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UNIQUE FACTORIZATION OF PSEUDO-ARITHMETIC FUNCTIONS

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ในปี ค.ศ.1959 แคชเวลล์ และ เอเวอเรต ได้พิสูจน์ว่าเซตของฟังก์ชันเลขคณิตค่าเชิงซ้อนประกอบกันเป็นโดเมนที่มีการแยกตัวประกอบได้อย่างเดียว ภายใต้การบวกและสังวัตนาการ ในวิทยานิพนธ์นี้เราขยายผลนี้ไปในสองทิศทาง

ทิศทางที่หนึ่ง เราแทนเซตของจำนวนธรรมชาติและสนามจำนวนเชิงซ้อนด้วยกึ่งกลุ่มเชิงเลขคณิตและโดเมน D ซึ่งแยกตัวประกอบได้อย่างเดียว ตามลำดับ และพิสูจน์ว่าเมื่อวงของอนุกรมกำลังรูปนัยในตัวไม่กำหนดจำนวนจำกัดเหนือ D เป็นโดเมนซึ่งแยกตัวประกอบได้อย่างเดียวแล้ว เซตของฟังก์ชันเลขคณิตเทียมทั้งหมดจาก S ไปยัง D เป็นโดเมนซึ่งแยกตัวประกอบได้อย่างเดียวภายใต้การบวกและสังวัตนาการ

ทิศทางที่สอง เราแทนสนามจำนวนเชิงซ้อนด้วย วงที่มีการแยกตัวประกอบได้อย่างเดียว $\mathcal{R}$ ซึ่งมีตัวหารของศูนย์ และพิสูจน์ว่าเมื่อวงของอนุกรมกำลังรูปนัยในตัวไม่กำหนดจำนวนจำกัดเหนือ $\mathcal{R}$ เป็นวงที่มีการแยกตัวประกอบได้อย่างเดียวซึ่งมีตัวหารของศูนย์ และ วงของอนุกรมกำลังรูปนัยในตัวไม่กำหนดจำนวนอนันต์นับได้เหนือ $\mathcal{R}$ เป็นวงกระชับแล้ว เซตของฟังก์ชันเลขคณิตค่าเหนือ $\mathcal{R}$ เป็นวงที่มีการแยกตัวประกอบได้อย่างเดียวซึ่งมีตัวหารของศูนย์ ภายใต้การบวกและสังวัตนาการ

วิธีการพิสูจน์ที่ใช้มาจากการวิเคราะห์วิธีการของ แคชเวลล์ และ เอเวอเรต (1959) และ ลู่(1965) อย่างละเอียด โดยมีการปรับปรุงเพิ่มเติม พร้อมด้วยแนวคิดเพิ่มเติมได้แก่ กฎการตัดออกอย่างอ่อน และความกระชับ

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AN ABSTRACT

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PATHIRA RUENGSINSUP : UNIQUE FACTORIZATION OF PSEUDO-ARITHMETIC FUNCTIONS. THESIS ADVISOR : ASST. PROF. PATANEE UDOMKAVANICH, PH.D., THESIS COADVISOR : ASSOC. PROF. VICHIAN LAOHAKOSOL, Ph.D., 29 PP. ISBN 974-334-063-7.

In 1959, Cashwell and Everett proved that the set of complex-valued arithmetic functions forms a unique factorization domain under addition and convolution. In this thesis we further this result in two directions.

In the first direction, we replace the set of all natural numbers and the complex field by an arithmetical semigroup S and a unique factorization domain D, respectively, and prove that when the ring of formal power series in any finite number of indeterminates over D is a unique factorization domain , then the set of all pseudo-arithmetic functions from S into D is a unique factorization domain under addition and convolution .

In the latter direction, we replace the complex field by a unique factorization ring $\mathcal{R}$ with zero divisors and prove that when the ring of formal power series over $\mathcal{R}$ in any finite number of indeterminates is a unique factorization ring with zero divisors and the ring of formal power series over $\mathcal{R}$ in a countably infinite number of indeterminates is compact, then the set of all arithmetic functions over $\mathcal{R}$ is a unique factorization ring with zero divisors under addition and convolution.

The proofs employed come from a detailed analysis of those used by Cashwell and Everett (1959) ,and Lu(1965) with a number of modifications together with an introduction of concepts such as weak cancellation law and compactness.

ภาควิชา คณิตศาสตร์	ลายมือชื่อนิสิต ....
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CONTENT

INTRODUCTION

The set of all functions from an arithmetical semigroup S into a commutative ring R with identity , denoted by ${}_{S}\Omega_{R}$, forms a commutative ring with identity under addition and convolution , see e.g. Berberian[1]. It was proved by Cashwell and Everett [3] in 1959 that when S is the set of all natural numbers and R the complex field , ${}_{\mathbb{N}}\Omega_{\mathbb{C}}$ is a unique factorization domain . The proof was based on the fact that ${}_{\mathbb{N}}\Omega_{\mathbb{C}}$ is isomorphic to $\mathbb{C}[[x_1,x_2, \ldots]]$, the ring of formal power series over $\mathbb{C}$ in countably many indeterminates and that the rings of formal power series over $\mathbb{C}$ in a finite number of indeterminates are unique factorization domains. In the sixties , Cashwell and Everett [4] , Lu[10] considered instead the case where the complex field is replaced by an integral domain D and proved that ${}_{\mathbb{N}}\Omega_{D}$ is a unique factorization domain if D is a unique factorization domain such that the rings of formal power series over D in a finite number of indeterminates are unique factorization domains .

In Chapter I , we introduce notation , definitions and prove auxiliary theorems used throughout this thesis.

In Chapter II , we prove subject to certain conditions the unique factorization theorem in ${}_{S}\Omega_{D}$, where S is an arithmetical semigroup and D a unique factorization domain. This extends the original works of Cashwell and Everett [3] in the direction of the domain involved. The proofs is divided into two parts. First , the case where the range is the complex field , it is proved that such arithmetical semigroup is isomorphic to the set of natural numbers and the result then follows from Cashwell and Everett theorem. This is essentially the proof adopted by Knopfmacher[8]. Second , the case where the range is any unique factorization domain, the proof is a modification of that used by Cashwell and Everett in [3].

In Chapter III , we prove subject to appropriate conditions the unique factorization theorem in ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$, where $\mathcal{R}$ is a unique factorization ring with zero divisors. This also extends the result of Cashwell and Everett in the direction of the range involved. The main proof is a combination of ideas used in Cashwell and Everett[3] , and Lu[10] . First , a characterization of unique factorization ring with zero divisors as a ring with greatest common divisors satisfying weak cancellation law is established. Passing to isomorphic setting in the ring of formal power series , a concept of compactness is introduced which enables us to complete the proof.

CHAPTER I

PRELIMIANRIES

In this chapter we give notation , definitions and theorems used in this thesis. The following symbols will be standard :

$\mathbb{Z}$ is the set of all integers,

$\mathbb{N}$ is the set of all positive integers,

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$\mathbb{C}$ is the complex field.

§1. ARITHMETICAL SEMIGROUPS

We take the same definition and examples of arithmetical semigroups as in Knopfmacher[9].

Definition 1.1 *Let $(S,\cdot)$ be a commutative semigroup with identity 1_S, S is called an **arithmetical semigroup** if and only if*

(i) S has a finite or countably infinite subset P (whose elements are called the primes of S) such that every element $s \neq 1_S$ in S has a unique factorization of the form $s = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$,where the p_i are distinct elements of P , the n_i are positive integers , k may be arbitary , and uniqeness is up to the order of the factors indicated

(ii) there exits a non-negative real-valued norm mapping $|.|$ on S such that

(1) $|1_S| = 1$, $|p| > 1$ for $p \in P$,

(2) $|ab| = |a||b|$ for all $a,b \in S$,

(3) the total number $N_S(x)$ of elements $a \in S$ of norm $|a| \leq x$ is finite , for each real $x > 0$.

Note *For all $a \in S$, $|a| \geq 1$ and the only one factor of 1_S is itself .*

Example 1.2 Define a norm on $\mathbb{N}$ by $|n| = n$ for all $n \in \mathbb{N}$ .Then for all real $x > 0$, $N_{\mathbb{N}}(x) = [x]$, the greatest integer not exceeding x. Thus $(\mathbb{N},\cdot)$ with its subset P of all rational primes is an arithmetical semigroup.

Example 1.3 Let D be an integral domain. Then the set G_D of all associate classes $\bar{a}$ of nonzero element $a \in D$ forms a commutative semigroup with identity under the multiplication $\bar{a} \cdot \bar{b} = \overline{ab}$.

In the case when D is a principal ideal domain , the content of the unique factorization for D is that each element $\bar{a} \neq \bar{1}$ of G_D admits unique factorization into power of the classes $\bar{p}$ of prime elements $p \in D$.

If D is a Euclidian domain with norm function $|\,.\,|$, then define a norm on G_D satisfying conditions (1) and (2) above by letting $|\bar{a}| = |a|$. In certain interesting cases this norm satisfies condition (3) above, and G_D forms an arithmetical semigroup. The followings are illustrations .

1.3.1) Let D be the ring $\mathbb{Z}[i]$ of all Gaussian integers m+ni ($m,n \in \mathbb{Z}$) .This ring is a Euclidian domain with the norm $|m+ni| = m^2+n^2$. Since $\mathbb{Z}[i]$ has only four units 1,-1, i, and -i ,

$$N_{G_D}(x) = \frac{1}{4}\sum_{k \le x} r(k) < \infty ,$$

where r(k) denotes the total number of lattice points (a,b) ($a,b \in \mathbb{Z}$) on the circle $y^2 + z^2 = k$ $(k \geq 1)$ in the Euclidian plane $\mathbb{R}^2$. Thus G_D forms an arithmetical semigroup.

1.3.2) Let F be a field and x an indeterminate.Then the polynomial ring F[x] is a Euclidian domain with the norm $|f| = 2^{\deg f}$ for $0 \neq f \in F[x]$, $|0| = 0$ .The units in F[x] are nonzero constant polynomials. In the case when F is a finite (Galois) field GF(q) , the total number of units in F[x] is q-1. Let $f = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in F[x]$,$a_n \neq 0$ Then $\bar{f} = \{ af : a \in F-\{0\}\}$ and $|\bar{f}| = |f| = 2^n$. Thus $\#\{ \bar{g} \in G_{F[x]} : |\bar{g}| = 2^n \} = \#\{ g = a_n x^n + b_{n-1}x^{n-1} + \ldots + b_0 : b_i \in F\} = q^n$. Let r be a positive real number . Thus

$$N_{G_{F[x]}}(r) = \sum_{n=0}^{[\log_2 r]} q^n .$$

Hence $G_{F[x]}$ forms an arithmetical semigroup.

