สาทิสสัณฐานของไฮเพอร์กรุปบางชนิด

นายวิทวัส พันธวิมล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2553

HOMOMORPHISMS OF SOME HYPERGROUPS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2010

Thesis Title	HOMOMORPHISMS OF SOME HYPERGROUPS
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Field of Study	Mathematics
Thesis Advisor	Professor Yupaporn Kemprasit, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

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วิทวัส พันธวิมล : สาทิสสัณฐานของไฮเพอร์กรุปบางชนิด. (HOMOMORPHISMS OF SOME HYPERGROUPS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ศ. ดร. ยุพาภรณ์ เข็มประสิทธิ์, 41 หน้า.

*สาทิสสัณฐาน*ของไฮเพอร์กรุป $(H,\circ)$ คือฟังก์ชัน $f:H\to H$ ซึ่ง $f(x\circ y)\subseteq f(x)\circ f(y)$ สำหรับทุก $x,y\in H$ ถ้าการเท่ากันเป็นจริง เราเรียก f ว่า*สาทิสสัณฐานดี* เราเรียกสาทิสสัณฐาน f ของ $(H,\circ)$ ซึ่ง $f(H)=H$ ว่า *สาทิสสัณฐานทั่วถึง* สำหรับไฮเพอร์กรุป $(H,\circ)$ เราให้สัญลักษณ์ $\mathrm{Hom}(H,\circ)$, $\mathrm{GHom}(H,\circ)$, $\mathrm{Epi}(H,\circ)$ และ $\mathrm{GEpi}(H,\circ)$ แทนเซตของสาทิสสัณฐานทั้งหมด เซตของสาทิสสัณฐานดีทั้งหมด เซตของสาทิสสัณฐานทั่วถึงทั้งหมด และเซตของสาทิสสัณฐานทั่วถึงดีทั้งหมดของ $(H,\circ)$ ตามลำดับ ถ้า G เป็นกรุป และ N เป็น กรุปย่อยปรกติของ G เราให้ $(G,\circ_N)$ เป็นไฮเพอร์กรุปโดยที่นิยามการดำเนินการไฮเพอร์ $\circ_N$ โดย $x\circ_N y=xyN$ สำหรับทุก $x,y\in G$ ได้มีการให้ลักษณะเฉพาะของสมาชิกของ $\mathrm{GHom}(\mathbb{Z},\circ_{m\mathbb{Z}})$ และ $\mathrm{GEpi}(\mathbb{Z},\circ_{m\mathbb{Z}})$ มาแล้ว ยังแสดงแล้วด้วยว่า $|\mathrm{GHom}(\mathbb{Z},\circ_{m\mathbb{Z}})| = |\mathrm{GEpi}(\mathbb{Z},\circ_{m\mathbb{Z}})| = 2^{\aleph_0}$ ถ้า $m\neq 0$

วัตถุประสงค์หลักของการวิจัยนี้ คือ การให้ลักษณะเฉพาะของสมาชิกของ $\mathrm{Hom}(\mathbb{Z},\circ_{m\mathbb{Z}})$, $\mathrm{Epi}(\mathbb{Z},\circ_{m\mathbb{Z}})$, $\mathrm{Hom}(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$, $\mathrm{GHom}(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$, $\mathrm{Epi}(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$ และ $\mathrm{GEpi}(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$ ยิ่งไปกว่านั้นเราให้จำนวนเชิงการนับของเซตเหล่านี้ด้วย การวิจัยนี้ยังมีผลบางอย่างเกี่ยวกับสาทิสสัณฐานของไฮเพอร์กรุปต่อไปนี้ P-ไฮเพอร์กรุป ไฮเพอร์กรุปที่นิยามจากกรุปสลับที่ซึ่งผลคูณไฮเพอร์เป็นกรุปย่อย และไฮเพอร์กรุปที่นิยามจาก $\mathbb{R}$ ซึ่งผลคูณไฮเพอร์เป็นช่วงปิด

ภาควิชา..............คณิตศาสตร์....... ลายมือชื่อนิสิต...

สาขาวิชา............คณิตศาสตร์....... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..............................

ปีการศึกษา..........2553.................

4973845023 : MAJOR MATHEMATICS

KEYWORDS : HOMOMORPHISM / HYPERGROUP

WITTHAWAS PHANTHAWIMOL : HOMOMORPHISMS OF SOME HYPERGROUPS. THESIS ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph.D., 41 pp.

A *homomorphism* of a hypergroup $(H,\circ)$ is a function $f:H \to H$ such that $f(x \circ y) \subseteq f(x) \circ f(y)$ for all $x, y \in H$. If the equality holds, f is called a *good homomorphism* of $(H,\circ)$. A homomorphism f of a hypergroup $(H,\circ)$ is called an *epimorphism* if $f(H) = H$. For a hypergroup $(H,\circ)$, denote by Hom$(H,\circ)$, GHom $(H,\circ)$, Epi$(H,\circ)$ and GEpi$(H,\circ)$ the set of all homomorphisms, the set of all good homomorphisms, the set of all epimorphisms and the set of all good epimorphisms of $(H,\circ)$, respectively. If G is a group and N is a normal subgroup of G, let $(G,\circ_N)$ be the hypergroup where the hyperoperation $\circ_N$ is defined by $x \circ_N y = xyN$ for all $x, y \in G$. The elements of GHom$(\mathbb{Z},\circ_{m\mathbb{Z}})$ and GEpi$(\mathbb{Z},\circ_{m\mathbb{Z}})$ have been characterized. It was also shown that | GHom$(\mathbb{Z},\circ_{m\mathbb{Z}})$| = | GEpi$(\mathbb{Z},\circ_{m\mathbb{Z}})$| = $2^{\aleph_0}$ if $m \neq 0$.

The main purpose of this research is to characterize the elements of Hom $(\mathbb{Z},\circ_{m\mathbb{Z}})$, Epi$(\mathbb{Z},\circ_{m\mathbb{Z}})$, Hom$(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$, GHom$(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$, Epi$(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$ and GEpi $(\mathbb{Z}_n,\circ_{m\mathbb{Z}_n})$. In addition, the cardinalities of these sets are given. This research also includes some results on homomorphisms of the following hypergroups : P-hypergroups, hypergroups defined from abelian groups whose hyperproducts are subgroups and the hypergroup defined from $\mathbb{R}$ whose hyperproducts are closed intervals.

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Academic Year : 2010

ACKNOWLEDGEMENTS

I greatly appreciate the help of Professor Dr.Yupaporn Kemprasit, my supervisor, for her untired suggestions and assistance during preparing and writing this dissertation. I would like to thank my committee for their useful comments to the research. I am also grateful to all the teachers for their teaching during my study.

Finally, I wish to express my deep gratitude to my beloved father and mother for their sincere encouragement throughout my study.

CONTENTS

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INTRODUCTION

The concept of homomorphism has been introduced and studied in every algebraic structure. As we know, the concept of group plays a crucial role in algebra. Hypergroups introduced in the area of algebraic structures are a nice generalization of groups. Hypergroup homomorphisms generalize group homomorphisms naturally (see [3], p.12 or [4], p.4). An important hypergroup homomorphism is a good homomorphism. Homomorphisms of various types were introduced by J. Jantosciak in [6]. Epimorphisms of hypergroups were defined in [7] analogously to that of groups. A relationship between homomorphisms of groups and good homomorphisms of certain hypergroups defined from those groups were studied in [9].

Hypergroups defined from groups and their normal subgroups are of our main interest. If G is a group and N is a normal subgroup of G, let $(G, \circ_N)$ be the hypergroup where $x \circ_N y = xyN$ for all $x, y \in G$ ([3], p.11). In [7], the authors characterized the good homomorphisms and the good epimorphisms of the hypergroup $(\mathbb{Z}, \circ_{m\mathbb{Z}})$ defined from the group $(\mathbb{Z}, +)$ and its subgroup $m\mathbb{Z}$. Then $x \circ_{m\mathbb{Z}} y = x + y + m\mathbb{Z}$ for all $x, y \in \mathbb{Z}$. For a hypergroup $(H, \circ)$, let $\mathrm{Hom}(H, \circ)$, $\mathrm{GHom}(H, \circ)$, $\mathrm{Epi}(H, \circ)$ and $\mathrm{GEpi}(H, \circ)$ denote the set of all homomorphisms, the set of all good homomorphisms, the set of all epimorphisms and the set of all good epimorphisms of $(H, \circ)$, respectively. Then the elements of $\mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $\mathrm{GEpi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ have been characterized in [7]. It was shown in [7] that both $\mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $\mathrm{GEpi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ have the same cardinality which is $2^{\aleph_0}$. In [8], the authors found a suitable equivalence relation δ on the semigroup $\mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ under composition such that $\mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})/\delta \cong (\mathbb{Z}_m, \cdot)$, the multiplicative semigroup of integers modulo m.

The purpose of Chapter II is to extend the results in [7] mentioned above. The elements of $\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $\mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ are characterized. We also show in this chapter that $|\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = |\mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}$ if $m \neq 0$.

Chapter III deals with the hypergroup $(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ defined from the group $(\mathbb{Z}_n, +)$ and its subgroup $m\mathbb{Z}_n$ as above. This chapter gives characterizations of the elements of $\mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, $\mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, $\mathrm{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ and $\mathrm{GEpi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$. The cardinalities of these sets are also provided.

Let $(G, \bullet_P)$ be the P-hypergroup defined from a group G and a nonempty subset P of G, i.e., $x \bullet_P y = xPy$ for all $x, y \in G$ ([5], p.37). Some results on homomorphisms and good homomorphisms of P-hypergroups defined from the groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ were given in [2]. Chapter IV deals with homomorphisms, good homomorphisms, epimorphisms and good epimorphisms of P-hypergroups defined from the group $(\mathbb{Q}, +)$. This chapter is also concerned with the hypergroup defined from a group G whose hyperproduct $x \circ y$ of $x, y \in G$ is the subgroup of G generated by x and y ([3], p.11). The groups which we are interested in are $(\mathbb{Z}, +)$, $(\mathbb{Z}_n, +)$ and $(\mathbb{Q}, +)$. Some relationships of $\mathrm{Hom}(A, +)$ and $\mathrm{GHom}(A, \circ)$ are determined where $(A, +)$ is one of $(\mathbb{Z}, +)$, $(\mathbb{Z}_n, +)$ and $(\mathbb{Q}, +)$. The hypergroup $(\mathbb{R}, \bullet)$ where $x \bullet y = y \bullet x = [x, y]$ if $x \leq y$ is considered. We show in this chapter that $\mathrm{Hom}(\mathbb{R}, \bullet)$ is the set of all monotone functions from $\mathbb{R}$ into itself and $\mathrm{GHom}(\mathbb{R}, \bullet)$ is the set of all monotone continuous functions from $\mathbb{R}$ into itself. In addition, we show that $\mathrm{Epi}(\mathbb{R}, \bullet)$ is contained in $\mathrm{GHom}(\mathbb{R}, \bullet)$. It then follows that $\mathrm{GEpi}(\mathbb{R}, \bullet) = \mathrm{Epi}(\mathbb{R}, \bullet)$.

The definitions and quoted results used in this research are provided in Chapter I.

CHAPTER I

PRELIMINARIES

The cardinality of a set X is denoted by $|X|$.

The set of integers, the set of rational numbers and the set of real numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively.

A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \to P(H) \setminus \{\varnothing\}$ where $P(H)$ is the power set of H. The value of $(x, y) \in H \times H$ under $\circ$ is denoted by $x \circ y$ which is called the *hyperproduct* of x and y. The system $(H, \circ)$ is called a *hypergroupoid*. For $A, B \subseteq H$ and $x \in H$, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ A = \{x\} \circ A.$$

The hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \text{ for all } x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ satisfying the condition

$$H \circ x = x \circ H = H \text{ for all } x \in H.$$

Then hypergroups are a generalization of groups.

Example 1.1. ([3], p.11) Let G be a group and N a normal subgroup of G. If $\circ_N$ is the hyperoperation defined on G by

$$x \circ_N y = xyN \text{ for all } x, y \in G,$$

then $(G, \circ_N)$ is a hypergroup.

Example 1.2. ([3], p.11) Let G be a group and P a nonempty subset of G. If $\bullet_P$ is the hyperoperation defined on G by

$$x \bullet_P y = xPy \ \text{ for all } \ x, y \in G,$$

then $(G, \bullet_P)$ is a hypergroup. It may be called a *P-hypergroup* (see [5], p.37).

Example 1.3. ([3], p.11) Let G be a group. For $x, y \in G$, define

$$x \circ y = \langle x, y \rangle \, , \ \text{ the subgroup of } G \text{ generated by } x \text{ and } y.$$

Then $(G, \circ)$ is a hypergroup. Note that if $(A, +)$ is an abelian group, then $x \circ y = \mathbb{Z}x + \mathbb{Z}y$ for all $x, y \in A$.

Example 1.4. ([5], p.39) Define the hyperoperation $\bullet$ on $\mathbb{R}$ as follows :

$$x \bullet x = \{x\} \ \text{ for all } x \in \mathbb{R},$$
$$x \bullet y = y \bullet x = (x, y) \ \text{ if } x < y.$$

Then $(\mathbb{R}, \bullet)$ is a commutative hypergroup.

