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FUNCTIONAL EQUATION ANALOGOUS TO THE 3-DIMENSIONAL LAPLACE EQUATION

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$$\frac{f(x+h_1,y,z)-2f(x,y,z)+f(x-h_1,y,z)}{h_1^2} + \frac{f(x,y+h_2,z)-2f(x,y,z)+f(x,y-h_2,z)}{h_2^2} + \frac{f(x,y,z+h_3)-2f(x,y,z)+f(x,y,z-h_3)}{h_3^2} = 0$$

ซึ่งเทียบคล้ายสมการลาปลาซ 3 มิติ

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f(x,y,z)=0$$

ภาควิชา.........คณิตศาสตร์.......... ลายมือชื่อนิสิต..
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In this thesis, we determine the general solution of a functional equation

$$\frac{f(x+h_1,y,z)-2f(x,y,z)+f(x-h_1,y,z)}{h_1^2} + \frac{f(x,y+h_2,z)-2f(x,y,z)+f(x,y-h_2,z)}{h_2^2} + \frac{f(x,y,z+h_3)-2f(x,y,z)+f(x,y,z-h_3)}{h_3^2} = 0$$

analogous to the 3-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f(x,y,z)=0.$$

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CONTENTS

CHAPTER I

INTRODUCTION

In 1969, J.A. Baker [2] studied the functional equation

$$f(x+t,y)+f(x-t,y)=f(x,y+t)f(x,y-t) \tag{1.1}$$

analogous to the well-known wave equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)f(x,y)=0$$

and found that all continuous solutions $f:\mathbb{R}\longrightarrow\mathbb{R}$ are of the form

$$f(x,y)=\alpha(x+y)+\beta(x-y), \tag{1.2}$$

where α and β are arbitrary continuous functions.

Next, in 1970, H. Haruki [6] also solved (1.1) by a different method and obtained results similar to those of J.A. Baker.

In the same year, M. Kucharzewski [8] studied (1.1) which satisfies the relation

$$f(x,y)-f(-x,y)-f(x,-y)+f(-x,-y)=\alpha_1(x+y)+\beta_1(x-y),$$

where α_1 and β_1 are arbitrary functions, and get the solutions of the form (1.2), where α and β are arbitrary functions.

In 1972, D.P. Flemming [4] solved (1.1) using the transformation of coordinates (x,y) into (ζ,η) according to the relations

$$\begin{cases}\zeta=\frac{k+1}{2}x+\frac{k-1}{2}y\\ \eta=\frac{k-1}{2}x+\frac{k+1}{2}y\end{cases}$$

and obtained the solutions of the form (1.2) when there are no assumptions regarding the continuity of α,β and f.

In the same year, M. A. McKiernan [9] gave the solutions of (1.1) in the following more general context which take the form of

$$f(x,y) = \alpha(x+y) + \beta(x-y) + A(x,y),$$

where α, β are arbitrary functions and A is an arbitrary skew-symmetric bi-additive function,

$$A(x,y) = -A(y,x)$$

and

$$A(x,y+z) = A(x,y) + A(x,z).$$

Next, in 1973, D. Girod [5] studied more general version of (1.1) and obtained a result similar to that of M. A. McKiernan.

In 1988, H. Haruki [7] gave the general solution of the functional equation

$$\frac{f(x+t,y) - 2f(x,y) + f(x-t,y)}{t^2} = \frac{f(x,y+s) - 2f(x,y) + f(x,y-s)}{s^2}$$

analogous to the wave equation with the aid of generalized polynomials when no regularity assumptions are imposed on f and obtained the general solution of the form

$$\begin{aligned} f(x,y) = a_0 + \alpha(x) + \delta(y) + \beta(x,y) + a_1(x^2+y^2) + a_2(x^3+3xy^2) \\ + a_3(y^3+3x^2y) + a_4(x^3y+xy^3). \end{aligned}$$

In 1968, J. Aczél, H. Haruki, M.A. McKiernan and G. N. Sakovič [1] gave the general solution of the so-called square functional equation

$$f(x+t,y+t) + f(x+t,y-t) + f(x-t,y+t) + f(x-t,y-t) = 4f(x,y)$$

which is equivalent to the functional equation

$$f(x+t,y) + f(x-t,y) + f(x,y+t) + f(x,y-t) = 4f(x,y), \qquad (1.3)$$

which in turn is analogous to the 2-dimensional Laplace's equation

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f(x, y) = 0.$$

The general solution of the functional equation (1.3) is in the form of generalized polynomials

$$\begin{aligned} f(x, y) = A^0 + A^1(x) + B^1(y) + A^{1,1}(x, y) + \big(A^2(x) - A^2(y)\big) + \big(A^3(x) - 3A_3(x, y, y)\big) \\ + \big(B^3(y) - 3B_3(x, x, y)\big) + \big(A^{1,3}(x; y) - A_{1,3}(x; x, x, y)\big), \end{aligned}$$

where each term represents a generalized polynomial of x and y.