Proposition 1.4 *If S is an arithmetical semigroup then for all $a \in S$,*
$\{ (x,y) \in S \times S : xy = a \}$ is finite.

Proof Let $a \in S$. For $x,y \in S$, if $xy = a$ then $|x||y| = |xy| = |a|$, so $|x| \leq |a|$ and $|y| \leq |a|$. Since $N_S(|a|)$ is finite, $\{ (x,y) \in S \times S : xy = a \}$ is finite. #

§2. POWER SERIES

We first recall some general definitions. Let R be a commutative ring with identity 1 . An element $u \in R$ is a unit if there is $v \in R$ such that $uv = 1$. For $r,s \in R$, r divides s, written $r|s$, if there exists $t \in R$ such that $rt = s$. Two elements r and s are associate ,written $r \sim s$, if there is a unit u of R such that $ru = s$ .An element $r \neq 0$ is a zero divisor if there is $s \neq 0$ in R such that $rs=0$. Let r be a nonzero non-unit element of R ; r is prime if, whenever $r|ab$ where $ab \neq 0$, then $r|a$ or $r|b$; r is reducible if there are nonzero non-unit $a,b \in R$ such that $r = ab$; r is irreducible if r is not reducible. It is easy to shown that if r and s are irreducible , then r divides s if and only if r and s are associates. An element $d \in R-\{0\}$ is called a greatest common divisor of $a_1, a_2,\dots,a_n \in R$, not all zero , if $d|a_i$ for all $i \in \{1,2,\dots,n\}$,and if $c \in R-\{0\}$ is such that $c|a_i$ for all $i \in \{1,2,\dots,n\}$, then $c|d$. If d_1 and d_2 are two greatest common divisors of $a_1, a_2,\dots,a_n \in R$ then $d_1 \sim d_2$. Denote by $(a_1, a_2,\dots,a_n)$ a greatest common divisor of $a_1, a_2,\dots,a_n \in R$.

Let R be a commutative ring with identity 1. Denote by $R_\omega = R[[x_1 , x_2 , \dots]]$ the set of all formal power series in a countably infinite number of indeterminates $x_1 , x_2 , \dots$ over R and for $j \geq 1$, by $R_j = R[[x_1 , x_2 , \dots , x_j]]$ the set of all formal power series in j indeterminates over R.

Definition 1.5 *We say that a nonzero monomial $c x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, $c \in R$ is **of weight r** if $1 \cdot n_1 + 2 \cdot n_2 + \dots + k \cdot n_k = r$.*

It is easy to see that the product of two monomials , whose weights are t_1 and t_2 respectively and whose coefficients are not zero divisors , is of weight $t_1 + t_2$.

Every element f of R_ω can be expressed in the form $f = (f_0 , f_1 , \dots , f_n , \dots)$, where each f_n is either zero or a finite or an infinite sum of monomials of weight n .

Definition 1.6 *Define addition and multiplication of two power series* $f = (f_0, f_1, \dots, f_n, \dots)$ *and* $g = (g_0, g_1, \dots, g_n, \dots)$ *as follows :*

$$f+g = (f_0+g_0,\ f_1+g_1,\ \dots,\ f_n+g_n,\ \dots)$$

$$fg = (h_0, h_1, \dots, h_n, \dots) \quad , \text{ where } h_n = \sum_{i+j=n} f_i g_j .$$

With these definitions of addition and multiplication , R_ω becomes a commutative ring with identity 1 .

Theorem 1.7 *An element* $f = (f_0, f_1, \dots, f_n, \dots)$ *of* R_ω *is a unit if and only if* f_0 *is a unit of* R .

Proof Let $f = (f_0, f_1, \dots, f_n, \dots) \in R_\omega$. Assume that f is a unit of R_ω . Then $fg = 1$ for some $g = (g_0, g_1, \dots, g_n, \dots) \in R_\omega$. Thus $f_0 g_0 = 1$, so f_0 is a unit of R .

Conversely , assume that f_0 is a unit of R . We will construct an element $g = (g_0, g_1, \dots, g_n, \dots)$ of R_ω, where each g_n is either zero or a form of weight n, such that $\sum_{i+j=n} f_i g_j = 0$ for all $n \neq 0$.

Define $g_0 = f_0^{-1}$. Let $n \geq 1$. Assume that $g_0, g_1, \dots, g_{n-1}$ have been defined and that each g_i is either zero or a form of weight i for $0 \leq i \leq n-1$. Define $g_n = -f_0^{-1}(\sum_{i+j=n, j\neq n} f_i g_j)$. It is clear that g_n is then either zero or a form of weight n and $\sum_{i+j=n} f_i g_j = 0$. Now set $g = (g_0, g_1, \dots, g_n, \dots)$. Then $g \in R_\omega$ and $fg = 1$, so f is a unit of R_ω . #

Definition 1.8 *Define an order function* $\mathcal{O}$ *on* R_ω *as follows :*

$$\mathcal{O}(0) = \infty \text{ and } \mathcal{O}(f) = \min\{ n \in \mathbb{N}_0 : f_n \neq 0 \} \text{ if } f \neq 0 .$$

Theorem 1.9 *Let f and g be power series in* R_ω *. Then*

(i) $\mathcal{O}(f+g) \geq \min\{ \mathcal{O}(f), \mathcal{O}(g) \}$

(ii) $\mathcal{O}(fg) \geq \mathcal{O}(f) + \mathcal{O}(g)$.

Proof If f or g is zero , then there is nothing to prove . Assume that f and g are both nonzero . Then $\mathcal{O}(f) = n$ and $\mathcal{O}(g) = m$ for some $n, m \in \mathbb{N}_0$.

(i) If $f+g = 0$, then the result is trivial. Assume that $f+g \neq 0$. Then $\mathcal{O}(f+g) = k$ for some $k \in \mathbb{N}_0$,so $f_k + g_k \neq 0$. Thus $f_k \neq 0$ or $g_k \neq 0$, so $k \geq \min\{n,m\} = \min\{\mathcal{O}(f), \mathcal{O}(g)\}$.

(ii) If $fg = 0$, then the result is trivial . Assume that $fg \neq 0$. Then $\mathcal{O}(fg) = s$ for some $s \in \mathbb{N}_0$, so $\sum_{i+j=s} f_i g_j \neq 0$. Thus $f_i \neq 0$ and $g_j \neq 0$ for some i,j such that $i+j = s$, so $i \geq n$ and $j \geq m$. Hence $\mathcal{O}(fg) = s = i+j \geq n+m = \mathcal{O}(f) + \mathcal{O}(g)$. #

Notation *Let* $\mathcal{B} = \{B_t(f) : f \in R_\omega \text{ and } t \in \mathbb{N}_0\}$ *,where* $B_t(f) = \{ g \in R_\omega : \mathcal{O}(g-f) \geq t \}$.

Cleary , $R_\omega = B_0(0)$, so R_ω is the union of the elements of $\mathcal{B}$.

Proposition 1.10 *Let* $f, g \in R_\omega$ *and* $s,t \in \mathbb{N}_0$ *. Assume that* $B_s(f) \cap B_t(g) \neq \varnothing$ *and* $s \leq t$ *. Then*

(i) $f_i = g_i$ *for all* $i \in \{ 0,1,\ldots, s-1 \}$.

(ii) $B_t(g) \subseteq B_s(f)$.

Proof (i) Let $h \in B_s(f) \cap B_t(g)$. Then $\mathcal{O}(h-f) \geq s$ and $\mathcal{O}(h-g) \geq t$. Thus for all $i \in \{ 0,1,\ldots, s-1 \}$, $h_i = f_i$ and for all $j \in \{ 0,1,\ldots, t-1 \}$, $h_j = g_j$. Hence for all $i \in \{ 0,1,\ldots, s-1 \}$, $f_i = g_i$.

(ii) Let $p \in B_t(g)$. Then for all $i \in \{0,1,\ldots, t-1\}$, $p_i = g_i$, so for all $i \in \{0,1,\ldots, s-1\}$, $p_i = g_i = f_i$. Then $\mathcal{O}(p-f) \geq s$, so $p \in B_s(f)$. Thus $B_t(g) \subseteq B_s(f)$. #

By Proposition 1.10 , $\mathcal{B}$ satisfies the condition : if U and V are any elements of $\mathcal{B}$ and if x is any point of $U \cap V$ then there is an element W of $\mathcal{B}$ such that $x \in W \subseteq U \cap V$. Then by Theorem 2.2(c) of Gautam and Narayan [6] , The class τ of all subset X of R_ω such that X is the union of a family of elements of $\mathcal{B}$ is a topology for R_ω and $\mathcal{B}$ is a basis of τ . This topology is called **the weight topoloty**.

Definition 1.11 *A sequence* $(f^{(n)})$ *of elements in* R_ω *is called a* ***Cauchy sequence*** *if for every* $i \geq 0$ *there exists a positive integer* $T(i)$ *such that for* $n,m \geq T(j)$ *,* $\mathcal{O}(f^{(m)} - f^{(n)}) > i$.

Definition 1.12 *We say that sequence* $(f^{(n)})$ *of elements in* R_ω ***converges*** *to an element* $f \in R_\omega$ *if for every* $i \geq 0$ *there exists a positive integer* $T(i)$ *such that for all* $n \geq T(i)$, $\mathcal{O}(f^{(n)} - f) > i$.