Remark 1.5. It can be shown that in Example 1.4 if (x, y) is replaced by $[x, y]$, we still have that $(\mathbb{R}, \bullet)$ is a commutative hypergroup. In this case

$$x \bullet y = y \bullet x = [x, y] \ \text{ if } x \leq y,$$

or equivalently,

$$x \bullet y = [\min\{x, y\}, \max\{x, y\}] \ \text{ for all } x, y \in \mathbb{R}.$$

To be sure that this is true, a proof is given as follows: By the definition of the hyperoperation $\bullet$, we have that $(\mathbb{R}, \bullet)$ is a commutative hypergroupoid. Let $x, y, z \in \mathbb{R}$. Claim that $(x \bullet y) \bullet z = [\min\{x, y, z\}, \max\{x, y, z\}] = x \bullet (y \bullet z)$. We have that

$$\begin{aligned}
(x \bullet y) \bullet z &= [\min\{x, y\}, \max\{x, y\}] \bullet z \\
&= \bigcup \{t \bullet z \mid t \in [\min\{x, y\}, \max\{x, y\}]\} \\
&= \Big(\bigcup \{[t, z] \mid t \in [\min\{x, y\}, \max\{x, y\}] \text{ and } t \leq z\} \Big) \bigcup \\
&\quad \Big(\bigcup \{[z, t] \mid t \in [\min\{x, y\}, \max\{x, y\}] \text{ and } t > z\} \Big)
\end{aligned}$$

$$= \begin{cases} \varnothing \bigcup [z, \max\{x,y\}] = [\min\{x,y,z\}, \max\{x,y,z\}] \\ \qquad\qquad\qquad\qquad \text{if } z < \min\{x,y\} \\ [\min\{x,y\}, z] \bigcup \varnothing = [\min\{x,y,z\}, \max\{x,y,z\}] \\ \qquad\qquad\qquad\qquad \text{if } z > \max\{x,y\} \\ [\min\{x,y\}, z] \bigcup [z, \max\{x,y\}] = [\min\{x,y,z\}, \max\{x,y,z\}] \\ \qquad\qquad\qquad\qquad \text{if } z \in [\min\{x,y\}, \max\{x,y\}], \end{cases}$$

so $(x \bullet y) \bullet z = [\min\{x,y,z\}, \max\{x,y,z\}]$. We can show similarly that $x \bullet (y \bullet z) = [\min\{x,y,z\}, \max\{x,y,z\}]$. Hence $(\mathbb{R}, \bullet)$ is a semihypergroup. We also have that for $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{R} \bullet x &= \bigcup_{t \in \mathbb{R}} t \bullet x \\ &= \left(\bigcup_{t \leq x} [t, x] \right) \bigcup \left(\bigcup_{t > x} [x, t] \right) \\ &= (-\infty, x] \bigcup [x, \infty) = \mathbb{R}. \end{aligned}$$

This proves that $(\mathbb{R}, \bullet)$ is a commutative hypergroup.

A function f from a hypergroup $(H, \circ)$ into a hypergroup $(H', \circ')$ is called a *homomorphism* if

$$f(x \circ y) \subseteq f(x) \circ' f(y) \text{ for all } x, y \in H.$$

If the equality is valid, f is called a *good homomorphism*. Denote by $\text{Hom}((H, \circ), (H', \circ'))$ and $\text{GHom}((H, \circ), (H', \circ'))$ the set of all homomorphisms and the set of all good homomorphisms from $(H, \circ)$ into $(H', \circ')$, respectively. Let $\text{Hom}(H, \circ)$ and $\text{GHom}(H, \circ)$ stand for $\text{Hom}((H, \circ), (H, \circ))$ and $\text{GHom}((H, \circ), (H, \circ))$, respectively. For $f \in \text{Hom}((H, \circ), (H', \circ'))$, f is called an *epimorphism* if $f(H) = H'$. Denote by $\text{Epi}((H, \circ), (H', \circ'))$ and $\text{GEpi}((H, \circ), (H', \circ'))$ the set of all epimorphisms and the set of all good epimorphisms from $(H, \circ)$ onto $(H', \circ')$, respectively and let $\text{Epi}(H, \circ)$ and $\text{GEpi}(H, \circ)$ stand for $\text{Epi}((H, \circ), (H, \circ))$ and

GEpi$((H,\circ),(H,\circ))$, respectively.

Let $\mathbb{Z}^+$ be the set of positive integers and $\mathbb{Z}_n$ the set of integers modulo $n \in \mathbb{Z}^+$. The equivalence class of $x \in \mathbb{Z}$ modulo n is denoted by $\overline{x}$. For $x, y \in \mathbb{Z}$, not both 0, let (x,y) denote the g.c.d. of x and y. Then

$$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\} = \{\overline{x} \mid x \in \mathbb{Z}\},\ |\mathbb{Z}_n| = n.$$

For $m \in \mathbb{Z}$, $m\mathbb{Z}$ and $m\mathbb{Z}_n$ are subgroups of $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+)$, respectively. We also have that

$$m\mathbb{Z}_n = (m,n)\mathbb{Z}_n = \left\{\overline{0}, \overline{(m,n)}, \dots, \left(\tfrac{n}{(m,n)} - 1\right)\overline{(m,n)}\right\},\ |m\mathbb{Z}_n| = \tfrac{n}{(m,n)},$$

$$\mathbb{Z} = \bigcup_{i=0}^{m-1} (i + m\mathbb{Z}) \text{ if } m \in \mathbb{Z}^+ \text{ and } \ \mathbb{Z}_n = \bigcup_{i=0}^{(m,n)-1} (\overline{i} + (m,n)\mathbb{Z}_n)$$

which are disjoint unions. We give a proof that $\mathbb{Z}_n = \bigcup_{i=0}^{(m,n)-1} (\overline{i} + (m,n)\mathbb{Z}_n)$ which is a disjoint union. Since $\frac{|\mathbb{Z}_n|}{|(m,n)\mathbb{Z}_n|} = \frac{n}{\frac{n}{(m,n)}} = (m,n)$, it follows that the index of the subgroup $(m,n)\mathbb{Z}_n$ in the group $(\mathbb{Z}_n,+)$ is (m,n). If $i, j \in \{0, 1, \dots, (m,n)-1\}$ are such that $\overline{i} + (m,n)\mathbb{Z}_n = \overline{j} + (m,n)\mathbb{Z}_n$, then $\overline{i} - \overline{j} = (m,n)\overline{s}$ for some $s \in \mathbb{Z}$, so $i - j - (m,n)s = nt$ for some $t \in \mathbb{Z}$. Since $(m,n)s + nt$ is divisible by (m,n), we have that $i - j$ is divisible by (m,n). Hence $i = j$, so the desired result follows.

Moreover, $x\mathbb{Z}_n = \mathbb{Z}\overline{x}$ for all $x \in \mathbb{Z}$ and $\mathbb{Z}\overline{x} + \mathbb{Z}\overline{y} = x\mathbb{Z}_n + y\mathbb{Z}_n = (x,y)\mathbb{Z}_n = \mathbb{Z}\overline{(x,y)}$ for all $x, y \in \mathbb{Z}$, not both 0. For $a \in \mathbb{Z}$, define

$$g_a(x) = ax \text{ and } h_{\overline{a}}(\overline{x}) = \overline{ax} \text{ for all } x \in \mathbb{Z}.$$

Then we have that $\mathrm{Hom}(\mathbb{Z},+) = \{g_a \mid a \in \mathbb{Z}\}$, $g_a \neq g_b$ if $a \neq b$ and $\mathrm{Hom}(\mathbb{Z}_n,+) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}$, $h_{\overline{a}} \neq h_{\overline{b}}$ if $\overline{a} \neq \overline{b}$. Notice that for $a \in \mathbb{Z}$, $g_a(\mathbb{Z}) = \mathbb{Z}$ if and only if $a = 1$ or $a = -1$. Since for $a \in \mathbb{Z}$, $\overline{a}\mathbb{Z}_n (= \mathbb{Z}\overline{a}) = \mathbb{Z}_n$ if and only if $\overline{a}$ is a generator of the group $(\mathbb{Z}_n,+)$, it follows that for $a \in \mathbb{Z}$, $h_{\overline{a}}$ is an epimorphism if and only if $(a,n) = 1$.

From Example 1.1, we have that $(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ are the hypergroups where

$$x \circ_{m\mathbb{Z}} y = x + y + m\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z},$$
$$\overline{x} \circ_{m\mathbb{Z}_n} \overline{y} = \overline{x} + \overline{y} + m\mathbb{Z}_n \quad \text{for all } x, y \in \mathbb{Z}.$$

Notice that $(-m)\mathbb{Z} = m\mathbb{Z}$, $(-m)\mathbb{Z}_n = m\mathbb{Z}_n$, $(\mathbb{Z}, \circ_{0\mathbb{Z}}) = (\mathbb{Z}, +)$ and $(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n}) = (\mathbb{Z}_n, +)$. Then $\mathrm{Hom}(\mathbb{Z}, \circ_{0\mathbb{Z}}) = \mathrm{GHom}(\mathbb{Z}, \circ_{0\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\}$, $\mathrm{Epi}(\mathbb{Z}, \circ_{0\mathbb{Z}}) = \mathrm{GEpi}(\mathbb{Z}, \circ_{0\mathbb{Z}}) = \mathrm{Epi}(\mathbb{Z}, +) = \{g_1, g_{-1}\}$, $\mathrm{Hom}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n}) = \mathrm{GHom}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n}) = \mathrm{Hom}(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}$ and $\mathrm{Epi}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n}) = \mathrm{GEpi}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n}) = \mathrm{Epi}(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z} \text{ and } (a, n) = 1\}$. This implies that $|\mathrm{Hom}(\mathbb{Z}, \circ_{0\mathbb{Z}})| = |\mathrm{GHom}(\mathbb{Z}, \circ_{0\mathbb{Z}})| = \aleph_0$, $|\mathrm{Epi}(\mathbb{Z}, \circ_{0\mathbb{Z}})| = |\mathrm{GEpi}(\mathbb{Z}, \circ_{0\mathbb{Z}})| = 2$, $|\mathrm{Hom}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n})| = |\mathrm{GHom}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n})| = n$ and $|\mathrm{Epi}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n})| = |\mathrm{GEpi}(\mathbb{Z}_n, \circ_{0\mathbb{Z}_n})| = \phi(n)$ where ϕ is the Euler-phi function. Recall that for a positive integer n, $\phi(n)$ is the number of $x \in \{1, 2, \ldots, n\}$ relatively prime to n.

Throughout this research, we assume that $m \in \mathbb{Z}^+$. However, some results we obtain are clearly true when $m = 0$. In [7], the authors characterized the good homomorphisms and good epimorphisms of $(\mathbb{Z}, \circ_{m\mathbb{Z}})$ by introducing the following general result.

Lemma 1.6. ([7]) *Let G be a group and N a normal subgroup of G. Then the following statements hold.*

(i) *For every $f \in \mathrm{GHom}(G, \circ_N)$, $f(N) = N$.*

(ii) *If $f \in \mathrm{GHom}(G, \circ_N)$, $x \in G$ and $k \in \mathbb{Z}$, then $f(x^k N) = (f(x))^k N$.*

Theorem 1.7. ([7]) *If $f : \mathbb{Z} \to \mathbb{Z}$, then the following statements are equivalent.*

(i) $f \in \mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$.

(ii) $f(x + m\mathbb{Z}) = xf(1) + m\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer a such that*

$$f(x + m\mathbb{Z}) = xa + m\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

We give a remark that if a satisfies (iii) of Theorem 1.7, then $a \equiv f(1)(\mathrm{mod}\ m)$.

The following facts are also used in our work.

Proposition 1.8. ([7]) *If G is a group, then* $\mathrm{GHom}(G, \circ_G) = \{f : G \to G \mid f(G) = G\} = \mathrm{GEpi}(G, \circ_G)$.

Theorem 1.9. ([7]) *If X is an infinite set, then*

$$|\{f : X \to X \mid f(X) = X\}| = 2^{|X|}.$$

In [7], the elements of $\mathrm{GEpi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ were characterized and $|\mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})|$ and $|\mathrm{GEpi}(\mathbb{Z}, \circ_{m\mathbb{Z}})|$ were determined as follows :

Theorem 1.10. ([7]) *For $f \in \mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$, $f \in \mathrm{GEpi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ if and only if $f(1)$ and m are relatively prime.*

Theorem 1.11. ([7]) $|\mathrm{GHom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = |\mathrm{GEpi}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}$.

In the proof of Theorem 1.11 the following fact of cardinal numbers was used. If p is an infinite cardinal number, then $p^p = 2^p$ ([10], p.161). In particular, $\aleph_0^{\aleph_0} = 2^{\aleph_0}$.

The following fact relating to a set of functions and its cardinality is used. If X and Y are nonempty sets, then

$$|\{f \mid f : X \to Y\}| = |Y|^{|X|}.$$

In particular, if X is an infinite set, then

$$|\{f \mid f : X \to X\}| = |X|^{|X|} = 2^{|X|}.$$

The following theorem of homomorphisms and good homomorphisms on P-hypergroups is known.

Theorem 1.12. ([2]) *Let G be a group and $\varnothing \neq P \subseteq G$. Then the following statements hold.*

(i) *For $f \in \mathrm{Hom}(G)$, $f \in \mathrm{Hom}(G, \bullet_P)$ if and only if $f(P) \subseteq P$.*

(ii) *For $f \in \mathrm{Hom}(G)$, $f \in \mathrm{GHom}(G, \bullet_P)$ if and only if $f(P) = P$.*

From Theorem 1.12 and the facts that

$$\mathrm{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\} \text{ and } \mathrm{Hom}(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\},$$

the following theorem is directly obtained.

Theorem 1.13. *The following statements hold.*

(i) *For $\varnothing \neq P \subseteq \mathbb{Z}$ and $a \in \mathbb{Z}$, $g_a \in \mathrm{Hom}(\mathbb{Z}, \bullet_P)$ if and only if $aP \subseteq P$.*

(ii) *For $\varnothing \neq P \subseteq \mathbb{Z}$ and $a \in \mathbb{Z}$, $g_a \in \mathrm{GHom}(\mathbb{Z}, \bullet_P)$ if and only if $aP = P$.*

(iii) *For* $\varnothing \neq P \subseteq \mathbb{Z}_n$ *and* $a \in \mathbb{Z}_n$, $h_{\overline{a}} \in \mathrm{Hom}(\mathbb{Z}_n, \bullet_P)$ *if and only if* $\overline{a}P \subseteq P$.

(iv) *For* $\varnothing \neq P \subseteq \mathbb{Z}_n$ *and* $a \in \mathbb{Z}_n$, $h_{\overline{a}} \in \mathrm{GHom}(\mathbb{Z}_n, \bullet_P)$ *if and only if* $\overline{a}P = P$.

The following theorem was given in [2]. In fact, it follows from Theorem 1.13(i) and (iii) and the fact that each of $\mathbb{Z}$ and $\mathbb{Z}_n$ contains a multiplicative identity.

Theorem 1.14. ([2]) *The following statements hold.*

(i) *For* $\varnothing \neq P \subseteq \mathbb{Z}$, $\mathrm{Hom}(\mathbb{Z}, +) \subseteq \mathrm{Hom}(\mathbb{Z}, \bullet_P)$ *if and only if* $\mathbb{Z}P = P$.

