพิจารณาโครงสร้างของเซตของการเชื่อมโยง เซตนี้มีสมสัณฐานอันดับเชิงสมมิติในแง่ของกรวยอันดับเชิงนอร์มกับเซตของฟังก์ชันทางเดียวเชิงตัวดำเนินการบนเซตของจำนวนจริงที่ไม่เป็นลบ เซตนี้มีสมสัณฐานเชิงสมมิติกับเซตของเมเชอร์โบเรลจำกัดบนภาคขยายของกึ่งเส้น ยิ่งกว่านั้นเราพิสูจน์สมบัติบางประการของการเชื่อมโยงที่เกี่ยวข้องกับอสมการของตัวดำเนินการ

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A connection is a binary operation assigned to each pair of positive operators on a Hilbert space satisfying monotonicity, transformer inequality and continuity from above. A mean is a normalized connection.

In this work, it is shown that the continuity assumption in the definition of a connection can be relaxed. We also provide various axiomatic characterizations of connections and means, involving concavity and betweenness properties. Each operator connection gives rise to a scalar connection. In fact, there is an affine order isomorphism between connections and induced scalar connections. We give an explicit description of a general connection by decomposing connections. Structures of the set of connections are also investigated. This set is isometrically order-isomorphic, as normed ordered cones, to the set of operator monotone functions on the nonnegative reals. It is isometrically isomorphic, as normed cones, to the set of finite Borel measures on the extended half-line. Moreover, we establish some properties of connections related to operator inequalities.

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CONTENTS

CHAPTER I

INTRODUCTION

The concept of means is one of the most familiar concepts in mathematics. It is proved to be a powerful tool from theoretical as well as practical points of view. The theory of scalar means was developed since the ancient Greeks by the Pythagoreans until the last century by many famous mathematicians. In the Pythagorean school, various means are defined via the method of proportions (in fact, they are solutions of certain algebraic equations). See the development of this subject in a survey article [30]. The theory of connections and means for matrices and operators started when the concept of parallel sum for matrices was introduced in [2] for analyzing electrical networks. The *parallel sum* of two positive definite matrices A and B is defined by

$$A : B = (A^{-1} + B^{-1})^{-1}.$$

The parallel sum for positive semidefinite matrices A and B is defined by forming the parallel sum of $A + \epsilon I$ and $B + \epsilon I$ for $\epsilon > 0$ and then take $\epsilon \to 0$ in the strong-operator topology. Subsequently, this notion was extended to positive operators on a Hilbert space in [5] via the same method.

In order to study operator means, the first step is to consider three classical means, namely, arithmetic, geometric and harmonic means. Arithmetic and harmonic means are easy to extend from positive real numbers to positive operators. The *harmonic mean*, denoted by !, for positive operators is the twice parallel sum. The *geometric mean* of two positive definite matrices A and B was defined in [7]:

$$A \# B = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

In [8], important properties of geometric and harmonic means are firmly established and, as applications, they played crucial roles in the study of concavity and monotonicity of many interesting maps between matrix spaces. Another important mean in mathematics, namely the power mean, was considered in [10]. See also [12, Chapter IV] for a systematic treatment on matrix means.

A study of operator means in an abstract way was given by Kubo and Ando [23]. Let $B(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Denote by $B(\mathcal{H})^+$ the set of positive operators on $\mathcal{H}$. A *connection* is a binary operation σ on $B(\mathcal{H})^+$ such that for all $A, B, C, D \in B(\mathcal{H})^+$:

(M1) *monotonicity*: $A \leqslant C, B \leqslant D \implies A\,\sigma\,B \leqslant C\,\sigma\,D$

(M2) *transformer inequality*: $C(A\,\sigma\,B)C \leqslant (CAC)\,\sigma\,(CBC)$

(M3) *joint–continuity from above*: for $A_n, B_n \in B(\mathcal{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\,\sigma\,B_n \downarrow A\,\sigma\,B$.

Typical examples of connections are the sum $(A, B) \mapsto A + B$ and the parallel sum. A *mean* is a connection σ such that $A\,\sigma\,A = A$ for any positive operator A. The followings are examples of means in practical usage:

- t-weighted arithmetic means: $A \triangledown_t B = (1-t)A + tB$
- t-weighted geometric means: $A \#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$
- t-weighted harmonic means: $A\,!_t\,B = [(1-t)A^{-1} + tB^{-1}]^{-1}$
- logarithmic mean: $(A, B) \mapsto A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$ where $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = (x-1)/\log x$.

A fundamental tool in Kubo-Ando theory of connections and means is the theory of operator monotone functions. Denote by $OM(\mathbb{R}^+)$ the set of operator monotone functions from $\mathbb{R}^+ = [0, \infty)$ to itself. In [23], a connection σ on $B(\mathcal{H})^+$ can be characterized as follows:

- There is an $f \in OM(\mathbb{R}^+)$ satisfying

$$f(x)I = I\,\sigma\,(xI), \quad x \in \mathbb{R}^+. \tag{1.1}$$

- There is an $f \in OM(\mathbb{R}^+)$ such that

$$A \,\sigma\, B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}, \quad A, B > 0. \tag{1.2}$$

- There is a finite Borel measure μ on $[0, \infty]$ such that

$$A \,\sigma\, B = \alpha A + \beta B + \int_{(0,\infty)} \frac{\lambda + 1}{2\lambda} \{(\lambda A) \,!\, B\} \, d\mu(\lambda) \tag{1.3}$$

  where the integral is the Bochner integral, $\alpha = \mu(\{0\})$ and $\beta = \mu(\{\infty\})$.

In fact, the functions f in (1.1) and (1.2) are unique and coincide, called the *representing function* of σ. From the integral representation (1.3), every connection σ is concave in the sense that

$$(tA + (1-t)B) \,\sigma\, (tA' + (1-t)B') \geqslant t(A \,\sigma\, A') + (1-t)(B \,\sigma\, B') \tag{1.4}$$

for all $A, B \geqslant 0$ and $t \in (0, 1)$. Moreover, the map $\sigma \mapsto f$ is an affine order-isomorphism.

The mean-theoretic approach has various applications. It can be used to obtain the monotonicity, concavity and convexity of interesting maps between matrix algebras or operator algebras (see the original idea in [8]). The order isomorphism $f \mapsto \sigma$ transforms suitable scalar inequalities to operator inequalities concerning means. For example, the arithmetic–logarithmic–geometric–harmonic means inequalities are obtained from applying this order isomorphism to the scalar inequalities

$$\frac{2x}{1+x} \leqslant x^{1/2} \leqslant \frac{x-1}{\log x} \leqslant \frac{1+x}{2}, \quad x > 0, x \neq 1.$$

The concavity of general connections serves simple proofs of operator versions of Hölder inequality, Cauchy-Schwarz inequality, Minkowski's inequality, Aczel's inequality, Popoviciu's inequality and Bellman's inequality (e.g. [27]). The famous Furuta's inequality and its generalizations are obtained from axiomatic properties of connections (e.g. [17, 18, 19]). Kubo-Ando theory can be applied to matrix and operator equations since harmonic and geometric means can be viewed as solutions of certain operator equations (e.g. [4, 24]). It also plays an important role in noncommutative information theory. A relative operator entropy was defined in [16] to be the connection corresponding to the operator monotone function $x \mapsto \log x$. See more information in [11, Chapter IV] and therein references.

Kubo-Ando definition of a connection is a binary operation satisfying axioms (M1), (M2) and (M3). In this work, we show that some of the axioms can be weakened. Moreover, we provide alternative sets of axioms involving concavity property.

Consider the following axioms:

(M3′) for each $A, X \in B(\mathcal{H})^+$, if $A_n \downarrow A$, then $A_n \,\sigma\, X \downarrow A \,\sigma\, X$ and $I \,\sigma\, A_n \downarrow I \,\sigma\, A$;

(M3″) for each $A, X \in B(\mathcal{H})^+$, if $A_n \downarrow A$, then $X \,\sigma\, A_n \downarrow X \,\sigma\, A$ and $A_n \,\sigma\, I \downarrow A \,\sigma\, I$;

(M4) *concavity*: $(tA + (1-t)B) \,\sigma\, (tA' + (1-t)B') \geqslant t(A \,\sigma\, A') + (1-t)(B \,\sigma\, B')$ for $t \in (0,1)$;

(M4′) *midpoint concavity*: $(A+B)/2 \;\sigma\; (A'+B')/2 \geqslant [(A \,\sigma\, A') + (B \,\sigma\, B')]/2$.

Note that condition (M3′) is one of the axiomatic properties of *solidarity* introduced in [15]. We will show that the axiom (M3) in the definition of a connection can be relaxed to (M3′) or (M3″). The conditions (M3′) and (M3″) are clearly weaker, and easier to verify, than the joint-continuity assumption (M3). Moreover, a connection can be axiomatically defined as follows. Fix the transformer inequality (M2). We can freely replace the monotonicity (M1) by the concavity (M4) or the midpoint concavity (M4′). At the same time, we can use (M3′) or (M3″) instead of the joint-continuity (M3). This result gives different viewpoints of connections. It shows the importance of the concavity property of a connection. Moreover, it asserts that the concepts of monotonicity and concavity are equivalent under suitable conditions. We also show that a connection is a mean if and only if it satisfies a usual property of scalar means on $\mathbb{R}^+$, namely,

betweenness: $A \leqslant B \Rightarrow A \leqslant A \sigma B \leqslant B$.

Each connection (mean) σ on $B(\mathcal{H})^+$ gives rise to a unique connection (mean, respectively) $\tilde{\sigma}$ on $\mathbb{R}^+$ satisfying $(xI) \,\sigma\, (yI) = (x \,\tilde{\sigma}\, y)I$ for $x, y \in \mathbb{R}^+$. Properties of $\tilde{\sigma}$ related to σ, its representing function and its representing measure are investigated. In fact, there is an affine order isomorphism between connections on $B(\mathcal{H})^+$ and induced connections on $\mathbb{R}^+$. This gives a natural way to define any named mean. For example, the geometric mean on $B(\mathcal{H})^+$ is the mean on $B(\mathcal{H})^+$ that corresponds to the usual geometric mean on $\mathbb{R}^+$.

We consider the relationship between connections and the representing measures. It is shown that a connection σ can be uniquely written as

$$\sigma = \sigma_{ac} + \sigma_{sd} + \sigma_{sc},$$

where σ_{ac}, σ_{sd} and σ_{sc} are connections, subject to suitable conditions. The "singularly discrete part" σ_{sd} is a countable sum of means of the form σ_λ with non-negative coefficients, where

$$A \, \sigma_\lambda \, B = \frac{\lambda+1}{2\lambda}(\lambda A \,!\, B), \quad A, B \geqslant 0$$

for each $\lambda \in [0, \infty]$, here $\sigma_0 : (A, B) \mapsto A$ and $\sigma_\infty : (A, B) \mapsto B$. The "absolutely continuous part" σ_{ac} has an integral representation with respect to Lebesgue measure m on the real line. The "singularly continuous part" σ_{sc} has an integral representation with respect to a continuous measure mutually singular to m.

Structures of the set of connections are also investigated. In fact, this set is isometrically order-isomorphic, as normed ordered cones, to the set of operator monotone functions on $\mathbb{R}^+$. Moreover, it is isometrically isomorphic, as normed cones, to the set of finite Borel measures on $[0, \infty]$.

Finally, we establish further properties of connections related to operator inequalities. A connection behaves nicely with any positive linear map Φ in the sense that

$$\Phi(A \sigma B) \leqslant \Phi(A) \, \sigma \, \Phi(B), \quad A, B \geqslant 0.$$

We also prove some properties of connections related to monotonicity and concavity of maps between operator algebras. These will generalize some results related to specific connections in the literature.

This thesis is organized as follows. Chapter II deals with the development of the theory of connections and means for positive operators, focused on the axiomatic theory of Kubo and Ando. In Chapter III, various axiomatic characterizations of connections and means are provided. The relationship between connections and their induced connections is also considered here. Chapter IV contains an explicit decomposition of an arbitrary connection and a mean. Chapter V is a discussion of structures of the set of connections. In Chapter VI, we establish some properties of connections involving operator inequalities. Some preliminaries and results needed for this research are collected in Appendix A. They cover the spectral theory for operators and the integration theory on Banach spaces.

CHAPTER II

KUBO–ANDO THEORY OF OPERATOR CONNECTIONS AND OPERATOR MEANS

This chapter contains the development of the theory of connections and means for positive operators on a Hilbert space. The beginning of the theory came from electrical networks as the presence of the parallel sum; see Section 2.1. This lead to a study of matrix/operator means in Section 2.2. A general theory of connections and means was investigated by Kubo and Ando in 1980. A major result of Kubo-Ando theory is the correspondences between connections, operator monotone functions and Borel measures; see Section 2.3. Equivalent definitions and practical examples of means in Kubo-Ando sense are provided in Section 2.4.

Throughout, let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Denote by $B(\mathcal{H})$ the von Neumann algebra of bounded linear operators acting on $\mathcal{H}$. The sets of self-adjoint operators, positive operators, strictly positive operators on $\mathcal{H}$ are written by $B(\mathcal{H})^{sa}$, $B(\mathcal{H})^{+}$ and $B(\mathcal{H})^{++}$, respectively. For $A, B \in B(\mathcal{H})^{sa}$, we define $A \leqslant B$ if $B - A \in B(\mathcal{H})^{+}$. If A is strictly positive, then we write $A > 0$. We always reserve A, B, C, D for positive operators. Write $A_n \to A$ to indicate that A_n converges strongly to A. If A_n is a sequence in $B(\mathcal{H})^{sa}$, the expression $A_n \downarrow A$ indicates that A_n is a decreasing sequence and $A_n \to A$. The set of nonnegative real numbers is denoted by $\mathbb{R}^{+}$.

2.1 Parallel Sum

In electrical engineering, Anderson and Duffin [2] defined the *parallel sum* of two positive definite matrices A and B by

$$A : B = (A^{-1} + B^{-1})^{-1} = A - A(A+B)^{-1}A. \tag{2.1}$$

Recall that the impedance of an electrical network can be represented by a positive (semi)definite matrix. If A and B are impedance matrices of multiport networks, then the parallel sum $A : B$ indicates the total impedance of two electrical networks connected in parallel. This notion plays a crucial role for analyzing multiport electrical networks. This is a starting point of the study of matrix and operator means. This notion can be extended to invertible positive operators by the same formula.

Lemma 2.1. *([2]) Let $A, B, C, D, A_n, B_n \in B(\mathcal{H})^{++}$ for each $n \in \mathbb{N}$.*

(1) If $A \leqslant C$ and $B \leqslant D$, then $A : B \leqslant C : D$.

(2) If $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n : B_n \downarrow A : B$.

(3) If $A_n \downarrow A$ and $B_n \downarrow B$, then $\lim A_n : B_n$ exists and does not depend on the choices of A_n, B_n.

Proof. The exsitence of limits follows from the order completeness of the von Neumann algebra $B(\mathcal{H})$ (see Appendix A.2). □

In the physical context, the normal situation is that the impedance matrices A and B are strictly positive. However, the case $A = 0$ or $B = 0$–a short circuit–can be handled by letting $A : B = 0$. This motivates us to define the *parallel sum* for arbitrary positive operators $A, B \in B(\mathcal{H})^+$:

$$A : B = \lim_{\epsilon \downarrow 0} (A + \epsilon I) : (B + \epsilon I) \tag{2.2}$$

where the limit is taken in the strong-operator topology. We have a variational description for the parallel sum as follows.

Lemma 2.2. *([2]) For each $x \in \mathcal{H}$,*

$$\langle (A : B)x, x \rangle = \inf\{\langle Ay, y \rangle + \langle Bz, z \rangle : y, z \in \mathcal{H}, y + z = x\}. \tag{2.3}$$

Proof. First, assume that A, B are invertible. Then for all $x, y \in \mathcal{H}$,

$$\begin{aligned}
&\langle Ay, y\rangle + \langle B(x-y), x-y\rangle - \langle (A:B)x, x\rangle \\
&= \langle Ay, y\rangle + \langle Bx, x\rangle - 2\operatorname{Re}\langle Bx, y\rangle + \langle By, y\rangle - \langle (B - B(A+B)^{-1}B)x, x\rangle \\
&= \langle (A+B)y, y\rangle - 2\operatorname{Re}\langle Bx, y\rangle + \langle (A+B)^{-1}Bx, Bx\rangle \\
&= \left\|(A+B)^{1/2}y\right\|^2 - 2\operatorname{Re}\langle Bx, y\rangle + \left\|(A+B)^{-1/2}Bx\right\|^2 \\
&\geqslant 0.
\end{aligned}$$

With $y = (A+B)^{-1}Bx$, we have

$$\langle Ay, y\rangle + \langle B(x-y), x-y\rangle - \langle (A:B)x, x\rangle = 0.$$

Hence, we have the claim for $A, B > 0$. For $A, B \geqslant 0$, consider $A + \epsilon I$ and $B + \epsilon I$ where $\epsilon \downarrow 0$. $\square$

Remark 2.3. This lemma has a physical interpretation, called the *Maxwell's power principle*. This principle governs the flow of currents through electrical circuits. Recall that a positive operator represents the impedance of a multiport electrical network while the power dissipation of a network with impedance A and current x is given by the inner product $\langle Ax, x\rangle$. Consider two electrical networks connected in parallel. For a given current input x, the current will divide $x = y + z$, where y and z are currents of each network, in such a way that the power dissipation is minimized.

Theorem 2.4. *([2]) The parallel sum satisfies*

(1) monotonicity: $A_1 \leqslant A_2, B_1 \leqslant B_2 \Rightarrow A_1 : B_1 \leqslant A_2 : B_2$.

(2) transformer inequality: $S^*(A:B)S \leqslant (S^*AS) : (S^*BS)$ *for* $S \in B(\mathcal{H})$.