Theorem 1.13([10]) R_ω *is complete under the weight topology .*

Proof Let $(f^{(n)})$ be a Cauchy sequence of elements in R_ω. Then for every integer $j \geq 0$, there exists an integer $T(j)$ such that $\mathcal{O}(f^{(m)} - f^{(n)}) > j$ if $n,m \geq T(j)$. Then $f_k^{(m)} = f_k^{(n)}$ for all $k \leq j$. Put $f = (f_0^{(T(0))}, f_1^{(T(1))}, \ldots, f_n^{(T(n))}, \ldots)$. Then $f \in R_\omega$ and for every j , $f_k = f_k^{(n)}$ for all $k \leq j$ if $n \geq T(j)$. Thus $\mathcal{O}(f^{(n)} - f) > j$ if $n \geq T(j)$. Therefore $(f^{(n)})$ converges to f. Hence R_ω is complete. #

Lemma 1.14 *If* $u \in R_\omega$ *is a limit of a convergent sequence of units in* R_ω *then* u *is a unit .*

Proof Let $(u^{(n)})$ be a convergent sequence of units in R_ω with a limit $u \in R_\omega$. Then for every integer $j \geq 0$, there exists an integer $T(j)$ such that for all $n \geq T(j)$, $\mathcal{O}(u^{(n)} - u) > j$. Therefore $u_0 = u_0^{(n)}$ for n sufficiently large, so u_0 is a unit of R . Hence u is a unit of R_ω . #

Definition 1.15 *Let* $f = f(x_1, x_2, \ldots, x_j, \ldots) \in R_\omega$ *; then for any integer* $j \geq 0$ *, the formal power series* $f(x_1, x_2, \ldots, x_j, 0, 0, \ldots)$ *in* R_j *, denoted by* $(f)_j$ *, is called* ***the projection*** *of* f *on* R_j *. We set* $R = R_0$ *.*

Clearly , the mapping $f \rightarrow (f)_j$ is a ring homomorphism from R_ω to R_j , i.e. , $(f+g)_j = (f)_j + (g)_j$ and $(fg)_j = (f)_j(g)_j$.

Definition1.16 *A sequence* $(f^{(0)}, f^{(1)}, \ldots, f^{(i)}, \ldots)$ *, where* $f^{(i)} \in R_i$ *is said to be a* ***telescopic chain*** *if* $f^{(i)} = (f^{(i+1)})_i$ *for each* i *.*

Lemma 1.17([10]) *Every infinite telescopic chain* $(f^{(0)}, f^{(1)}, \ldots, f^{(i)}, \ldots)$ *is a Cauchy sequence for the weight topology, and hence has a limit in* R_ω .

<u>Proof</u> Since the sequence is telescopic , for every integer $i \geq 0$ and $j > 0$, each monomial of $f^{(i+j)} - f^{(i)}$ is either zero or contains at least one x_k with $k > i$ as a factor. Hence $\mathscr{O}(f^{(i+j)} - f^{(i)}) > i$. Thus the sequence is a Cauchy sequence . Since R_ω is complete , the sequence has a limit in R_ω. #

Note *Every* $f \in R_\omega$ *is a limit of a finite or an infinite telescopic chain* $((f)_0, (f)_1, \ldots, (f)_i, \ldots)$.

Definition 1.18 R_ω *is said to be* ***compact*** *(with respect to weight topology) if every sequence of units in* R_ω *has a convergent subsequence.*

Definition 1.19 *A sequence* $(f^{(0)}, f^{(1)}, \ldots, f^{(i)}, \ldots)$ *, where* $f^{(i)} \in R_i$ *is said to be a* ***pseudo-telescopic chain*** *if* $(f^{(i+1)})_i$ *is associate to* $f^{(i)}$ *in* R_i *for every i .*

Lemma 1.20 *If* R_ω *is compact , then any pseudo-telescopic chain has a convergent subchain .*

<u>Proof</u> Let $(f^{(0)}, f^{(1)}, \ldots, f^{(i)}, \ldots)$, where $f^{(i)} \in R_i$ be a pseudo-telescopic chain. Then for each $j \geq 0$, $f^{(j)} = u^{(j)}(f^{(j+1)})_j = (u^{(j)}f^{(j+1)})_j$, where $u^{(j)}$ is a unit in R_j . Put $F^{(0)} = f^{(0)}$, $F^{(1)} = u^{(0)}f^{(1)}$, $F^{(2)} = u^{(0)}u^{(1)}f^{(2)}$, ... , $F^{(j)} = u^{(0)}u^{(1)}\ldots u^{(j-1)}f^{(j)}$. Then $f^{(j)} = v^{(j)}F^{(j)}$, where $v^{(j)} = (u^{(0)}u^{(1)}\ldots u^{(j-1)})^{-1}$ is a unit in R_j and $(F^{(0)}, F^{(1)}, \ldots)$ is a telescopic chain , which has a limit in R_ω , say F .Since R_ω is compact , there is a subsequence $(v^{(j_k)})$ of $(v^{(j)})$ which converges to a unit v in R_ω . Therefore $\lim_{k\to\infty} f^{(j_k)} = \lim_{k\to\infty} v^{(j_k)}F^{(j_k)} = \lim_{k\to\infty} v^{(j_k)} \lim_{k\to\infty} F^{(j_k)} = vF$. Hence the $(f^{(j_k)})$ is a convergent subchain of $(f^{(j)})$. #

Lemma 1.21([3]) *Let f be a nonzero non-unit element in R_ω .Then there is a least positive integer $L = L(f)$, hereby called **the index of f**, for which $(f)_j$ is a nonzero non-unit element in R_j , for all $j \geq L$.*

Proof Since f is a nonzero non-unit element of R_ω , then f must contain some monomial term $x_1^{n_1} x_2^{n_2} \cdots$ with nonzero coefficient and n_i not all zero . If in this term x_k is the last variable with $n_k > 0$,then $(f)_k \neq 0$. Hence there is a least positive integer L with $(f)_L \neq 0$, so $(f)_j$ is a nonzero non-unit element in R_j for $j \geq L$. #

Lemma 1.22([3]) *Let f be a nonzero non-unit element in R_ω with index L . If $(f)_j$ is irreducible in R_j for some $j \geq L$, then $(f)_m$ is irreducible in R_m for all $m \geq j$, and f is irreducible in R_ω .*

Proof Assume that $(f)_j$ is irreducible for some $j \geq L$. Let $m \geq j$. Let $g^{(m)}, h^{(m)} \in R_\omega$ be such that $(f)_m = g^{(m)} h^{(m)}$. Then $(f)_j = ((f)_m)_j = (g^{(m)} h^{(m)})_j = (g^{(m)})_j (h^{(m)})_j$. Thus $(g^{(m)})_j$ or $(h^{(m)})_j$ is a unit in R_j , so $g^{(m)}$ or $h^{(m)}$ is a unit in R_m . Hence $(f)_m$ is irreducible in R_m. Similarly , we can show that f is irreducible in R_ω . #

Definition 1.23 *Let f be a nonzero non-unit in R_ω. A **true factor** of $(f)_j$, $j \in \mathbb{N}_0$, is a non-unit proper divisor of $(f)_j$ in R_j .*

Definition 1.24 *A nonzero non-unit element f in R_ω is said to be **finitely irreducible** if there is a least integer $P \geq$ the index L of f such that $(f)_j$ is irreducible in R_j for all $j \geq P$.*

§3. PSEUDO-ARITHMETIC FUNCTION

In this section S stands for an arithmetical semigroup and R for a commutative ring with identity.

Definition 1.25 *A **pseudo-arithmetic function** is a function from S to R. Let ${}_S\Omega_R$ be the set of all pseudo-arithmetic functions. Define addition and multiplication $*$, which is called a **convolution** , in ${}_S\Omega_R$ as follows :*

For $\alpha, \beta \in {}_S\Omega_R$, $(\alpha+\beta)(a) = (\alpha)(a) + (\beta)(a)$ *and*

$$(\alpha*\beta)(a) = \sum_{xy=a} \alpha(x)\beta(y) \qquad \text{for all } a \in S .$$

The summation is well-defined by Proposition1.4 .

It is easy to show that (${}_S\Omega_R$, + , $*$) is a commutative ring with identity . The zero θ and additive inverse $-\alpha$ of $\alpha \in {}_S\Omega_R$ are the pseudo-arithmetic functions defined by $\theta(a) = 0$ and $(-\alpha)(a) = -\alpha(a)$ for all $a \in S$. The convolution identity ε is defined by $\varepsilon(1) = 1$ and $\varepsilon(a) = 0$ for all $a \in S-\{1\}$.

Definition 1.26 *Define the order function* $\langle . \rangle$ *on* ${}_S\Omega_R$ *by ,*

$\langle\theta\rangle = 0$ *and* $\langle\alpha\rangle = \min\{\ |a| : \alpha(a) \neq 0\ \}$ *if* $\alpha \neq \theta$.

Proposition 1.27 *(i) For* $\alpha \in {}_S\Omega_R$, $\langle\alpha\rangle \geq 0$ *and* $\langle\alpha\rangle = 0$ *if and only if* $\alpha = \theta$.

(ii) For $\alpha, \beta \in {}_S\Omega_R$, $\langle\alpha*\beta\rangle \geq \langle\alpha\rangle\langle\beta\rangle$. *In particular , if R has no zero divisor then* $\langle\alpha*\beta\rangle = \langle\alpha\rangle\langle\beta\rangle$ *for all* $\alpha, \beta \in {}_S\Omega_R$.

Proof (i) is obvious from the definitiion.

(ii) Let $\alpha, \beta \in {}_S\Omega_R$ .The case $\alpha = \theta$ or $\beta = \theta$ is trivial . Assume that α and β are nonzero. Then $\langle\alpha\rangle = |a|$ and $\langle\beta\rangle = |b|$ for some $a,b \in S$. Then for all $c \in S$ such that $|c| < |ab|$, $(\alpha*\beta)(c) = \sum_{xy=c}\alpha(x)\beta(y) = 0$ and $(\alpha*\beta)(ab) = \sum_{xy=ab}\alpha(x)\beta(y) = \alpha(a)\beta(b)$.

Then $\langle\alpha*\beta\rangle \geq |ab|$. If R has no zero divisor then $\alpha(a)\beta(b) \neq 0$, so $\langle\alpha*\beta\rangle = \langle\alpha\rangle\langle\beta\rangle$. #

Proposition 1.28 *Let* $\alpha \in {}_S\Omega_R$. *Then* α *is a unit if and only if* $\alpha(1)$ *is a unit in R .*

Proof Assume that α is a unit. Then there is a nonzero element β in ${}_S\Omega_R$ such that $\alpha*\beta = \varepsilon$. Thus $1 = \varepsilon(1) = (\alpha*\beta)(1) = \alpha(1)\beta(1)$, so $\alpha(1)$ is a unit in R .