(ii) *For* $\varnothing \neq P \subseteq \mathbb{Z}_n$, $\mathrm{Hom}(\mathbb{Z}_n, +) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \bullet_P)$ *if and only if* $\mathbb{Z}_nP = P$.

CHAPTER II

HOMOMORPHISMS OF HYPERGROUPS DEFINED FROM THE GROUP $(\mathbb{Z}, +)$ AND ITS SUBGROUPS

In this chapter, we characterize the homomorphisms and the epimorphisms of the hypergroup $(\mathbb{Z}, \circ_{m\mathbb{Z}})$ which is defined from the group $(\mathbb{Z}, +)$ and its subgroup $m\mathbb{Z}$ as in Example 1.1. The cardinalities of $\text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $\text{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ are also provided. The purpose is to extend Theorem 1.7, Theorem 1.10 and Theorem 1.11.

2.1 Characterizations of Homomorphisms and Epimorphisms

First we recall that $x \circ_{m\mathbb{Z}} y = x + y + m\mathbb{Z}$ for all $x, y \in \mathbb{Z}$. We characterize the elements of $\text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. The proof of Lemma 1.6 gives us an idea of proving the following general results which will be used for our characterization.

Lemma 2.1.1. *Let G be a group, N a normal subgroup of G. Then the following statements hold for $f \in \text{Hom}(G, \circ_N)$.*

(i) $f(N) \subseteq N$.

(ii) *For all $x \in G$, $f(xN) \subseteq f(x)N$.*

(iii) *For all $x, y \in G$, $f(xyN) \subseteq f(xy)N = f(x)f(y)N$.*

(iv) *For all $x \in G$, $f(x^{-1}N) \subseteq f(x^{-1})N = f(x)^{-1}N$.*

(v) *For all $x \in G$ and $k \in \mathbb{Z}$, $f(x^kN) \subseteq f(x^k)N = f(x)^kN$.*

Proof. First, we recall that for all $x, y \in G$, $xN \cap yN \neq \varnothing$ implies $xN = yN$.

(i) We have that

$$f(N) = f(eeN) = f(e \circ_N e) \subseteq f(e) \circ_N f(e) = f(e)f(e)N.$$

Then $f(e) \in f(N) \subseteq f(e)f(e)N$. Since G is cancellative, we have $e \in f(e)N$

which implies that $N = f(e)N$, so $f(N) \subseteq f(e)f(e)N = N$.

(ii) By (i), $f(e) \in N$. If $x \in G$, then

$$\begin{aligned} f(xN) = f(xeN) &= f(x \circ_N e) \\ &\subseteq f(x) \circ_N f(e) \\ &= f(x)f(e)N \\ &= f(x)N. \end{aligned}$$

(iii) Let $x, y \in G$. Then by (ii),

$$f(xyN) \subseteq f(xy)N.$$

We also have that

$$\begin{aligned} f(xyN) &= f(x \circ_N y) \\ &\subseteq f(x) \circ_N f(y) \\ &= f(x)f(y)N. \end{aligned}$$

Then $f(xyN) \subseteq f(xy)N \bigcap f(x)f(y)N$ which implies that $f(xy)N = f(x)f(y)N$. Hence (iii) holds.

(iv) If $x \in G$, then

$$f(N) = f(xx^{-1}N) = f(x \circ_N x^{-1}) \subseteq f(x)f(x^{-1})N.$$

But $f(N) \subseteq N$ by (i), so $f(N) \subseteq N \cap f(x)f(x^{-1})N$. Then $N = f(x)f(x^{-1})N$ which implies that $f(x^{-1})N = f(x)^{-1}N$. By (ii), $f(x^{-1}N) \subseteq f(x^{-1})N$. Hence (iv) holds.

(v) Let $x \in G$. Then by (ii), for all $k \in \mathbb{Z}$, $f(x^kN) \subseteq f(x^k)N$. It remains to show that $f(x^k)N = f(x)^kN$ for all $k \in \mathbb{Z}$. This is true for $k = 1$, and by (i), this is true for $k = 0$. Assume that $k \in \mathbb{Z}^+$ and $f(x^k)N = f(x)^kN$. Then

$$\begin{aligned} f(x^{k+1})N &= f(xx^k)N \\ &= f(x)f(x^k)N && \text{by (iii)} \\ &= f(x)(f(x^k)N) \\ &= f(x)(f(x)^kN) && \text{by assumption} \\ &= f(x)^{k+1}N. \end{aligned}$$

This shows that $f(y^l)N = f(y)^l N$ for all $y \in G$ and $l \in \mathbb{Z}^+$. If $k \in \mathbb{Z}^+$, then

$$
\begin{aligned}
f(x^{-k})N &= f((x^{-1})^k)N \\
&= f(x^{-1})^k N \\
&= (f(x^{-1})N)\dots(f(x^{-1})N) \qquad (k \text{ brackets}) \\
&= (f(x)^{-1}N)\dots(f(x)^{-1}N) \qquad \text{by (iv)} \\
&= (f(x)^{-1})^k N \\
&= f(x)^{-k}N.
\end{aligned}
$$

Hence (v) is proved. □

Theorem 2.1.2. *For $f : \mathbb{Z} \to \mathbb{Z}$, the following statements are equivalent.*

(i) $f \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$.

(ii) $f(x + m\mathbb{Z}) \subseteq xf(1) + m\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer a such that*

$$f(x + m\mathbb{Z}) \subseteq xa + m\mathbb{Z} \ \textit{ for all } x \in \mathbb{Z}.$$

Proof. (i)⇒(ii) follows directly from Lemma 2.1.1(v).

(ii)⇒(iii) is evident.

(iii)⇒(i). Let $x, y \in \mathbb{Z}$. Then $f(x) \in f(x) + m\mathbb{Z}$ and $f(y) \in f(y) + m\mathbb{Z}$. Since $f(x) \in f(x + m\mathbb{Z}) \subseteq xa + m\mathbb{Z}$ and $f(y) \in f(y + m\mathbb{Z}) \subseteq ya + m\mathbb{Z}$, it follows that $f(x) + m\mathbb{Z} = xa + m\mathbb{Z}$ and $f(y) + m\mathbb{Z} = ya + m\mathbb{Z}$. Consequently,

$$
\begin{aligned}
f(x \circ_{m\mathbb{Z}} y) &= f(x + y + m\mathbb{Z}) \\
&\subseteq (x + y)a + m\mathbb{Z} \\
&= xa + m\mathbb{Z} + ya + m\mathbb{Z} \\
&= f(x) + m\mathbb{Z} + f(y) + m\mathbb{Z} \\
&= f(x) + f(y) + m\mathbb{Z} \\
&= f(x) \circ_{m\mathbb{Z}} f(y).
\end{aligned}
$$

Hence $f \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$, as desired. □

Remark 2.1.3. For $f : \mathbb{Z} \to \mathbb{Z}$ and $a \in \mathbb{Z}$, if f and a satisfy (iii) of Theorem 2.1.2, then $a \equiv f(1) (\text{mod } m)$ since $f(1) \in f(1 + m\mathbb{Z}) \subseteq a + m\mathbb{Z}$.

Next, we provide the following general fact. It is used to characterize the elements of $\text{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$.

Lemma 2.1.4. *Let G be a group and N a normal subgroup of G. If the index $[G : N]$ of N in G is finite and $f \in \text{Epi}(G, \circ_N)$, then $f(xN) = f(x)N$ for all $x \in G$.*

Proof. Let $[G : N] = n$. Then there are $x_1, \ldots, x_n \in G$ such that $G = \bigcup_{i=1}^{n} x_i N$. Then $x_1N, \ldots, x_nN$ are mutually disjoint. By Lemma 2.1.1(ii), $f(x_iN) \subseteq f(x_i)N$ for all $i \in \{1, \ldots, n\}$. Hence

$$G = f(\bigcup_{i=1}^{n} x_i N) = \bigcup_{i=1}^{n} f(x_i N) \subseteq \bigcup_{i=1}^{n} f(x_i)N,$$

which implies that

$$G = \bigcup_{i=1}^{n} f(x_i N) = \bigcup_{i=1}^{n} f(x_i)N.$$

Since $[G : N] = n$, it follows that $f(x_1)N, \ldots, f(x_n)N$ are mutually disjoint. But $f(x_iN) \subseteq f(x_i)N$ for all $i \in \{1, \ldots, n\}$, thus we have

$$f(x_iN) = f(x_i)N \text{ for all } i \in \{1, \ldots, n\}.$$

Next, let $x \in G$. Then $xN = x_jN$ for some $j \in \{1, \ldots, n\}$. By Lemma 2.1.1(ii), $f(xN) \subseteq f(x)N$. Hence

$$f(x_j)N = f(x_jN) = f(xN) \subseteq f(x)N$$

which implies that $f(x)N = f(x_j)N$. Consequently,

$$f(xN) = f(x_jN) = f(x_j)N = f(x)N.$$

Hence $f(xN) = f(x)N$ for all $x \in G$. □

Theorem 2.1.5. *For $f : \mathbb{Z} \to \mathbb{Z}$, $f \in \mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ if and only if*

(i) $f(x + m\mathbb{Z}) = xf(1) + m\mathbb{Z}$ *for all* $x \in \mathbb{Z}$ *and*

(ii) $f(1)$ *and* m *are relatively prime.*

Proof. First, assume that $f \in \mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. By Lemma 2.1.4, $f(x+m\mathbb{Z}) = f(x)+m\mathbb{Z}$ for all $x \in \mathbb{Z}$. But by Lemma 2.1.1(v), $f(x)+m\mathbb{Z} = xf(1)+m\mathbb{Z}$ for all $x \in \mathbb{Z}$. Thus (i) holds. The fact that $f(\mathbb{Z}) = \mathbb{Z}$ and (i) yield

$$\mathbb{Z} = f\big(\bigcup_{x\in\mathbb{Z}} (x + m\mathbb{Z})\big) = \bigcup_{x\in\mathbb{Z}} (xf(1) + m\mathbb{Z}).$$

Then $1 \in yf(1) + m\mathbb{Z}$ for some $y \in \mathbb{Z}$. Thus $1 = yf(1) + tm$ for some $t \in \mathbb{Z}$ which implies that $f(1)$ and m are relatively prime. Therefore (ii) holds.

For the converse, assume that (i) and (ii) hold. Then from (i) and Theorem 2.1.2, $f \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. From (ii), $sf(1) + tm = 1$ for some $s, t \in \mathbb{Z}$. But since

$$\begin{aligned} \text{for every } x \in \mathbb{Z},\ \ x + m\mathbb{Z} &= x(sf(1) + tm) + m\mathbb{Z} \\ &= xsf(1) + m\mathbb{Z} \\ &= f(xs + m\mathbb{Z}) \qquad\qquad \text{by (i)} \\ &\subseteq f(\mathbb{Z}), \end{aligned}$$

it follows that $f(\mathbb{Z}) = \mathbb{Z}$. Hence $f \in \mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. □

Remark 2.1.6. We have that $\mathrm{Hom}(H, \circ)$ is a semigroup under composition where $(H, \circ)$ is a hypergroup. Note that 1_H, the identity function on H, is clearly an element of $\mathrm{Hom}(H, \circ)$. Let $f, g \in \mathrm{Hom}(H, \circ)$ and $x, y \in H$. Then

$$\begin{aligned} (gf)(x \circ y) = g(f(x \circ y)) &\subseteq g(f(x) \circ f(y)) \\ &\subseteq g(f(x)) \circ g(f(y)) \\ &= (gf)(x) \circ (gf)(y). \end{aligned}$$

We know that $F(H)$ is a semigroup under composition where $F(H)$ is the set of all functions from H into itself. It follows that $\mathrm{Hom}(H, \circ)$ is a subsemigroup of $F(H)$. It is clearly seen that $\mathrm{GHom}(H, \circ)$, $\mathrm{Epi}(H, \circ)$ and $\mathrm{GEpi}(H, \circ)$ are subsemigroups of the semigroup $\mathrm{Hom}(H, \circ)$.

As mentioned above, we have that $\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ is a semigroup having GHom $(\mathbb{Z}, \circ_{m\mathbb{Z}})$, $\mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $\mathrm{GEpi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ as its subsemigroups. By Theorem 2.1.2, for all $f \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$,

$$f(x + m\mathbb{Z}) \subseteq xf(1) + m\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

If $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$, then

$$(gf)(1 + m\mathbb{Z}) = g(f(1 + m\mathbb{Z})) \subseteq g(f(1) + m\mathbb{Z}) \subseteq f(1)g(1) + m\mathbb{Z}$$

and

$$(gf)(1 + m\mathbb{Z}) \subseteq (gf)(1) + m\mathbb{Z}.$$

This implies that $f(1)g(1) + m\mathbb{Z} = (gf)(1) + m\mathbb{Z}$. It follows that

$$(gf)(1) \equiv f(1)g(1) \equiv g(1)f(1) \equiv (fg)(1) (\mathrm{mod}\ m).$$

Next, we claim that $(\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}}), +)$ is an abelian group. First, we note that $\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}}) \subseteq F(\mathbb{Z})$ and $(F(\mathbb{Z}), +)$ is an abelian group where $F(\mathbb{Z})$ is the set of all functions from $\mathbb{Z}$ into itself. Let $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $x \in \mathbb{Z}$. Then

$$\begin{aligned}(f + g)(x + m\mathbb{Z}) &\subseteq f(x + m\mathbb{Z}) + g(x + m\mathbb{Z}) \\ &\subseteq xf(1) + m\mathbb{Z} + xg(1) + m\mathbb{Z} \\ &= x(f(1) + g(1)) + m\mathbb{Z} \\ &= x((f + g)(1)) + m\mathbb{Z},\end{aligned}$$

so by Theorem 2.1.2, $f + g \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. Since

$$\begin{aligned}(-f)(x + m\mathbb{Z}) = -(f(x + m\mathbb{Z})) &\subseteq -(xf(1) + m\mathbb{Z}) \\ &= x(-f(1)) + (-m\mathbb{Z}) \\ &= x((-f)(1)) + m\mathbb{Z},\end{aligned}$$

$-f \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. Hence we have the claim.

2.2 Results on Cardinalities

This section is concerned with the cardinalities of $\text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ and $\text{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$.