In this thesis, we gave the general solution of a functional equation

$$\begin{aligned} &\frac{f(x + h_1, y, z) - 2f(x, y, z) + f(x - h_1, y, z)}{h_1^2} \\ &+ \frac{f(x, y + h_2, z) - 2f(x, y, z) + f(x, y - h_2, z)}{h_2^2} \\ &+ \frac{f(x, y, z + h_3) - 2f(x, y, z) + f(x, y, z - h_3)}{h_3^2} = 0 \end{aligned}$$

analogous to the 3-dimensional Laplace's equation

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})f(x, y) = 0.$$

In Chapter II, we will provide background knowledges used in this thesis. In Chapter III, we will study the Haruki's Lemma and some extensions. In Chapter IV, we will solve the functional equation analogous to the 2-dimensional Poisson equation. Finally, in Chapter V, we give the general solution of the functional equation analogous to the 3-dimensional Laplace equation.

CHAPTER II
PRELIMINARIES

Let X, Y be linear spaces over a field rational number, let n be any positive integer, and let $f : X^n \to Y$ be an arbitrary function. For any $x_1, x_2, \dots, x_n, t \in X$, we define the central difference operator $\Delta_{i,t}$ with respect to the i^{th} argument for any $i = 1, 2, 3, \dots, n$ by

$$\Delta_{i,t} f(x_1, \dots, x_i, \dots, x_n) := f(x_1, \dots, x_i + \frac{t}{2}, \dots, x_n) - f(x_1, \dots, x_i - \frac{t}{2}, \dots, x_n).$$

The iterates $\Delta^n_{i,t}$ of $\Delta_{i,t}$ for any positive integer n, are defined by the natural recurrence

$$\Delta^1_{i,t} f = \Delta_{i,t} f$$

and

$$\Delta^{n+1}_{i,t} f = \Delta_{i,t}(\Delta^n_{i,t} f).$$

So, the 3-dimensional Laplace functional equation

$$\begin{aligned} &\frac{f(x + h_1, y, z) - 2f(x, y, z) + f(x - h_1, y, z)}{h_1^2} \\ &+ \frac{f(x, y + h_2, z) - 2f(x, y, z) + f(x, y - h_2, z)}{h_2^2} \\ &+ \frac{f(x, y, z + h_3) - 2f(x, y, z) + f(x, y, z - h_3)}{h_3^2} = 0 \end{aligned}$$

can be written in the form

$$\Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2} + \frac{1}{h_3^2}\Delta^2_{3,h_3}\Big) f(x, y, z) = 0.$$

Given a positive integer n, a function $f : X^n \to Y$ which satisfies the equation

$$f(x_1, \dots, x_i + x'_i, \dots, x_n) = f(x_1, \dots, x_n) + f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n), \quad (2.1)$$

where $x_i, x_i' \in X$ for all $i = 1, 2, 3, \ldots, n$, will be called an *n-additive function*. A function $f : X^n \to Y$ which satisfies the equation

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n), \qquad (2.2)$$

for all $x_i, x_j \in X$ and $i, j = 1, 2, 3, \ldots, n$, will be called a *symmetric function*.

Next, we will give an example of an n-additive symmetric function.

Example 2.1. *Given a positive integer n and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function defined by*

$$f(x, y, z) = cx_1 \cdots x_n$$

for all $x_1, \ldots, x_n \in \mathbb{R}$ and c is a constant. Prove that f is additive and symmetric.

Proof. For all $y \in \mathbb{R}$ and $x_i \in \mathbb{R}, i = 1, 2, 3, \ldots, n$,

$$\begin{aligned} f(x_1, \ldots, x_{i-1}, x_i + y, x_{i+1}, \ldots, x_n) &= cx_1 \cdots x_{i-1}(x_i + y)x_{i+1} \cdots x_n \\ &= (cx_1 \cdots x_n) + (cx_1 \cdots x_{i-1} y x_{i+1} \cdots x_n) \\ &= f(x_1, \ldots, x_n) + f(x_1, \ldots, y, x_{i+1}, \ldots, x_n). \end{aligned}$$