Conversely, assume that $\alpha(1)$ is a unit in R. We will construct a nonzero element β in ${}_S\Omega_R$ such that $\alpha*\beta = \varepsilon$. Define $\beta(1) = \alpha(1)^{-1}$. Let $a \in S$ be such that $a \neq 1$. Assume that for all $b \in S$ such that $|b| < |a|$, $\beta(b)$ has been defined and satisfied the condition $(\alpha*\beta)(b) = \varepsilon(b)$. Define $\beta(a) = \alpha(1)^{-1}(-\sum_{bc=a}\alpha(b)\beta(c))$. Then

$(\alpha*\beta)(a) = \sum_{bc=a}\alpha(b)\beta(c) = \sum_{bc=a,b\neq 1}\alpha(b)\beta(c) + \alpha(1)\beta(c) = 0 = \varepsilon(a)$. Thus $(\alpha*\beta)(a) = \varepsilon(a)$ for all $a \in S$, so $\alpha*\beta = \varepsilon$. Hence α is a unit. #

Now let the primes p of S be listed in the order $p_1 , p_2 ,\ldots$, where $|p_i| \leq |p_{i+1}|$. Then every element a of S may be written uniquely in the form $a = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ for some k , where each n_i is a non-negative integer. Hence every pseudo-arithmetic function α may be associated with a definite formal power series in R_ω by means of the correspondence :

$$(*) \qquad \alpha \to f^{(\alpha)} = \sum\alpha(a)\, x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k} \ ,$$

where the summation extends over all $a = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ of S ; obviously the sum $f^{(\alpha)}$ can be identified with some formal power series in R_ω . It is easy to verify that the correspondence is an isomorphism between ${}_S\Omega_R$ and R_ω .

Definition1.29 *Define* S_k *to be the set consisting of all elements of S of the form* $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$, $n_i \geq 0$ *for each* $i = 1,2,3,\ldots,k$.

Then clearly $S_1 \subset S_2 \subset S_3 \subset \ldots \subset S_k \subset \ldots$ and $\bigcup_{k=1}^{\infty} S_k = S$.

Let $({}_S\Omega_R)_k$ be the subset of ${}_S\Omega_R$ consisting of those pseudo-arithmetic functions α such that $\alpha(a) = 0$ for all $a \notin S_k$. Then , the set is a collection of all functions on S_k into R . It can easily verified that $({}_S\Omega_R)_k \cong R[[x_1 , x_2 , \ldots , x_k]]$ under the correspondence $(*)$.

§4. UNIQUE FACTORIZATION RINGS WITH ZERO DIVISORS.

We take essentially the same definition of unique factorization rings with zero divisors as in Galovich[5] .

Definition 1.30 *A commutative ring R with identity and zero divisors is a* ***unique factorization ring with zero divisors (UFRZ)*** *if*

(i) every nonzero non-unit of R can be written as a finite product of irreducible factors and

(ii) if $a_1a_2\cdots a_n = b_1b_2\cdots b_m$ *, where each of the elements* a_i *and* b_j *is irreducible , then* $n = m$ *and* a_i *can be renumbered so that* $a_i \sim b_i$ $(i = 1,\ldots,n)$.

Example 1.31 $\mathbb{Z} / p^m\mathbb{Z}$, where p is a rational prime and m is a positive integer ≥ 2, is a UFRZ , see Billis[2].

Definition 1.32 *A commutative ring R with identity is said to satisfies a* ***weak cancellation law*** *whenever* $ax = ay \neq 0$ *, where* $a,x,y \in R$ *, then* $x \sim y$ *.*

Proposition 1.33 *UFRZ R satisfies a weak cancellation law.*

<u>Proof</u> Let $a,x,y \in R$. Assume that $ax = ay \neq 0$. Then a ,x and y are nonzero elements. By the unique factorization property on R , $a = a_1a_2\cdots a_s$, $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_m$, where each of the elements a_i , x_j and y_k is irreducible , so $a_1a_2\cdots a_s\, x_1x_2\cdots x_n = a_1a_2\cdots a_s\, y_1y_2\cdots y_m$. Then $n = m$ and after some renumbering $x_i \sim y_i$ $(i = 1,\ldots,n)$,so $x \sim y$. #

Lemma 1.34([7]) *Let R be a commutative ring with identity and zero divisors. Assume that any two elements of R , not both zero, have a greatest common divisor and R satisfies a weak cancellation law. Then*

(i) *For* $a_1,a_2,\ldots,a_n \in R$ *, a greatest common divisor of* $a_1, a_2,\ldots,a_n$ *exists.*

(ii) *For* $a , b, c \in R$*,* $((a,b),c) \sim (a,(b,c))$*.*

(iii) *For* $a , b, c \in R$*,* $c(a,b) \sim (ca,cb)$*.*

(iv) *For* $a , b, c \in R$*, if* $(a,b) \sim 1$ *and* $(a,c) \sim 1$ *then* $(a,bc) \sim 1$*.*

<u>Proof</u> (i) Let $a_1,a_2,\ldots,a_n \in R$. Let $d_1 = (a_1,a_2)$, $d_{i+1} = (d_i ,a_{i+1})$ $1 \leq i \leq n-1$ and $d = d_n$. Then $d \mid d_{n-1}$ and $d \mid a_n$, so $d \mid a_i$ for $1 \leq i \leq n$. Let e be a common divisor of $a_1,a_2,\ldots,a_n$. Then $e \mid d_1$, so $e \mid d_i$ for $1 \leq i \leq n$. Thus $e \mid d_n = d$. Hence d is a greatest common divisor of $a_1,a_2,\ldots,a_n$.

(ii) Let $a, b, c \in R$ and $g = ((a,b),c)$. Then $g \mid (a,b)$ and $g \mid c$. Thus $g \mid a$, $g \mid b$ and $g \mid c$, so g is common divisor of a ,b and c . Let d be a common divisor of a ,b and c. Then $d \mid a$, $d \mid b$ and $d \mid c$. Thus $d \mid (a,b)$, so $d \mid ((a,b),c)$. Thus $((a,b),c) = g \sim (a,b,c)$. Similarly , we can show that $(a,(b,c)) \sim (a,b,c)$. Hence $((a,b),c) \sim (a,(b,c))$.

(iii) Let $a, b, c \in R$, $d = (a,b)$ and $g = (ca,cb)$. Then $d \mid a$ and $d \mid b$, so $cd \mid ca$ and $cd \mid cb$. Then $cd \mid g$, so $g = cdx$ for some $x \in R$. Since $g \mid ca$, $ca = gy$ for some $y \in R$. Then $ca = gy = cdxy$. Since R satisfies a weak cancellation law , $a = dxyu$ for some unit $u \in R$, so $dx \mid a$. Similarly , we can show that $dx \mid b$, so $dx \mid d$. Then x is a unit and $(ca,cb) = g \sim cd = c(a,b)$.

(iv) Let $a, b, c \in R$. Assume that $(a,b) \sim 1$ and $(a,c) \sim 1$. By (iii) , $(ac,bc) \sim c$ and $(a,ac) \sim a$. Then $1 \sim (a,c) \sim (a,(ac,bc)) \sim ((a,ac),bc) \sim (a,bc)$. #

Proposition 1.35 *Let R be a commutative ring with identity and zero divisors. Assume that every nonzero non-unit element of R can be written as a finite product of irreducible elements of R. Then the following assertions are equivalent :*

(i) R is a UFRZ .

(ii) Any two elements of R ,not both zero, have a greatest common divisor and R satisfies a weak cancellation law .

Proof Assume that (i) holds.By Proposition 1.33,R satisfies a weak cancellation law. Let $x,y \in R$. Since R is a UFRZ , we can write $x = r_1^{n_1} r_2^{n_2} \cdots r_k^{n_k}$ and $y = r_1^{m_1} r_2^{m_2} \cdots r_k^{m_k}$, where r_i are distinct irreducible elements of R and n_i ,m_i are non-negative integers. Let $d = r_1^{\min\{n_1,m_1\}} r_2^{\min\{n_2,m_2\}} \cdots r_k^{\min\{n_k,m_k\}}$ .Then $d \mid a$ and $d \mid b$, so d is a common divisor of a and b.Let e be any other common divisor of a and b. Then $e = r_1^{s_1} r_2^{s_2} \cdots r_k^{s_k}$, where $0 \le s_i \le n_i$ and $0 \le s_i \le m_i$. Thus $s_i \le \min\{ n_i,m_i \}$, so $e \mid d$. Hence d is a greatest common divisor of a and b.

Assume that (ii) holds. We will show that every irreducible element of R is a prime. Let p be an irreducible element of R such that $p \mid ab$, $a,b \in R$. Suppose that p does not divide a and b . Then $(p,a) \sim 1 \sim (p,b)$. For , if there is a non-unit $d \in R$ such that $d = (p,a)$, then $p = dx$ for some $x \in R$, since p is irreducible , we have x is a unit , so $p \mid d$; hence $p \mid a$, a contradiction. By Proposition 1.34(iv) , $(p,ab) \sim 1$ which contradicts $p \mid ab$. Hence $p \mid a$ or $p \mid b$, so p is a prime in R.

Now we will show the uniqueness of factorization in R. Let a be a nonzero non-unit element of R . Suppose that $r_1 r_2 \cdots r_n = a = s_1 s_2 \cdots s_m$, where r_i and s_j are irreducible . Since irreducible elements are primes , $r_1 \mid s_j$ for some j , say s_1 . Since r_1 and s_1 are irreducible , $r_1 \sim s_1$, so $r_1 = u_1 s_1$ for some unit u_1 in R. Then $u_1 s_1 r_2 \cdots r_n = s_1 s_2 \cdots s_m$. Since R satisfies a weak cancellation law, $u_1 r_2 \cdots r_n = v_1 s_2 \cdots s_m$ for some unit v_1 . Continue this process. If $n \neq m$, we have a product of irreducible elements equal to a unit , which is impossible . Then $n = m$ and after renumbering $r_i \sim s_j$ for all i . #

CHAPTER II

CASE OF NO ZERO DIVISORS

E.D.Cashwell and C.J.Everett [3] have proved that the ring of complex-valued arithmetic functions is a unique factorization domain. In this chapter, we consider the case where the set of natural numbers and the complex field are replaced by an arithmetical semigroup S and a unique factorization domain D , respectively, and call such functions pseudo-arithmetic functions. We shall prove that the ring of all pseudo-arithmetic functions from S to D is a unique factorization domain. The proofs is divides into two parts. First , the case where the range is the complex field. Second , the case where the range is any unique factorization domain .

Throughout this chapter , S denotes an arithmetical semigroup and D a unique factorization domain.