If $a \in \mathbb{Z}$, then for $x, y \in \mathbb{Z}$,

$$
\begin{aligned}
g_a(x \circ_{m\mathbb{Z}} y) &= g_a(x + y + m\mathbb{Z}) \\
&= a(x + y + m\mathbb{Z}) \\
&= ax + ay + am\mathbb{Z} \\
&\subseteq ax + ay + m\mathbb{Z} \\
&= ax \circ_{m\mathbb{Z}} ay \\
&= g_a(x) \circ_{m\mathbb{Z}} g_a(y).
\end{aligned}
$$

This shows that $g_a \in \text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ for all $a \in \mathbb{Z}$. Hence $\text{Hom}(\mathbb{Z}, +) \subseteq \text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. Observe that $g_a(m\mathbb{Z}) = am\mathbb{Z} \subseteq m\mathbb{Z}$ for all $a \in \mathbb{Z}$. In general, we have that if N is a normal subgroup of a group G and $f \in \text{Hom}(G)$ such that $f(N) \subseteq N$, then $f \in \text{Hom}(G, \circ_N)$. The proof is given as follows: For $x, y \in G$,

$$
\begin{aligned}
f(x \circ_N y) &= f(xyN) \\
&= f(x)f(y)f(N) \\
&\subseteq f(x)f(y)N \\
&= f(x) \circ_N f(y).
\end{aligned}
$$

Hence we have

Theorem 2.2.1. *If G is a group and N is a normal subgroup of G, then*

$$\{f \in \text{Hom}(G) \mid f(N) \subseteq N\} \subseteq \text{Hom}(G, \circ_N).$$

From the fact that $\text{Hom}(\mathbb{Z}, +) \subseteq \text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$, we have $|\text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| \geq \aleph_0$. It will be shown that,

$$|\text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = |\text{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}.$$

To show that $|\text{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}$, we need the following lemma.

Lemma 2.2.2. *If G is a group, then* $\text{Hom}(G, \circ_G) = \{f \mid f : G \to G\}$.

Proof. If $f : G \to G$, then for all $x, y \in G$,

$$f(x \circ_G y) = f(xyG) = f(G) \subseteq G = f(x)f(y)G = f(x) \circ_G f(y),$$

so $f \in \mathrm{Hom}(G, \circ_G)$.

Hence the result follows. □

Theorem 2.2.3. $|\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}$.

Proof. By Lemma 2.2.2, $\mathrm{Hom}(\mathbb{Z}, \circ_{1\mathbb{Z}}) = \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$. Then

$$|\mathrm{Hom}(\mathbb{Z}, \circ_{1\mathbb{Z}})| = |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

Next, assume that $m > 1$. Let $K = \{g \mid g : m\mathbb{Z} \to m\mathbb{Z}\}$. Then $|K| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. Recall that for each $x \in \mathbb{Z}$, there are unique $q_x \in \mathbb{Z}$ and $r_x \in \{0, 1, \ldots, m-1\}$ such that $x = mq_x + r_x$. For each $g \in K$, define $\bar{g} : \mathbb{Z} \to \mathbb{Z}$ by

$$\bar{g}(x) = r_x + g(mq_x) \text{ for all } x \in \mathbb{Z}.$$

Then for every $g \in K$, $\bar{g}_{|m\mathbb{Z}} = g$ and for $x \in \mathbb{Z}$,

$$\begin{aligned}
\bar{g}(x + m\mathbb{Z}) &= \bar{g}(r_x + mq_x + m\mathbb{Z}) \\
&= \bar{g}(r_x + m\mathbb{Z}) \\
&= r_x + g(m\mathbb{Z}) \\
&\subseteq r_x + m\mathbb{Z} \\
&= r_x + mq_x + m\mathbb{Z} \\
&= x + m\mathbb{Z}.
\end{aligned}$$

By Theorem 2.1.2, we have that $\bar{g} \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ for all $g \in K$. It follows that

$$\begin{aligned}
2^{\aleph_0} = |K| &= |\{\bar{g} \mid g \in K\}| \\
&\leq |\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| \\
&\leq |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}
\end{aligned}$$

which implies that $|\mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}$.

Hence the theorem is proved. □

Next we show that $|\mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}$. Theorem 1.9 is also needed to prove this fact.

Theorem 2.2.4. $|\mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})| = 2^{\aleph_0}$.

Proof. By Lemma 2.2.2, we have that $\mathrm{Epi}(\mathbb{Z}, \circ_{1\mathbb{Z}}) = \{f : \mathbb{Z} \to \mathbb{Z} \mid f(\mathbb{Z}) = \mathbb{Z}\}$. Then by Theorem 1.9, $|\mathrm{Epi}(\mathbb{Z}, \circ_{1\mathbb{Z}})| = 2^{\aleph_0}$.

Assume that $m > 1$. Let $L = \{g : m\mathbb{Z} \to m\mathbb{Z} \mid g(m\mathbb{Z}) = m\mathbb{Z}\}$. Also, by Theorem 1.9, $|L| = 2^{\aleph_0}$. For each $x \in \mathbb{Z}$, let $q_x, r_x \in \mathbb{Z}$ be such that $r_x \in \{0, 1, \ldots, m-1\}$ and $x = mq_x + r_x$. Note that q_x and r_x are unique. For each $g \in L$, define $\bar{g} : \mathbb{Z} \to \mathbb{Z}$ by

$$\bar{g}(x) = r_x + g(mq_x) \text{ for all } x \in \mathbb{Z}.$$

Then for $g \in L, \bar{g}_{|m\mathbb{Z}} = g$ and we can see from the proof of Theorem 2.2.3 and the fact that $g(m\mathbb{Z}) = m\mathbb{Z}$ that

$$\bar{g}(x + m\mathbb{Z}) = x + m\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}.$$

It follows from Theorem 2.1.2 that $\bar{g} \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ for all $g \in L$. We also have that

$$\bar{g}(\mathbb{Z}) = \bar{g}\big(\bigcup_{x\in\mathbb{Z}} (x + m\mathbb{Z})\big) = \bigcup_{x\in\mathbb{Z}} \bar{g}(x + m\mathbb{Z}) = \bigcup_{x\in\mathbb{Z}} (x + m\mathbb{Z}) = \mathbb{Z}.$$

Hence $\bar{g} \in \mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ for all $g \in L$. Consequently,

$$\begin{aligned} 2^{\aleph_0} = |L| &= |\{\bar{g} \mid g \in L\}| \\ &\le |\mathrm{Epi}(\mathbb{Z}, \circ_{m\mathbb{Z}})| \\ &\le |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}, \end{aligned}$$

so the desired result follows. □

CHAPTER III

HOMOMORPHISMS OF HYPERGROUPS DEFINED FROM THE GROUP $(\mathbb{Z}_n, +)$ AND ITS SUBGROUPS

In this chapter, we characterize the homomorphisms, the good homomorphisms, the epimorphisms and the good epimorphisms of the hypergroup $(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ (see Example 1.1). The cardinalities of $\text{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, $\text{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, Epi $(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ and $\text{GEpi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ are also determined.

3.1 Characterizations of Homomorphisms, Good Homomorphisms, Epimorphisms and Good Epimorphisms

Let us recall that $\overline{x} \circ_{m\mathbb{Z}_n} \overline{y} = \overline{x} + \overline{y} + m\mathbb{Z}_n$ for all $x, y \in \mathbb{Z}$. Lemma 2.1.1 is needed to characterize the elements of $\text{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$.

Theorem 3.1.1. *For $f : \mathbb{Z}_n \to \mathbb{Z}_n$, the following statements are equivalent.*

(i) $f \in \text{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$.

(ii) $f(\overline{x} + m\mathbb{Z}_n) \subseteq xf(\overline{1}) + m\mathbb{Z}_n$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer a such that*

$$f(\overline{x} + m\mathbb{Z}_n) \subseteq x\overline{a} + m\mathbb{Z}_n \ \ \textit{for all } x \in \mathbb{Z}.$$

Proof. (i)⇒(ii) follows directly from Lemma 2.1.1(v).

(ii)⇒(iii) is evident.

(iii)⇒(i). Let $x, y \in \mathbb{Z}$. Then $f(\overline{x}) \in f(\overline{x}) + m\mathbb{Z}_n$ and $f(\overline{y}) \in f(\overline{y}) + m\mathbb{Z}_n$. Since $f(\overline{x}) \in f(\overline{x} + m\mathbb{Z}_n) \subseteq x\overline{a} + m\mathbb{Z}_n$ and $f(\overline{y}) \in f(\overline{y} + m\mathbb{Z}_n) \subseteq y\overline{a} + m\mathbb{Z}_n$, it follows that $f(\overline{x}) + m\mathbb{Z}_n = x\overline{a} + m\mathbb{Z}_n$ and $f(\overline{y}) + m\mathbb{Z}_n = y\overline{a} + m\mathbb{Z}_n$. Therefore we have that

$$f(\overline{x} \circ_{m\mathbb{Z}_n} \overline{y}) = f(\overline{x} + \overline{y} + m\mathbb{Z}_n)$$

$$
\begin{aligned}
&\subseteq (x+y)\overline{a} + m\mathbb{Z}_n \\
&= x\overline{a} + m\mathbb{Z}_n + y\overline{a} + m\mathbb{Z}_n \\
&= f(\overline{x}) + m\mathbb{Z}_n + f(\overline{y}) + m\mathbb{Z}_n \\
&= f(\overline{x}) + f(\overline{y}) + m\mathbb{Z}_n \\
&= f(\overline{x}) \circ_{m\mathbb{Z}_n} f(\overline{y}).
\end{aligned}
$$

Hence $f \in \mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, as desired. □

We can see easily from Lemma 1.6(ii) and the proof of Theorem 3.1.1 that the following result holds.

Theorem 3.1.2. *For $f : \mathbb{Z}_n \to \mathbb{Z}_n$, the following statements are equivalent.*

(i) $f \in \mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$.

(ii) $f(\overline{x} + m\mathbb{Z}_n) = xf(\overline{1}) + m\mathbb{Z}_n$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer a such that*

$$f(\overline{x} + m\mathbb{Z}_n) = x\overline{a} + m\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

We need Lemma 2.1.4 to characterize the elements of $\mathrm{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$.

Theorem 3.1.3. *For $f : \mathbb{Z}_n \to \mathbb{Z}_n$, $f \in \mathrm{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ if and only if the following conditions hold.*

(i) $f(\overline{x} + m\mathbb{Z}_n) = xf(\overline{1}) + m\mathbb{Z}_n$ *for all* $x \in \mathbb{Z}$.

(ii) *If $f(\overline{1}) = \overline{a}$ for $a \in \mathbb{Z}$, then a and (m, n) are relatively prime.*

Proof. Assume that $f \in \mathrm{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$. The condition (i) follows directly from Lemma 2.1.4 and Lemma 2.1.1(v). Let $f(\overline{1}) = \overline{a}$ where $a \in \mathbb{Z}$. Since $f(\mathbb{Z}_n) = \mathbb{Z}_n$, it follows from Lemma 2.1.1(v) that

$$\mathbb{Z}_n = f\big(\bigcup_{x\in\mathbb{Z}} (\overline{x} + (m,n)\mathbb{Z}_n)\big) \subseteq \bigcup_{x\in\mathbb{Z}} (xf(\overline{1}) + (m,n)\mathbb{Z}_n).$$

Then $\overline{1} \in yf(\overline{1}) + (m,n)\mathbb{Z}_n$ for some $y \in \mathbb{Z}$, so $\overline{1} = y\overline{a} + (m,n)\overline{z}$ for some $z \in \mathbb{Z}$. Hence $1 = ya + (m,n)z + nw$ for some $w \in \mathbb{Z}$, so $ya + (m,n)(z + \frac{n}{(m,n)}w) = 1$ which implies that a and (m,n) are relatively prime. Hence (ii) holds.

For the converse, assume that (i) and (ii) hold. Then from (i) and Theorem 3.1.1, $f \in \operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$. From (ii), we have that there are $y, z \in \mathbb{Z}$ such that $ay + (m,n)z = 1$. Then

$$\overline{1} = y\overline{a} + (m,n)\overline{z} \in yf(\overline{1}) + (m,n)\mathbb{Z}_n.$$

Hence from (i), we have that for $x \in \mathbb{Z}$,

$$\begin{aligned} \overline{x} = x\overline{1} &\in x(yf(\overline{1}) + (m,n)\mathbb{Z}_n) \\ &\subseteq xyf(\overline{1}) + (m,n)\mathbb{Z}_n = f(\overline{xy} + (m,n)\mathbb{Z}_n) \subseteq f(\mathbb{Z}_n) \end{aligned}$$

which implies that $f(\mathbb{Z}_n) = \mathbb{Z}_n$. Thus $f \in \operatorname{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$.

Hence the theorem is proved. □

The following result follows directly from Theorem 3.1.2 and Theorem 3.1.3.

Corollary 3.1.4. $\operatorname{GEpi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) = \operatorname{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) \subseteq \operatorname{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$.

Remark 3.1.5. From Remark 2.1.6, we have that $\operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ is a semigroup under composition having $\operatorname{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, $\operatorname{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})(= \operatorname{GEpi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}))$ as its subsemigroups. We can see from the proof given in Remark 2.1.6 that for all $f, g \in \operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$,

$$(gf)(1) + m\mathbb{Z}_n = f(1)g(1) + m\mathbb{Z}_n = g(1)f(1) + m\mathbb{Z}_n = (fg)(1) + m\mathbb{Z}_n,$$

Moreover, $(\operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}), +)$ is also an abelian group.

3.2 Combinatorial Results

In this section, we determine the cardinalities of the sets $\operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, GHom $(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ and $\operatorname{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})(= \operatorname{GEpi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}))$.

For $a \in \mathbb{Z}$, we have that $h_{\overline{a}}(m\mathbb{Z}_n) = am\mathbb{Z}_n \subseteq m\mathbb{Z}_n$. It follows from Theorem 2.2.1 that $\operatorname{Hom}(\mathbb{Z}_n, +) \subseteq \operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, so $\operatorname{Epi}(\mathbb{Z}_n, +) \subseteq \operatorname{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$. Consequently, $|\operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| \geq n$ and $|\operatorname{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| \geq \phi(n)$.