(3) continuity from above: if $A_n \downarrow A$ *and* $B_n \downarrow B$*, then* $A_n : B_n \downarrow A : B$.

Proof. (1) The monotonicity follows from the formula (2.2) and Lemma 2.1(1).

(2) For each $x, y, z \in \mathcal{H}$ such that $x = y + z$, by Lemma 2.2,

$$\begin{aligned}
\langle S^*(A:B)Sx, x\rangle &= \langle (A:B)Sx, Sx\rangle \\
&\leqslant \langle ASy, Sy\rangle + \langle BSz, Sz\rangle \\
&= \langle S^*ASy, y\rangle + \langle S^*BSz, z\rangle.
\end{aligned}$$

Again, Lemma 2.2 assures $S^*(A:B)S \leqslant (S^*AS):(S^*BS)$.

(3) Apply Lemma 2.1(2) to $A_n+\epsilon I, B_n+\epsilon I, A+\epsilon I, B+\epsilon I$ and use (2.2). $\square$

Remark 2.5. The positive operator S^*AS represents the impedance of a network connected to a transformer. The transformer inequality states that the impedance of parallel connection with transformer first is greater than that with transformer last.

2.2 Matrix and Operator Means

Some desired properties of any object that is called a "mean" M on $B(\mathcal{H})^+$ should have are given here.

(A1) *positivity*: $A, B \geqslant 0 \Rightarrow M(A,B) \geqslant 0$;

(A2) *monotonicity*: $A \geqslant A', B \geqslant B' \Rightarrow M(A,B) \geqslant M(A',B')$;

(A3) *positive homogeneity*: $M(kA,kB) = kM(A,B)$ for $k \in \mathbb{R}^+$;

(A4) *transformer inequality*: $X^*M(A,B)X \leqslant M(X^*AX, X^*BX)$ for $X \in B(\mathcal{H})$;

(A5) *congruence invariance*: $X^*M(A,B)X = M(X^*AX, X^*BX)$ for invertible $X \in B(\mathcal{H})$;

(A6) *concavity*: $M(tA+(1-t)B, tA'+(1-t)B') \geqslant tM(A,A')+(1-t)M(B,B')$ for $t \in [0,1]$;

(A7) *continuity from above*: if $A_n \downarrow A$ and $B_n \downarrow B$, then $M(A_n,B_n) \downarrow M(A,B)$;

(A8) *fixed point property*: $M(A,A) = A$;

(A9) *betweenness*: if $A \leqslant B$, then $A \leqslant M(A,B) \leqslant B$.

In order to study matrix or operator means in general, the first step is to consider three classical means in mathematics, namely, arithmetic, geometric and harmonic means. The *arithmetic mean* of $A, B \in B(\mathcal{H})^+$ is defined by

$$A \triangledown B = \frac{1}{2}(A+B). \tag{2.4}$$

Then the arithmetic mean satisfies the properties (A1)–(A9). In fact, the properties (A4) and (A5) can be replaced by a stronger condition:

$$X^*(A \triangledown B)X = (X^*AX) \triangledown (X^*BX), \quad X \in B(\mathcal{H}).$$

Moreover, the arithmetic mean is *affine* in the sense that

$$(kA + C) \triangledown (kB + C) = k(A \triangledown B) + C, \quad k \in \mathbb{R}^+.$$

Define the *harmonic mean* of positive operators $A, B \in B(\mathcal{H})^+$ by

$$A \,!\, B = 2(A : B) = \lim_{\epsilon \downarrow 0} 2(A_\epsilon^{-1} + B_\epsilon^{-1})^{-1} \tag{2.5}$$

where $A_\epsilon \equiv A + \epsilon I$ and $B_\epsilon \equiv B + \epsilon I$. This mean satisfies (A1)–(A9); see [2, 8].

The geometric mean for matrices or operators was firstly defined by Pusz and Woronowicz [29]:

$$A \,\#\, B = \max\{T \geqslant 0 : |\langle Tx, y\rangle| \leqslant \|A^{1/2}x\| \, \|B^{1/2}y\| \ \forall x, y \in \mathcal{H}\}, \quad A, B \geqslant 0.$$

This definition coincides with the following formula given by Ando [7]:

$$A \,\#\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad A, B > 0. \tag{2.6}$$

This formula comes from two natural requirements. This definition should coincide with the usual geometric mean on $\mathbb{R}^+$: $A \,\#\, B = (AB)^{1/2}$ provided that $AB = BA$. The second condition is that, for any invertible $T \in B(\mathcal{H})$,

$$T^*(A \,\#\, B)T = (T^*AT) \,\#\, (T^*BT). \tag{2.7}$$

The geometric mean of $A, B > 0$ can be equivalently defined by iterative process [3] as follows:

$$A_0 = A, B_0 = B, \quad A_n = A_{n-1} \triangledown B_{n-1}, B_n = A_{n-1} \,!\, B_{n-1}.$$

Indeed, A_n is decreasing while B_n is increasing. Since A_n and B_n are bounded below and bounded above, respectively, they converge to positive operators by the order completeness of $B(\mathcal{H})$. In fact, they have a common limit, namely, the geometric mean of A and B.

It was also pointed out in [6] that the geometric mean of $A, B > 0$ is the unique positive solution to the *Riccati equation*:

$$XA^{-1}X = B.$$

This equation plays an important role in circuit and system theory.

We define the *geometric mean* of $A, B \geqslant 0$ by

$$A \# B = \lim_{\epsilon \downarrow 0} (A + \epsilon I) \# (B + \epsilon I). \tag{2.8}$$

The geometric mean enjoys the properties (A1)–(A9); see e.g. [8]. Moreover, it is *self-duality* in the sense that

$$(A \# B)^{-1} = A^{-1} \# B^{-1}.$$

The power mean for operators was considered in [10]. The *power mean* or *Hölder mean* with exponent $p \in \mathbb{R}$ of $A, B \in B(\mathcal{H})^+$ is defined to be

$$\left(\frac{A^p + B^p}{2} \right)^{1/p}.$$

Here, the case $p = 0$ is understood that we take limit as $p \to 0$ and we get the geometric mean. The case $p = 2$ is called the *quadratic mean* or *root mean square*.

The *arithmetic-geometric mean* or *Gaussian mean* is defined in [3] as follows:

$$A_0 = A, B_0 = B, \quad A_n = A_{n-1} \triangledown B_{n-1}, \quad B_n = A_{n-1} \# B_{n-1}.$$

Indeed, A_n is decreasing while B_n is increasing. Both sequences converge to a common limit, namely, the Gaussian mean of A and B.

2.3 Connections, Operator Monotone Functions and Borel Measures

The notion of parallel sum was characterized via a set of axioms in [28]. This result lead naturally to a study of connections and means in an abstract way. In an influential paper [23], Kubo and Ando proposed an axiomatic definition of a connection as follows.

Definition 2.6. A *connection* is a binary operation σ on $B(\mathcal{H})^+$such that for all positive operators A, B, C, D:

(M1) *monotonicity*: $A \leqslant C, B \leqslant D \implies A \,\sigma\, B \leqslant C \,\sigma\, D$

(M2) *transformer inequality*: $C(A \,\sigma\, B)C \leqslant (CAC) \,\sigma\, (CBC)$

(M3) *continuity from above*: for $A_n, B_n \in B(\mathcal{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$ then $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$.

Typical examples of connections are the sum and the parallel sum. Connections with the fixed point property $(A, A) \mapsto A$ will be discussed in the next section. We introduce algebraic operations on connections as follows:

$$\sigma_1 + \sigma_2 : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+ : (A, B) \mapsto (A \,\sigma_1\, B) + (A \,\sigma_2\, B),$$
$$k\sigma : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+ : (A, B) \mapsto k(A \,\sigma\, B), \quad k \in \mathbb{R}^+.$$

Every nonnegative linear combination of connections is a connection.

Example 2.7. (Transpose, adjoint and dual of connections). Given a connection σ, we can construct a new connection as follows. The *transpose* of σ is the connection $(A, B) \mapsto B \,\sigma\, A$. The *adjoint* of σ is the connection defined by

$$(A, B) \mapsto (A^{-1} \,\sigma\, B^{-1})^{-1}.$$

The *dual* of σ is the transpose of the adjoint of σ.

Example 2.8. (Composition of connections). If σ_1, σ_2 and η are connections, then the binary operation

$$\sigma_1(\eta)\sigma_2 : (A, B) \mapsto (A \,\sigma_1\, B) \,\eta\, (A \,\sigma_2\, B)$$

is also a connection.

This axiomatic approach has many applications in operator inequalities (e.g. [17, 27]), operator equations (e.g. [4, 24]) and operator entropy ([16]).

A fundamental tool in Kubo-Ando theory is an important class of real-valued functions, introduced by Löwner in a seminal paper [25], namely:

Definition 2.9. Let I be an interval. A function $f : I \to \mathbb{R}$ is said to be *operator monotone* if for all Hilbert spaces $\mathcal{H}$ and for all Hermitian operators A, B on $\mathcal{H}$ whose spectra are contained in I,

$$A \leqslant B \implies f(A) \leqslant f(B),$$

where $f(A)$ is the functional calculus of f at A.

Example 2.10.

(1) Any straight line with nonnegative slope is operator monotone on $\mathbb{R}$.

(2) The function $x \mapsto -1/x$ is operator monotone on $(0, \infty)$.

(3) For each $p \in [0, 1]$, the function $x \mapsto x^p$ is operator monotone on $\mathbb{R}^+$. This is known as the Löwner-Heinz inequality ([25]).

(4) The logarithmic function is operator monotone on $(0, \infty)$.

(5) The functions $f(x) = (x-1)/\log x$ and $g(x) = (x \log x)/(x-1)$ are operator monotone on $\mathbb{R}^+$. Here, we use the L'Hôpital's rule for $x = 0, 1$.

(6) The function $x \mapsto [(1 + x^p)/2]^{1/p}$ is operator monotone on $\mathbb{R}^+$ if and only if $p \in [-1, 1]$. Here, when $p = 0$, we take limit as p tends to 0.

The set of operator monotone functions is closed under taking nonnegative linear combinations and pointwise limits. See more information in [11, 13, 21].

Theorem 2.11. *([22]) The following statements are equivalent for a continuous function $f : I \to \mathbb{R}$:*

(i) $A \leqslant B \implies f(A) \leqslant f(B)$ *for all Hermitian matrices A, B of all orders whose spectra are contained in I;*

(ii) $A \leqslant B \implies f(A) \leqslant f(B)$ *for all Hermitian operators $A, B \in B(\mathcal{H})$ whose spectra are contained in I and for an infinite-dimensional Hilbert space $\mathcal{H}$;*

(iii) $A \leqslant B \implies f(A) \leqslant f(B)$ *for all Hermitian operators $A, B \in B(\mathcal{H})$ whose spectra are contained in I and for all Hilbert spaces $\mathcal{H}$.*

Theorem 2.12. *([25]) A continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if there is a finite Borel measure μ on $[0,\infty]$ such that*

$$f(x) = \int_{[0,\infty]} \phi_\lambda(x)\, d\mu(\lambda), \quad x \in \mathbb{R}^+ \tag{2.9}$$

where

$$\phi_\lambda(x) = \frac{x(1+\lambda)}{x+\lambda} \; for \; \lambda \in (0,\infty), \quad \phi_0(x) = 1, \quad \phi_\infty(x) = x.$$

Moreover, the measure μ is unique and we can write

$$f(x) = a + bx + \int_{(0,\infty)} \frac{x(1+\lambda)}{x+\lambda}\, d\mu(\lambda), \quad x \in \mathbb{R}^+$$

where $a := \mu(\{0\}) = f(0)$ and $b := \mu(\{\infty\}) = \lim_{x\to\infty} f(x)/x$.

A major result in Kubo-Ando theory is that there are correspondences between connections on $B(\mathcal{H})^+$, operator monotone functions on $\mathbb{R}^+$ and finite Borel measures on $[0,\infty]$. Define the relation $\leqslant$ for connections on $B(\mathcal{H})^+$ by $\sigma_1 \leqslant \sigma_2$ if $A\,\sigma_1\,B \leqslant A\,\sigma_2\,B$ for all $A, B \in B(\mathcal{H})^+$. Equip $OM(\mathbb{R}^+)$ with the pointwise order relation.

Theorem 2.13. *([23]) For each connection σ, there exists a unique operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$f(x)I = I\,\sigma\,(xI), \quad x \in \mathbb{R}^+.$$

In fact, the map $\sigma \mapsto f$ is an affine order-isomorphism between connections and operator monotone functions on $\mathbb{R}^+$. Here, the order-isomorphism means that when $\sigma_i \mapsto f_i$ for $i = 1, 2$, we have $\sigma_1 \leqslant \sigma_2$ if and only if $f_1 \leqslant f_2$.

Moreover, every connection σ takes the form

$$A\,\sigma\,B = \lim_{\epsilon \downarrow 0} A_\epsilon^{1/2} f(A_\epsilon^{-1/2} B_\epsilon A_\epsilon^{-1/2}) A_\epsilon^{1/2}$$

where $A_\epsilon \equiv A + \epsilon I$ and $B_\epsilon \equiv B + \epsilon I$.

We call f the *representing function* of σ.

Theorem 2.14. *([23]) For every connection σ, there is a unique finite Borel measure μ on $[0,\infty]$ such that for each $A, B \in B(\mathcal{H})^+$*

$$A\,\sigma\,B = aA + bB + \int_{(0,\infty)} \frac{1+\lambda}{\lambda}\{(\lambda A) : B\}d\mu(\lambda), \tag{2.10}$$

where $a = \mu(\{0\})$ and $b = \mu(\{\infty\})$. The map $\sigma \mapsto \mu$ is an affine bijection between connections and finite Borel measures on $[0,\infty]$.

We call μ the *representing measure* of σ.

Remark 2.15. Let us consider operator connections from electrical circuit viewpoint. A general connection represents a formulation of making a new impedance from two given impedances. The integral representation (2.10) shows that such a formulation can be described as a series of (infinite) weighted parallel sums. From this point of view, the theory of operator connections can be regarded as a mathematical theory of electrical circuits.

Corollary 2.16. *([23]) Every connection satisfies the properties (A1)–(A7) in Section 2.2.*

2.4 Kubo-Ando Means

Let σ be a connection on $B(\mathcal{H})^+$ with representing function f and representing measure μ. By [23], the followings are equivalent:

(i) $I\,\sigma\,I = I$;

(ii) σ satisfies the fixed point property, i.e., $A\,\sigma\,A = A$ for all $A \in B(\mathcal{H})^+$;

(iii) f is normalized, i.e., $f(1) = 1$;

(iv) μ is normalized, i.e., μ is a probability measure.

A (Kubo-Ando) *mean* is defined to be a connection satisfying one (thus, all) of the above properties. Every convex combination of means is a mean. The transpose, the adjoint and the dual of a mean are also means.

Example 2.17. (Trivial means). The *left-trivial mean* $(A,B) \mapsto A$ and the *right-trivial mean* $(A,B) \mapsto B$ are means. Their representing functions are given by the normalized operator monotone functions $x \mapsto 1$ and $x \mapsto x$, respectively.

Example 2.18. (Pythagorean means and their weighted versions). Let $\alpha \in [0,1]$. The *α-weighted arithmetic mean* is the mean defined by

$$A \,\triangledown_\alpha\, B = (1-\alpha)A + \alpha B, \quad A, B \geqslant 0.$$

This mean has the normalized operator monotone funtion $x \mapsto (1-\alpha) + \alpha x$ as the representing function. The *α-weighted harmonic mean* is defined by

$$A \,!_\alpha\, B = [(1-\alpha)A^{-1} + \alpha B^{-1}]^{-1}, \quad A, B > 0.$$

The representing function of $!_\alpha$ is given by $x \mapsto x/((1-\alpha)x + \alpha)$. The *$\alpha$-weighted geometric mean* is defined to be

$$A \,\#_\alpha\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}, \quad A, B > 0.$$

Its representing function is given by $x \mapsto x^\alpha$.

Example 2.19. (Quasi-arithmetic power means). For each $p \in [-1,1]$ and $\alpha \in [0,1]$, the operator monotone function

$$x \mapsto [(1-\alpha) + \alpha x^p]^{1/p}$$

gives rise to the *quasi-arithmetic power mean* with exponent p and weight α:

$$A \,\#_{p,\alpha}\, B = [(1-\alpha)A^p + \alpha B^p]^{1/p}, \quad A, B \in B(\mathcal{H})^+.$$

The special case $\#_{1,\alpha}$ of this mean gives the α-weighted arithmetic mean. The case $\#_{0,\alpha}$ is the α-weighted geometric mean. The case $\#_{-1,\alpha}$ is the α-weighted harmonic mean. The mean $\#_{p,1/2}$ is the power mean with exponent p. These means satisfy the property that

$$A \,\#_{p,\alpha}\, B \;=\; B \,\#_{p,1-\alpha}\, A.$$

Moreover, they are interpolated in the sense that

$$(A \,\#_{p,s}\, B) \,\#_{p,\alpha}\, A \,\#_{p,t}\, B) \;=\; A \,\#_{p,(1-\alpha)s+\alpha t}\, B.$$

Example 2.20. The *logarithmic mean* is the mean given by

$$(A, B) \mapsto A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

where $f(x) = (x-1)/\log x$. For $A, B > 0$ such that $A \neq B$ and $AB = BA$, the logarithmic mean of A and B is

$$(A - B)(\log A - \log B)^{-1}.$$

This mean is important in the consideration of heat flow in chemical engineering (see e.g. [11, Chapter IV]). The dual of the logarithmic mean is the mean corresponding to the operator monotone function $x \mapsto (x \log x)/(x-1)$.