Proposition 2.1 *If the set P of all primes of S is countably infinite, then S is isomorphic to* $\mathbb{N}$*.*

<u>Proof</u> Since P is countably infinite,we can write $P = \{p_1, p_2, \ldots\}$ where $|p_i| \leq |p_{i+1}|$. Let the primes of $\mathbb{N}$ be listed in any definite order $q_1, q_2, \ldots$. Define $\varphi : S \to \mathbb{N}$ as follows : for $1 \neq s \in S$, $s = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ for some $k \geq 1$, $n_i \geq 0$, let $\varphi(s) = q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k}$ and $\varphi(1) = 1$.

Since the unique factorization holds in S and $\mathbb{N}$, φ is well-defined and one-to-one . Clearly , φ is onto. Let $s,t \in S$. If $s = 1$ or $t = 1$ then $\varphi(st) = \varphi(s)\varphi(t)$. Assume that $s \neq 1$ and $t \neq 1$. Then $s = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ and $t = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$ for some $k,r \geq 1$, n_i , $m_j \geq 0$. Suppose that $k \leq r$. Then $\varphi(st) = \varphi(p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}\, p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}) = \varphi(p_1^{n_1+m_1} p_2^{n_2+m_2} \cdots p_k^{n_k+m_k} p_{k+1}^{m_{k+1}} \cdots p_r^{m_r}) = q_1^{n_1+m_1} q_2^{n_2+m_2} \cdots q_k^{n_k+m_k} q_{k+1}^{m_{k+1}} \cdots q_r^{m_r} = q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k}\, q_1^{m_1} q_2^{m_2} \cdots q_r^{m_r} = \varphi(s)\varphi(t)$. Thus φ is a homomorphism . Hence φ is an isomorphism. #

Corollary 2.2 ([8]) *The set of all pseudo-arithmetic functions from S to $\mathbb{C}$ is a unique factorization domain* .

Proof If the set P of all primes of S is countably infinite , then $S \cong \mathbb{N}$ by proposition 2.1, so ${}_S\Omega_{\mathbb{C}} \cong {}_{\mathbb{N}}\Omega_{\mathbb{C}}$; hence ${}_S\Omega_{\mathbb{C}}$ is a unique factorization domain. On the other hand if the set P is finite,say $P = \{p_1,p_2 , \ldots,p_k\}$, then by chapter I §3 , ${}_S\Omega_{\mathbb{C}}$ is isomophic to a formal power series of the form $\mathbb{C}[[x_1,x_2 ,\ldots,x_k]]$ which is a unique factorization domain , so ${}_S\Omega_{\mathbb{C}}$ is a unique factorization domain. #

Next , we shall prove that ${}_S\Omega_D$ is a unique factorization domain.

Proposition 2.3 *${}_S\Omega_D$ is an integral domain.*

Proof Since D has no zero divisors , $\langle\alpha*\beta\rangle = \langle\alpha\rangle\langle\beta\rangle$ for all $\alpha,\beta \in {}_S\Omega_D$ by Proposition 1.27(ii) . Let $\alpha,\beta \in {}_S\Omega_D$ be such that $\alpha*\beta = 0$. Then $\langle\alpha\rangle\langle\beta\rangle = \langle\alpha*\beta\rangle = 0$. Then $\langle\alpha\rangle = 0$ or $\langle\beta\rangle = 0$, so $\alpha = 0$ or $\beta = 0$. #

The next corollary follows directly from Proposition 2.3 and ${}_S\Omega_D \cong D_\omega$.

Corollary 2.4 *D_ω is an integral domain.*

Recall that a commutative ring R with identity is said to satisfy the ascending chain condition for principal ideals (ACCP) if for every ascending chain $(a_1) \subseteq (a_2) \subseteq \ldots$ of principal ideals of R there is an integer n such that $(a_i) = (a_n)$ for all $i \geq n$.

Theorem 2.5 *Let R be a commutative ring with identity. Assume that R satisfies the ACCP. Then every nonzero non-unit element in R is a product of a finite number of irreducible factors.*

Proof Let a be a nonzero non-unit element in R. If a is irreducible , we are done. Assume that a is reducible. Then there exist nonzero non-unit elements b_1,c_1 in R such that $a = b_1c_1$, so $(a) \subset (b_1)$. If both b_1 and c_1 are irreducible , we are done. If not at least one of them is reducible , say b_1 . By definition , there are nonzero non-unit b_2,c_2 in R such that $b_1= b_2c_2$, so $(b_1) \subset (b_2)$. Continuing this process, we get a strictly ascending chain of principal ideals $(a) \subset (b_1) \subset (b_2) \subset \ldots$. By the ACCP of R , this chain terminates at some (b_n) , and b_n must then be irreducible.

We have proved that for an element a which is neither zero nor a unit in R , either a is irreducible or $a = r_1s_1$ for r_1 an irreducible and s_1 is not a unit. By an argument similar to the one just made , in the latter case we can conclude $(a) \subset (s_1)$. If s_1 is reducible , then $s_1= r_2s_2$ for an irreducible element r_2 with s_2 not a unit. Continuing , we get a strictly ascending chain $(a) \subset (s_1) \subset (s_2) \subset \ldots$. By assumption the chain must terminates at some (s_m) and s_m must be irreducible . Then

$a = r_1 r_2\ldots r_m s_m$, where $r_1 ,r_2, \ldots,r_m,s_m$ are irreducible . #

Lemma 2.6 *D satisfies the ACCP.*

Proof Let $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \ldots$ be an ascending chain of principal ideals of D . Without loss of generality we may assume that $a_1 \neq 0$; then $a_i \neq 0$ for all $i \geq 1$. Then $a_i \mid a_1$ for all $i \geq 2$. Since unique factorization holds in D , the number of factors of a_1 is finite , so the chain terminates. #

Lemma 2.7 *Every nonzero non-unit element of ${}_S\Omega_D$ is a product of a finite number of irreducible factors.*

Proof First we shall show that ${}_S\Omega_D$ satisfies the ACCP. Let $(\alpha_1) \subseteq (\alpha_2) \subseteq (\alpha_3) \subseteq \ldots$ be an ascending chain of principal ideals of ${}_S\Omega_D$. Without loss of generality we may assume that $\alpha_1 \neq 0$; then $\alpha_i \neq 0$ for all $i \geq 1$. Fix $i \geq 1$. Since $(\alpha_i) \subseteq (\alpha_{i+1})$, $\alpha_i = \alpha_{i+1} * \beta_i$ for some nonzero element α_i of ${}_S\Omega_D$.Then $\langle \alpha_i \rangle = \langle \alpha_{i+1} * \beta_i \rangle = \langle \alpha_{i+1} \rangle \langle \beta_i \rangle \geq \langle \alpha_{i+1} \rangle$. Since $\langle \alpha_i \rangle = |a_i|$ for some $a_i \in S$, we have a nonincreasing chain $|a_1| \geq |a_2| \geq |a_3| \geq \ldots$. Since $N_S(x)$ is finite for all $x > 0$, $N_S(|a_1|)$ if finite , so there is $n \in \mathbb{N}_0$ such that for $j \geq n$, $|a_j| = |a_n|$ i.e. $\langle \alpha_j \rangle = \langle \alpha_n \rangle = |a_n|$. Let $j \geq n$. Then $0 \neq \alpha_j(a_n) = (\alpha_{j+1} * \beta_j)(a_n)$ $= \sum_{xy=a_n} \alpha_{j+1}(x)\beta_j(y) = \sum_{\substack{xy=a_n \\ y\neq 1}} \alpha_{j+1}(x)\beta_j(y) + \alpha_{j+1}(a_n)\beta_j(1) = \alpha_{j+1}(a_n)\beta_j(1)$. Thus $(\alpha_j(a_n)) \subseteq (\alpha_{j+1}(a_n))$. Then we have an ascending chain $(\alpha_n(a_n)) \subseteq (\alpha_{n+1}(a_n)) \subseteq \ldots$ of nonzero principal ideals of D . By Lemma 2.6, D satisfies the ACCP , so there is an integer m such that $(\alpha_m(a_n)) = (\alpha_j(a_n))$ for all $j \geq m$. Let $j \geq m$. Then $\alpha_m(a_n) = u_j \cdot \alpha_j(a_n)$ for some unit u_j of D. Since $(\alpha_m) \subseteq (\alpha_j)$, there is a nonzero element γ_j of ${}_S\Omega_D$ such that $\alpha_m = \alpha_j * \gamma_j$. Then $u_j \cdot \alpha_j(a_n) = \alpha_m(a_n) = \sum_{xy=a_n} \alpha_j(x)\gamma_j(y) = \alpha_j(a_n)\gamma_j(1)$. By unique factorization in D, $u_j = \gamma_j(1)$.By Proposition 1.28 , γ_j is a unit of ${}_S\Omega_D$ .Then $(\alpha_j) = (\alpha_m)$. Then ${}_S\Omega_D$ satisfies the ACCP. By Theorem 2.5, every nonzero non-unit element of ${}_S\Omega_D$ is a product of a finite number of irreducible factors. #

Theorem 2.8 *If unique factorization property fails in ${}_S\Omega_D$, there exists an element of the form $\alpha*\beta = \gamma*\delta$, where $\alpha, \beta, \gamma, \delta$ are irreducible elements having the same order and α not associated with either γ or δ.*

Proof Assume that unique factorization property fails in ${}_S\Omega_D$ .Let A be the set of all nonzero non-unit elements of ${}_S\Omega_D$ whose factorization into irreducible elements is unique and let B be the set of all nonzero non-unit elements of ${}_S\Omega_D$ which can be factored into irreducible elements into two essentially different ways. Clearly every irreducible element of ${}_S\Omega_D$ is in A by definition.

We shall prove that if α is an element of B of minimum order $\langle\alpha\rangle$, and $\alpha = \beta_1*\beta_2*\ldots*\beta_n = \gamma_1*\gamma_2*\ldots*\gamma_m$ are two essentially different factorizations of α into irreducible elements , then necessarily $n = m = 2$ and $\beta_1, \beta_2, \gamma_1, \gamma_2$ all have the same order.