Lemma 2.2.2 is also needed to determine $|\operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})|$.

Theorem 3.2.1. $|\operatorname{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = n\left(\frac{n}{(m,n)}\right)^{n-1}$.

Proof. Recall that $|m\mathbb{Z}_n| = \frac{n}{(m,n)}$,

$$\mathbb{Z}_n = \bigcup_{i=0}^{(m,n)-1} (\bar{i} + (m,n)\mathbb{Z}_n)$$

which is a disjoint union and note that for nonempty sets $A, B, |\{f \mid f : A \to B\}| = |B|^{|A|}$.

Case 1 : $(m,n) = 1$. Then $m\mathbb{Z}_n = \mathbb{Z}_n$ and so $(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) = (\mathbb{Z}_n, \circ_{\mathbb{Z}_n})$. By Lemma 2.2.2, $|\mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = n^n$. Hence $|\mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = n\left(\frac{n}{(m,n)}\right)^{n-1}$.

Case 2 : $(m,n) > 1$. Then $n > 1$. By Theorem 3.1.1, we have that

$$\mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) = \{f : \mathbb{Z}_n \to \mathbb{Z}_n \mid f(\overline{x} + m\mathbb{Z}_n) \subseteq xf(\overline{1}) + m\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}\}.$$

It follows that for $f : \mathbb{Z}_n \to \mathbb{Z}_n$,

$$\begin{aligned} f \in \mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) \Longleftrightarrow & f((m,n)\mathbb{Z}_n) \subseteq (m,n)\mathbb{Z}_n, \\ & f(\overline{1} + (m,n)\mathbb{Z}_n) \subseteq f(\overline{1}) + (m,n)\mathbb{Z}_n, \\ & f(\overline{2} + (m,n)\mathbb{Z}_n) \subseteq 2f(\overline{1}) + (m,n)\mathbb{Z}_n, \\ & \ldots \\ & f(\overline{(m,n)-1} + (m,n)\mathbb{Z}_n) \subseteq ((m,n)-1)f(\overline{1}) + (m,n)\mathbb{Z}_n. \end{aligned}$$

For $f : \mathbb{Z}_n \to \mathbb{Z}_n$, all the possibilities of $f(\overline{1})$ are $\overline{0}, \overline{1}, \ldots, \overline{n-1}$. We have that $f(\overline{1}) \in f(\overline{1} + (m,n)\mathbb{Z}_n)$. From these facts, we have

$$\begin{aligned} |\mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| &= n \times \left(\frac{n}{(m,n)}\right)^{\frac{n}{(m,n)}} \times \left(\frac{n}{(m,n)}\right)^{\frac{n}{(m,n)}-1} \\ &\quad \times \underbrace{\left(\frac{n}{(m,n)}\right)^{\frac{n}{(m,n)}} \times \cdots \times \left(\frac{n}{(m,n)}\right)^{\frac{n}{(m,n)}}}_{(m,n)-2 \text{ copies}} \\ &= n \times \left(\frac{n}{(m,n)}\right)^{\frac{n}{(m,n)} \times (m,n) - 1} \\ &= n\left(\frac{n}{(m,n)}\right)^{n-1}. \end{aligned}$$

Hence the proof is complete. □

Next, $|\mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})|$ is determined by using Proposition 1.8 and Theorem 3.1.2

Theorem 3.2.2. $|\mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = n\left(\left(\frac{n}{(m,n)} - 1\right)!\right)\left(\left(\frac{n}{(m,n)}\right)!\right)^{(m,n)-1}$.

Proof. Recall that

$$\mathbb{Z}_n = \bigcup_{i=0}^{(m,n)-1} (\bar{i} + (m,n)\mathbb{Z}_n)$$

which is a disjoint union and $|\bar{i} + (m,n)\mathbb{Z}_n| = |(m,n)\mathbb{Z}_n| = \left|\frac{n}{(m,n)}\right|$ for all $i \in \{0, 1, \ldots, (m,n) - 1\}$. First we note that for finite nonempty sets A, B with $|A| = |B|$,

$$|\{f : A \to B \mid f(A) = B\}| = |A|!.$$

If $a \in A$ and $b \in B$, then

$$|\{f : A \to B \mid f(a) = b \text{ and } f(A) = B\}| = (|A| - 1)!.$$

Case 1 : $(m,n) = 1$. Then $m\mathbb{Z}_n = \mathbb{Z}_n$, so $(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) = (\mathbb{Z}_n, \circ_{\mathbb{Z}_n})$. By Proposition 1.8, $\mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) = \{f : \mathbb{Z}_n \to \mathbb{Z}_n \mid f(\mathbb{Z}_n) = \mathbb{Z}_n\}$. But since $\mathbb{Z}_n$ is finite, it follows that $|\mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = n!$, so the result follows for this case.

Case 2 : $(m,n) > 1$. By Theorem 3.1.2,

$$\mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) = \{f : \mathbb{Z}_n \to \mathbb{Z}_n \mid \ f(\bar{x} + m\mathbb{Z}_n) = xf(\bar{1}) + m\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}\}.$$

This implies that for $f : \mathbb{Z}_n \to \mathbb{Z}_n$,

$$\begin{aligned} f \in \mathrm{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) \Longleftrightarrow & f((m,n)\mathbb{Z}_n) = (m,n)\mathbb{Z}_n, \\ & f(\bar{1} + (m,n)\mathbb{Z}_n) = f(\bar{1}) + (m,n)\mathbb{Z}_n, \\ & f(\bar{2} + (m,n)\mathbb{Z}_n) = 2f(\bar{1}) + (m,n)\mathbb{Z}_n, \\ & \ldots \\ & f(\overline{(m,n) - 1} + (m,n)\mathbb{Z}_n) = ((m,n) - 1)f(\bar{1}) + (m,n)\mathbb{Z}_n. \end{aligned}$$

For $f : \mathbb{Z}_n \to \mathbb{Z}_n$, all the possibilities of $f(\overline{1})$ are $\overline{0}, \overline{1}, \ldots, \overline{n-1}$. Notice that $f(\overline{1}) \in f(\overline{1} + (m,n)\mathbb{Z}_n)$. From these facts, we have that

$$\begin{aligned}
|\text{GHom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| &= n \times \left(\frac{n}{(m,n)}\right)! \times \left(\frac{n}{(m,n)} - 1\right)! \\
&\qquad \times \underbrace{\left(\frac{n}{(m,n)}\right)! \times \cdots \times \left(\frac{n}{(m,n)}\right)!}_{(m,n)-2 \text{ copies}} \\
&= n \times \left(\frac{n}{(m,n)} - 1\right)! \times \left(\left(\frac{n}{(m,n)}\right)!\right)^{(m,n)-1}. \\
&= n \left(\left(\frac{n}{(m,n)} - 1\right)!\right) \left(\left(\frac{n}{(m,n)}\right)!\right)^{(m,n)-1}.
\end{aligned}$$

Therefore the proof is complete. $\square$

Finally, we determine $|\text{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})|$ by the following theorem.

Theorem 3.2.3. *The following statements hold.*

(i) *If* $(m,n) = 1$, *then* $|\text{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = n!$.

(ii) *If* $(m,n) > 1$, *then* $|\text{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = \phi((m,n)) \left(\left(\frac{n}{(m,n)} - 1\right)!\right) \left(\left(\frac{n}{(m,n)}\right)!\right)^{(m,n)-1}$.

Proof. (i) If $(m,n) = 1$, it follows from Lemma 2.2.2 that

$$\text{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) = \text{Epi}(\mathbb{Z}_n, \circ_{\mathbb{Z}_n}) = \{f : \mathbb{Z}_n \to \mathbb{Z}_n \mid f(\mathbb{Z}_n) = \mathbb{Z}_n\},$$

so $|\text{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| = n!$.

(ii) Assume that $(m,n) > 1$. It follows from Theorem 3.1.3 that for $f : \mathbb{Z}_n \to \mathbb{Z}_n$,

$$\begin{aligned}
f \in \text{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n}) \iff\ & f(\overline{1}) = \overline{a} \ \text{ where } a \text{ and } (m,n) \text{ are relatively prime}, \\
& f((m,n)\mathbb{Z}_n) = (m,n)\mathbb{Z}_n, \\
& f(\overline{1} + (m,n)\mathbb{Z}_n) = f(\overline{1}) + (m,n)\mathbb{Z}_n, \\
& f(\overline{2} + (m,n)\mathbb{Z}_n) = 2f(\overline{1}) + (m,n)\mathbb{Z}_n, \\
& \cdots \\
& f(\overline{(m,n)-1} + (m,n)\mathbb{Z}_n) = ((m,n)-1)f(\overline{1}) + (m,n)\mathbb{Z}_n.
\end{aligned}$$

For $f \in \mathrm{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, the number of all possibilities of $f(\overline{1})$ is $\phi((m,n))$. Notice that $f(\overline{1}) \in f(\overline{1} + (m,n)\mathbb{Z}_n)$. These facts yield the following result.

$$
\begin{aligned}
|\mathrm{Epi}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})| &= \phi((m,n)) \times \left(\frac{n}{(m,n)}\right)! \times \left(\frac{n}{(m,n)} - 1\right)! \\
&\qquad \times \underbrace{\left(\frac{n}{(m,n)}\right)! \times \cdots \times \left(\frac{n}{(m,n)}\right)!}_{(m,n)-2 \text{ copies}} \\
&= \phi((m,n)) \left(\left(\frac{n}{(m,n)} - 1\right)!\right) \left(\left(\frac{n}{(m,n)}\right)!\right)^{(m,n)-1}.
\end{aligned}
$$

□

Example 3.2.4. From Theorem 3.2.1, Theorem 3.2.2 and Theorem 3.2.3, we have respectively that

$$
|\mathrm{Hom}(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6})| = 6 \times \left(\frac{6}{(4,6)}\right)^{6-1} = 6 \times 3^5 = 1,458,
$$

$$
\begin{aligned}
|\mathrm{GHom}(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6})| &= 6\left(\left(\frac{6}{(4,6)} - 1\right)\right)! \left(\left(\frac{6}{(4,6)}\right)!\right)^{(4,6)-1} \\
&= 6 \times 2! \times 3! = 72 \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
|\mathrm{Epi}(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6})| &= \phi((4,6)) \left(\left(\frac{6}{(4,6)} - 1\right)!\right) \left(\left(\frac{6}{(4,6)}\right)!\right)^{(4,6)-1} \\
&= 1 \times 2! \times 3! = 12.
\end{aligned}
$$

Then the number of the homomorphisms in $(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6})$ which are not good homomorphisms is $1,458 - 72 = 1,386$ and the number of the homomorphisms of $(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6})$ which are not epimorphisms is $1,458 - 12 = 1,446$. Recall that $\mathrm{Epi}(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6}) \subseteq \mathrm{GHom}(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6})$(Corollary 3.1.4). Then the number of the good homomorphisms of $(\mathbb{Z}_6, \circ_{4\mathbb{Z}_6})$ which are not epimorphisms is $72 - 12 = 60$. Notice that the number of all functions from $\mathbb{Z}_6$ into itself is $6^6 = 46,656$.

CHAPTER IV

HOMOMORPHISMS OF SOME OTHER HYPERGROUPS

In this chapter, we are concerned with the following hypergroups: $(\mathbb{Q}, \bullet_P)$ defined as in Example 1.2, $(\mathbb{Z}, \circ)$, $(\mathbb{Z}_n, \circ)$ and $(\mathbb{Q}, \circ)$ defined as in Example 1.3 and $(\mathbb{R}, \bullet)$ defined in Remark 1.5. Some results concerning homomorphisms of $(\mathbb{Q}, \bullet_P)$, $(\mathbb{Z}, \circ)$, $(\mathbb{Z}_n, \circ)$ and $(\mathbb{Q}, \circ)$ are provided. Characterizations of the elements of $\mathrm{Hom}(\mathbb{R}, \bullet)$, $\mathrm{GHom}(\mathbb{R}, \bullet)$, $\mathrm{Epi}(\mathbb{R}, \bullet)$ and $\mathrm{GEpi}(\mathbb{R}, \bullet)$ are given.

4.1 P-hypergroups

In this section, we deal with the P-hypergroup $(\mathbb{Q}, \bullet_P)$ defined from the group $(\mathbb{Q}, +)$ and $\varnothing \neq P \subseteq \mathbb{Q}$. Recall that $x \bullet_P y = x + P + y$ for all $x, y \in \mathbb{Q}$.

First, we give a general result on homomorphisms of $(\mathbb{Q}, +)$.

Lemma 4.1.1. *For $a \in \mathbb{Q}$, define $k_a : \mathbb{Q} \to \mathbb{Q}$ by*

$$k_a(x) = ax \text{ for all } x \in \mathbb{Q}.$$

Then $\mathrm{Hom}(\mathbb{Q}, +) = \{k_a \mid a \in \mathbb{Q}\}$.

Proof. It is clear that $k_a \in \mathrm{Hom}(\mathbb{Q}, +)$ for all $a \in \mathbb{Q}$. For the reverse inclusion, let $f \in \mathrm{Hom}(\mathbb{Q}, +)$. Claim that $f = k_{f(1)}$. Let $m \in \mathbb{Z}^+$ and $l \in \mathbb{Z}$. Then

$$f(1) = f(m(\tfrac{1}{m})) = mf(\tfrac{1}{m})$$

which implies that $f(\frac{1}{m}) = \frac{f(1)}{m}$. Hence

$$f(\tfrac{l}{m}) = f(l(\tfrac{1}{m})) = lf(\tfrac{1}{m}) = \tfrac{l}{m}f(1) = k_{f(1)}(\tfrac{l}{m}),$$

so we have the claim.

Therefore $\mathrm{Hom}(\mathbb{Q}, +) = \{k_a \mid a \in \mathbb{Q}\}$, as desired. $\square$

The following theorem analogous to Theorem 1.13 is directly obtained from Theorem 1.12 and the definition of k_a for $a \in \mathbb{Q}$ defined in Lemma 4.1.1.