Example 2.21. If σ_1 and σ_2 are means such that $\sigma_1 \leqslant \sigma_2$, then there is a family of means that interpolates between σ_1 and σ_2, namely, $(1-\alpha)\sigma_1 + \alpha\sigma_2$ for all $\alpha \in [0, 1]$. Note that the map $\alpha \mapsto (1-\alpha)\sigma_1 + \alpha\sigma_2$ is increasing. For instance, the *Heron mean* with weight $\alpha \in [0, 1]$ is defined to be $h_\alpha = (1-\alpha)\,\#\, + \,\alpha\,\triangledown$. This family is the linear interpolations between the geometric mean and the arithmetic mean. The representing function of h_α is

$$x \mapsto (1-\alpha)x^{1/2} + \frac{\alpha}{2}(1+x).$$

The case $\alpha = 2/3$ is named the *Heronian mean* after Hero of Alexandria and it is used in finding the volume of a frustum of a pyramid.

Example 2.22. If σ_1, σ_2 and η are means, then the composition $\sigma_1(\eta)\sigma_2$ is also a mean. Recall that the Gaussian mean of $A, B \geqslant 0$ is defined by the iterative process as follows:

$$A_0 = A, B_0 = B, \quad A_n = A_{n-1} \,\triangledown\, B_{n-1}, B_n = A_{n-1} \,\#\, B_{n-1}.$$

Since each step is a mean, the limit of this iteration is also a mean.

CHAPTER III

CHARACTERIZATIONS OF CONNECTIONS AND MEANS

Recall that connection in Kubo-Ando sense is a binary operation σ on $B(\mathcal{H})^+$ satisfying the following axioms:

(M1) *monotonicity*: $A \geqslant A', B \geqslant B' \Rightarrow A \,\sigma\, B \geqslant A' \,\sigma\, B'$;

(M2) *transformer inequality*: $C(A \,\sigma\, B)C \leqslant (CAC) \,\sigma\, (CBC)$;

(M3) *continuity from above*: if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$.

A mean is a connection σ such that $I \,\sigma\, I = I$. In this chapter, we provide various axiomatic characterizations of connections and means. In Section 3.1, it is shown that the axiom (M3) can be relaxed. We also provide alternative sets of axioms for a connection involving concavity property in Section 3.2. Characterizations of means are given in Section 3.3. An interesting result is that a connection is a mean if and only if it satisfies a usual property of scalar means, namely, the betweenness property. Each operator connection on $B(\mathcal{H})^+$ induces a unique scalar connection on $\mathbb{R}^+$. The correspondence between connections and induced connections will be discussed in details in Section 3.4.

Throughout this chapter, σ is a binary operation on $B(\mathcal{H})^+$. Consider the following properties:

(M3$'$) for each $A, X \in B(\mathcal{H})^+$, if $A_n \downarrow A$, then $A_n \,\sigma\, X \downarrow A \,\sigma\, X$ and $I \,\sigma\, A_n \downarrow I \,\sigma\, A$;

(M3$''$) for each $A, X \in B(\mathcal{H})^+$, if $A_n \downarrow A$, then $X \,\sigma\, A_n \downarrow X \,\sigma\, A$ and $A_n \,\sigma\, I \downarrow A \,\sigma\, I$;

(M4) *concavity*: $(tA + (1-t)B) \,\sigma\, (tA' + (1-t)B') \geqslant t(A \,\sigma\, A') + (1-t)(B \,\sigma\, B')$ for $t \in (0, 1)$;

(M4$'$) *midpoint concavity*: $(A+B)/2 \;\sigma\; (A'+B')/2 \geqslant [(A \,\sigma\, A') + (B \,\sigma\, B')]/2$;

(P) if a projection $P \in B(\mathcal{H})^+$ commutes with $A, B \in B(\mathcal{H})^+$, then

$$P(A \,\sigma\, B) = (PA) \,\sigma\, (PB) = (A \,\sigma\, B)P;$$

in particular, P commutes with $A \,\sigma\, B$.

The set of binary operations having property (A) is denoted by $BO(A)$. Note that the condition (M3$'$) is one of the axiomatic properties of *solidarity* introduced in [15]. The property (P) will play an important role in relaxing and characterizing connections in Sections 3.1 and 3.2.

3.1 Improvement of the Definition of a Connection

The definition of a connection can be relaxed as follows.

Theorem 3.1. *Let σ be a binary operation on $B(\mathcal{H})^+$. Then the followings are equivalent:*

(i) σ is a connection;

(ii) σ satisfies (M1), (M2) and (M3$'$);

(iii) σ satisfies (M1), (M2) and (M3$''$).

The condition (M3$'$) or (M3$''$) is clearly weaker than the joint-continuity assumption (M3) in Kubo-Ando definition. We divide the proof of this theorem into several lemmas. Each lemma is of interest in its own right.

Lemma 3.2. *The transformer inequality (M2) implies*

- congruence invariance*: $C(A\sigma B)C = (CAC) \,\sigma\, (CBC)$ for $A, B \geqslant 0$ and $C > 0$;*
- positive homogeneity*: $\alpha(A \,\sigma\, B) = (\alpha A) \,\sigma\, (\alpha B)$ for $A, B \geqslant 0$ and $\alpha \in \mathbb{R}^+$.*

Proof. By (M2), we have $C(A\sigma B)C \leqslant (CAC)\,\sigma\,(CBC)$. Since $C > 0$, we get

$$C^{-1}[(CAC)\,\sigma\,(CBC)]C^{-1} \leqslant (C^{-1}CACC^{-1})\,\sigma\,(C^{-1}CBCC^{-1}) = A\,\sigma\,B,$$

i.e. $(CAC)\,\sigma\,(CBC) \leqslant C(A\sigma B)C$. For $\alpha \in (0,\infty)$, by setting $C = \sqrt{\alpha}I > 0$ we have $\alpha(A\,\sigma\,B) = (\alpha A)\,\sigma\,(\alpha B)$. For each $n \in \mathbb{N}$, we have

$$nI(0\,\sigma\,0)nI \leqslant (nI)0(nI)\,\sigma\,(nI)0(nI) = 0\,\sigma\,0$$

and, hence, $0\,\sigma\,0 \leqslant (1/n^2)(0\,\sigma\,0)$. Taking $n \to \infty$ in the norm topology yields $0\,\sigma\,0 = 0$ by Proposition A.3(4). $\square$

Lemma 3.3. *If σ satisfies (M1) and (M2), then σ satisfies (P).*

Proof. Let P be a projection commuting with A and B. By Theorems A.7 and A.8, P commutes with $A^{1/2}$. Since $\mathrm{Sp}(P) \subseteq \{0,1\}$, we have $P \leqslant I$ by Theorem A.2 and hence

$$PAP = AP^2 = AP = A^{1/2}PA^{1/2} \leqslant A^{1/2}IA^{1/2} = A.$$

Similarly, $PBP \leqslant B$. By (M1) and (M2), we have

$$P(A\,\sigma\,B)P \leqslant (PAP)\,\sigma\,(PBP) \leqslant A\,\sigma\,B. \tag{3.1}$$

Consider $X \equiv (A\,\sigma\,B) - P(A\,\sigma\,B)P \geqslant 0$. Then

$$\left|X^{1/2}P\right|^2 = (X^{1/2}P)^*X^{1/2}P = PXP = P(A\,\sigma\,B)P - PP(A\,\sigma\,B)PP = 0.$$

Hence, $X^{1/2}P = 0$ and $XP = 0$, meaning that $[(A\,\sigma\,B) - P(A\,\sigma\,B)P]P = 0$ or $(A\,\sigma\,B)P = P(A\,\sigma\,B)P$. Similarly, $P(A\,\sigma\,B) = P(A\,\sigma\,B)P$.

From (3.1) and the fact that P commutes with PA and PB, we have

$$PP(A\,\sigma\,B)PP \leqslant P(PA\,\sigma\,PB)P = P(PA\,\sigma\,PB) \leqslant P(A\,\sigma\,B)P$$

and hence $P(A\,\sigma\,B)P = P(PA\,\sigma\,PB)$. Note that, by (M2), we have

$$(I-P)(PA\,\sigma\,PB)(I-P) \leqslant (I-P)PA(I-P)\,\sigma\,(I-P)PB(I-P) = 0\,\sigma\,0 = 0.$$

Now, since $I - P$ commutes with PA and PB, we get

$$PA\,\sigma\,PB = P(PA\,\sigma\,PB) + (I-P)(PA\,\sigma\,PB)(I-P) = P(PA\,\sigma\,PB).$$

Thus, $P(A\,\sigma\,B) = P(PA\,\sigma\,PB) = PA\,\sigma\,PB$. $\square$

Lemma 3.4. *Assume that $\sigma \in BO(M3')$ satisfies the positive homogeneity. If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $f(x)I = I\,\sigma\,(xI)$ for all $x \in \mathbb{R}^+$, then f is continuous.*

Proof. To show that f is right continuous at each $x \in \mathbb{R}^+$, consider a sequence $\{x_n\}$ in $\mathbb{R}^+$ such that $x_n \downarrow x$. Then by (M3′)

$$f(x_n)I = I\,\sigma\,(x_nI) \downarrow I\,\sigma\,(xI) = f(x)I,$$

i.e. $f(x_n) \downarrow f(x)$. To show that f is left continuous at each $x > 0$, consider a sequence $x_n > 0$ such that x_n is increasing and $x_n \to x$. Then $x_n^{-1} \downarrow x^{-1}$ and

$$\begin{aligned}\lim x_n^{-1} f(x_n)I &= \lim x_n^{-1}(I\,\sigma\,x_nI) = \lim(x_n^{-1}I)\,\sigma\,I = (x^{-1}I)\,\sigma\,I \\ &= x^{-1}(I\,\sigma\,xI) = x^{-1}f(x)I\end{aligned}$$

That is $t \mapsto x^{-1}f(x)$ is left continuous and so is f. □

Lemma 3.5. *Assume that $\sigma \in BO(M3')$ satisfies (P). If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function such that $f(x)I = I\,\sigma\,(xI)$ for all $x \in \mathbb{R}^+$, then $f(A) = I\,\sigma\,A$ for all $A \in B(\mathcal{H})^+$.*

Proof. First consider $A \in B(\mathcal{H})^+$ in the form $\sum_{i=1}^m \lambda_i P_i$ where $\{P_i\}_{i=1}^m$ is an orthogonal family of projections with sum I and $\lambda_i > 0$ for all $i = 1, \ldots, m$. Since each P_i commutes with A, we have by the property (P) that

$$\begin{aligned}I\,\sigma\,A &= \sum P_i(I\,\sigma\,A) = \sum P_i\,\sigma\,P_iA = \sum P_i\,\sigma\,\lambda_iP_i \\ &= \sum (I\,\sigma\,\lambda_iI)P_i = \sum f(\lambda_i)P_i = f(A).\end{aligned}$$

Now, consider $A \in B(\mathcal{H})^+$. Then there is a sequence $\{A_n\}$ of strictly positive operators in the above form such that $A_n \downarrow A$. By (M3′) and Theorem A.9, we have $I\,\sigma\,A = \lim I\,\sigma\,A_n = \lim f(A_n) = f(A)$. □

From now on, $\mathcal{H}$ is assumed to be an infinite-dimensional Hilbert space.

Proposition 3.6. *If $\sigma \in BO(M1, M2, M3')$, then there is a unique operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x)I = I\,\sigma\,(xI)$ for all $x \in \mathbb{R}^+$.*

Proof. Note that I and xI commutes with any projection $P \in B(\mathcal{H})$. By Lemma 3.3, $I\,\sigma\,(xI)$ commutes with any projection $P \in B(\mathcal{H})$. By Theorems A.7 and A.8, $I\,\sigma\,(xI)$ commutes with any self-adjoint operators. Note that any bounded linear operator T can be wriiten as $T = T_1 + iT_2$ for some self-adjoint operators T_1 and T_2. Hence, $I\,\sigma\,(xI)$ commutes with every bounded linear operator on $\mathcal{H}$. Since the center of $B(\mathcal{H})$ is trivial by Proposition A.5, $I\,\sigma\,(xI)$ is a scalar multiple of identity. Hence, there is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x)I = I\,\sigma\,(xI)$ for all $x \in \mathbb{R}^+$. Since σ is monotone, f is increasing. By Lemma 3.2, σ satisfies the positive homogeneity. Lemma 3.4 implies that f is continuous. We obtain from Lemma 3.5 and (M1) that for $A \leqslant B$ in $B(\mathcal{H})^+$,

$$f(A) = I\,\sigma\,A \leqslant I\,\sigma\,B = f(B).$$

Since $\mathcal{H}$ is of infinite dimensional, f is operator monotone by Theorem 2.11. If there is another $g \in OM(\mathbb{R}^+)$ such that $f(x)I = I\,\sigma\,(xI)$ for all $x \in \mathbb{R}^+$, then $f(x)I = I\,\sigma\,(xI) = g(x)I$ for each $x \in \mathbb{R}^+$, i.e. $f = g$. $\square$

Proposition 3.7. *Given an operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$, there exists a $\sigma \in BO(M1, M2, M3')$ on $B(\mathcal{H})^+$ such that $f(x)I = I\,\sigma\,(xI)$ for $x \in \mathbb{R}^+$.*

Proof. Let μ be the corresponding finite Borel measure on $[0,\infty]$ of the function f given by Theorem 2.12. We define a binary operation σ on $B(\mathcal{H})^+$ by

$$A\,\sigma\,B = \alpha A + \beta B + \int_{(0,\infty)} \frac{\lambda+1}{2\lambda}\{(\lambda A)\,!\,B\}d\mu(\lambda), \quad A, B \geqslant 0 \tag{3.2}$$

where $\alpha = \mu(\{0\})$, $\beta = \mu(\{\infty\})$ and the integral is the Bochner integral. Consider $A, B \geqslant 0$ and set $F_\lambda = \frac{\lambda+1}{2\lambda}(\lambda A\,!\,B)$. Since $A \leqslant \|A\|\,I$ and $B \leqslant \|B\|\,I$, we get

$$\lambda A\,!\,B \leqslant \lambda\,\|A\|\,I\,!\,\|B\|\,I = (\lambda\,\|A\|\,!\,\|B\|)I$$

and hence for any $\lambda > 0$

$$\|F_\lambda\| \leqslant \frac{\lambda+1}{2\lambda}(\lambda\,\|A\|\,!\,\|B\|) \leqslant \max\{\|A\|\,,\|B\|\} \equiv M.$$

It follows that

$$\int_{(0,\infty)} \|F_\lambda\|\;d\mu(\lambda) \leqslant \int_{(0,\infty)} M\,d\mu(\lambda) < \infty.$$

By Theorem A.10, F_λ is Bochner integrable. Since F_λ is a positive operator for all $\lambda > 0$, the operator $\int_{(0,\infty)} F_\lambda \, d\mu(\lambda)$ is also positive by Proposition A.14.

The midpoint concavity (M1) and the transformer inequality (M2) come from passing those properties of harmonic mean through the integral. To show (M3′), let $A, X \in B(\mathcal{H})^+$ and consider a sequence $A_n \in B(\mathcal{H})^+$ such that $A_n \downarrow A$. Then $\lambda A_n \,!\, X \downarrow \lambda A \,!\, X$ for $\lambda > 0$ by Theorem 2.4. The sequence $A_n \sigma X$ is decreasing by Proposition A.14. Let $\xi \in H$. Define a bounded linear map $\Phi : B(\mathcal{H}) \to \mathbb{C}$ by $\Phi(T) = \langle T\xi, \xi \rangle$. Put $T_\infty(\lambda) = \frac{\lambda+1}{2\lambda}(\lambda A \,!\, X)$ and set

$$T_n(\lambda) = \frac{\lambda+1}{2\lambda}(\lambda A_n \,!\, X), \quad n \in \mathbb{N}.$$

By Theorem A.11, $\Phi \circ T_n$ is Bochner integrable and

$$\langle \int T_n(\lambda)\, d\mu(\lambda)\xi, \xi \rangle = \Phi(\int T_n(\lambda)\, d\mu(\lambda)) = \int \Phi \circ T_n(\lambda)\, d\mu(\lambda)$$

for each $n \in \mathbb{N} \cup \{\infty\}$. Since $T_n(\lambda)$ converges strongly to $T_\infty(\lambda)$, we have that

$$\langle T_n(\lambda)\xi, \xi \rangle \to \langle T_\infty(\lambda)\xi, \xi \rangle, \quad \text{as } n \to \infty$$

for each $\lambda > 0$. We obtain from the dominated convergence theorem that

$$\begin{aligned}
\lim_{n\to\infty} \langle (A_n \,\sigma\, X)\xi, \xi \rangle &= \lim_{n\to\infty} \langle (\alpha A_n + \beta X)\xi, \xi \rangle + \lim_{n\to\infty} \langle \int T_n(\lambda)\, d\mu(\lambda)\xi, \xi \rangle \\
&= \langle (\alpha A + \beta X)\xi, \xi \rangle + \lim_{n\to\infty} \int \langle T_n(\lambda)\xi, \xi \rangle \, d\mu(\lambda) \\
&= \langle (\alpha A + \beta X)\xi, \xi \rangle + \int \langle T_\infty(\lambda)\xi, \xi \rangle \, d\mu(\lambda) \\
&= \langle (\alpha A + \beta X)\xi, \xi \rangle + \langle \int T_\infty(\lambda) d\mu(\lambda)\xi, \xi \rangle \\
&= \langle (A \,\sigma\, X)\xi, \xi \rangle.
\end{aligned}$$

Hence, $A_n \sigma X$ converges weakly to $A\sigma X$. By Theorem A.6, $A_n \,\sigma\, X \downarrow A \,\sigma\, X$. Similarly, $I \,\sigma\, A_n \downarrow I \,\sigma\, A$. Thus, σ is a connection on $B(\mathcal{H})^+$. A direct computation using Proposition A.12 shows that $I \,\sigma\, (xI) = f(x)I$ for $x \geqslant 0$. □

Proof of Theorem 3.1: Clearly, (i) ⇒ (ii) and (i) ⇒ (iii).