Note first that neither n nor m is 1 since an irreducible element is in A. Moreover , no β_i is the associate of any γ_j , for if so , cancellation would produce an element in B of order less that $\langle\alpha\rangle$. Without loss of generality , we may assume that $\langle\beta_1\rangle \leq \langle\beta_2\rangle \leq \ldots \leq \langle\beta_n\rangle$, $\langle\beta_1\rangle \leq \langle\gamma_1\rangle$ and $\langle\gamma_1\rangle \leq \langle\gamma_2\rangle \leq \ldots \leq \langle\gamma_m\rangle$. Then $\langle\beta_1*\gamma_1\rangle = \langle\beta_1\rangle\langle\gamma_1\rangle \leq \langle\gamma_1\rangle\langle\gamma_1\rangle \leq \langle\gamma_1\rangle\langle\gamma_2\rangle \leq \langle\alpha\rangle$. Suppose that $\langle\beta_1*\gamma_1\rangle < \langle\alpha\rangle$. Let $\delta = \alpha - \beta_1*\gamma_1$. If $\alpha = \beta_1*\gamma_1$ then $\beta_2*\ldots*\beta_n = \gamma_1$ and since γ_1 is irreducible , $n = 2$ and $\gamma_1 \sim \beta_2$, a contradiction . Then $\alpha \neq \beta_1*\gamma_1$ i.e. $\delta \neq 0$. Since β_1 is irreducible and $\beta_1 | \delta$, δ is a non-unit .Let $a \in S$ be such that $|a| = \langle\beta_1*\gamma_1\rangle < \langle\alpha\rangle$. Then $(\beta_1*\gamma_1)(a) \neq 0$ and $\alpha(a) = 0$, so $\delta(a) = \alpha(a) - (\beta_1*\gamma_1)(a) = -(\beta_1*\gamma_1)(a) \neq 0$. For all $b \in S$ such that $|b| < |a|$, $(\beta_1*\gamma_1)(b) = 0 = \alpha(b)$, so $\delta(b) = 0$. Thus $\langle\delta\rangle = |a| < \langle\alpha\rangle$, so $\delta \in A$. Since the non associates β_1 ,γ_1 both divide δ, $\lambda_1\beta_1 = \delta = \lambda_2\gamma_1$ for some $\lambda_1, \lambda_2 \in {}_S\Omega_D$. Since $\delta \in A$, $\beta_1 \sim \lambda_2$, so $\delta = \mu\beta_1\gamma_1$

for some unit $\mu \in {}_S\Omega_D$. Then $\beta_1*\gamma_1 \mid \delta$, so $\beta_1*\gamma_1 \mid \alpha$ i.e. $\alpha = \beta_1*\gamma_1*\sigma$ for some $\sigma \in {}_S\Omega_D$. Thus $\beta_2*\ldots*\beta_n = \gamma_1*\sigma$. But $\langle \beta_2*\ldots*\beta_n \rangle < \langle\alpha\rangle$ and $\beta_2*\ldots*\beta_n$ is nonzero non-unit so $\beta_2*\ldots*\beta_n = \gamma_1*\sigma$ is in A and γ_1 is associated with some β_i, a contradiction. Then $\langle\beta_1*\gamma_1\rangle = \langle\alpha\rangle$. Thus $\langle\beta_1*\gamma_1\rangle = \langle\beta_1\rangle\langle\gamma_1\rangle = \langle\gamma_1\rangle\langle\gamma_1\rangle = \langle\gamma_1\rangle\langle\gamma_2\rangle = \langle\alpha\rangle$, so $\langle\beta_1\rangle = \langle\gamma_1\rangle = \langle\gamma_2\rangle$ and m = 2. Thus $\langle\gamma_1\rangle^2 = \langle\alpha\rangle = \langle\beta_1\rangle\langle\beta_2\rangle \cdots \langle\beta_n\rangle \geq \langle\beta_1\rangle^n = \langle\gamma_1\rangle^n$ implies $n \leq 2$. But $n > 1$, so $n = 2$ and $\langle\beta_1\rangle = \langle\beta_2\rangle$. Hence $\beta_1*\beta_2 = \alpha = \gamma_1*\gamma_2$. #

Lemma 2.9 *Let D be a unique factorization domain. Assume that D_j is a unique factorization domain for all $j \geq 1$. Then all irreducible elements of D_ω are finitely irreducible.*

Proof We first show that if f is a nonzero non-unit element in D_ω and $(f)_j$ reducible in D_j for all j ≥ the index L of f, then f is reducible in D_ω.

Let f be a nonzero non-unit element in D_ω with index L and suppose that for every $j \geq L$, $(f)_j = g_j h_j$ where g_j and h_j are true factors of $(f)_j$ in D_j. Now observe that any true factorization $(f)_m = g_m h_m$, $m > L$ induces a true factorization of $(f)_{m-1} = (f_m)_{m-1} = (g_m)_{m-1}(h_m)_{m-1} = g_{m-1}h_{m-1}$ and so down to $(f)_L = g_L h_L$, where the sequence of true factors $(g_L, g_{L+1}, \ldots, g_m)$ is telescopic. From the assumption of f, we have the existence of a sequence $K_0 = (g_{00})$, $K_1 = (g_{10}, g_{11})$, $K_2 = (g_{20}, g_{21}, g_{22})$, … of telescopic chains K_i of true factors g_{ij} (j = 0,1,…,i) of $(f)_{L+j}$. Since unique factorization holds in D_j for all j ≥1, the number of true factors of $(f)_j$ is finite. Then there is a true factor T_0 of $(f)_L$ such that there is an infinite set of the chains K_i having their first entries associate to T_0. Choose one of this set and call it K_0'. Of this infinite set, there is an infinite subset of K_i whose second entries are associate to some one true factor T_1 of $(f)_{L+1}$. Choose one and call it K_1'. Continuing in this way we have a sequence of telescopic chains $K_0' = (g_{00}', \ldots)$, $K_1' = (g_{10}', g_{11}' \ldots)$, … each of which extends at least to the main diagonal, such that the entries on the diagonal and below have the property that, for each j = 0,1,2,…, $g_{ij}' \sim T_j$ for all i ≥ j.

Now we construct a telescopic infinite chain K^* working only with the main diagonal and the diagonal next below it, as follows:

Define $G^{(L)} = g_{00}'$. Since $g_{10}' \sim T_0 \sim g_{00}'$ in D_L, there is a unit u_L of D_L such that $G^{(L)} = g_{10}'\, u_L = (g_{11}' u_L)_L$. Define $G^{(L+1)} = g_{11}' u_L$ in D_{L+1}. Then $G^{(L)} = (G^{(L+1)})_L$,

$G^{(L+1)}$ is a true factor of $(f)_{L+1}$, and $G^{(L+1)} \sim T_1$ in D_{L+1} . Since $g'_{21} \sim T_1 \sim G^{(L+1)}$ in D_{L+1} , there is a unit u_{L+1} in D_{L+1} such that $G^{(L+1)} = g'_{21}\ u_{L+1} = (g'_{22}\ u_{L+1})_{L+1}$. Define $G^{(L+2)} = g'_{22}\ u_{L+1}$ in D_{L+2}. Then $G^{(L+1)} = (G^{(L+2)})_{L+1}$, $G^{(L+2)}$ is a true factor of $(f)_{L+2}$ and $G^{(L+2)} \sim T_2$ in D_{L+2} . Continuing in this way , we have an infinite telescopic chain of true factors $K^* = (G^{(L)}, G^{(L+1)}, G^{(L+2)}, \ldots)$.Then for all $j \geq 0$, $(f)_{L+j} = G^{(L+j)}H^{(L+j)}$, where $H^{(L+j)}$ is a nonzero non-unit element in D_{L+j} . Thus for all $j \geq 0$, $G^{(L+j)}H^{(L+j)} = (f)_{L+j} = ((f)_{L+j+1})_{L+j} = (G^{(L+j+1)})_{L+j}(H^{(L+j+1)})_{L+j} = G^{(L+j)}(H^{(L+j+1)})_{L+j}$, so $H^{(L+j)} = (H^{(L+j+1)})_{L+j}$ by unique factorization of D_{L+j} and $G^{(L+j)} \neq 0$. Then the sequence $(H^{(L)}, H^{(L+1)}, H^{(L+2)}, \ldots)$ is also a telescopic chain . By Lemma 1.17 , the chains $(G^{(L)}, G^{(L+1)}, G^{(L+2)}, \ldots)$ and $(H^{(L)}, H^{(L+1)}, H^{(L+2)}, \ldots)$ have limits in D_ω , say G and H , respectively. Then $(G)_j = G^{(j)}$ or $(G^{(L)})_j$, and $(H)_j = H^{(j)}$ or $(H^{(L)})_j$, according as $j \geq L$ or $j < L$ for $j \geq 0$. Then for $j \geq L$, $(f)_j = G^{(j)}H^{(j)} = (G)_j(H)_j = (GH)_j$, and for $j < L$, $(f)_j = ((f)_L)_j = (G^{(L)}H^{(L)})_j = (G^{(L)})_j(H^{(L)})_j = (G)_j(H)_j = (GH)_j$. Hence for every $j \geq 0$, $(f)_j = (GH)_j$. It follows that $f = \lim_{j\to\infty} (f)_j = \lim_{j\to\infty} (GH)_j = GH$ for the weight topology. Clearly, G and H are non-units of D_ω , so f is reducible in D_ω . Therefore if f is irreducible in D_ω then there is a least integer $P \geq L$ such that $(f)_P$ is irreducible in D_P and for all $j \geq P$, $(f)_j$ is irreducible in D_j by Lemma 1.22 , so f is finitely irreducible. #

Theorem 2.10 *Let D be a unique factorization domain . Assume that D_j is a unique factorization domain for all $j \geq 1$. Then ${}_S\Omega_D$ is a unique factorization domain.*

Proof Suppose that unique factorization into irreducible elements fails in ${}_S\Omega_D$ which is isomorphic to D_ω . By Theorem 2.8, we must have an element in D_ω of the form $fg = pq$ where f, g, p, q are irreducible in D_ω and f is not associated with p or q . By Lemma 2.9 , f, g, p, q are finitely irreducible , so there exists an integer $k \geq 0$ such that $(f)_j(g)_j = (fg)_j = (pq)_j = (p)_j(q)_j$, where $(f)_j$,$(g)_j$,$(p)_j$, and $(q)_j$ are irreducible in D_j for all $j \geq k$. Since unique factorization holds in D_j for all $j \geq k$, $(f)_j$ must be associated with either $(p)_j$ or $(q)_j$ in D_j . Then there is an infinite increasing subsequence $\mathcal{M}$ of integers $m \geq k$ such that either $(f)_m \sim (p)_m$ or $(f)_m \sim (q)_m$ in D_m for all $m \in \mathcal{M}$. Without loss of generality , we may assume the first case. Then for each $m \in \mathcal{M}$, $(f)_m = u_m(p)_m$, where u_m is a unit of D_m . If $m < n$ are any two integers in $\mathcal{M}$, then $u_m(p)_m = (f)_m = ((f)_n)_m = (u_n)_m((p)_n)_m = (u_n)_m(p)_m$, so $u_m = (u_n)_m$ by unique factorization of D_m. Thus the sequence $(u_m)_{m\in\mathcal{M}}$, is telescopic and so has a limit in D_ω, say u. Clearly , u is a unit of D_ω. Then $f = \lim_{m\to\infty}(f)_m = \lim_{m\to\infty}(up)_m = \lim_{m\to\infty}(u)_m \lim_{m\to\infty}(p)_m = up$. Therefore $f \sim p$, a contradiction . Hence unique factorization holds in D_ω, so does ${}_S\Omega_D$. #

The next corollary follows from Theorem 2.10 and $({}_S\Omega_D)_k \cong D[[x_1, \ldots ,x_k]] = D_k$.