Theorem 4.1.2. *Let $\varnothing \neq P \subseteq \mathbb{Q}$. The following statements hold.*

(i) *For $a \in \mathbb{Q}$, $k_a \in \mathrm{Hom}(\mathbb{Q}, \bullet_P)$ if and only if $aP \subseteq P$.*

(ii) *For $a \in \mathbb{Q}$, $k_a \in \mathrm{GHom}(\mathbb{Q}, \bullet_P)$ if and only if $aP = P$.*

From Theorem 4.1.2 and the fact that $a\mathbb{Q} = \mathbb{Q}$ if and only if $a \in \mathbb{Q} \setminus \{0\}$, we obtain the following theorem.

Theorem 4.1.3. *Let $\varnothing \neq P \subseteq \mathbb{Q}$. Then the following statements hold.*

(i) *For $a \in \mathbb{Q}$, $k_a \in \mathrm{Epi}(\mathbb{Q}, \bullet_P)$ if and only if $a \neq 0$ and $aP \subseteq P$.*

(ii) *For $a \in \mathbb{Q}$, $k_a \in \mathrm{GEpi}(\mathbb{Q}, \bullet_P)$ if and only if $a \neq 0$ and $aP = P$.*

Example 4.1.4. Let $\mathbb{Z}^- = \{x \in \mathbb{Z} \mid x < 0\}$, $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ and $\mathbb{Q}^- = \{x \in \mathbb{Q} \mid x < 0\}$.

The following results are clearly obtained from Theorem 4.1.2 and Theorem 4.1.3.

$$\begin{aligned}
\{a \in \mathbb{Q} \mid k_a \in \mathrm{Hom}(\mathbb{Q}, \bullet_{\mathbb{Z}})\} &= \mathbb{Z}, \\
\{a \in \mathbb{Q} \mid k_a \in \mathrm{Epi}(\mathbb{Q}, \bullet_{\mathbb{Z}})\} &= \mathbb{Z} \setminus \{0\}, \\
\{a \in \mathbb{Q} \mid k_a \in \mathrm{GHom}(\mathbb{Q}, \bullet_{\mathbb{Z}})\} &= \{-1, 1\} \\
&= \{a \in \mathbb{Q} \mid k_a \in \mathrm{GEpi}(\mathbb{Q}, \bullet_{\mathbb{Z}})\}, \\
\{a \in \mathbb{Q} \mid k_a \in \mathrm{Hom}(\mathbb{Q}, \bullet_{\mathbb{Z}^+})\} &= \mathbb{Z}^+ \\
&= \{a \in \mathbb{Q} \mid k_a \in \mathrm{Epi}(\mathbb{Q}, \bullet_{\mathbb{Z}^+})\}, \\
\{a \in \mathbb{Q} \mid k_a \in \mathrm{GHom}(\mathbb{Q}, \bullet_{\mathbb{Z}^+})\} &= \{1\} \\
&= \{a \in \mathbb{Q} \mid k_a \in \mathrm{GEpi}(\mathbb{Q}, \bullet_{\mathbb{Z}^+})\}, \\
\{a \in \mathbb{Q} \mid k_a \in \mathrm{Hom}(\mathbb{Q}, \bullet_{\mathbb{Z}^-})\} &= \mathbb{Z}^+ \\
&= \{a \in \mathbb{Q} \mid k_a \in \mathrm{Epi}(\mathbb{Q}, \bullet_{\mathbb{Z}^-})\},
\end{aligned}$$

$$\{a \in \mathbb{Q} \mid k_a \in \mathrm{GHom}(\mathbb{Q}, \bullet_{\mathbb{Z}^-})\} = \{1\}$$
$$= \{a \in \mathbb{Q} \mid k_a \in \mathrm{GEpi}(\mathbb{Q}, \bullet_{\mathbb{Z}^-})\},$$
$$\{a \in \mathbb{Q} \mid k_a \in \mathrm{Hom}(\mathbb{Q}, \bullet_{\mathbb{Q}^+})\} = \mathbb{Q}^+ = \{a \in \mathbb{Q} \mid k_a \in \mathrm{Epi}(\mathbb{Q}, \bullet_{\mathbb{Q}^+})\}$$
$$= \{a \in \mathbb{Q} \mid k_a \in \mathrm{GHom}(\mathbb{Q}, \bullet_{\mathbb{Q}^+})\}$$
$$= \{a \in \mathbb{Q} \mid k_a \in \mathrm{GEpi}(\mathbb{Q}, \bullet_{\mathbb{Q}^+})\},$$
$$\{a \in \mathbb{Q} \mid k_a \in \mathrm{Hom}(\mathbb{Q}, \bullet_{\mathbb{Q}^-})\} = \mathbb{Q}^+ = \{a \in \mathbb{Q} \mid k_a \in \mathrm{Epi}(\mathbb{Q}, \bullet_{\mathbb{Q}^-})\}$$
$$= \{a \in \mathbb{Q} \mid k_a \in \mathrm{GHom}(\mathbb{Q}, \bullet_{\mathbb{Q}^-})\}$$
$$= \{a \in \mathbb{Q} \mid k_a \in \mathrm{GEpi}(\mathbb{Q}, \bullet_{\mathbb{Q}^-})\}.$$

The next theorem is analogous to Theorem 1.14. It is obtained from Lemma 4.1.1, Theorem 4.1.2(i) and a property of $\mathbb{Q}$.

Theorem 4.1.5. *For $\varnothing \neq P \subseteq \mathbb{Q}$, $\mathrm{Hom}(\mathbb{Q}, +) \subseteq \mathrm{Hom}(\mathbb{Q}, \bullet_P)$ if and only if either $P = \{0\}$ or $P = \mathbb{Q}$.*

Proof. Assume that $\mathrm{Hom}(\mathbb{Q}, +) \subseteq \mathrm{Hom}(\mathbb{Q}, \bullet_P)$. By Lemma 4.1.1 and Theorem 4.1.2(i), $\mathbb{Q}P \subseteq P$. If $a \in P$ for some $a \in \mathbb{Q} \smallsetminus \{0\}$, then

$$\mathbb{Q} = \mathbb{Q}a \subseteq \mathbb{Q}P \subseteq P \subseteq \mathbb{Q},$$

so $P = \mathbb{Q}$. This implies that either $P = \{0\}$ or $P = \mathbb{Q}$.

For the converse, assume that $P = \{0\}$ or $P = \mathbb{Q}$. Then $aP \subseteq P$ for all $a \in \mathbb{Q}$. It then follows from Lemma 4.1.1 and Theorem 4.1.2(i) that $\mathrm{Hom}(\mathbb{Q}, +) \subseteq \mathrm{Hom}(\mathbb{Q}, \bullet_P)$. □

Remark 4.1.6. Let G be a group and $\varnothing \neq P \subseteq G$. We know from Remark 2.1.6 that $\mathrm{Hom}(G, \bullet_P)$ is a semigroup under composition having $\mathrm{GHom}(G, \bullet_P)$, $\mathrm{Epi}(G, \bullet_P)$ and $\mathrm{GEpi}(G, \bullet_P)$ as its subsemigroups. Let $(A, +)$ be an abelian group and P a subsemigroup of $(A, +)$. We claim that $\mathrm{Hom}(A, \bullet_P)$ is a commutative semigroup under addition. We have that $(F(A), +)$ is an abelian group where $F(A)$ is the set of all functions from A into itself. Next, let $g, f \in \mathrm{Hom}(A, \bullet_P)$ and $x, y \in A$. Then

$$
\begin{aligned}
(g+f)(x \bullet_P y) &= (g+f)(x+P+y) \\
&\subseteq g(x+P+y)+f(x+P+y) \\
&= g(x \bullet_P y)+f(x \bullet_P y) \\
&\subseteq (g(x) \bullet_P g(y))+(f(x) \bullet_P f(y)) \\
&= (g(x)+P+g(y))+(f(x)+P+f(y)) \\
&= g(x)+f(x)+P+P+g(y)+f(y) \\
&\subseteq g(x)+f(x)+P+g(y)+f(y) \\
&= (g(x)+f(x)) \bullet_P (g(y)+f(y)) \\
&= (g+f)(x) \bullet_P (g+f)(y).
\end{aligned}
$$

This shows that $\mathrm{Hom}(A, \bullet_P)$ is a subsemigroup of $(F(A), +)$.

If P is a subgroup of $(A, +)$, then we have that $(\mathrm{Hom}(A, \bullet_P), +)$ is an abelian group. It remains to show that for $f \in \mathrm{Hom}(A, \bullet_P)$, $-f \in \mathrm{Hom}(A, \bullet_P)$. Since P is a subgroup of $(A, +)$, we have $-P = P$. Let $f \in \mathrm{Hom}(A, \bullet_P)$. Then for $x, y \in A$,

$$
\begin{aligned}
(-f)(x \bullet_P y) &= (-f)(x+P+y) \\
&= -(f(x+P+y)) \\
&\subseteq -(f(x)+P+f(y)) \\
&= -f(x)-P-f(y) \\
&= (-f)(x)+P+(-f)(y) \\
&= (-f)(x) \bullet_P (-f)(y).
\end{aligned}
$$

It follows from the above facts that $(\mathrm{Hom}(\mathbb{Q}, \bullet_{\mathbb{Z}^+}), +)$ is a commutative semigroup and $(\mathrm{Hom}(\mathbb{Q}, \bullet_{\mathbb{Z}}), +)$ is an abelian group.

4.2 Hypergroups Defined from Abelian Groups Whose Hyperproducts Are Subgroups

In this section, let $(A,+)$ be an abelian group and $(A,\circ)$ the hypergroup under the hyperoperation $\circ$ defined by $x \circ y = \mathbb{Z}x + \mathbb{Z}y$ for all $x, y \in A$.

First, we give some necessary conditions for $f \in \mathrm{GHom}(A,\circ)$.

Proposition 4.2.1. *For* $f \in \mathrm{GHom}(A,\circ)$,

(i) $f(0) = 0$ *and*

(ii) $f(\mathbb{Z}x) = \mathbb{Z}f(x)$ *for all* $x \in A$.

(iii) *If* $(A,+)$ *is the cyclic group generated by an element* $a \in A$, *then* $f(A) = \mathbb{Z}f(a)$, *the cyclic subgroup of* A *generated by* $f(a)$.

Proof. (i) Since
$$\{f(0)\} = f(\mathbb{Z}0 + \mathbb{Z}0) = f(0 \circ 0) = f(0) \circ f(0) = \mathbb{Z}f(0) + \mathbb{Z}f(0) = \mathbb{Z}f(0) \supseteq \{0\},$$

it follows that $f(0) = 0$.

(ii) If $x \in A$, then

$$\begin{aligned} f(\mathbb{Z}x) = f(\mathbb{Z}x + \mathbb{Z}0) &= f(x \circ 0) \\ &= f(x) \circ f(0) \\ &= f(x) \circ 0 \quad \text{by (i)} \\ &= \mathbb{Z}f(x) + \mathbb{Z}0 \\ &= \mathbb{Z}f(x), \end{aligned}$$

so (ii) hold.

(iii) Since $A = \mathbb{Z}a$, (iii) follows from (ii). □

The following results follow directly from Proposition 4.2.1(iii).

Corollary 4.2.2. *The following statements hold.*

(i) *If* $f \in \mathrm{GHom}(\mathbb{Z}, \circ)$*, then* $f(\mathbb{Z}) = \mathbb{Z}f(1)$*, and* $f \in \mathrm{GEpi}(\mathbb{Z}, \circ)$ *if and only if either* $f(1) = 1$ *or* $f(1) = -1$*.*

(ii) *If* $f \in \mathrm{GHom}(\mathbb{Z}_n, \circ)$*, then* $f(\mathbb{Z}_n) = \mathbb{Z}f(\overline{1}) = \mathbb{Z}_n f(\overline{1})$*, and* $f \in \mathrm{GEpi}(\mathbb{Z}_n, \circ)$ *if and only if* a *and* n *are relatively prime where* $\overline{a} = f(\overline{1})$*.*

The next theorem shows that every homomorphism of $(A, +)$ is a good homomorphism of $(A, \circ)$ when A is any of $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$.

Theorem 4.2.3. $\mathrm{Hom}(\mathbb{Z}, +) \subseteq \mathrm{GHom}(\mathbb{Z}, \circ)$*,* $\mathrm{Hom}(\mathbb{Z}_n, +) \subseteq \mathrm{GHom}(\mathbb{Z}_n, \circ)$ *and* $\mathrm{Hom}(\mathbb{Q}, +) \subseteq \mathrm{GHom}(\mathbb{Q}, \circ)$*.*

Proof. If $a, x, y \in \mathbb{Z}$, then

$$\begin{aligned} g_a(x \circ y) &= g_a(\mathbb{Z}x + \mathbb{Z}y) \\ &= a(\mathbb{Z}x + \mathbb{Z}y) \\ &= \mathbb{Z}ax + \mathbb{Z}ay \\ &= ax \circ ay \\ &= g_a(x) \circ g_a(y), \end{aligned}$$

so $g_a \in \mathrm{GHom}(\mathbb{Z}, \circ)$. Since $\mathrm{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\}$, we have $\mathrm{Hom}(\mathbb{Z}, +) \subseteq \mathrm{GHom}(\mathbb{Z}, \circ)$.

Recall that $\mathrm{Hom}(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}$ and $\mathrm{Hom}(\mathbb{Q}, +) = \{k_a \mid a \in \mathbb{Q}\}$ (by Lemma 4.1.1). We can show similarly that $\mathrm{Hom}(\mathbb{Z}_n, +) \subseteq \mathrm{GHom}(\mathbb{Z}_n, \circ)$ and $\mathrm{Hom}(\mathbb{Q}, +) \subseteq \mathrm{GHom}(\mathbb{Q}, \circ)$. □

From Corollary 4.2.2 and Theorem 4.2.3, we have

Corollary 4.2.4. *The following statements hold.*

(i) $\mathrm{Hom}(\mathbb{Z}, +) \bigcap \mathrm{GEpi}(\mathbb{Z}, \circ) = \{g_1, g_{-1}\}$*.*

(ii) $\mathrm{Hom}(\mathbb{Z}_n, +) \bigcap \mathrm{GEpi}(\mathbb{Z}_n, \circ) = \{h_{\overline{a}} \mid a \in \mathbb{Z}$ *and* $(a, n) = 1\}$*.*

The following theorem shows that $\mathrm{Hom}(\mathbb{Z}, +) \subsetneq \mathrm{GHom}(\mathbb{Z}, \circ)$, $\mathrm{Hom}(\mathbb{Q}, +) \subsetneq \mathrm{GHom}(\mathbb{Q}, \circ)$ and gives a necessary and sufficient conditions for n guaranteeing that $\mathrm{Hom}(\mathbb{Z}_n, +) \subsetneq \mathrm{GHom}(\mathbb{Z}_n, \circ)$ holds.