(ii) ⇒ (i). Our aim is to construct a bijection from $BO(M1, M2, M3')$ to $OM(\mathbb{R}^+)$. By Proposition 3.6, there is a function

$$\sigma \in BO(M1, M2, M3') \mapsto f \in OM(\mathbb{R}^+)$$

such that $f(x)I = I\,\sigma\,(xI)$ for all $x \in \mathbb{R}^+$. This map is surjective by Proposition 3.7. To show the injectivity of this map, let $\sigma_1, \sigma_2 \in BO(M1, M2, M3')$ be such that $\sigma_i \mapsto f$ where, for each $t \geqslant 0$,

$$I\,\sigma_i\,(xI) = f(x)I, \quad i = 1, 2.$$

Since σ_i satisfies the property (P) by Lemma 3.3, we have $I\,\sigma_i\,A = f(A)$ for $A \geqslant 0$ by Lemma 3.5. Lemma 3.2 assures that σ_1 and σ_2 satisfy the congruence invariance. Then, for $A > 0$ and $B \geqslant 0$,

$$A\,\sigma_i\,B = A^{1/2}(I\,\sigma_i\,A^{-1/2}BA^{-1/2})A^{1/2} = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}.$$

For each $A, B \geqslant 0$, we obtain by (M3′) that

$$\begin{aligned} A\,\sigma_1\,B &= \lim_{\epsilon\downarrow 0} A_\epsilon\,\sigma_1\,B = \lim_{\epsilon\downarrow 0} A_\epsilon^{1/2}(I\,\sigma_1\,A_\epsilon^{-1/2}BA_\epsilon^{-1/2})A_\epsilon^{1/2} \\ &= \lim_{\epsilon\downarrow 0} A_\epsilon^{1/2}f(A_\epsilon^{-1/2}BA_\epsilon^{-1/2})A_\epsilon^{1/2} \\ &= \lim_{\epsilon\downarrow 0} A_\epsilon^{1/2}(I\,\sigma_2\,A_\epsilon^{-1/2}BA_\epsilon^{-1/2})A_\epsilon^{1/2} \\ &= \lim_{\epsilon\downarrow 0} A_\epsilon\,\sigma_2\,B = A\,\sigma_2\,B, \end{aligned}$$

where $A_\epsilon \equiv A + \epsilon I$. That is $\sigma_1 = \sigma_2$.

Thus, there is a bijection between $OM(\mathbb{R}^+)$ and $BO(M1, M2, M3')$. Every element in $BO(M1, M2, M3')$ has an integral representation (3.2). Since the harmonic mean possesses (M3) by Theorem 2.4, so is any element in $BO(M1, M2, M3')$.

(iii) $\Rightarrow$ (i). We can develop the similar results when (M3′) is replaced by (M3″) by swapping "left" and "right." Indeed, given $\sigma \in BO(M1, M2, M3'')$ there is a unique $f \in OM(\mathbb{R}^+)$ such that

$$f(x)I = (xI)\,\sigma\,I, \quad x \in \mathbb{R}^+.$$

On the other hand, given $f \in OM(\mathbb{R}^+)$, we construct σ by setting

$$A\,\sigma\,B = \alpha B + \beta A + \int_{(0,\infty)} \frac{\lambda + 1}{2\lambda}\{A\,!\,(\lambda B)\}d\mu(\lambda)$$

where μ is the corresponding measure of f, $\alpha = \mu(\{0\})$ and $\beta = \mu(\{\infty\})$. $\square$

Remark 3.8. In the proof of (iii) $\Rightarrow$ (i) in Theorem 3.1, σ corresponds to the representing function of the transpose of σ. In fact, there is a one-to-one correspondence between operator monotone functions f on $\mathbb{R}^+$ and operator monotone functions on $\mathbb{R}^+$ in the form $x \mapsto xf(1/x)$. The representing function of a connection σ in Kubo-Ando theory can be shown to be the function $f \in OM(\mathbb{R}^+)$ satisfying one of the following equivalent conditions for each $x \in \mathbb{R}^+$:

(i) $f(x)I = I\,\sigma\,(xI)$;

(ii) $f(x)P = P\,\sigma\,(xP)$ for all projection P on $\mathcal{H}$;

(iii) $f(x)A = A\,\sigma\,(xA)$ for all $A > 0$;

(iv) $f(x)A = A\,\sigma\,(xA)$ for all $A \geqslant 0$.

3.2 Axiomatic Characterizations of Connections

A connection can be axiomatically defined as follows. Fix the transformer inequality (M2). We can freely replace the monotonicity (M1) by the concavity (M4) or the midpoint concavity (M4′). At the same time, we can use (M3′) or (M3″) instead of the joint-continuity (M3).

Theorem 3.9. *Let σ be a binary operation on $B(\mathcal{H})^+$ satisfying (M2). Then the followings are equivalent:*

(i) σ satisfies (M1) and (M3), i.e., σ is a connection;

(ii) σ satisfies (M4) and (M3);

(iii) σ satisfies (M4) and (M3′);

(iv) σ satisfies (M4) and (M3″);

(v) σ satisfies (M4′) and (M3);

(vi) σ satisfies (M4′) and (M3′);

(vii) σ satisfies (M4′) and (M3″).

In order to prove this theorem, we use the following lemmas.

Lemma 3.10. *If $\sigma \in BO(M2, M4')$, then for each $A, B, C, D \geqslant 0$,*

(1) $(A \,\sigma\, B) + (C \,\sigma\, D) \leqslant (A + C) \,\sigma\, (B + D)$*;*

(2) $A \leqslant B$ *implies* $A \,\sigma\, I \leqslant B \,\sigma\, I$ *and* $I \,\sigma\, A \leqslant I \,\sigma\, B$*.*

Proof. As in Lemma 3.2, (M2) implies the positive homogeneity. The fact (2) follows from the midpoint concavity (M4′) and positive homogeneity. If $A \leqslant B$, then by (1),

$$I \,\sigma\, B = (I + 0) \,\sigma\, (A + B - A) \geqslant (I \,\sigma\, A) + (0 \,\sigma\, (B - A)) \geqslant I \,\sigma\, A$$

and similarly $B \,\sigma\, I \geqslant A \,\sigma\, I$. □

Lemma 3.11. *If $\sigma \in BO(M2, M4')$, then σ satisfies (P).*

Proof. Let P be a projection such that $AP = PA$ and $BP = PB$. We have $A = PAP + (I - P)A(I - P)$ and $B = PBP + (I - P)B(I - P)$. Then by Lemma 3.10(1) and (M2)

$$A \,\sigma\, B \geqslant (PAP \,\sigma\, PBP) + ((I - P)A(I - P) \,\sigma\, (I - P)B(I - P)) \tag{3.3}$$

$$\geqslant P(A \,\sigma\, B)P + (I - P)(A \,\sigma\, B)(I - P). \tag{3.4}$$

Consider $C \equiv A \,\sigma\, B - P(A \,\sigma\, B)P - (I - P)(A \,\sigma\, B)(I - P) \geqslant 0$. We have

$$PCP = 0 = (I - P)C(I - P)$$

which implies $C^{1/2}P = 0 = C^{1/2}(I - P)$. Hence, $CP = 0 = C(I - P)$ and $C = 0$, meaning that

$$A \,\sigma\, B = P(A \,\sigma\, B)P + (I - P)(A \,\sigma\, B)(I - P).$$

It follows that $P(A \,\sigma\, B) = P(A \,\sigma\, B)P = (A \,\sigma\, B)P$. The inequalities (3.3) and (3.4) become equalities, which implies $P(A \,\sigma\, B)P = (PAP) \,\sigma\, (PBP) = (PA) \,\sigma\, (PB)$. □

Lemma 3.12. *If $\sigma \in BO(M2, M4')$, then there is a unique binary operation $\tilde{\sigma}$ on $\mathbb{R}^+$ subject to the same properties and*

$$(xI)\,\sigma\,(yI) = (x\,\tilde{\sigma}\,y)I, \quad x, y \in \mathbb{R}^+. \tag{3.5}$$

Proof. Note that any projection on $\mathcal{H}$ commutes with xI and yI for any $x, y \in \mathbb{R}^+$. By Lemma 3.11, $(xI)\,\sigma\,(yI)$ commutes with any projection in $B(\mathcal{H})$. Theorems A.7 and A.8 imply that $(xI)\,\sigma\,(yI)$ commutes with every bounded linear operators on $\mathcal{H}$. It follows from Proposition A.5 that there exists a $k \in \mathbb{R}^+$ such that $(xI)\,\sigma\,(yI) = kI$. If there is a $k' \in \mathbb{R}^+$ such that $(xI)\,\sigma\,(yI) = k'I$, then $k' = k$. Hence, each connection σ on $B(\mathcal{H})^+$ induces a unique binary operation $\tilde{\sigma} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (3.5). It is routine to check that $\tilde{\sigma}$ satisfies (M2) and (M4′). □

Proposition 3.13. *If $\sigma \in BO(M2, M3', M4')$, then there is a unique $f \in OM(\mathbb{R}^+)$ such that $f(x)I = I\,\sigma\,(xI)$ for all $x \in \mathbb{R}^+$. In fact, $f(x) = 1\,\tilde{\sigma}\,x$ for all $x \in \mathbb{R}^+$.*

Proof. Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $x \mapsto 1\,\tilde{\sigma}\,x$ by Lemma 3.12. If $0 \leqslant t_1 \leqslant t_2$, then Lemma 3.10(2) implies

$$f(x_1)I = I\,\sigma\,(x_1 I) \leqslant I\,\sigma\,(x_2 I) = f(x_2)I,$$

i.e. $f(x_1) \leqslant f(x_2)$. The continuity of f is assured by Lemma 3.4. Then Lemma 3.5 implies $f(A) = I\sigma A$ for all $A \geqslant 0$. If $A, B \in B(\mathcal{H})^+$ are such that $A \leqslant B$, then $f(A) = I\,\sigma\,A \leqslant I\,\sigma\,B = f(B)$, again by Lemma 3.10(2). Since $\mathcal{H}$ is infinite dimensional, f is operator monotone by Theorem 2.11. □

Proof of Theorem 3.9: By Corollary 2.16, (i) implies (ii)-(vii). It suffices to show that (vi) implies (i). Assume that $\sigma \in BO(M2, M3', M4')$. Our aim is to construct a bijection between $BO(M2, M3', M4')$ and $OM(\mathbb{R}^+)$. Proposition 3.13 assures that the map $\sigma \in BO(M2, M3', M4') \mapsto f \in OM(\mathbb{R}^+)$, where $f(x)I = I\sigma(xI)$ for all $x \in \mathbb{R}^+$, is well-defined. This map is surjective via the same method as the construction in Theorem 3.1. The injectivity of this map can be proved by using the same argument as the proof of that in Theorem 3.1, here the property

(P) of $\sigma \in BO(M2, M3', M4')$ is fulfilled by Lemma 3.11. Hence, we are allowed to consider only the binary operations constructed from operator monotone functions on $\mathbb{R}^+$. Thus, σ admits an integral representation (3.2). By passing the properties (M1) and (M3) of the harmonic mean through the integral representation, σ also satisfies those properties. □

3.3 Characterizations of Means

Recall that a (Kubo-Ando) mean is a connection σ satisfying $I\sigma I = I$ or, equivalenthly, the fixed point property: $A\,\sigma\,A = A$ for all $A \geqslant 0$. According to the definition of mean for positive real numbers in [30], a mean M is defined to be a binary operation $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ satisfying

- *betweenness*: $x \leqslant y \implies x \leqslant M(x,y) \leqslant y$.

In fact, the betweenness property is a necessary and sufficient condition for a connection to be a mean:

Theorem 3.14. *The followings are equivalent for a connection σ with representing function f:*

(i) *σ is a mean;*

(ii) *σ satisfies the* betweenness property, *i.e., $A \leqslant B \implies A \leqslant A\,\sigma\,B \leqslant B$;*

(iii) $0 \leqslant A \leqslant I \implies A \leqslant A\,\sigma\,I \leqslant I$;

(iv) $I \leqslant A \implies I \leqslant I\,\sigma\,A \leqslant A$;

(v) $0 \leqslant x \leqslant 1 \implies x \leqslant f(x) \leqslant 1$;

(vi) $1 \leqslant x \implies 1 \leqslant f(x) \leqslant x$.

Proof. (i) ⇒ (ii). Use the fixed point property and the monotonicity.

(ii) ⇒ (i). We have $I \leqslant I\sigma I \leqslant I$, i.e., $I\sigma I = I$.

(ii) ⇒ (iii). Clear.

(iii) $\Rightarrow$ (ii). Consider $0 \leqslant A \leqslant B$ with $B > 0$. Since $B^{-1/2}AB^{-1/2} \leqslant I$, we get

$$B^{-1/2}AB^{-1/2} \leqslant B^{-1/2}AB^{-1/2} \,\sigma\, I \leqslant I.$$

The transformer inequality implies that $A \leqslant A \,\sigma\, B \leqslant B$. Now, assume that $0 \leqslant A \leqslant B$. Then for all $\epsilon > 0$, $A \leqslant B + \epsilon I$ and hence, by the previous claim,

$$A \leqslant A \,\sigma\, (B + \epsilon I) \leqslant B + \epsilon I.$$

Hence, $A \leqslant A \,\sigma\, B \leqslant B$ by the continuity from above.

(ii) $\Leftrightarrow$ (iv). It is similar to (ii) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (v). If $x \geqslant 1$, then $I \leqslant I \,\sigma\, (xI) \leqslant xI$ which is $I \leqslant f(x)I \leqslant xI$, i.e. $1 \leqslant f(x) \leqslant x$.

(v) $\Rightarrow$ (i). We have $f(1) = 1$.

(i) $\Leftrightarrow$ (vi). It is similar to (i) $\Leftrightarrow$ (v). □

Hence, every (Kubo-Ando) mean satisfies all desired properties in Section 2.2.

Remark 3.15. For a connection σ and $A, B \geqslant 0$, the operators A, B and $A \,\sigma\, B$ need not be comparable. The previous theorem tells us that if σ is a mean, then the condition $0 \leqslant A \leqslant B$ guarantees the comparability between A, B and $A \sigma B$.

3.4 Induced Connections

Each connection σ on $B(\mathcal{H})^+$, thanks to Lemma 3.12, induces a unique connection $\tilde{\sigma}$ on $\mathbb{R}^+ = B(\mathbb{C})^+$ satisfying

$$(x \,\tilde{\sigma}\, y)I = (xI) \,\sigma\, (yI), \quad x, y \in \mathbb{R}^+.$$

We call $\tilde{\sigma}$ the *induced connection* of σ.

Proposition 3.16. *Let $x, y \in \mathbb{R}^+$. If $x > 0$, then $x \,\tilde{\sigma}\, y = xf(y/x)$. If $y > 0$, then $x \,\tilde{\sigma}\, y = yf(x/y)$.*

Proof. Use the positive homogeneity of σ and Lemma 3.5. □

We can restate Proposition 3.13 as follows.

Proposition 3.17. *Each connection σ on $B(\mathcal{H})^+$ gives rise to an operator monotone function $x \mapsto 1\,\tilde{\sigma}\,x$ on $\mathbb{R}^+$. Moreover, any operator monotone function on $\mathbb{R}^+$ arises in this form.*

Theorem 3.18. *The map $\sigma \mapsto \tilde{\sigma}$ from the connections on $B(\mathcal{H})^+$ to the connections on $\mathbb{R}^+$ such that*

$$(x\,\tilde{\sigma}\,y)I = (xI)\,\sigma\,(yI), \quad x, y \in \mathbb{R}^+ \tag{3.6}$$

is an affine order-isomorphism. Hence, there exists an affine order isomorphism between the set of connections for positive operators acting on different infinite-dimensional Hilbert spaces.

Proof. To show that this map is surjective, let η be a connection on $\mathbb{R}^+$. Then the function $f(x) = 1\,\eta\,x$ is operator monotone on $\mathbb{R}^+$ by Proposition 3.17. Theorem 2.13 implies that there is a connection σ on $B(\mathcal{H})^+$ such that $f(x)I = I\sigma(xI)$ for all $x \in \mathbb{R}^+$. For $x, y > 0$ we have by Proposition 3.16 that

$$(x\,\eta\,y)I = x(1\,\eta\,(y/x))I = xf(y/x)I = x(I\,\sigma\,(y/x)I) = (xI)\,\sigma\,(yI).$$

Hence $(x\,\eta\,y)I = (xI)\,\sigma\,(yI)$ for all $x, y \in \mathbb{R}^+$ by continuity.

Now, suppose $\sigma_i \mapsto \eta$ for $i = 1, 2$. Let f_i be the representing function of σ_i for $i = 1, 2$. Then for $x \in \mathbb{R}^+$

$$f_1(x)I = I\,\sigma_1\,(xI) = (1\,\eta\,x)I = I\,\sigma_2\,(xI) = f_2(x)I,$$

i.e. $f_1 = f_2$. Hence, $\sigma_1 = \sigma_2$ by Theorem 2.13.

It is straightforward to check that the map $\sigma \mapsto \tilde{\sigma}$ and its inverse are affine (i.e. it preserves nonnegative linear combinations) and order-preserving. $\square$

Corollary 3.19. *A connection on $B(\mathcal{H})^+$ and its induced connection have the same representing function, the same representing measure and the same formula. More precisely, given an operator monotone function*

$$f(x) = \alpha + \beta x + \int_{(0,\infty)} \frac{\lambda + 1}{2\lambda}(\lambda \,!\, t)\, d\mu(\lambda), \tag{3.7}$$

then for each $A, B \in B(\mathcal{H})^+$ and $x, y \in \mathbb{R}^+$,

$$A\,\sigma\,B = \alpha A + \beta B + \int_{(0,\infty)} \frac{\lambda+1}{2\lambda} (\lambda A \,!\, B)\, d\mu(\lambda), \tag{3.8}$$

$$x\,\tilde{\sigma}\,y = \alpha x + \beta y + \int_{(0,\infty)} \frac{\lambda+1}{2\lambda} (\lambda x \,\tilde{!}\, y)\, d\mu(\lambda), \tag{3.9}$$

where $\tilde{!}$ is the harmonic mean on $\mathbb{R}^+$.