Corollary 2.11 *Let D be a unique factorization domain such that the subrings* $({}_S\Omega_D)_k$ *of* ${}_S\Omega_D$ *are unique factorization domains for all* $k \geq 1$. *Then* ${}_S\Omega_D$ *is a unique factorization domain .*

CHAPTER III
CASE OF ZERO DIVISORS

Chin Pi-Lu[10] has proved that under appropriate conditions the ring of all arithmetic functions over a unique factorization domain is a unique factorization domain. In this chapter , we consider the case where the unique factorization domain is replaced by a unique factorization ring $\mathcal{R}$ with zero divisors . We shall prove that under similar conditions the ring of all arithmetic function over $\mathcal{R}$ is a unique factorization ring with zero divisors.

Proposition 3.1 *${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ has zero divisors .*

Proof Let x be a zero divisor of $\mathcal{R}$ .Then there exists a $y \in \mathcal{R}-\{0\}$ such that $xy = 0$. Define two functions $\alpha,\beta \in {}_{\mathbb{N}}\Omega_{\mathcal{R}}$ by $\alpha(1) = x$, $\beta(1) = y$ and $\alpha(a) = 0 = \beta(a)$ for all $a \neq 1$. Then $(\alpha*\beta)(1) = \alpha(1)\beta(1) = xy = 0$ and for all $a \neq 1$, $(\alpha*\beta)(a) = \sum_{bc=a}\alpha(b)\beta(c) = 0$. Thus α and β are nonzero elements in ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ and $\alpha*\beta = 0$. Hence ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ has zero divisors. #

Proposition 3.2 *The ring $\mathcal{R}_\omega$ of formal power series over $\mathcal{R}$ has zero divisors.*

Proof Since $\mathcal{R}$ is a subring of $\mathcal{R}_\omega$, zero divisors in $\mathcal{R}$ are zero divisors in $\mathcal{R}_\omega$ too. #

Lemma 3.3 *Let A and B be any commutative rings with identity . Assume that A is isomorphic to B. If A satisfies the ACCP then so does B.*

Proof Let φ be an isomorphism between A and B. Let $(\beta_1) \subseteq (\beta_2) \subseteq (\beta_3) \subseteq \ldots$ be an ascending chain of principal ideals in B. Since $\beta_{i+1} \mid \beta_i$ in B , there is $\delta_i \in B$ such that $\beta_i = \beta_{i+1}\delta_i$. Since φ is onto , $\beta_i = \varphi(\alpha_i)$ and $\delta_i = \varphi(\gamma_i)$ for some α_i ,$\gamma_i \in A$. Then $\varphi(\alpha_i) = \beta_i = \beta_{i+1}\delta_i = \varphi(\alpha_{i+1})\varphi(\gamma_i) = \varphi(\alpha_{i+1}\gamma_i)$. By injectivity of φ , $\alpha_i = \alpha_{i+1}\gamma_i$. Then for all $i \geq 1$, $(\alpha_i) \subseteq (\alpha_{i+1})$. By the ACCP in A , there is an $r \geq 1$ such that for all $i \geq 0$, $(\alpha_r) = (\alpha_{r+i})$. Then for all $i \geq 0$, $\alpha_{r+i} = \alpha_r \mu_i$ for some unit μ_i in A . Thus for i ≥ 0 , $\beta_{r+i} = \varphi(\alpha_{r+i}) = \varphi(\alpha_r \mu_i) = \varphi(\alpha_r)\ \varphi(\mu_i) = \beta_r \varphi(\mu_i)$; obviously , $\varphi(\mu_i)$ is a unit in B. Then $(\beta_r) = (\beta_{r+i})$ for all i ≥ 0 . Hence B satisfies the ACCP. #

Lemma 3.4 *$\mathcal{R}$ satisfies the ACCP.*

Proof Let $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \ldots$ be an ascending chain of principal ideals of $\mathcal{R}$. Without loss of generality we may assume that $a_1 \neq 0$; then $a_i \neq 0$ for all i ≥ 1 . Then $a_i \mid a_1$ for all i ≥ 2 . Since unique factorization holds in $\mathcal{R}$, the number of factors of a_1 is finite , so the chain is finite. #

Proposition 3.5 *Every nonzero non-unit element in $\mathcal{R}_\omega$ is a product of a finite number of irreducible factors.*

Proof First we shall show that ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ satisfies the ACCP. Let $(\alpha_1) \subseteq (\alpha_2) \subseteq (\alpha_3) \subseteq \ldots$ be an ascending chain of principal ideals of $\mathcal{R}_\omega$ .Without loss of generality we may assume that $\alpha_1 \neq 0$; then $\alpha_i \neq 0$ for all i ≥ 1. Fix i ≥ 1. Since $(\alpha_i) \subseteq (\alpha_{i+1})$, $\alpha_i = \alpha_{i+1} * \beta_i$ for some nonzero element β_i of ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ .Then $\langle\alpha_i\rangle = \langle\alpha_{i+1} * \beta_i\rangle \geq \langle\alpha_{i+1}\rangle\ \langle\beta_i\rangle \geq \langle\alpha_{i+1}\rangle$. Thus we have a decending chain of positive integers $\langle\alpha_1\rangle \geq \langle\alpha_2\rangle \geq \ldots$, so there are n,k $\in \mathbb{N}$ such that for $j \geq n$, $\langle\alpha_j\rangle = \langle\alpha_n\rangle = k$. Let $j \geq n$. Then $0 \neq \alpha_j(k) = (\alpha_{j+1} * \beta_j)(k) = \sum\limits_{xy=k} \alpha_{j+1}(x)\beta_j(y) = \sum\limits_{\substack{xy=k \\ y\neq 1}} \alpha_{j+1}(x)\beta_j(y) + \alpha_{j+1}(k)\beta_j(1) = \alpha_{j+1}(k)\beta_j(1)$.Thus $(\alpha_j(k)) \subseteq (\alpha_{j+1}(k))$.

Then we have the ascending chain $(\alpha_n(k)) \subseteq (\alpha_{n+1}(k)) \subseteq \ldots$ of nonzero principal ideals of $\mathcal{R}$ .By Lemma 3.4 , $\mathcal{R}$ satisfies the ACCP , so there is an integer m such that $(\alpha_m(k)) = (\alpha_j(k))$ for all $j \geq m$. Let $j \geq m$. Then $\alpha_m(k) = u_j \cdot \alpha_j(k)$ for some unit u_j of $\mathcal{R}$. Since $(\alpha_m) \subseteq (\alpha_j)$, there is a nonzero element γ_j of ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ such that $\alpha_m = \alpha_j * \gamma_j$. Then $u_j \cdot \alpha_j(k) = \alpha_m(k) = \sum\limits_{xy=k} \alpha_j(x)\gamma_j(y) = \alpha_j(k)\ \gamma_j(1)$. Since $\mathcal{R}$ satisfies a weak cancellation law, $u_j \sim \gamma_j(1)$,so γ_j is a unit of ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ by Proposition 1.28.Thus $(\alpha_j) = (\alpha_m)$. Then ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ satisfies the ACCP, so $\mathcal{R}_\omega$ satisfies the ACCP by Lemma 3.3 . By Theorem 2.5,every nonzero non-unit element of $\mathcal{R}_\omega$ is a product of a finite number of irreducible factors. #

Lemma 3.6 *Let $\mathcal{R}$ be a UFRZ such that $\mathcal{R}_j$ is a UFRZ for all $j \geq 1$ and $\mathcal{R}_\omega$ is compact. Then all irreducible elements of $\mathcal{R}_\omega$ are finitely irreducible .*

Proof We claim that if f is a nonzero non-unit element in $\mathcal{R}_\omega$ and $(f)_j$ reducible in $\mathcal{R}_j$ for all j ≥ the index L of f, then f is reducible in $\mathcal{R}_\omega$.

Let f be a nonzero non-unit element in $\mathcal{R}_\omega$ with index L and suppose that for every $j \geq L$,$(f)_j$ is reducible in $\mathcal{R}_j$. Then we can construct an infinite telescopic chain $(G^{(L)}, G^{(L+1)}, G^{(L+2)}, \ldots)$, where $G^{(L+j)}$ is a true factor of $(f)_{L+j}$ in $\mathcal{R}_{L+j}$, by the same method as in Lemma 2.9 . This chain has a limit in $\mathcal{R}_\omega$, say G . Then for all $j \geq 0$, $(f)_{L+j} = G^{(L+j)}H^{(L+j)}$, where $H^{(L+j)}$ is a nonzero non-unit element in $\mathcal{R}_{L+j}$, so $G^{(L+j)}H^{(L+j)} = (f)_{L+j} = ((f)_{L+j+1})_{L+j} = (G^{(L+j+1)})_{L+j}(H^{(L+j+1)})_{L+j} = G^{(L+j)}(H^{(L+j+1)})_{L+j}$. Then $H^{(L+j)} \sim (H^{(L+j+1)})_{L+j}$ by the weak cancellation law of $\mathcal{R}_{L+j}$. Thus the sequence $(H^{(L)}, H^{(L+1)}, H^{(L+2)}, \ldots)$ is a pseudo-telescopic chain . By Lemma 1.20 , there is a subchain $(H^{(L+j_k)})$ of $(H^{(L+j)})$ which converges to a $H \in \mathcal{R}_\omega$. Then $f = \lim_{k\to\infty} (f)_{j_k} = \lim_{k\to\infty}(G^{(L+j_k)}H^{(L+j_k)}) = \lim_{k\to\infty} G^{(L+j_k)} \lim_{k\to\infty} H^{(L+j_k)} = GH$. Clearly, G and H are non-units of $\mathcal{R}_\omega$, so f is reducible in $\mathcal{R}_\omega$. Thus we have the claim . Therefore if f is irreducible in $\mathcal{R}_\omega$ then there is a least integer $P \geq L$ such that $(f)_P$ is irreducible in $\mathcal{R}_P$ and for all $j \geq P$, $(f)_j$ is irreducible in $\mathcal{R}_j$ by Lemma 1.22 , so f is finitely irreducible. #