Theorem 4.2.5. *The following statements hold.*

(i) $\mathrm{Hom}(\mathbb{Z},+) \subsetneq \mathrm{GHom}(\mathbb{Z},\circ)$.

(ii) $\mathrm{Hom}(\mathbb{Q},+) \subsetneq \mathrm{GHom}(\mathbb{Q},\circ)$.

(iii) *For* $n \in \mathbb{Z}^+$, $\mathrm{Hom}(\mathbb{Z}_n,+) \subsetneq \mathrm{GHom}(\mathbb{Z}_n,\circ)$ *if and only if* $n \geq 4$.

Proof. Define $f : \mathbb{Z} \to \mathbb{Z}$ and $\overline{f} : \mathbb{Q} \to \mathbb{Q}$ by

$$f(1) = \overline{f}(1) = -1,\ f(-1) = \overline{f}(-1) = 1,$$
$$f(x) = x \text{ for all } x \in \mathbb{Z} \smallsetminus \{1,-1\} \text{ and}$$
$$\overline{f}(x) = x \text{ for all } x \in \mathbb{Q} \smallsetminus \{1,-1\}.$$

It is easily seen that $f \neq g_a$ for all $a \in \mathbb{Z}$ and $\overline{f} \neq k_a$ for any $a \in \mathbb{Q}$. We have that

$$f(x \circ y) = f(\mathbb{Z}x + \mathbb{Z}y),\ f(x) \circ f(y) = \mathbb{Z}f(x) + \mathbb{Z}f(y) \text{ for all } x, y \in \mathbb{Z}$$

and

$$\overline{f}(x \circ y) = \overline{f}(\mathbb{Z}x + \mathbb{Z}y),\ \overline{f}(x) \circ \overline{f}(y) = \mathbb{Z}\overline{f}(x) + \mathbb{Z}\overline{f}(y) \text{ for all } x, y \in \mathbb{Q}.$$

Since for $x, y \in \mathbb{Q},\ 1 \in \mathbb{Z}x + \mathbb{Z}y \Longleftrightarrow -1 \in \mathbb{Z}x + \mathbb{Z}y$, it follows that

$$f(\mathbb{Z}x + \mathbb{Z}y) = \mathbb{Z}x + \mathbb{Z}y \text{ for all } x, y \in \mathbb{Z}$$

and

$$\overline{f}(\mathbb{Z}x + \mathbb{Z}y) = \mathbb{Z}x + \mathbb{Z}y \text{ for all } x, y \in \mathbb{Q}.$$

By the definitions of f and $\overline{f}$ and the fact that $\mathbb{Z}(1) = \mathbb{Z}(-1)$, we have that

$$\mathbb{Z}f(x) + \mathbb{Z}f(y) = \mathbb{Z}x + \mathbb{Z}y \text{ for all } x, y \in \mathbb{Z}$$

and

$$\mathbb{Z}\overline{f}(x) + \mathbb{Z}\overline{f}(y) = \mathbb{Z}x + \mathbb{Z}y \text{ for all } x, y \in \mathbb{Q}.$$

These show that $f \in \mathrm{GHom}(\mathbb{Z},\circ)$ and $\overline{f} \in \mathrm{GHom}(\mathbb{Q},\circ)$. Thus $f \in \mathrm{GHom}(\mathbb{Z},\circ) \smallsetminus \mathrm{Hom}(\mathbb{Z},+)$ and $\overline{f} \in \mathrm{GHom}(\mathbb{Q},\circ) \smallsetminus \mathrm{Hom}(\mathbb{Q},+)$. This proves (i) and (ii).

To prove (iii), assume that $n \geq 4$.

Case 1 : $n = 4$. Define $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ by $f(\overline{0}) = \overline{0}$ and $f(\overline{1}) = f(\overline{2}) = f(\overline{3}) = \overline{2}$. It is clear that $f \neq h_{\overline{a}}$ for all $\overline{a} \in \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Thus $f \notin \text{Hom}(\mathbb{Z}_4, +)$. To show $f \in \text{GHom}(\mathbb{Z}_4, \circ)$, we first note that if A is a subset of $\mathbb{Z}_4$ containing $\overline{0}$ and a nonzero element, then $f(A) = \{\overline{0}, \overline{2}\}$. It is evident that $f(\overline{0} \circ \overline{0}) = \{\overline{0}\} = f(\overline{0}) \circ f(\overline{0})$. Next, let $\overline{x}, \overline{y} \in \mathbb{Z}_4$, not both $\overline{0}$, say $\overline{x} \neq \overline{0}$. Then $\overline{x} \circ \overline{y} = \mathbb{Z}_4\overline{x} + \mathbb{Z}_4\overline{y} \supseteq \{\overline{0}, \overline{x}\}$. Thus $f(\overline{x} \circ \overline{y}) = \{\overline{0}, \overline{2}\}$. Since

$$f(\overline{x}) \circ f(\overline{y}) = \begin{cases} \overline{2} \circ \overline{0} = \mathbb{Z}_4\overline{2} + \mathbb{Z}_4\overline{0} = \{\overline{0}, \overline{2}\} & \text{if } \ \overline{y} = 0, \\ \overline{2} \circ \overline{2} = \mathbb{Z}_4\overline{2} + \mathbb{Z}_4\overline{2} = \{\overline{0}, \overline{2}\} & \text{if } \ \overline{y} \neq 0, \end{cases}$$

it follows that $f(\overline{x} \circ \overline{y}) = f(\overline{x}) \circ f(\overline{y})$, so $f \in \text{GHom}(\mathbb{Z}_4, \circ)$, as desired. Hence $\text{Hom}(\mathbb{Z}_4, +) \subsetneq \text{GHom}(\mathbb{Z}_4, \circ)$.

Case 2 : $n \geq 5$. Then 1 and $n-1$ are relatively primes to n. Then $\mathbb{Z}(\overline{1}) = \mathbb{Z}(\overline{n-1}) = \mathbb{Z}_n$. Define $f : \mathbb{Z}_n \to \mathbb{Z}_n$ by

$$f(\overline{1}) = \overline{n-1}, \quad f(\overline{n-1}) = \overline{1} \ \text{ and}$$
$$f(\overline{x}) = \overline{x} \ \text{ for all } \ \overline{x} \in \mathbb{Z}_n \smallsetminus \{\overline{1}, \overline{n-1}\}.$$

Then $f(\overline{1}+\overline{n-2}) = f(\overline{n-1}) = \overline{1}$ and $f(\overline{1})+f(\overline{n-2}) = \overline{n-1}+\overline{n-2} = \overline{2n-3} = \overline{-3} = \overline{n-3}$. But since $n \geq 5$, $\overline{1} \neq \overline{n-3}$, so $f(\overline{1} + \overline{n-2}) \neq f(\overline{1}) + f(\overline{n-2})$, it follows that $f \notin \text{Hom}(\mathbb{Z}_n, +)$. To show that $f \in \text{GHom}(\mathbb{Z}_n, \circ)$, let $x, y \in \mathbb{Z}$. Then

$$f(\overline{x} \circ \overline{y}) = f(\mathbb{Z}\overline{x} + \mathbb{Z}\overline{y}) \text{ and } f(\overline{x}) \circ f(\overline{y}) = \mathbb{Z}f(\overline{x}) + \mathbb{Z}f(\overline{y}).$$

It is evident that $f(\overline{0} \circ \overline{0}) = f(\overline{0}) = \overline{0} = \mathbb{Z}\overline{0} + \mathbb{Z}\overline{0} = f(\overline{0}) \circ f(\overline{0})$. Assume that $\overline{x} \neq \overline{0}$ or $\overline{y} \neq \overline{0}$.

Subcase 2.1 : $\overline{x} = \overline{1}$ or $\overline{x} = \overline{n-1}$. Then $f(\mathbb{Z}\overline{x} + \mathbb{Z}\overline{y}) = f(\mathbb{Z}_n) = \mathbb{Z}_n$ and

$$\mathbb{Z}f(\overline{x}) + \mathbb{Z}f(\overline{y}) = \begin{cases} \mathbb{Z}(\overline{n-1}) + \mathbb{Z}f(\overline{y}) = \mathbb{Z}_n & \text{if } \ \overline{x} = \overline{1}, \\ \mathbb{Z}(\overline{1}) + \mathbb{Z}f(\overline{y}) = \mathbb{Z}_n & \text{if } \ \overline{x} = \overline{n-1}, \end{cases}$$

thus $f(\mathbb{Z}\overline{x} + \mathbb{Z}\overline{y}) = \mathbb{Z}_n = \mathbb{Z}f(\overline{x}) + \mathbb{Z}f(\overline{y})$.

Subcase 2.2 : $\overline{y} = \overline{1}$ or $\overline{y} = \overline{n-1}$. It follows similarly to Case 1 that $f(\mathbb{Z}\overline{x} + \mathbb{Z}\overline{y}) = \mathbb{Z}_n = \mathbb{Z}f(\overline{x}) + \mathbb{Z}f(\overline{y})$.

Subcase 2.3 : $\overline{x}, \overline{y} \in \mathbb{Z}_n \smallsetminus \{\overline{1}, \overline{n-1}\}$. Then $f(\mathbb{Z}\overline{x} + \mathbb{Z}\overline{y}) = f(\mathbb{Z}\overline{(x, y)})$ and

$\mathbb{Z}f(\overline{x}) + \mathbb{Z}f(\overline{y}) = \mathbb{Z}\overline{x} + \mathbb{Z}\overline{y} = (x,y)\mathbb{Z}_n = \mathbb{Z}\overline{(x,y)}$. Since $\mathbb{Z}\overline{(x,y)}$ is a subgroup of $(\mathbb{Z}_n, +)$ and $\overline{1}$ and $\overline{n-1}$ are inverses of each other in $(\mathbb{Z}_n, +)$, it follows that $\overline{1} \in \mathbb{Z}\overline{(x,y)} \Longleftrightarrow \overline{n-1} \in \mathbb{Z}\overline{(x,y)}$. Hence $f(\mathbb{Z}\overline{(x,y)}) = \mathbb{Z}\overline{(x,y)}$, so $f(\mathbb{Z}\overline{x} + \mathbb{Z}\overline{y}) = \mathbb{Z}f(\overline{x}) + \mathbb{Z}f(\overline{y})$.

Therefore we have that $f \in \mathrm{GHom}(\mathbb{Z}_n, \circ) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +)$.

To prove that if $\mathrm{Hom}(\mathbb{Z}_n, +) \subsetneq \mathrm{GHom}(\mathbb{Z}_n, \circ)$, then $n \geq 4$, it is equivalent to show that if $n < 4$, then $\mathrm{Hom}(\mathbb{Z}_n, +) = \mathrm{GHom}(\mathbb{Z}_n, \circ)$ by Theorem 4.2.3. Recall that by Proposition 4.2.1, $f(\overline{0}) = \overline{0}$ for all $f \in \mathrm{GHom}(\mathbb{Z}_n, \circ)$ and by Corollary 4.2.2(ii) for $f \in \mathrm{GHom}(\mathbb{Z}_n, \circ)$, $f(\mathbb{Z}_n) = \mathbb{Z}_n f(\overline{1})$, and $f(\mathbb{Z}_n) = \mathbb{Z}_n$ if and only if a and n are relatively prime where $f(\overline{1}) = \overline{a}$. We also have that for $f \in \mathrm{GHom}(\mathbb{Z}_n, \circ)$, $f(\overline{1}) = \overline{0}$ if and only if $f = h_{\overline{0}}$. It is evident that $\mathrm{Hom}(\mathbb{Z}_n, +) = \mathrm{GHom}(\mathbb{Z}_n, \circ)$ if $n = 1$.

Let $f \in \mathrm{GHom}(\mathbb{Z}_2, \circ)$. Then $f(\overline{0}) = \overline{0}$. If $f(\overline{1}) = \overline{0}$, then $f = h_{\overline{0}}$. If $f(\overline{1}) = \overline{1}$, then $f = h_{\overline{1}}$.

Next, let $f \in \mathrm{GHom}(\mathbb{Z}_3, \circ)$. Then $f(\overline{0}) = \overline{0}$. If $f(\overline{1}) = \overline{0}$, then $f = h_{\overline{0}}$. If $f(\overline{1}) = \overline{1}$, then $f(\mathbb{Z}_3) = \mathbb{Z}_3$ which implies that $f(\overline{2}) = \overline{2}$, so $f = h_{\overline{1}}$. If $f(\overline{1}) = \overline{2}$, then $f(\mathbb{Z}_3) = \mathbb{Z}_3$ which implies that $f(\overline{2}) = \overline{1}$, thus $f = h_{\overline{2}}$.

The proof is thereby complete. $\square$

Remark 4.2.6. Let $(A, +)$ be an abelian group. We know from Remark 2.1.6 that $\mathrm{Hom}(A, \circ)$ is a semigroup under composition having $\mathrm{GHom}(A, \circ)$, $\mathrm{Epi}(A, \circ)$ and $\mathrm{GEpi}(A, \circ)$ as its subsemigroups. If $f \in \mathrm{Hom}(A, \circ)$ and $x, y \in A$. Then

$$\begin{aligned}
(-f)(x \circ y) &= -(f(x \circ y)) \\
&\subseteq -(f(x) \circ f(y)) \\
&= -(\mathbb{Z}f(x) + \mathbb{Z}f(y)) \\
&= -\mathbb{Z}f(x) + (-\mathbb{Z}f(y)) \\
&= \mathbb{Z}((-f)(x)) + \mathbb{Z}((-f)(y)) \\
&= (-f)(x) \circ (-f)(y).
\end{aligned}$$

This shows that $-f \in \mathrm{Hom}(A, \circ)$ for all $f \in \mathrm{Hom}(A, \circ)$. We can see from the above proof that if $f \in \mathrm{GHom}(A, \circ)$, then $-f \in \mathrm{GHom}(A, \circ)$. Since $-A = A$, it follows that for $f \in \mathrm{Epi}(A, \circ)$, $-f \in \mathrm{Epi}(A, \circ)$ and for $f \in \mathrm{GEpi}(A, \circ)$, $-f \in \mathrm{GEpi}(A, \circ)$.