Proof. Let σ be a connection and $\tilde{\sigma}$ its induced connection. Then the correspondences between connections, induced connections, finite Borel measures and operator monotone functions imply that σ and $\tilde{\sigma}$ have the same representing function and the same representing measure. Then σ has the integral representation (3.8). The formula (3.9) of $\tilde{\sigma}$ can be computed by using Proposition 3.16. $\square$

From this corollary, a connection and its induced connection can be written by the same notation.

Corollary 3.20. *A connection is a mean if and only if the induced connection is a mean on $\mathbb{R}^+$.*

Recall that the class of means on $B(\mathcal{H})^+$ is a convex set. We say that a map between convex sets is convex if it preserves convex combinations.

Corollary 3.21. *Given an infinite-dimensional Hilbert space $\mathcal{H}$, the map $\sigma \mapsto \tilde{\sigma}$ is a convex order-isomorphism from the means on $B(\mathcal{H})^+$ to the means on $\mathbb{R}^+$. Hence, there exists a convex order-isomorphism between the means on the positive operators acting on different infinite-dimensional Hilbert spaces.*

Proof. It is an immediate consequence of Theorem 3.18 and Theorem 3.20. $\square$

Remark 3.22. According to Corollary 3.21, we can naturally define any named means on $B(\mathcal{H})^+$ to be the corresponding ones on $\mathbb{R}^+$. On the other hand, there is one and only one "good" such extension from means on $\mathbb{R}^+$ to Kubo-Ando means on $B(\mathcal{H})^+$.

For example, the binary operation

$$\sigma : (A, B) \mapsto \frac{1}{2}(A^{1/4}B^{1/2}A^{1/4} + B^{1/4}A^{1/2}B^{1/4})$$

satisfies $A\sigma B = (AB)^{1/2}$, provided that $AB = BA$. The *spectral geometric mean* of two positive definite matrices A, B is defined in [14] by

$$(A^{-1}\#B)^{1/2}A(A^{-1}\#B)^{1/2}.$$

Both of them can be viewed as extensions of the geometric mean on $\mathbb{R}^+$. However, this corollary says that they are not Kubo-Ando means. In fact, they lack the monotonicity.

CHAPTER IV

CONNECTIONS AND BOREL MEASURES

In this chapter, we investigate the relationship between connections and its representing measures. We use the decomposition of a measure in order to decompose a connection and also study properties of each part in such decomposition. As an illustation, we compute the representing measures of connections and means in practical usage.

4.1 Representing Measures

Recall that there is a one-to-one correspondence between connections and finite Borel measures on the extended half-line. Let us compute the representing measures of practical operator connections.

Example 4.1. For each $0 < \lambda < \infty$, the binary operation σ_λ defined by

$$A \,\sigma_\lambda\, B = \frac{\lambda+1}{2\lambda}(\lambda A \,!\, B), \quad A, B \geqslant 0,$$

is a mean since its representing function is the normalized operator monotone function

$$\phi_\lambda(x) = \frac{x(1+\lambda)}{x+\lambda} \quad \text{for } x > 0, \quad \phi_\lambda(0) = 1.$$

Note that $\phi_0(x) \equiv \lim_{\lambda \to 0^+} \phi_\lambda(x) = 1$ and $\phi_\infty(x) \equiv \lim_{\lambda\to\infty} \phi_\lambda(x) = x$ for each $x \in \mathbb{R}^+$. Hence, we denote σ_0 and σ_∞ to be the left- and the right-trivial means, respectively. The mean σ_λ has the representing measure given by the Dirac measure concentrated at λ:

$$\delta_\lambda(E) = \begin{cases} 1, & \lambda \in E; \\ 0, & \lambda \notin E, \end{cases}$$

for each Borel set E in $[0,\infty]$. Indeed, for each $\lambda \in [0,\infty]$,

$$\int_{[0,\infty]} A\,\sigma_t\,B\,d\delta_\lambda(t) = \int_{\{\lambda\}} A\,\sigma_t\,B\,d\delta_\lambda(t) = A\,\sigma_\lambda\,B.$$

In particular, the representing measures of the left- and the right-trivial means are given respectively by δ_0 and δ_∞. The representing measure of the harmonic mean is δ_1. By affinity of the map $\sigma \mapsto \mu$, the representing measures of the sum and the parallel sum are given by $\delta_0 + \delta_\infty$ and $\frac{1}{2}\delta_1$, respectively.

The means σ_λ's for $\lambda \in [0,\infty]$ are extreme points of the convex sets of means on $B(\mathcal{H})^+$. This result follows from the fact that the Dirac measures δ_λ's are extreme points of the convex set of probability Borel measures on $[0,\infty]$.

Example 4.2. The representing measure of the α-weighted arithmetic mean is given by $(1-\alpha)\delta_0+\alpha\delta_\infty$ for each $\alpha \in [0,1]$. More generally, the measure $\sum_{i=1}^n a_i\,\delta_{t_i}$, where $t_i \in [0,\infty]$ and $a_i \geqslant 0$, represents the connection $\sum_{i=1}^n a_i\,\sigma_{t_i}$. In particular, the probability measure $(1-\alpha)\delta_t + \alpha\delta_s$, when $\alpha \in [0,1]$ and $s,t \in [0,\infty]$, is associated to the α-weighted arithmetic mean between σ_s and σ_t.

Example 4.3. Consider the representing measure of the α-weighted geometric mean for $0 < \alpha < 1$. From contour integration, we have

$$x^\alpha = \int_{[0,\infty]} \frac{x\lambda^{\alpha-1}}{x+\lambda} \cdot \frac{\sin\alpha\pi}{\pi}\,d\lambda.$$

Hence, the weighted geometric mean $\#_\alpha$ has the integral representation

$$A \,\#_\alpha\, B = \int_{[0,\infty]} A\,\sigma_\lambda\,B\,d\mu(\lambda)$$

where the representing measure μ is given by

$$d\mu(\lambda) = \frac{\sin\alpha\pi}{\pi} \cdot \frac{\lambda^{\alpha-1}}{1+\lambda}\,dm(\lambda).$$

Here, m denotes Lebesgue measure on $\mathbb{R}$.

Remark 4.4. Even though the map $\mu \mapsto \sigma$ is order-preserving, the inverse map $\sigma \mapsto \mu$ is not order-preserving in general. For example, the representing measures of the harmonic mean $!$ and the arithmetic mean $\triangledown$ are given by δ_1 and $(\delta_0+\delta_\infty)/2$, respectively. We have $! \leqslant \triangledown$ but $\delta_1 \nleqslant (\delta_0+\delta_\infty)/2$.

4.2 Decomposition of Connections and Means

In this section, we decompose arbitrary connection into three parts using decomposition of Borel measures on the extended half-line $[0, \infty]$.

Recall that a complex Borel measure μ on $\mathbb{R}^d$ is called *discrete* if there are countable family $\{x_n\}$ in $\mathbb{R}^d$ and $\{c_n\}$ in $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \text{and} \quad \mu = \sum_{n=1}^{\infty} c_n \delta_{x_n}.$$

The second condition means that $\mu(E) = \sum_{n=1}^{\infty} c_n \delta_{x_n}(E)$ for each Borel set $E \subseteq [0, \infty]$. On the other hand, μ is called *continuous* if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^d$. We have the following facts:

- Any complex Borel measure μ on $\mathbb{R}^d$ can be written uniquely as $\mu = \mu_d + \mu_c$ where μ_d is discrete and μ_c is continuous.
- If μ is discrete, then μ is mutually singular to Lebesgue measure m.
- If $\mu \ll m$, i.e. μ is absolutely continuous with respect to m, then μ is continuous.

Using these facts and Radon-Nikodym theorem, we have:

- Any complex Borel measure μ on $\mathbb{R}^d$ can be written uniquely as

  $$\mu = \mu_{sd} + \mu_{ac} + \mu_{sc}$$

  where μ_d is discrete, $\mu_{ac} \ll m$ and μ_{sc} is *singularly continuous*, i.e. μ_{sc} is a continuous measure mutually singular to m.

Note that a Borel measure μ on $\mathbb{R}^+$ can be uniquely extended to a Borel measure on $[0, \infty]$ by setting

$$\mu(E) = \begin{cases} \mu(E), & \infty \notin E; \\ \mu(E - \{\infty\}) + \mu(\{\infty\}), & \infty \in E \end{cases}$$

for each Borel set E.

Theorem 4.5. *Let σ be a connection on $B(\mathcal{H})^+$. Then there is a unique triple $(\sigma_{ac}, \sigma_{sc}, \sigma_{sd})$ of connections on $B(\mathcal{H})^+$ such that*

$$\sigma = \sigma_{ac} + \sigma_{sc} + \sigma_{sd} \tag{4.1}$$

and

(1) there are a countable set $D \subset [0, \infty]$ and a family $\{a_\lambda\}_{\lambda \in D} \subset \mathbb{R}^+$ such that $\sum_{\lambda \in D} a_\lambda < \infty$ and

$$\sigma_{sd} = \sum_{\lambda \in D} a_\lambda \sigma_\lambda,$$

i.e. for each $A, B \geqslant 0$, $A \,\sigma_{sd}\, B = \sum_{\lambda \in D} a_\lambda (A \,\sigma_\lambda B)$ and the series converges in the norm topology;

(2) there is a (unique m-a.e.) integrable function $g : [0, \infty] \to \mathbb{R}^+$ such that

$$A \,\sigma_{ac}\, B = \int_{(0,\infty)} g(\lambda)(A \,\sigma_\lambda\, B)\, dm(\lambda); \tag{4.2}$$

(3) its representing measure of σ_{sc} is a continuous measure mutually singular to m.

Moreover, the representing functions of σ_{ac}, σ_{sd} and σ_{sc} are given respectively by

$$\begin{aligned} f_{ac}(x) &= \int_{[0,\infty]} \phi_\lambda(x)\, d\mu_{ac}(\lambda), \\ f_{sd}(x) &= \int_{[0,\infty]} \phi_\lambda(x)\, d\mu_{sd}(\lambda) = \sum_{\lambda \in D} a_\lambda \phi_\lambda(x) \\ f_{sc}(x) &= \int_{[0,\infty]} \phi_\lambda(x)\, d\mu_{sc}(\lambda) \end{aligned}$$

and the representing measure of σ_{sd} is given by $\sum_{\lambda \in D} a_\lambda \,\delta_\lambda$.

Proof. Let μ be the representing measure of σ. Then there is a unique triple $(\mu_{ac}, \mu_{sc}, \mu_{sd})$ of finite Borel measures on $[0, \infty]$ such that $\mu = \mu_{ac} + \mu_{sc} + \mu_{sd}$ where μ_{sd} is a discrete measure, $\mu_{ac} \ll m$ and μ_{sc} is a continuous measure mutually singular to m. Define

$$A\,\sigma_{ac}\,B = \int_{[0,\infty]} A\,\sigma_\lambda\,B\,d\mu_{ac}(\lambda),$$

$$A\,\sigma_{sc}\,B = \int_{[0,\infty]} A\,\sigma_\lambda\,B\,d\mu_{sc}(\lambda),$$

$$A\,\sigma_{sd}\,B = \int_{[0,\infty])} A\,\sigma_\lambda\,B\,d\mu_{sd}(\lambda) = \sum_{\lambda\in D} a_\lambda(A\,\sigma_\lambda\,B)$$

for each $A, B \geqslant 0$. The series $\sum_{\lambda\in D} a_\lambda(A\,\sigma_\lambda\,B)$ converges in norm. Indeed, the fact that, for each $n < m$ in $\mathbb{N}$ and $t_i \in [0,\infty]$,

$$\begin{aligned} \left\| \sum_{i=1}^{n} a_{t_i}(A\,!_{t_i}\,B) - \sum_{i=1}^{m} a_{t_i}(A\,!_{t_i}\,B) \right\| &\leqslant \sum_{i=n+1}^{m} a_{t_i}\,\|A\,!_{t_i}\,B\| \\ &\leqslant \sum_{i=n+1}^{m} a_{t_i}(\|A\|\,!_{t_i}\,\|B\|) \\ &\leqslant \sum_{i=n+1}^{m} a_{t_i}\max\{\|A\|\,,\|B\|\} \end{aligned}$$

together with the convergence of $\sum_{i=1}^{\infty} a_{t_i}$ implies the convergence of the series $\sum_{i=1}^{\infty} a_{t_i}(A\,!_{t_i}\,B)$. The one-to-one correspondence between operator monotone functions on $\mathbb{R}^+$ and finite Borel measures on $[0,\infty]$ (Theorem 2.14) shows that the representing measures of σ_{ac}, σ_{sd} and σ_{sc} are given by μ_{ac}, μ_{sd} and μ_{sc}, respectively. The condition (1) comes from the fact that the representing measure of σ_λ is δ_λ for each $\lambda \in [0,\infty]$ in Example 4.1. The condition $\mu_{ac} \ll m$ means precisely the condition (2) by Radon-Nikodym theorem. The decomposition (4.1) is unique since the decomposition of the representing measure is unique. $\square$

This theorem says that every connection σ consists of three parts. The "singularly discrete part" σ_{sd} is a countable sum of means in the form σ_λ. Such type of connections include the weighted arithmetic means, the weighted harmonic means, the sum and the parallel sum. The "absolutely continuous part" σ_{ac} arises as an integral with respect to Lebesgue measure, given by the formula (4.2). Example 4.3 shows that weighted geometric means are typical examples of such connections. The "singularly continuous part" σ_{sc} has an integral representation with respect to a continuous measure mutually singular to Lebesgue measure. Examples of such measures correspond to nonconstant continuous functions $F : \mathbb{R} \to \mathbb{C}$

such that F is of bounded variation, $F(-\infty) = 0$ and $F' = 0$ almost everywhere. One such function is the Cantor function (which gives rise to the Cantor measure). Hence (aside singularly continuous part) this theorem gives an explicit description of arbitrary connections.

Proposition 4.6. *The connection σ_{ac} defined by* (4.2) *is a mean if and only if the average of a density function g on $[0,\infty]$ is* 1*, i.e.*

$$\int_{[0,\infty]} g(\lambda)\, d\lambda = 1.$$

Proof. Use the fact that a connection σ is a mean if and only if $I \sigma I = I$. ☐

The next result is a decomposition of means as a convex combination of means.

Corollary 4.7. *Let σ be a mean on $B(\mathcal{H})^+$. Then there are unique triples $(\widetilde{\sigma_{ac}}, \widetilde{\sigma_{sc}}, \widetilde{\sigma_{sd}})$ of means or zero connections on $B(\mathcal{H})^+$ and (k_{ac}, k_{sc}, k_{sd}) of real numbers in $[0,1]$ such that*

$$\sigma = k_{ac}\widetilde{\sigma_{ac}} + k_{sc}\widetilde{\sigma_{sc}} + k_{sd}\widetilde{\sigma_{sd}}, \quad k_{ac} + k_{sc} + k_{sd} = 1$$

and

(1) *there are a countable set $D \subset [0,\infty]$ and a family $\{a_\lambda\}_{\lambda \in D} \subseteq \mathbb{R}^+$ such that $\sum_{\lambda \in D} a_\lambda = 1$ and $\widetilde{\sigma_{sd}} = \sum_{\lambda \in D} a_\lambda\, \sigma_\lambda$;*

(2) *there is a (unique m-a.e.) integrable function $g : [0,\infty] \to \mathbb{R}^+$ with average* 1 *such that*

$$A\,\widetilde{\sigma_{ac}}\,B = \int_{[0,\infty]} g(\lambda)(A\,\sigma_\lambda\,B)\,d\lambda, \quad A, B \geqslant 0;$$

(3) *its associated measure of $\widetilde{\sigma_{sc}}$ is continuous and mutually singular to m.*

Proof. Let μ be the associated probability measure of $\sigma = \sigma_{ac} + \sigma_{sd} + \sigma_{sc}$ and write $\mu = \mu_{ac} + \mu_{sd} + \mu_{sc}$. Suppose that μ_{ac}, μ_{sd} and μ_{sc} are nonzero. Then

$$\mu = \mu_{ac}([0,\infty])\frac{\mu_{ac}}{\mu_{ac}([0,\infty])} + \mu_{sd}([0,\infty])\frac{\mu_{sd}}{\mu_{sd}([0,\infty])} + \mu_{sc}([0,\infty])\frac{\mu_{sc}}{\mu_{sc}([0,\infty])}.$$

Set

$$\begin{aligned}
\widetilde{\mu_{ac}} &= \frac{\mu_{ac}}{\mu_{ac}([0,\infty])}, \quad \widetilde{\mu_{sd}} = \frac{\mu_{sd}}{\mu_{sd}([0,\infty])}, \quad \widetilde{\mu_{sc}} = \frac{\mu_{sc}}{\mu_{sc}([0,\infty])}, \\
k_{ac} &= \mu_{ac}([0,\infty]), \quad k_{sd} = \mu_{sd}([0,\infty]), \quad k_{sc} = \mu_{sc}([0,\infty]).
\end{aligned}$$

Define $\widetilde{\sigma_{ac}}, \widetilde{\sigma_{sd}}, \widetilde{\sigma_{sc}}$ to be the means corresponding to the measures $\widetilde{\mu_{ac}}, \widetilde{\mu_{sd}}, \widetilde{\mu_{sc}}$, respectively. Now, apply Theorem 4.5 and Proposition 4.6. $\square$

CHAPTER V

STRUCTURES OF THE SET OF CONNECTIONS

We investigate the algebraic, order and topological structure of the set of connections. It is shown that this set is a normed ordered cone. In fact, it is isometrically order-isomorphic to the set of operator monotone functions on $\mathbb{R}^+$. Moreover, it is isometrically isomorphic, as normed cones, to the set of finite Borel measures on $[0, \infty]$.