Lemma 3.7 *Let $\mathcal{R}$ be a UFRZ such that $\mathcal{R}_j$ is a UFRZ for all $j \geq 1$, f ,g any elements of $\mathcal{R}_\omega$ and $D^{(j)}$ a greatest common divisor of $(f)_j$ and $(g)_j$ in $\mathcal{R}_j$. Then $(D^{(j+1)})_j \sim D^{(j)}$ for all $j \geq$ a certain non-negative integer, $J(f,g)$.*

Proof If f or g is zero then the assertion is trivial. Assume that f and g are nonzero. Let n be the least positive integer such that $(f)_n \neq 0$, $(g)_n \neq 0$ and i any integer ≥ n. Since $\mathcal{R}_i$ is a UFRZ , we can represent $D^{(i)}$ as a finite product of irreducible elements of $\mathcal{R}_i$; denote by $\lambda(D^{(i)})$ the number of all irreducible factors (not necessarily distinct) of $D^{(i)}$. Since $(D^{(i+1)})_i$ is a factor of $D^{(i)}$, $\lambda(D^{(i)}) \geq \lambda(D^{(i+1)})$. Note that the projection of each irreducible factor of $D^{(i+1)}$ on $\mathcal{R}_i$ may not be irreducible in $\mathcal{R}_i$. Thus we have the following decending chain of non-negative integers :

$\lambda(D^{(n)}) \geq \lambda(D^{(n+1)}) \geq \ldots$. Then there exists integers j and k such that $k = \lambda(D^{(n+j+m)})$ for all $m \geq 0$. This means that for every $m \geq n+j$, the projection of each irreducible factor of $D^{(m+1)}$ on $\mathcal{R}_m$ is also irreducible and moreover $(D^{(m+1)})_m \sim D^{(m)}$. We donote n+j by J(f,g) . #

Theorem 3.8 *Let $\mathcal{R}$ be a UFRZ such that $\mathcal{R}_j$ is a UFRZ for all $j \geq 1$ and $\mathcal{R}_\omega$ is compact . Then $\mathcal{R}_\omega$ is a UFRZ .*

Proof Since $\mathcal{R}$ is a UFRZ , $\mathcal{R}$ satisfies the ACCP by Lemma 3.4. By Theorem 2.5 , every nonzero non-unit element of $\mathcal{R}_\omega$ is a finite product of irreducible factors. We shall apply Proposition 1.35 to show that $\mathcal{R}_\omega$ is a UFRZ. First we shall prove that any two elements f and g of $\mathcal{R}_\omega$, not both zero , have a greatest common divisor. Since the assertion is trivial for the case where either $f = 0$ or $g = 0$, we assume that f and g are nonzero. Let n be the least positive integer such that $(f)_n \neq 0$ and $(g)_n \neq 0$. Let $D^{(i)}$ be a greatest common divisor of $(f)_i$ and $(g)_i$ for all $i \geq n$. We construct an infinite telescopic chain $(E^{(J)}, E^{(J+1)}, \ldots)$ with the initial term in $\mathcal{R}_J$, where $J = J(f,g)$ is as in Lemma 3.7, as follows. Put $E^{(J)} = D^{(J)}$. By Lemma 3.7 , $D^{(J)} \sim (D^{(J+1)})_J$ in $\mathcal{R}_J$, so there exists a unit $u^{(J)}$ in $\mathcal{R}_J$ such that $D^{(J)} = u^{(J)}(D^{(J+1)})_J = (u^{(J)}D^{(J+1)})_J$, we take $E^{(J+1)} = u^{(J)}D^{(J+1)}$. Then $(E^{(J+1)})_J = E^{(J)}$ and $E^{(J+1)}$ is a greatest common divisor of $(f)_{J+1}$ and $(g)_{J+1}$ in $\mathcal{R}_{J+1}$. By Lemma 3.7, $E^{(J+1)} \sim (D^{(J+2)})_{J+1}$ in $\mathcal{R}_{J+1}$, so there exists a unit $u^{(J+1)}$ in $\mathcal{R}_{J+1}$ such that $E^{(J+1)} = u^{(J+1)}(D^{(J+2)})_{J+1} = (u^{(J+1)}D^{(J+2)})_{J+1}$. We take $E^{(J+2)} = u^{(J+1)}D^{(J+2)}$. Then $(E^{(J+2)})_{J+1} = E^{(J+1)}$ and $E^{(J+2)}$ is a greatest common divisor of $(f)_{J+2}$ and $(g)_{J+2}$ in $\mathcal{R}_{J+2}$. Continuing in this way , we have a telescopic chain $(E^{(J)}, E^{(J+1)}, E^{(J+2)}, \ldots)$. This chain has a limit E in $\mathcal{R}_\omega$. Let $F^{(j)}$ and $G^{(j)}$ be two elements in $\mathcal{R}_j$ such that $(f)_j = E^{(j)}F^{(j)}$ and $(g)_j = E^{(j)}G^{(j)}$ for all $j \geq J$; then $E^{(j)}F^{(j)} = (f)_j = ((f)_{j+1})_j = (E^{(j+1)})_j(F^{(j+1)})_j = E^{(j)}(F^{(j+1)})_j$, so $(F^{(j+1)})_j \sim F^{(j)}$ by the weak cancellation law in $\mathcal{R}_j$ .Similarly , we have $(G^{(j+1)})_j \sim G^{(j)}$ for all $j \geq J$. Then the sequences $(F^{(J)}, F^{(J+1)}, F^{(J+2)}, \ldots)$ and $(G^{(J)}, G^{(J+1)}, G^{(J+2)}, \ldots)$ are pseudo-telescopic chains . Since $\mathcal{R}_\omega$ is compact , there are convergent subchains $(F^{(J+j_k)})$ and $(G^{(J+i_k)})$ of $(F^{(J+j)})$ and $(G^{(J+i)})$, respectively. These two subchains have limits in $\mathcal{R}_\omega$, say F and G , respectively. Then $f = \lim_{k\to\infty}(f)_{j_k} = \lim_{k\to\infty}(E^{(J+j_k)}F^{(J+j_k)}) = \lim_{k\to\infty}E^{(J+j_k)}\lim_{k\to\infty}F^{(J+j_k)} = EF$ and $g = \lim_{k\to\infty}(g)_{i_k} = \lim_{k\to\infty}(E^{(J+i_k)}G^{(J+i_k)}) = \lim_{k\to\infty}E^{(J+i_k)}\lim_{k\to\infty}G^{(J+i_k)} = EG$. Thus E is a common divisor of f and g . To show that E is a greatest common divisor of f and g in $\mathcal{R}_\omega$, we let E^* be any common divisor of f and g in $\mathcal{R}_\omega$. Let $j \geq J$. Then $(E^*)_j$ is a common divisor of $(f)_j$ and $(g)_j$ in $\mathcal{R}_j$. Since $(E)_j$ is a greatest common divisor $(f)_j$ and $(g)_j$ in $\mathcal{R}_j$, $(E)_j = h^{(j)}(E^*)_j$, where $h^{(j)} \in \mathcal{R}_j$. Then $h^{(j)}(E^*)_j = (E)_j = ((E)_{j+1})_j = (h^{(j+1)}(E^*)_{j+1})_j = (h^{(j+1)})_j(E^*)_j$, so $h^{(j)} \sim (h^{(j+1)})_j$ by the weak cancellation law in $\mathcal{R}_j$. Thus we get a pseudo-telescopic chain $(h^{(J)}, h^{(J+1)}, \dots)$, so by compactness this chain has a subchain $(h^{(J+j_k)})$ which converges to a limit h in $\mathcal{R}_\omega$ by Lemma1.20 .Then $E = \lim_{k\to\infty}(E)_{j_k} = \lim_{k\to\infty}(h^{(J+j_k)}(E^*)_{J+j_k}) = \lim_{k\to\infty} h^{(J+j_k)}\lim_{k\to\infty}(E^*)_{J+j_k} = hE^*$. Hence E is a greatest common divisor of f and g in $\mathcal{R}_\omega$. Lastly we show that $\mathcal{R}_\omega$ satisfies a weak cancellation law. Let f , g and $h \in \mathcal{R}_\omega$ be such that $fg = fh \neq 0$.Then there are integers n and k for which $(fg)_j = (fh)_j \neq 0$ for all $j \geq n$ and $(f)_i \neq 0$ for all $i \geq k$. Choose $m = \max\{n,k\}$. Let $j \geq m$. Then we get $(f)_j(g)_j = (fg)_j = (fh)_j = (f)_j(h)_j \neq 0$. By the weak cancellation law in $\mathcal{R}_j$, $(g)_j \sim (h)_j$, so there is a unit $u^{(j)}$ in $\mathcal{R}_j$ such that $(g)_j = u^{(j)}(h)_j$. Thus we get a sequence $(u^{(m)}, u^{(m+1)}, \dots)$ of units in $\mathcal{R}_\omega$. The definition of compactness of $\mathcal{R}_\omega$ implies that this sequence have a subsequence $(u^{(m+j_k)})$, which converges to a unit u in $\mathcal{R}_\omega$. Then $g = \lim_{k\to\infty}(g)_{j_k} = \lim_{k\to\infty}(u^{(m+j_k)}(h)_{m+j_k}) = \lim_{k\to\infty} u^{(m+j_k)}\lim_{k\to\infty}(h)_{m+j_k} = uh$. Therefore $\mathcal{R}_\omega$ satisfies a weak cancellation law . By Proposition 1.35 , $\mathcal{R}_\omega$ is a UFRZ . #

The following Corollaries are consequences of Theorem 3.8 .

Corollary 3.9 *If $\mathcal{R}$ is a UFRZ such that $\mathcal{R}_j$ is a UFRZ for each positive integer j and $\mathcal{R}_\omega$ is compact , then ${}_{N}\Omega_{\mathcal{R}}$ is a UFRZ .*

Corollary 3.10 *If $\mathcal{R}$ is a UFRZ such that the subring $({}_{\mathbb{N}}\Omega_{\mathcal{R}})_k$ of ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ is a UFRZ for each positive integer k, then ${}_{\mathbb{N}}\Omega_{\mathcal{R}}$ is a UFRZ.*

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