4.3 The Hypergroup Defined from $\mathbb{R}$ Whose Hyperproducts Are Closed Intervals

In this section, we consider the hypergroup $(\mathbb{R}, \bullet)$ where

$$x \bullet y = y \bullet x = [x, y] \text{ if } x \leq y.$$

We first characterize the homomorphisms of the hypergroup $(\mathbb{R}, \bullet)$.

Theorem 4.3.1. *Let $f : \mathbb{R} \to \mathbb{R}$. Then $f \in \mathrm{Hom}(\mathbb{R}, \bullet)$ if and only if f is monotone.*

Proof. To prove that $f \in \mathrm{Hom}(\mathbb{R}, \bullet)$ implies that f is monotone by contrapositive, assume that f is not monotone. Then there are $x, y, z \in \mathbb{R}$ such that $x < y < z$ and either $f(x) < f(y) > f(z)$, or $f(x) > f(y) < f(z)$. Thus $f(x \bullet z) = f([x, z]) = \{f(t) \mid t \in [x, z]\} \ni f(y)$.

Case 1 : $f(x) < f(y) > f(z)$.

Subcase 1.1 : $f(x) = f(z)$. Then $f(x) \bullet f(z) = \{f(x)\}$ and $f(x) \neq f(y)$, so $f(x \bullet z) \nsubseteq f(x) \bullet f(z)$.

Subcase 1.2 : $f(x) < f(z)$. Then $f(x) \bullet f(z) = [f(x), f(z)] \not\ni f(y)$, so $f(x \bullet z) \nsubseteq f(x) \bullet f(z)$.

Subcase 1.3 : $f(x) > f(z)$. Then $f(x) \bullet f(z) = [f(z), f(x)] \not\ni f(y)$, so $f(x \bullet z) \nsubseteq f(x) \bullet f(z)$.

Case 2 : $f(x) > f(y) < f(z)$. We can prove similarly to Case 1, that $f(x \bullet z) \nsubseteq f(x) \bullet f(z)$.

From Case 1 and Case 2, we conclude that $f \notin \mathrm{Hom}(\mathbb{R}, \bullet)$.

Conversely, assume that f is monotone. Then f is increasing or decreasing.

First assume that f is increasing. Let $x, y \in \mathbb{R}$ be such that $x \leq y$. Then $f(x) \leq f(y)$. Since $f(x \bullet y) = f(y \bullet x) = f([x, y]) = \{f(t) \mid t \in [x, y]\}$ and $f(x) \leq f(t) \leq f(y)$ for all $t \in [x, y]$, it follows that

$$f(x \bullet y) \subseteq [f(x), f(y)] = f(x) \bullet f(y) = f(y) \bullet f(x).$$

This proves that $f \in \mathrm{Hom}(\mathbb{R}, \bullet)$. We can see from above proof that if f is decreasing and $x, y \in \mathbb{R}$ such that $x \leq y$, then

$$f(x \bullet y) = f(y \bullet x) \subseteq [f(y), f(x)] = f(y) \bullet f(x) = f(x) \bullet f(y),$$

so we have that $f \in \mathrm{Hom}(\mathbb{R}, \bullet)$.

Hence the theorem is proved. □

Example 4.3.2. From Theorem 4.3.1, the following functions from $\mathbb{R}$ into itself are homomorphisms of a hypergroup $(\mathbb{R}, \bullet)$.

(1) For $a, b \in \mathbb{R}$, $f(x) = ax + b$ for all $x \in \mathbb{R}$.

(2) For an odd integer $n \in \mathbb{Z}^+$, $g(x) = x^n$ for all $x \in \mathbb{R}$.

(3) $h(x) = \begin{cases} 0 & \text{if } \ x \leq 0, \\ x + 1 & \text{if } \ x > 0. \end{cases}$

We can see f and g are continuous functions but h is not continuous.

Recall a fact in Analysis that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and I is an interval in $\mathbb{R}$, then $f(I)$ is an interval ([1], p.162).

The following theorem gives a characterization determining when a function $f : \mathbb{R} \to \mathbb{R}$ is a good homomorphism of $(\mathbb{R}, \bullet)$.

Theorem 4.3.3. *For $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathrm{GHom}(\mathbb{R}, \bullet)$ if and only if f is monotone and continuous on $\mathbb{R}$.*

Proof. Assume that $f \in \mathrm{GHom}(\mathbb{R}, \bullet)$. By Theorem 4.3.1, f is monotone. First, assume that f is increasing. Then we have that for $x \leq y$,

$$f([x, y]) = f(x \bullet y) = f(y \bullet x) = f(x) \bullet f(y) = f(y) \bullet f(x) = [f(x), f(y)].$$

To show that f is continuous on $\mathbb{R}$, that is, to show that

$$\forall a \in \mathbb{R}\ \forall \epsilon > 0\ \exists \delta > 0,\ f((a-\delta, a+\delta)) \subseteq (f(a)-\epsilon, f(a)+\epsilon).$$

If f is a constant function, then f is continuous. Assume that f is not a constant function, let $a \in \mathbb{R}$ and $\epsilon > 0$ be given.

Case 1 : $f(a) = \max f(\mathbb{R})$. Then $f(x) = f(a)$ for all $x \geq a$ since f is increasing. Suppose that $(f(a)-\epsilon, f(a)) \bigcap f(\mathbb{R}) = \varnothing$. Since f is not a constant function and f is increasing, there exists $b \in \mathbb{R}$ such that $f(b) \leq f(a) - \epsilon$. Then $b < a$ and

$$(f(a)-\epsilon, f(a)) \subseteq [f(b), f(a)] = f([b,a]) \subseteq f(\mathbb{R}).$$

which is a contradiction. This implies that $(f(a)-\epsilon, f(a)) \bigcap f(\mathbb{R}) \neq \varnothing$. Then there exists $e \in \mathbb{R}$ such that $f(e) \in (f(a)-\epsilon, f(a))$, so $e < a$. Let $\delta = a - e$. Then

$$\begin{aligned} f((a-\delta, a+\delta)) &= f((e, a+\delta)) \\ &= f((e, a]) \\ &\subseteq f([e, a]) \\ &= [f(e), f(a)] \\ &\subseteq (f(a)-\epsilon, f(a)] \\ &\subseteq (f(a)-\epsilon, f(a)+\epsilon). \end{aligned}$$

Case 2 : $f(a) = \min f(\mathbb{R})$. We can show similarly that there exists $\delta > 0$ such that $f((a-\delta, a+\delta)) \subseteq (f(a)-\epsilon, f(a)+\epsilon)$.

Case 3 : $f(a)$ is neither a maximum of $f(\mathbb{R})$ nor a minimum of $f(\mathbb{R})$. Suppose that $(f(a)-\epsilon, f(a)) \bigcap f(\mathbb{R}) = \varnothing$. Since $f(a)$ is not a minimum of $f(\mathbb{R})$ and f is increasing there exists $b \in \mathbb{R}$ such that $f(b) \leq f(a) - \epsilon$. Then $b < a$ and

$$(f(a)-\epsilon, f(a)) \subseteq [f(b), f(a)] = f([b,a]) \subseteq f(\mathbb{R}),$$

a contradiction. Then $(f(a)-\epsilon, f(a)) \bigcap f(\mathbb{R}) \neq \varnothing$. Since $f(a)$ is not a maximum of $f(\mathbb{R})$ and f is increasing, we can show similarly that $(f(a), f(a)+\epsilon) \bigcap f(\mathbb{R}) \neq \varnothing$. Let $e_1, e_2 \in \mathbb{R}$ be such that $f(e_1) \in (f(a)-\epsilon, f(a))$ and $f(e_2) \in (f(a), f(a)+\epsilon)$. Then $e_1 < a < e_2$. Let $\delta = \min\{a - e_1, e_2 - a\}$. Then we have

$$\begin{aligned} f((a-\delta, a+\delta)) &\subseteq f([a-\delta, a+\delta]) \\ &\subseteq f([e_1, e_2]) \\ &= [f(e_1), f(e_2)] \\ &\subseteq (f(a)-\epsilon, f(a)+\epsilon). \end{aligned}$$

This shows that f is a continuous at a. But a is arbitrary in $\mathbb{R}$, so f is continuous on $\mathbb{R}$. If f is decreasing, it can be shown similarly that f is continuous on $\mathbb{R}$.

For the converse, assume that f is monotone and continuous. First assume that f is increasing. Let $x, y \in \mathbb{R}$ be such that $x \leq y$. Then $f(x \bullet y) = f(y \bullet x) = f([x, y])$ and $f(x) \leq f(t) \leq f(y)$ for all $t \in [x, y]$. Since f is continuous on $\mathbb{R}$, $f([x, y])$ is an interval in $\mathbb{R}$. It follows that

$$f([x, y]) = [f(x), f(y)] = f(x) \bullet f(y) = f(y) \bullet f(x).$$

This shows that $f \in \mathrm{GHom}(\mathbb{R}, \bullet)$. We can see from the above proof that if f is decreasing, then $f \in \mathrm{GHom}(\mathbb{R}, \bullet)$.

The proof is thereby complete. □

Example 4.3.4. From Example 4.3.2, we have that the functions f and g belong to $\mathrm{GHom}(\mathbb{R}, \bullet)$ but h is not in $\mathrm{GHom}(\mathbb{R}, \bullet)$. Then h is an element of $\mathrm{Hom}(\mathbb{R}, \bullet) \setminus \mathrm{GHom}(\mathbb{R}, \bullet)$.

The next theorem shows that an epimorphism of $(\mathbb{R}, \bullet)$ is a good homomorphism.

Theorem 4.3.5. $\mathrm{Epi}(\mathbb{R}, \bullet) \subseteq \mathrm{GHom}(\mathbb{R}, \bullet)$.

Proof. Let $f \in \mathrm{Epi}(\mathbb{R}, \bullet)$ be given. Then $f \in \mathrm{Hom}(\mathbb{R}, \bullet)$ and $f(\mathbb{R}) = \mathbb{R}$. By Theorem 4.3.1, f is monotone. Assume that f is increasing. To show that $f \in \mathrm{GHom}(\mathbb{R}, \bullet)$, let $x, y \in \mathbb{R}$ be such that $x \leq y$. Since $f \in \mathrm{Hom}(\mathbb{R}, \bullet)$ and f is increasing, it follows that

$$f([x, y]) = f(x \bullet y) = f(y \bullet x) \subseteq f(y) \bullet f(x) = f(x) \bullet f(y) = [f(x), f(y)].$$

Suppose that $f([x,y]) \subsetneq [f(x), f(y)]$. Let $a \in [f(x), f(y)] \smallsetminus f([x,y])$. But $f(x), f(y) \in f([x,y])$, so $f(x) < a < f(y)$. Since f is increasing, we have that

$$f(t) \le f(x) \text{ for all } t \in (-\infty, x) \text{ and } f(t) \ge f(y) \text{ for all } t \in (y, \infty).$$

This implies that $a \notin f((-\infty, x))$ and $a \notin f((y, \infty))$. Since $a \notin f([x,y])$. We deduce that

$$a \notin f((-\infty, x)) \bigcup f([x,y]) \bigcup f((y,\infty)) = f(\mathbb{R}) = \mathbb{R}$$

which is a contradiction. Hence $f([x,y]) = [f(x), f(y)]$ and thus $f(x \bullet y) = f(x) \bullet f(y)$. Hence $f \in \text{GHom}(\mathbb{R}, \bullet)$. If f is decreasing, we can show similarly that $f \in \text{GHom}(\mathbb{R}, \bullet)$.

Hence the theorem is proved. □

Remark 4.3.6. It follows directly from Theorem 4.3.5 that $\text{GEpi}(\mathbb{R}, \bullet) = \text{Epi}(\mathbb{R}, \bullet)$.

Remark 4.3.7. From Theorem 4.3.3 and Theorem 4.3.5, it indicates a fact in Analysis that if $f : \mathbb{R} \to \mathbb{R}$ is monotone and $f(\mathbb{R}) = \mathbb{R}$, then f is continuous.

Example 4.3.8. From Example 4.3.2, we have that $f \in \text{Epi}(\mathbb{R}, \bullet)$ if $a \neq 0$ and $f(x) = b$ for all $x \in \mathbb{R}$ is an element of $\text{GHom}(\mathbb{R}, \bullet) \smallsetminus \text{Epi}(\mathbb{R}, \bullet)$. In addition, we have that $g \in \text{Epi}(\mathbb{R}, \bullet)$.

Remark 4.3.9. Let $c \in \mathbb{R}$ be given. For $f : \mathbb{R} \to \mathbb{R}$, if f is increasing [decreasing] and $c \ge 0$, then cf is increasing [decreasing] and if f is increasing [decreasing] and $c < 0$, then cf is decreasing [increasing]. It follows from Theorem 4.3.1 that if $f \in \text{Hom}(\mathbb{R}, \bullet)$, then $cf \in \text{Hom}(\mathbb{R}, \bullet)$, so $-f \in \text{Hom}(\mathbb{R}, \bullet)$. For $f : \mathbb{R} \to \mathbb{R}$, if f is continuous, then so is cf. Thus we conclude Theorem 4.3.3 that if $f \in \text{GHom}(\mathbb{R}, \bullet)$, then $cf \in \text{GHom}(\mathbb{R}, \bullet)$, so $-f \in \text{GHom}(\mathbb{R}, \bullet)$. If $c \neq 0$, then $c\mathbb{R} = \mathbb{R}$, so $cf \in \text{Epi}(\mathbb{R}, \bullet)$ for all $f \in \text{Epi}(\mathbb{R}, \bullet)$. In particular, $-f \in \text{Epi}(\mathbb{R}, \bullet)$ for all $f \in \text{Epi}(\mathbb{R}, \bullet)$. Therefore we conclude that for $c \neq 0$, $c\text{Hom}(\mathbb{R}, \bullet) = \text{Hom}(\mathbb{R}, \bullet)$, $c\text{GHom}(\mathbb{R}, \bullet) = \text{GHom}(\mathbb{R}, \bullet)$ and $c\text{Epi}(\mathbb{R}, \bullet) = \text{Epi}(\mathbb{R}, \bullet)$.

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