5.1 Algebraic and Order Structures

Definition 5.1. A *cone* is a set C endowed with an addition $+ : C \times C \to C$ and a scalar multiplication $\cdot : \mathbb{R}^+ \times C \to C$ such that

(i) $(C, +)$ is a commutative monoid: the addition is associative, commutative and admits a neutral element 0 such that $x + 0 = x$ for all $x \in C$.

(ii) For each $x, y \in C$ and $r, s \in \mathbb{R}^+$,

$$r \cdot (x + y) = r \cdot x + r \cdot y,$$

$$(r + s) \cdot x = r \cdot x + s \cdot x,$$

$$(rs) \cdot x = r \cdot (s \cdot x),$$

$$1 \cdot x = x,$$

$$0 \cdot x = 0.$$

For convenience, we write rx instead of $r \cdot x$ for $r \in \mathbb{R}^+$ and $x \in C$.

A cone C is called

- *strict* if for each $x, y \in C$, $x + y = 0 \implies x = y = 0$,
- *cancellative* if for each $x, y, z \in C$, $x + y = x + z \implies y = z$.

Definition 5.2. An *ordered cone* is a cone C equipped with a partial order $\leqslant$ such that the addition and the scalar multiplication are order preserving, i.e., for each $x, y, z \in C$ and $r \in \mathbb{R}^+$,

$$x \leqslant y \implies x + z \leqslant y + z \text{ and } rx \leqslant ry.$$

An ordered cone C is

- *pointed* if $x \geqslant 0$ for all $x \in C$,
- *order cancellative* if for each $x, y, z \in C$, $x + y \leqslant x + z \implies y \leqslant z$.

It is easy to see that the order cancellability implies the cancellability. A pointed ordered cone is always strict. An element a in an ordered cone satisfies $a \geqslant 0$ if and only if the map $k \mapsto ka$ is order-preserving.

The ordered cone of connections

Recall that for connections σ and η on $B(\mathcal{H})^+$, we define

$$\sigma + \eta : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+ : (A, B) \mapsto (A \,\sigma\, B) + (A \,\eta\, B),$$
$$k\sigma : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+ : (A, B) \mapsto k(A \,\sigma\, B), \quad k \in \mathbb{R}^+.$$

Denote by $C(B(\mathcal{H})^+)$ the set of connections on $B(\mathcal{H})^+$. Define a partial order $\leqslant$ for connections on $B(\mathcal{H})^+$ by $\sigma_1 \leqslant \sigma_2$ if $A \,\sigma_1\, B \leqslant A \,\sigma_2\, B$ for all $A, B \in B(\mathcal{H})^+$. It is straightforward to show that the set $C(B(\mathcal{H})^+)$ is an ordered cone in which the neutral element is the zero connection $0 : (A, B) \mapsto 0$. This cone is pointed and order cancellative.

The ordered cone of operator monotone functions

The set $OM(\mathbb{R}^+)$ of operator monotone functions from $\mathbb{R}^+$ to itself is equipped with usual addition and the scalar multiplication. The partial order on $OM(\mathbb{R}^+)$ is defined pointwise. It is routine to show that $OM(\mathbb{R}^+)$ is an ordered cone in which the zero function $0 : x \mapsto 0$ is the neutral element. This cone is also pointed and order cancellative.

The ordered cone of finite Borel measures

Let $BM([0,\infty])$ be the set of finite Borel measures on $[0,\infty]$. Then $BM([0,\infty])$ is a cone under usual addition and scalar multiplication:

$$(\mu+\nu)(E) = \mu(E) + \nu(E), \quad (k\mu)(E) = k\mu(E)$$

for each $\mu, \nu \in BM([0,\infty])$, $k \in \mathbb{R}^+$ and Borel set E in $[0,\infty]$. Define $\mu \leqslant \nu$ if $\mu(E) \leqslant \nu(E)$ for all Borel sets E in $[0,\infty]$. It is easy to see that $BM([0,\infty])$ is an ordered cone in which the zero measure $0 : E \mapsto 0$ is the neutral element. This cone is also pointed and order cancellative.

5.2 Topological Structure

Definition 5.3. A *normed cone* is a cone $(C, +, \cdot)$ equipped with a function $\|\cdot\| : C \to \mathbb{R}^+$ such that for each $x, y \in C$ and $k \in \mathbb{R}^+$,

(i) $\|x\| = 0 \implies x = 0$,

(ii) $\|kx\| = k\,\|x\|$,

(iii) $\|x+y\| \leqslant \|x\| + \|y\|$.

Definition 5.4. A *normed ordered cone* is an ordered cone $(C, \leqslant)$ which is also a normed cone such that for each $x, y \in C$, $x \leqslant y \implies \|x\| \leqslant \|y\|$.

Define a function $\|\cdot\| : C(B(\mathcal{H})^+) \to \mathbb{R}^+$ by

$$\|\sigma\| = \sup\{\|A\,\sigma\,B\| : A, B \geqslant 0, \|A\| = \|B\| = 1\}$$

for each connection σ.

Lemma 5.5. *([9]) For each connection σ, we have $\|A\,\sigma\,B\| \leqslant \|A\|\ \sigma\ \|B\|$ for all $A, B \geqslant 0$.*

Proposition 5.6. *For each connection σ, we have $\|\sigma\| = \|I\sigma I\|$.*

Proof. Clearly, $\|\sigma\| \geqslant \|I\sigma I\|$. For each $A, B \geqslant 0$ with $\|A\| = \|B\| = 1$, we have by Lemma 5.5 that

$$\|A\,\sigma\,B\| \leqslant \|A\|\;\sigma\;\|B\| = 1\,\sigma\,1 = \|(1\,\sigma\,1)I\| = \|I\,\sigma\,I\|\,.$$

Hence, $\|\sigma\| \leqslant \|I\,\sigma\,I\|$. $\square$

Proposition 5.7. *The pair $(C(B(\mathcal{H})^+), \|\cdot\|)$ is a normed ordered cone.*

Proof. For each $\sigma, \eta \in C(B(\mathcal{H})^+)$ and $k \in \mathbb{R}^+$, by Proposition 5.6 we have

$$\|k\sigma\| = \|I\,(k\sigma)\,I\| = \|k(I\sigma I)\| = k\,\|I\sigma I\| = k\,\|\sigma\|\,,$$
$$\|\sigma+\eta\| = \|I\,(\sigma+\eta)\,I\| = \|(I\sigma I)+(I\eta I)\| \leqslant \|I\sigma I\| + \|I\eta I\| = \|\sigma\| + \|\eta\|\,.$$

Suppose now that $\|\sigma\| = 0$, i.e. $I\sigma I = 0$. For each projection P, we have $P \leqslant I$ and hence $I\sigma P \leqslant I\sigma I = 0$, i.e. $I\sigma P = 0$. Similarly, $(xI)\sigma I = 0$ for each $x \in [0,1]$. Then for each $x > 1$,

$$I\sigma(xI) = x\left(\frac{1}{x}I\,\sigma\,I\right) = 0.$$

Consider $A \in B(\mathcal{H})^+$ in the form $A = \sum_{i=1}^m \lambda_i P_i$ where $\lambda_i > 0$ and P_i's are projections such that $P_iP_j = 0$ for $i \neq j$ and $\sum_{i=1}^m P_i = I$. By Lemma 3.3, we get

$$I\,\sigma\,A = \sum_{i=1}^m (I\,\sigma\,A)P_i = \sum P_i\,\sigma\,AP_i = \sum P_i\,\sigma\,\lambda_iP_i = \sum P_i(I\,\sigma\,\lambda_iI) = 0.$$

For general $A \in B(\mathcal{H})^+$, let $\{A_n\}$ be a sequence in $B(\mathcal{H})^{++}$ such that $A_n \downarrow A$. Then $I\,\sigma\,A = \lim_{n\to\infty} I\,\sigma\,A_n = 0$ for all $A \geqslant 0$. Hence, for $A, B \in B(\mathcal{H})^+$,

$$A\,\sigma\,B = \lim_{\epsilon\downarrow 0} A_\epsilon\,\sigma\,B = \lim_{\epsilon\downarrow 0} A_\epsilon^{1/2}(I\,\sigma\,A_\epsilon^{-1/2}BA_\epsilon^{-1/2})A_\epsilon^{1/2} = 0,$$

i.e. $\sigma = 0$.

If $\sigma \leqslant \eta$, then $\|\sigma\| = \|I\,\sigma\,I\| \leqslant \|I\,\eta\,I\| = \|\eta\|$ since $I\,\sigma\,I \leqslant I\,\eta\,I$. $\square$

Definition 5.8. A function f from a cone C into a cone D is called *linear* or *affine* if $f(rx+sy) = rf(x) + sf(y)$ for each $x, y \in C$ and $r, s \in \mathbb{R}^+$.

Define a function $\|\cdot\| : OM(\mathbb{R}^+) \to \mathbb{R}^+$ by $\|f\| = f(1)$ for each $f \in OM(\mathbb{R}^+)$.

Proposition 5.9. *The pair $(OM(\mathbb{R}^+), \|\cdot\|)$ is a normed ordered cone. Moreover, the function $\|\cdot\|$ is linear.*

Proof. The only non-trivial part is to show that $\|f\| = 0$ implies $f = 0$. Consider $f \in OM(\mathbb{R}^+)$ such that $f(1) = 0$. Suppose that there is an $a > 0$ such that $f(a) = 0$. Then $f(x) = 0$ for $0 \leqslant x \leqslant a$. Since $f \in OM(\mathbb{R}^+)$, f is a concave function by [20]. The concavity of f implies that $f = 0$. $\square$

Assign to each measure $\mu \in BM([0, \infty])$ its total variation:

$$\|\mu\| = \mu([0, \infty]) < \infty.$$

Proposition 5.10. *The pair $(BM([0, \infty]), \|\cdot\|)$ is a normed ordered cone. Moreover, the function $\|\cdot\|$ is linear.*

Given any normed cone, we can equip it with a natural topology as follows.

Proposition 5.11. *Let $(C, \|\cdot\|)$ be a normed cone. Then*

(1) the function $d : C \times C \to \mathbb{R}^+$, $d(x, y) = \big|\, \|x\| - \|y\| \,\big|$ is a pseudo metric; in particular, C is a 1st-countable topological space with respect to the topology induced by d.

(2) the functions $\|\cdot\|$ and d are continuous, where the topology on $C \times C$ is given by the product topology.

(3) C becomes a topological cone *in the sense that the addition and the scalar multiplication are continuous.*

Proof. The proof is very similar to the case of normed linear spaces. Note that the topology induced by a pseudo metric satisfies the 1st-countability axiom. In this topology, a function is continuous if and only if it is sequentially continuous. $\square$

Hence the cones $C(B(\mathcal{H})^+)$, $OM(\mathbb{R}^+)$ and $BM([0, \infty])$ are toplological cones.

Definition 5.12. Let C and D be normed cones. A function $\varphi : C \to D$ is called an *isomorphism* if it is a continuous linear bijection whose inverse is continuous. In this case, we say that C and D are *isomorphic*.

By an *isometry*, we mean a linear function $\phi : C \to D$ such that $\|\phi(c)\| = \|c\|$ for all $c \in C$. If $\varphi : C \to D$ is an isomorphism which is also an isometry, we say that φ is an *isometric isomorphism*. In this case, C and D are said to be *isometrically isomorphic*.

Note that every isometry between normed cones is continuous and injective. The inverse of an isometry is an isometry.

Definition 5.13. Let C and D be normed ordered cones. A function $\varphi : C \to D$ is called an *order isomorphism* if it is an isomorphism (between normed cones) such that φ and φ^{-1} are order-preserving. In this case, we say that C and D are *order isomorphic*. If, in addition, φ is an isometry, we say that φ is an *isometric order-isomorphism*. In this case, C and D are said to be *isometrically order-isomorphic*.

Theorem 5.14. *(1) The normed ordered cones $C(B(\mathcal{H})^+)$ and $OM(\mathbb{R}^+)$ are isometrically order-isomorphic via the isometric order-isomorphism $\sigma \mapsto f_\sigma$, where f_σ is the representing function of σ.*

(2) The normed cones $C(B(\mathcal{H})^+)$ and $BM([0,\infty])$ are isometrically isomorphic via the isometric isomorphism $\sigma \mapsto \mu_\sigma$, where μ_σ is the representing measure of σ.

Proof. The function $\Phi : \sigma \mapsto f_\sigma$ is an order isomorphism by Theorem 2.13. For each connection σ, since $f_\sigma(1)I = I\,\sigma\,I$, we have

$$\|\Phi(\sigma)\| = \|f_\sigma\| = f_\sigma(1) = \|I\,\sigma\,I\| = \|\sigma\|\,.$$

The function $\Psi : \sigma \mapsto \mu_\sigma$ is an isomorphism by Theorem 2.14. For each connection σ, we have

$$\|\Psi(\sigma)\| = \|\mu_\sigma\| = \mu([0,\infty]) = \|I\,\sigma\,I\| = \|\sigma\|$$

since $I\,\sigma\,I = \int_{[0,\infty]} I\,\sigma_\lambda\,I\,d\mu(\lambda) = \mu([0,\infty])I$. $\square$

The next corollaries are immediate consequences.

Corollary 5.15. *The norm for connections is linear.*

Proof. It follows from the fact that the map $\sigma \mapsto f_\sigma$ is an isometric isomorphism and the norm on $OM(\mathbb{R}^+)$ is linear. $\square$

We obtain the following characterizations of a mean as follows.

Corollary 5.16. *A connection is a mean if and only if it has norm one.*

Proof. It follows from Proposition 5.6, Theorem 5.14 and the fact that a connection is a mean if and only if its representing function (measure) is normalized. $\square$

This corollary tells us that a mean is a normalized connection. Every mean arises as a normalization of a nonzero connection. The convex set of means is the unit sphere in the normed cone of connections.

Corollary 5.17. *The limit of a sequence of means is a mean.*

Proof. Use the fact that the norm for connections is continuous by Proposition 5.11 and the norm of a mean is 1 by Corollary 5.16. $\square$

The topologies of the cones $C(B(\mathcal{H})^+)$, $OM(\mathbb{R}^+)$ and $BM([0,\infty])$ are compatible with the isometric isomorphisms $\sigma \mapsto f_\sigma$ and $\sigma \mapsto \mu_\sigma$ in Theorem 5.14 as follows.

Corollary 5.18. *For each $n \in \mathbb{N}$, let σ_n be a connection with representing function f_n and representing measure μ_n. Then the followings are equivalent for a connection σ with representing function f and representing measure μ:*

(i) $\sigma_n \to \sigma$;

(ii) $f_n \to f$;

(iii) $\mu_n \to \mu$.

CHAPTER VI

CONNECTIONS AND OPERATOR INEQUALITIES

In this chapter, we generalize some results in the literature about positive linear maps, monotonicity and concavity involving connections such that specific connections are replaced by general connections. In Section 6.1, it is shown that a connection behaves nicely with any positive linear map. We consider monotonicity and concavity of certain maps between operator algebras related to connections in Section 6.2.

6.1 Positive Linear Maps

Definition 6.1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is said to be *positive* if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. It is called *unital* if $\Phi(I) = I$. It is *strictly positive* if $\Phi(A) > 0$ when $A > 0$.

It is easy to see that a positive linear map Φ is strictly positive if and only if $\Phi(I) > 0$. In particular, every unital positive linear map is strictly positive.

Example 6.2.

(1) For each $x \in \mathcal{H}$, the map $A \mapsto \langle Ax, x \rangle$ is a positive linear functional on $B(\mathcal{H})^+$.

(2) Every linear functional on $M_n(\mathbb{C})$ takes the form $\varphi(A) = \operatorname{tr} AX$ for some $X \in M_n(\mathbb{C})$. Moreover, φ is positive if and only if X is positive semidefinite. The special case that $\varphi(A)$ is the sum of all entries of A is obtained when X is a matrix whose all entries are 1.

(3) For each $X \in B(\mathcal{K}, \mathcal{H})$, the map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$, $\Phi(A) = X^*AX$ is a positive linear map. This map is unital if and only if X is unitary. A special

case of this map is the *compression map* that takes an n-by-n matrix to a k-by-k matrix on the top left corner. Let $P_1, \ldots, P_r$ be mutually orthogonal projections on $M_n(\mathbb{C})$ with the condition $P_1 \oplus \cdots \oplus P_r = I$. The map $A \mapsto \sum_{i=1}^r P_i A P_i$, called a *pinching*, is a positive linear map.

(4) For each $X \in B(\mathcal{K})^+$, the map $\Phi : B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{K})$, $\Phi(A) = A \otimes X$ or $\Phi(A) = X \otimes A$ is positive. The tensor product $\otimes$ is also called the *Kronecker product* in matrix case.

(5) For each $X \in M_n(\mathbb{C})^+$, the map $\Psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $A \mapsto A \circ X$ is positive. Here, $\circ$ denotes the *Hadamard product*, i.e. the entrywise product.