So, f is an n-additive function. For all $i, j = 1, 2, 3, \ldots, n$, we have

$$\begin{aligned} f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) &= cx_1 \cdots x_i \cdots x_j \cdots x_n \\ &= cx_1 \cdots x_j \cdots x_i \cdots x_n \\ &= f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n). \end{aligned}$$

Thus, f is a symmetric function. □

Theorem 2.2. *For a positive integer n, let $f : X^n \to Y$ be an n-additive function. For each $x_i \in X$, $i = 1, 2, 3, \ldots, n$, the following statements hold;*

(i) $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$,

(ii) $f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) = -f(x_1, \ldots, x_n)$,

(iii) $f(x_1, \ldots, x_{i-1}, mx_i, x_{i+1}, \ldots, x_n) = mf(x_1, \ldots, x_n)$ *for every integer m,*

(iv) $f(x_1, \ldots, x_{i-1}, rx_i, x_{i+1}, \ldots, x_n) = rf(x_1, \ldots, x_n)$ *for every rational number r.*

Proof. (i) It is obvious to see that

$$\begin{aligned} f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) &= f(x_1, \ldots, x_{i-1}, 0+0, x_{i+1}, \ldots, x_n) \\ &= f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \\ &\quad + f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n). \end{aligned}$$

So, we have

$$f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0.$$

(ii) By referring to (i), we obtain

$$\begin{aligned} f(x_1, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) &= f(x_1, \ldots, x_i - x_i, \ldots, x_n) \\ &= f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \\ &= 0. \end{aligned}$$

Thus, we get

$$f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) = -f(x_1, \ldots, x_n).$$

(iii) Let m be any integer.
Case 1; $m = 0$. Using (i), we get

$$f(x_1, \ldots, 0 \cdot x_i, \ldots, x_n) = 0 = 0 \cdot f(x_1, \ldots, x_n)$$

Case 2; $m > 0$. We obtain

$$\begin{aligned} f(x_1, \ldots, x_{i-1}, mx_i, x_{i+1}, \ldots, x_n) &= f(x_1, \ldots, x_{i-1}, \underbrace{x_i + \cdots + x_i}_{m}, x_{i+1}, \ldots, x_n) \\ &= \underbrace{f(x_1, \ldots, x_n) + \cdots + f(x_1, \ldots, x_n)}_{m} \\ &= mf(x_1, \ldots, x_n). \end{aligned}$$

Case 3; $m < 0$. That is $-m > 0$ and we get

$$f(x_1, \ldots, x_{i-1}, mx_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, -m(-x_i), x_{i+1}, \ldots, x_k). \quad (2.3)$$

By referring to Case 2, we have

$$f(x_1,\ldots,x_{i-1},-m(-x_i),x_{i+1},\ldots,x_k) = -mf(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n). \tag{2.4}$$

By using (ii), we have

$$-mf(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n) = mf(x_1,\ldots,x_n). \tag{2.5}$$

So, by (2.3), (2.4) and (2.5), we obtain

$$f(x_1,\ldots,x_{i-1},mx_i,x_{i+1},\ldots,x_k) = mf(x_1,\ldots,x_k).$$

(iv) Let r be any rational number. There exist an integer n and a non-zero integer m such that

$$r = \frac{n}{m}.$$

That is

$$\begin{aligned}
f(x_1,\ldots,x_{i-1},rx_i,x_{i+1},\ldots,x_n) &= f(x_1,\ldots,x_{i-1},\frac{n}{m}x_i,x_{i+1},\ldots,x_n) \\
&= nf(x_1,\ldots,x_{i-1},\frac{1}{m}x_i,x_{i+1},\ldots,x_n) \\
&= \frac{n}{m}\cdot mf(x_1,\ldots,x_{i-1},\frac{1}{m}x_i,x_{i+1},\ldots,x_n) \\
&= \frac{n}{m}f(x_1,\ldots,x_n) \\
&= rf(x_1,\ldots,x_n).
\end{aligned}$$

□

For any positive integer n, a function $f : X \to Y$ fulfilling the condition

$$\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}f\big(x+(n+1-i)t\big) = 0$$

for all $x, t \in X$, will be called *a polynomial function of order* n. For an arbitrary positive integer k, let $A_k : X^k \to Y$ be a symmetric k-additive function and let $A^k : X \to Y$ be the diagonalization of A_k defined by

$$A^k(x) := A_k(\underbrace{x,\ldots,x}_{k})$$

for any $x \in X$.

Theorem 2.3. *Let $f : X \to Y$ be a polynomial function of order n. Then there exist symmetric k-additive functions $A_k : X^k \to Y$, $k = 1, 2, \ldots, n$, such that*

$$f(x) = A^0 + A^1(x) + \cdots + A^n(x),$$

where A^0 is a constant and $A^k : X \to Y$ is the diagonalization of A_k for each $k = 1, 2, 3, \ldots, n$ (For more details, please refer to [3], pp. 65-79).