Recall the following fact:

Lemma 6.3 (Choi's inequality, [12])**.** *If $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is linear, strictly positive and unital, then for every $A > 0$, $\Phi(A)^{-1} \leqslant \Phi(A^{-1})$.*

Proposition 6.4. *If $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is linear and strictly positive, then*

$$\Phi(A)\Phi(B)^{-1}\Phi(A) \leqslant \Phi(AB^{-1}A), \quad A, B > 0. \tag{6.1}$$

Proof. For each $X \in B(\mathcal{H})$, set $\Psi(X) = \Phi(A)^{-1/2}\Phi(A^{1/2}XA^{1/2})\Phi(A)^{-1/2}$. Then Ψ is a unital strictly-positive linear map. For each $A, B > 0$, we have

$$\begin{aligned} \Phi(A)^{1/2}\Phi(B)^{-1}\Phi(A)^{1/2} &= \Psi(A^{-1/2}BA^{-1/2})^{-1} \\ &\leqslant \Psi\left((A^{-1/2}BA^{-1/2})^{-1}\right) \\ &= \Phi(A)^{-1/2}\Phi(AB^{-1}A)\Phi(A)^{-1/2} \end{aligned}$$

by Lemma 6.3. $\square$

This result was proved under an additional condition that Φ is unital in [26].

Theorem 6.5. *If $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a positive linear map, then for any connection σ on $B(\mathcal{H})^+$ and for each $A, B \geqslant 0$,*

$$\Phi(A \,\sigma\, B) \leqslant \Phi(A) \,\sigma\, \Phi(B). \tag{6.2}$$

Proof. First, asssume that Φ is strictly positive and consider $A, B > 0$. The formula (2.1) and Proposition 6.4 imply that

$$\begin{aligned}
\Phi(A : B) &= \Phi(A - A(A+B)^{-1}A) \\
&= \Phi(A) - \Phi\left(A(A+B)^{-1}A\right) \\
&\leqslant \Phi(A) - \Phi(A)\Phi(A+B)^{-1}\Phi(A) \\
&= \Phi(A) - \Phi(A)\left[\Phi(A) + \Phi(B)\right]^{-1}\Phi(A) \\
&= \Phi(A) : \Phi(B).
\end{aligned}$$

For $A, B \geqslant 0$, since $A_\epsilon \equiv A + \epsilon I > 0$ and $B_\epsilon \equiv B + \epsilon I > 0$, we have

$$\Phi(A_\epsilon : B_\epsilon) \leqslant \Phi(A_\epsilon) : \Phi(B_\epsilon).$$

Recall that a positive linear map between C*-algebras is norm-continuous. Since the parallel sum is also norm-continuous ([2]), we have $\Phi(A : B) \leqslant \Phi(A) : \Phi(B)$ for all $A, B \geqslant 0$.

For general case of Φ, consider the family $\Phi_\epsilon : A \mapsto \Phi(A) + \epsilon A$ where $\epsilon > 0$. Then each Φ_ϵ is linear and stricly positive. The previous result implies that

$$\Phi_\epsilon(A : B) \leqslant \Phi_\epsilon(A) : \Phi_\epsilon(B)$$

for all $A, B \geqslant 0$ and $\epsilon > 0$. Taking $\epsilon \to 0$, we have $\Phi(A : B) \leqslant \Phi(A) : \Phi(B)$ for all $A, B \geqslant 0$. Since Φ is a bounded linear operator, we have

$$\begin{aligned}
\Phi(A \,\sigma\, B) &= \Phi(\int_{[0,\infty]} \frac{\lambda+1}{\lambda}(\lambda A : B)\, d\mu(\lambda)) \\
&= \int_{[0,\infty]} \Phi\left(\frac{\lambda+1}{\lambda}(\lambda A : B)\right) d\mu(\lambda) \\
&\leqslant \int_{[0,\infty]} \frac{\lambda+1}{\lambda}(\lambda\Phi(A) : \Phi(B))\, d\mu(\lambda) \\
&= \Phi(A) \,\sigma\, \Phi(B)
\end{aligned}$$

by Theorem A.11. □

This theorem shows that a connection behaves nicely with any positive linear map. Note that the situation when $\Phi(A) = X^*AX$ is, of course, the transformer inequality. This result was proved under the condition that Φ is a unital strictly-positive linear map and A, B are strictly positive operators in [26].

Corollary 6.6. *If $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a positive linear map, then for any connection σ and for any positive semidefinite matrices $A, B \in M_n(\mathbb{C})$, we have*

$$\Phi(A\sigma B) \leqslant \Phi(A)\,\sigma\,\Phi(B).$$

The results for the cases that σ is the harmonic mean and the geometric mean were obtained in [8].

Corollary 6.7. *For any connection σ on $M_n(\mathbb{C})^+$ and $A, B, C \in M_n(\mathbb{C})^+$, we have*

$$A \circ (B\,\sigma\,C) \leqslant (A \circ B)\,\sigma\,(A \circ C). \tag{6.3}$$

6.2 Monotonicity and Concavity

Recall some preliminaries about monotonicity and concavity of maps between operator algebras.

Definition 6.8. Let I be an interval. A continuous function $f : I \to \mathbb{R}$ is called an *operator concave function* if for all $A, B \in B(\mathcal{H})^{sa}$ whose spectra are contained in I and for all Hilbert spaces $\mathcal{H}$, we have

$$f(tA + (1-t)B) \geqslant tf(A) + (1-t)f(B), \quad t \in [0,1].$$

A well-known result is that a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ or $f : (0,\infty) \to (0,\infty)$ is operator monotone if and only if it is operator concave ([20]). Hence, the map $t \mapsto \log t$ is operator concave on $(0,\infty)$ and the map $t \mapsto t^r$ is operator concave on $\mathbb{R}^+$ for any $r \in [0,1]$.

Definition 6.9. Let $\mathcal{C}$ be a convex subset of $B(\mathcal{H})^{sa}$. A map $\Psi : \mathcal{C} \to B(\mathcal{K})^{sa}$ is called *concave* if for each $A, B \in \mathcal{C}$ and $t \in [0,1]$,

$$\Psi(tA + (1-t)B) \geqslant t\Psi(A) + (1-t)\Psi(B).$$

A map $\Psi : \mathcal{C} \to B(\mathcal{K})^{sa}$ is called *monotone* if $A \leqslant B$ assures $\Psi(A) \leqslant \Psi(B)$.

If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone, then the map $A \mapsto f(A)$ is monotone and concave on $B(\mathcal{H})^+$.

Proposition 6.10. *If $\Phi_1, \Phi_2 : B(\mathcal{H})^+ \to B(\mathcal{K})^+$ are monotone, then the map*

$$(A_1, A_2) \mapsto \Phi_1(A_1)\,\sigma\,\Phi_2(A_2) \tag{6.4}$$

is monotone for any connection σ on $B(\mathcal{K})^+$.

Proof. Assume $A_i \leqslant A_i'$ for $i = 1, 2$. Then $\Phi_i(A_i) \leqslant \Phi_i(A_i')$ by the monotonicity of Φ_i for each $i = 1, 2$. Now, the monotonicity of σ implies $\Phi_1(A_1)\,\sigma\,\Phi_2(A_2) \leqslant \Phi_1(A_1')\,\sigma\,\Phi_2(A_2')$. □

Corollary 6.11. *Let σ be a connection. Then, for any operator monotone functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, the map $(A, B) \mapsto f(A)\,\sigma\,g(B)$ is monotone.*

Proposition 6.12. *If $\Phi_1, \Phi_2 : B(\mathcal{H})^+ \to B(\mathcal{K})^+$ are concave, then the map*

$$(A_1, A_2) \mapsto \Phi_1(A_1)\,\sigma\,\Phi_2(A_2) \tag{6.5}$$

is concave for any connection σ on $B(\mathcal{K})^+$.

Proof. Let $A_1, A_1', A_2, A_2' \geqslant 0$ and $t \in [0, 1]$. Since Φ_1 and Φ_2 are concave,

$$\Phi_i(tA_i + (1-t)A_i') \geqslant t\Phi_i(A_i) + (1-t)\Phi_i(A_i'), \quad i = 1, 2.$$

It follows from the monotonicity and concavity of σ that

$$\begin{aligned}
&\Phi_1(tA_1 + (1-t)A_1')\,\sigma\,\Phi_2(tA_2 + (1-t)A_2') \\
&\qquad\qquad \geqslant [t\Phi_1(A_1) + (1-t)\Phi_1(A_1')]\,\sigma\,[t\Phi_2(A_2) + (1-t)\Phi_2(A_2')] \\
&\qquad\qquad \geqslant t[\Phi_1(A_1)\,\sigma\,\Phi_2(A_2)] + (1-t)[\Phi_1(A_1)\,\sigma\,\Phi_2(A_2)].
\end{aligned}$$

This shows the concavity of the map $(A_1, A_2) \mapsto \Phi_1(A_1)\,\sigma\,\Phi_2(A_2)$. □

Corollary 6.13. *Let σ be a connection. Then, for any operator monotone functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, the map $(A, B) \mapsto f(A)\,\sigma\,g(B)$ is concave.*

The results in Propositions 6.10 and 6.12 for the case that σ is the harmonic mean or the geometric mean of matrices were considered in [8].

From Corollaries 6.11 and 6.13, the map $(A, B) \mapsto A^r\,\sigma\,B^s$ is monotone and concave on $B(\mathcal{H})^+$ for any $r, s \in [0, 1]$. The map $(A, B) \mapsto (\log A)\,\sigma\,(\log B)$ is monotone and concave on $B(\mathcal{H})^{++}$.

Corollary 6.14. *Let σ be a connection on $B(\mathcal{H})^+$. If $\Phi_1, \Phi_2 : B(\mathcal{H})^+ \to B(\mathcal{H})^+$ is monotone and strongly continuous, then the map*

$$(A, B) \mapsto \Phi_1(A) \,\sigma\, \Phi_2(B) \tag{6.6}$$

satisfies the continuity from above (M3). In particular, the map

$$(A, B) \mapsto f(A) \,\sigma\, g(B) \tag{6.7}$$

satisfies (M3) for any operator monotone functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. Suppose $A_n \downarrow A$ and $B_n \downarrow B$. The monotonicity of Φ_1 implies that the sequence $\Phi_1(A_n)$ is decreasing. Since Φ_1 strongly continuous, $\Phi_1(A_n)$ converges strongly to $\Phi_1(A)$. Similarly, $\Phi_2(B_n) \downarrow \Phi_2(B)$. Since σ satisfies (M3), we obtain

$$\Phi_1(A_n) \,\sigma\, \Phi_2(B_n) \downarrow \Phi_1(A) \,\sigma\, \Phi_2(B).$$

The last statement follows from the fact that if $A_n \downarrow A$, then $\mathrm{Sp}(A_n) \subseteq [0, \|A_1\|]$ for all n and hence $f(A_n)$ converges strongly to $f(A)$ by Theorem A.9. $\square$

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APPENDIX A

A.1 Banach Algebras and C*-Algebras

A *normed algebra* is an associative algebra equipped with a submultiplicative norm. A *Banach algebra* is a complete normed algebra. A Banach algebra $\mathcal{A}$ is *unital* if it has a multiplicative identity, denoted by $1_{\mathcal{A}}$. The spectrum of an element a in a unital Banach algebra $\mathcal{A}$ is defined by

$$\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}.$$

Then $\mathrm{Sp}(a)$ is a nonempty compact subset of $\mathbb{C}$. The *spectral radius* of a is defined to be $r(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(a)\}$. We have $r(a) \leqslant \|a\|$.

A $*$-*algebra* is an algebra $\mathcal{A}$ equipped with a conjugate-linear map $*$ on $\mathcal{A}$, called an *involution*, such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. An element a in a $*$-algebra is called

- *normal* if $a^*a = aa^*$,
- *self-adjoint* if $a^* = a$,
- *projection* if $a^2 = a = a^*$.

A $*$-*homomorphism* between $*$-algebras is a multiplicative linear map preserving involutions.

A *normed* $*$-*algebra* is a $*$-algebra $\mathcal{A}$ which is also a normed algebra such that $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$. A *C*-algebra* is a complete normed $*$-algebra $\mathcal{A}$ such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. If a is a normal element in a unital C*-algebra, then $r(a) = \|a\|$ and

(1) it is self-adjoint if and only if $\mathrm{Sp}(a) \subseteq \mathbb{R}$,

(2) it is a projection if and only if $\mathrm{Sp}(a) \subseteq \{0, 1\}$.

A typical example of a non-commutative (unital) C*-algebra is the algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ when $\dim \mathcal{H} > 1$. A typical example of a commutative (unital) C*-algebra is the algebra $C(X)$ of complex-valued continuous functions on a compact Hausdorff space X.

Theorem A.1. *Let a be a normal element in a unital C*-algebra $\mathcal{A}$. Denote by $z : \mathrm{Sp}(a) \to \mathbb{C}$ the inclusion map. Then there is a unique unital $*$-homomorphism Φ from $C(\mathrm{Sp}(a))$ to the C*-algebra generated by $1_{\mathcal{A}}$ and a such that $\Phi(z) = a$. Moreover, Φ is norm-preserving.*

Hence, if $f : \mathrm{Sp}(a) \to \mathbb{C}$ is a continuous function, we denote the corresponding element $\Phi(f)$ in $\mathcal{A}$ by $f(a)$. We call the unique unital $*$-homomorphism that sending $f \in C(\mathrm{Sp}(a))$ to an element $f(a) \in \mathcal{A}$ the *(continuous) functional calculus* of a. Note that $f(a)$ is normal.

Theorem A.2. *Let a be a normal element in a unital C*-algebra. For each $f \in C(\mathrm{Sp}(a))$, we have $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ where $f(\mathrm{Sp}(a)) = \{f(\lambda) : \lambda \in \mathrm{Sp}(a)\}$.*

An element a in a unital C*-algebra $\mathcal{A}$ is said to be *positive* if a is self-adjoint and $\mathrm{Sp}(a) \subseteq \mathbb{R}^+ = [0, \infty)$. In fact, $a \in \mathcal{A}$ is positive if and only if there is a (unique) self-adjoint element $b \in \mathcal{A}$ such that $b^2 = a$. The set of positive elements in $\mathcal{A}$ forms a closed positive cone of $\mathcal{A}$, denoted by $\mathcal{A}^+$. If a, b are self-adjoint, define $a \leqslant b$ if and only if $b - a \in \mathcal{A}^+$. We write $a > b$ when $a - b$ is positive and invertible. For each $a \in \mathcal{A}$, define $|a| = (a^*a)^{1/2}$.

Proposition A.3. *Let $\mathcal{A}$ be a unital C*-algebra.*

(1) An element $a \in \mathcal{A}$ is positive if and only if $a + \epsilon 1_{\mathcal{A}} > 0$ for all $\epsilon \in (0, \infty)$.

(2) Let $a \in \mathcal{A}$ be self-adjoint and $k \in \mathbb{R}^+$. Then $-k1_{\mathcal{A}} \leqslant a \leqslant k1_{\mathcal{A}}$ if and only if $\|a\| \leqslant k$. Similarly, we have that $-k1_{\mathcal{A}} < a < k1_{\mathcal{A}}$ if and only if $\|a\| < k$.

(3) If $\{a_\alpha\}_{\alpha \in \Lambda}$ is a net in $\mathcal{A}^+$ such that $a_\alpha \to a \in \mathcal{A}$, then $a \in \mathcal{A}^+$.

(4) Let $\{a_\alpha\}_{\alpha \in \Lambda}$, $\{b_\alpha\}_{\alpha \in \Lambda}$ and $\{c_\alpha\}_{\alpha \in \Lambda}$ be nets of self-adjoint elements in $\mathcal{A}$ such that $a_\alpha \leqslant b_\alpha \leqslant c_\alpha$ for all $\alpha \in \Lambda$. If a_α and c_α converge to $a \in \mathcal{A}$, then b_α also converges to a.

Proof. (1). Use Theorem A.2.

(2). Assume that $-k1_{\mathcal{A}} \leqslant a \leqslant k1_{\mathcal{A}}$. Since $k1_{\mathcal{A}} - a \geqslant 0$, we have $r(k1_{\mathcal{A}} - a) \geqslant 0$. By Theorem A.2, we have $\|a\| = r(a) \leqslant r(k1_{\mathcal{A}}) = k$.

Now, assume that $\|a\| \leqslant k$. Then $\mathrm{Sp}(a) \subseteq [-r(a), r(a)] \subseteq [-k, k]$. Theorem A.2 imples $\mathrm{Sp}(a - k1_{\mathcal{A}}) \subseteq [-2k, 0]$, i.e. $a - k1_{\mathcal{A}} \leqslant 0$. Similarly, $a + k1_{\mathcal{A}} \geqslant 0$.

(3). The limit a is self-adjoint since the involution is continuous. Consider $\epsilon \in (0, \infty)$. Since $a_\alpha \to a$, there is an $N \in \Lambda$ such that $\|a_N - a\| < \epsilon$. Note that $a_N - a$ is normal and $\mathrm{Sp}(a_N - a) \subseteq [-\|a_N - a\|, \|a_N - a\|] \subseteq (-\epsilon, \epsilon)$. Theorem A.2 implies that $\mathrm{Sp}(a_N - a - \epsilon 1_{\mathcal{A}}) \subseteq (-2\epsilon, 0)$. Hence, $a + \epsilon 1_{\mathcal{A}} > a_N \geqslant 0$. By (1), $a \geqslant 0$.

(4). Let $\epsilon \in (0, \infty)$. Then there are $m, k \in \Lambda$ such that $\|a_\alpha - x\| < \epsilon$ for all $\alpha \geqslant m$ and $\|c_\alpha - x\| < \epsilon$ for all $\alpha \geqslant k$. Choose $l \in \Lambda$ such that $l \geqslant m$ and $l \geqslant k$. Then, for $\alpha \geqslant l$, we have $\alpha \geqslant m$ and $\alpha \geqslant k$ which imply

$$x - \epsilon 1_{\mathcal{A}} < a_\alpha < x + \epsilon 1_{\mathcal{A}} \text{ and } x - \epsilon 1_{\mathcal{A}} < c_\alpha < x + \epsilon 1_{\mathcal{A}}$$

by (2). Hence, $-\epsilon 1_{\mathcal{A}} < a_\alpha - x \leqslant b_\alpha - x \leqslant c_\alpha - x < \epsilon 1_{\mathcal{A}}$. By (2), we have $\|b_\alpha - x\| < \epsilon$ for all $\alpha \geqslant l$. Thus, $b_\alpha \to x$. $\square$

A.2 The Von Neumann Algebra of Bounded Linear Operators on a Hilbert Space

Let $B(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. The sets of self-adjoint operators and positive operators on $\mathcal{H}$ are written by $B(\mathcal{H})^{sa}$ and $B(\mathcal{H})^+$, respectively. For $A, B \in B(\mathcal{H})^{sa}$, we define $A \leqslant B$ if $B - A \in B(\mathcal{H})^+$. By an increasing sequence in $B(\mathcal{H})$, we mean a sequence $\{A_n\}$ in $B(\mathcal{H})^{sa}$ such that $A_n \leqslant A_{n+1}$ for all $n \in \mathbb{N}$. A decreasing sequence in $B(\mathcal{H})$ is defined similarly.

Strong-operator topology

For each $x \in \mathcal{H}$, the function

$$\rho_x : B(\mathcal{H}) \to \mathbb{R}^+, T \mapsto \|Tx\|$$

is a seminorm on $B(\mathcal{H})$. Then the family $\{\rho_x\}_{x\in\mathcal{H}}$ separates points of $B(\mathcal{H})$, i.e. for each $T \in B(\mathcal{H}) - \{0\}$, we can find an $x \in \mathcal{H}$ such that $\rho_x(T) \neq 0$. The locally convex topology on $B(\mathcal{H})$ induced by this family is called the *strong-operator topology* on $B(\mathcal{H})$. Under this topology, $B(\mathcal{H})$ becomes a topological vector space. A net $\{T_\lambda\}_{\lambda\in\Lambda}$ *converges strongly* (i.e. converges in this topology) to $T \in B(\mathcal{H})$ if and only if

$$T(x) = \lim_{\lambda} T_\lambda(x)$$

for all $x \in \mathcal{H}$. This topology is smaller than the norm topology. The norm convergence implies the strong-operator convergence. If $\dim \mathcal{H} < \infty$, then both convergences are equivalent. An important feature of $B(\mathcal{H})$ is the order completeness:

Theorem A.4. *If T_α is an increasing (decreasing) net of self-adjoint operators in $B(\mathcal{H})$ that is bounded above (below, respectively), then T_α converges strongly to a self-adjoint operator in $B(\mathcal{H})$.*

A *von Neumann algebra* is a C*-subalgebra of $B(\mathcal{H})$ which is closed under strong-operator topology. In particular, $B(\mathcal{H})$ is a von Neumann algebra.

Proposition A.5. *The center of $B(\mathcal{H})$ is trivial, i.e. $B(\mathcal{H})$ is a factor.*

Proof. Let $T \in B(\mathcal{H})$ be such that $TS = ST$ for any $S \in B(\mathcal{H})$. Choose a bounded linear functional f in the topological dual of $\mathcal{H}$ such that $f(w) \neq 0$ for some $w \in \mathcal{H}$. For each $x \in \mathcal{H}$, consider $S_x \in B(\mathcal{H})$ defined by $S_x(v) = f(v)x$ for each $v \in \mathcal{H}$. Then

$$f(Tw)x = S_x(Tw) = TS_x(w) = T(f(w)x) = f(w)T(x)$$

and $T(x) = \alpha x$ where $\alpha = f(Tw)/f(w)$. Hence $T = \alpha I$. □

Weak-operator topology

The *weak-operator topology* on $B(\mathcal{H})$ is defined to be the locally convex topology on $B(\mathcal{H})$ induced by the separating family of seminorms defined by

$$\rho_{x,y} : B(\mathcal{H}) \to \mathbb{R}^+, \rho_{x,y}(T) = |\langle Tx, y\rangle|, \quad x, y \in \mathcal{H}.$$

Hence, a net $\{T_\alpha\}_{\alpha\in\Lambda}$ in $B(\mathcal{H})$ *converges weakly* (i.e. converges in this topology) to $T \in B(\mathcal{H})$ if and only if $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$ or, equivalently, $\langle T_\alpha x, x\rangle \to \langle Tx, x\rangle$ for all $x \in \mathcal{H}$ (by polarization identity).

Theorem A.6. *Let $\mathcal{A}$ be a von Neumann algebra. For a sequence $A_n \in \mathcal{A}$, we have that A_n converges weakly to $A \in \mathcal{A}$ if and only if A_n converges strongly to A.*

A.3 Spectral Theorem and Functional Calculus

Let Ω be a compact Hausdorff space and $\mathcal{H}$ a Hilbert space. A *spectral measure* relative to $(\Omega, \mathcal{H})$ is a map E from the Borel σ-algebra on Ω to the set of projections in $B(\mathcal{H})$ such that

- $E(\emptyset) = 0, E(\Omega) = 1$;
- $E(A \cap B) = E(A)E(B)$ for all Borel sets A, B of Ω;
- for each $x, y \in \mathcal{H}$, the map $E_{x,y} : A \mapsto \langle E(A)x, y\rangle$ is a regular Borel complex measure on Ω.

Let $B_\infty(\Omega)$ be the C*-algebra of all bounded Borel-measurable complex-valued functions on Ω. Then for each $f \in B_\infty(\Omega)$, there is a unique $T \in B(\mathcal{H})$ such that

$$\langle T(x), y\rangle = \int_\Omega f\, dE_{x,y}, \quad x, y \in \mathcal{H}.$$

We write $\int f\, dE$ for T. Moreover, the map

$$B_\infty(\Omega) \to B(\mathcal{H}), \quad f \mapsto \int f\, dE,$$

is a unital $*$-homomorphism.

Theorem A.7. *Let $\Phi : C(\Omega) \to B(\mathcal{H})$ be a $*$-homomorphism. Then there is a unique spectral measure E relative to $(\Omega, \mathcal{H})$ such that*

$$\Phi(f) = \int f\, dE, \quad f \in C(\Omega).$$

Moreover, $T \in B(\mathcal{H})$ commutes with $\Phi(f)$ for all $f \in B_\infty(\Omega)$ if and only if T commutes with $E(S)$ for all Borel sets S of Ω.

Theorem A.8. *Let A be a normal operator on $\mathcal{H}$. Then there is a unique spectral measure E relative to* $(\mathrm{Sp}(A), \mathcal{H})$ *such that*

$$A = \int z \, dE$$

where $z : \mathrm{Sp}(A) \to \mathbb{C}$ is the inclusion.

Since $f(A) = \int f \, dE$ for each $f \in C(\mathrm{Sp}(A))$, we can define

$$f(A) = \int f \, dE, \quad f \in B_\infty(\mathrm{Sp}(A)).$$

The unital $*$-homomorphism $B_\infty(\mathrm{Sp}(A)) \to B(\mathcal{H}), f \mapsto f(A)$ is called the *Borel functional calculus* at A. If a normal operator T commutes with A and A^*, then T commutes with $f(A)$ for all $f \in B_\infty(\mathrm{Sp}(A))$.

Theorem A.9. *([21]) Let A_n be a sequence of positive operators on $\mathcal{H}$ such that* $\mathrm{Sp}(A_n) \subseteq [\alpha, \beta]$ *for all $n \in \mathbb{N}$. Suppose that A_n converges strongly to a positive operator A. Then* $\mathrm{Sp}(A) \subseteq [\alpha, \beta]$ *and $f(A_n)$ converges strongly to $f(A)$ for any continuous function $f : [\alpha, \beta] \to \mathbb{C}$.*

A.4 Bochner Integral

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and V a vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. A function $f : \Omega \to V$ is called a *V-simple function* if there are pairwise disjoint $\mathcal{M}$-measurable sets $E_1, \ldots, E_n$ and nonzero vectors $v_1, \ldots, v_n \in V$ such that

$$f = \sum_{i=1}^{n} \chi_{E_i} v_i.$$

If $\mu(E_i) < \infty$ for all $i = 1, \ldots, n$, then such f is called a *V-step function.* The *integral* of f on Ω with respect to μ is defined to be

$$\int_\Omega f \, d\mu = \sum_{i=1}^{n} \mu(E_i) v_i.$$

Let $(X, \|\cdot\|)$ be a Banach space. For each function $f : \Omega \to X$, we define the *norm function* of f to be

$$\|f\| : \Omega \to \mathbb{R}, \quad \omega \mapsto \|f(\omega)\|.$$

A function $f : \Omega \to X$ is called *strongly measurable* if it is a pointwise (a.e.) limit of a sequence of X-simple functions. If f is strongly measurable, then $\|f\|$ is measurable. A strongly measurable function $f : \Omega \to X$ is *Bochner integrable* if there is a sequence (s_n) of X-step functions such that the measurable function $\|f - s_n\|$ is Lebesgue integrable for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \int_\Omega \|f - s_n\| \, d\mu = 0.$$

In this case, we define the *Bochner integral* of f on Ω with respect to μ by

$$\int_\Omega f \, d\mu = \lim_{n\to\infty} \int_\Omega s_n \, d\mu$$

in the norm topology on X. The above integral is well-defined.

Theorem A.10. *([1, p. 426]) Let Ω be a finite measure space and X a Banach space. Then a measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\|f\|$ is Lebesgue integrable.*

The set of Bochner integrable functions from Ω to X forms a vector space over $\mathbb{F}$. The Bochner integral acts as a linear operator.

Theorem A.11. *([1, p. 427]) Let (Ω, μ) be a measure space and let X, Y be Banach spaces. If $f : \Omega \to X$ is Bochner integrable and if $T : X \to Y$ is a bounded linear operator, then $T \circ f$ is Bochner integrable and*

$$\int_\Omega T \circ f \, d\mu = T(\int_\Omega f \, d\mu).$$

Proposition A.12. *Let (Ω, μ) be a measure space and let $\mathcal{A}$ be a Banach algebra with $a, b \in \mathcal{A}$. If $\phi : \Omega \to \mathcal{A}$ is Bochner integrable, then the map $\tilde{\phi} : \Omega \to \mathcal{A}$, $\tilde{\phi}(\omega) = a\phi(\omega)b$ is Bocher integrable and*

$$\int_\Omega \tilde{\phi} \, d\mu = a \left(\int_\Omega \phi \, d\mu \right) b.$$

Proof. Since ϕ is strongly measurable, there is a sequence $\{\phi_n\}$ of $\mathcal{A}$-simple functions such that $\phi_n(\omega) \to \phi(\omega)$ for μ-a.e. $\omega \in \Omega$. Define $\tilde{\phi_n}(\omega) = a\phi_n(\omega)b$ for $\omega \in \Omega$. Then each $\tilde{\phi_n}$ is an $\mathcal{A}$-simple function and $\tilde{\phi_n}(\omega) \to \phi_n(\omega)$ for μ-a.e. $\omega \in \Omega$ by continuity of the multiplication. Hence, $\tilde{\phi}$ is strongly measurable.

Let $\{\phi_n\}$ be a sequence of $\mathcal{A}$-step functions such that $\|\phi_n - \phi\|$ is Lebesgue integrable for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \int_\Omega \|\phi_n - \phi\| \, d\mu = 0.$$

For each $n \in \mathbb{N}$, define $\tilde{\phi_n}(\omega) = a\phi_n(\omega)b$ for $\omega \in \Omega$. Then $\tilde{\phi_n}$ is $\mathcal{A}$-step and $\tilde{\phi_n} - \tilde{\phi}$ is strongly measurable for each n. Hence, $\left\|\tilde{\phi_n} - \tilde{\phi}\right\|$ is measurable for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
\int_\Omega \left\|\tilde{\phi_n} - \tilde{\phi}\right\| \, d\mu &= \int_\Omega \|a\phi_n(\omega)b - a\phi(\omega)b\| \, d\mu(\omega) \\
&\leqslant \int_\Omega \|a\| \cdot \|\phi_n(\omega) - \phi(\omega)\| \cdot \|b\| \, d\mu(\omega) \\
&= \|a\| \int_\Omega \|\phi_n(\omega) - \phi(\omega)\| \, d\mu(\omega) \, \|b\| \\
&< \infty.
\end{aligned}$$

That is $\left\|\tilde{\phi_n} - \tilde{\phi}\right\|$ is Lebesgue integrable for all $n \in \mathbb{N}$. Now,

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega \left\|\tilde{\phi_n} - \tilde{\phi}\right\| \, d\mu &\leqslant \lim_{n\to\infty} \int_\Omega \|a\| \cdot \|\phi_n - \phi\| \cdot \|b\| \, d\mu \\
&= \|a\| \lim_{n\to\infty} \int_\Omega \|\phi_n - \phi\| \, d\mu \, \|b\| \\
&= 0.
\end{aligned}$$

Thus, $\tilde{\phi}$ is Bochner integrable. If ϕ is an $\mathcal{A}$-step function, it is easy to see that

$$\int a\phi b \, d\mu = a \left(\int \phi \, d\mu \right) b.$$

The case that ϕ is strongly measurable follows from the previous case and a continuity argument. $\square$

Lemma A.13. *Let (Ω, μ) be a measure space and $\mathcal{A}$ a unital C^*-algebra. If $\phi : \Omega \to \mathcal{A}$ is strongly measurable and $\phi(\omega) \geqslant 0$ for all $\omega \in \Omega$, then there is a sequence $\{\phi_n\}$ of $\mathcal{A}$-simple functions such that $0 \leqslant \phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi$ and*

$$\phi_n(\omega) \to \phi(\omega), \quad \forall \omega \in \Omega.$$

If, in addition, $\mu(\Omega) < \infty$ or ϕ is Bochner integrable, we can choose ϕ_n's to be $\mathcal{A}$-step.

Proof. For each $n \in \mathbb{N}$, set

$$\begin{aligned} E_{n,k} &= \{\omega \in \Omega : (k-1)/2^n \leqslant \|\phi(\omega)\| < k/2^n\}, \quad k = 1, \dots, n2^n, \\ F_n &= \{\omega \in \Omega : \|\phi(\omega)\| \geqslant n\}, \\ \phi_n &= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} 1_{\mathcal{A}} + n\chi_{F_n} 1_{\mathcal{A}}. \end{aligned}$$

Since ϕ is strongly measurable, we have that $\|\phi\|$ is measurable and, thus, each $E_{n,k}$ and F_n are measurable. Then ϕ_n is $\mathcal{A}$-simple and $\phi_n \leqslant \phi_{n+1} \leqslant \phi$ for each $n \in \mathbb{N}$. If $\|\phi(\omega)\| < n$, then $\chi_{F_n} = 0$ and there is a $k \in \{1, \dots, n2^n\}$ such that $(k-1)/2^n \leqslant \|\phi(\omega)\| < k/2^n$ which implies $\phi(\omega) \leqslant k/2^n \cdot 1_{\mathcal{A}}$ by Proposition A.3(2) and hence

$$0 \leqslant \phi(\omega) - \phi_n(\omega) = \phi(\omega) - \frac{k-1}{2^n} 1_{\mathcal{A}} \leqslant \frac{k}{2^n} 1_{\mathcal{A}} - \frac{k-1}{2^n} 1_{\mathcal{A}} = \frac{1}{2^n} 1_{\mathcal{A}}.$$

Thus, $\phi_n(\omega) \to \phi(\omega)$ as $n \to \infty$ by Proposition A.3(4). If $\mu(\Omega) < \infty$, then each ϕ_n is $\mathcal{A}$-step. Suppose that ϕ is Bochner integrable. If $\phi_n = \sum_{i=1}^{k_n} \chi_{E_{i,n}} a_{i,n}$ where each $E_{i,n}$ is measurable and $a_{i,n} \geqslant 0$ for all i and n, then

$$\infty > \int_\Omega \phi \, d\mu = \lim_{n\to\infty} \int_\Omega \phi_n \, d\mu = \lim_{n\to\infty} \sum_{i=1}^{k_n} \mu(E_{i,n}) a_{i,n}.$$

Hence, for n large enough, $\mu(E_{i,n}) < \infty$ for all $i = 1, \dots, k_n$. $\square$

Proposition A.14. *Let (Ω, μ) be a measure space and $\mathcal{A}$ a unital C^*-algebra. Assume that $\phi, \psi : \Omega \to \mathcal{A}$ are Bochner integrable.*

(1) If $\phi(\omega) \geqslant 0$ for all $\omega \in \Omega$, then $\int \phi \, d\mu \geqslant 0$.

(2) If $\phi(\omega) \geqslant \psi(\omega)$ for all $\omega \in \Omega$, then $\int \phi \, d\mu \geqslant \int \psi \, d\mu$.

Proof. We prove only the first assertion since it implies to the second one. Consider ϕ of the form $\sum_{i=1}^n \chi_{E_i} a_i$ where $\{E_i\}$ is a collection of pairwise disjoint measurable sets in Ω with $\mu(E_i) < \infty$ and $a_i \geqslant 0$ for all $i = 1, \dots, n$. We have

$$\int_\Omega \phi \, d\mu = \int_\Omega \sum_{i=1}^n \chi_{E_i} a_i \, d\mu = \sum_{i=1}^n \mu(E_i) a_i \geqslant 0.$$

Now, consider the case that ϕ is strongly measurable. By Lemma A.13, we can choose a sequence $\{\phi_n\}$ of $\mathcal{A}$-step functions such that $\phi_n(\omega) \geqslant 0$ for all $\omega \in \Omega$, $\|\phi_n - \phi\|$ is Lebesgue integrable for all n and

$$\lim_{n\to\infty} \int_\Omega \|\phi_n - \phi\| \, d\mu = 0.$$

It follows that

$$\int_\Omega \phi \, d\mu = \lim_{n\to\infty} \int_\Omega \phi_n \, d\mu \geqslant 0$$

by Proposition A.3(3). □

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