Example 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the relation*

$$f(x + 3t) - 3f(x + 2t) + 3f(x + t) - f(x) = 0 \tag{2.6}$$

for all $x \in \mathbb{R}$.

In Example 2.4, we can write (2.6) of the form

$$\sum_{i=0}^{3} (-1)^i \binom{3}{i} f\big(x + (3 - i)t\big) = 0.$$

So, f is a polynomial function of order 2. By Theorem 2.3, there exist symmetric k-additive functions $A_k : X^k \to Y$, $k = 1, 2$, such that

$$f(x) = A^0 + A^1(x) + A^2(x),$$

where A^0 is a constant and $A^k : X \to Y$ is the diagonalization of A_k for each $k = 1, 2$.

CHAPTER III
HARUKI'S LEMMA AND SOME EXTENSIONS

In this chapter, we studied the Haruki's Lemma (see [7]) and some extensions. Firstly, we will prove the Haruki's Lemma using a different method.

Lemma 3.1. *Two functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy equation*

$$\frac{1}{t^2}\Delta^2_{1,t} f(x) = g(x) \tag{3.1}$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\{0\}$ if and only if there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ and constants a, b, c such that

$$f(x) = a + A(x) + bx^2 + cx^3 \tag{3.2}$$

and

$$g(x) = 2b + 6cx \tag{3.3}$$

for all $x \in \mathbb{R}$.

Proof. Suppose $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy (3.1). Then, we have

$$\frac{1}{t^2}\big(f(x+t) - 2f(x) + f(x-t)\big) = g(x). \tag{3.4}$$

Replacing t by $2t$ in (3.4) to obtain

$$\frac{1}{4t^2}\big(f(x+2t) - 2f(x) + f(x-2t)\big) = g(x). \tag{3.5}$$

By (3.4) and (3.5), we get

$$f(x+2t) - 4f(x+t) + 6f(x) - 4f(x-t) + f(x-2t) = 0. \tag{3.6}$$

Replacing x by $x + 2t$ in (3.6) will give

$$f(x+4t) - 4f(x+3t) + 6f(x+2t) - 4f(x+t) + f(x) = 0$$

which can be written of the form

$$\sum_{i=0}^{4}(-1)^i\binom{4}{i}f\big(x+(4-i)t\big)=0.$$

So, f is a polynomial function of order 3. By Theorem 2.3, we get

$$f(x)=a+A(x)+A^2(x)+A^3(x). \tag{3.7}$$

Substituting (3.7) in (3.1) to obtain

$$\frac{1}{t^2}\big(2A_2(t,t)+6A_3(x,t,t)\big)=g(x). \tag{3.8}$$

Setting $x=0$ in (3.8), we have

$$A_2(t,t)=bt^2, \tag{3.9}$$

where $b=\frac{1}{2}g(0)$ is a constant. In addition, equation (3.9) also holds for $t=0$. Thus, we obtain

$$A^2(x)=bx^2 \tag{3.10}$$

for all $x\in\mathbb{R}$. Substituting (3.10) into (3.8) will give

$$2b+\frac{1}{t^2}6A_3(x,t,t)=g(x). \tag{3.11}$$

Setting $t=1$ in (3.11) to get

$$g(x)=2b+6A_3(x,1,1). \tag{3.12}$$

By (3.11) and (3.12), we have

$$A_3(x,t,t)=t^2A_3(x,1,1). \tag{3.13}$$

Replacing x by t in (3.13) will give

$$A^3(t)=t^2A_3(t,1,1). \tag{3.14}$$

In addition, equation (3.14) also holds for $t=0$. Thus, we get

$$A^3(x)=x^2A_3(x,1,1) \tag{3.15}$$

for all $x \in \mathbb{R}$. Hence, equation (3.7) becomes

$$f(x) = a + A(x) + bx^2 + x^2 A_3(x, 1, 1). \tag{3.16}$$

Substituting (3.12) and (3.16) in (3.1) to obtain

$$A_3(x, 1, 1) = \frac{x}{t} A_3(t, 1, 1). \tag{3.17}$$

Next, setting $t = 1$ in (3.17) will give

$$A_3(x, 1, 1) = cx,$$

where $c = A_3(1, 1, 1)$ is a constant. Therefore, equation (3.16) reduces to

$$f(x) = a + A(x) + bx^2 + cx^3$$

and equation (3.12) becomes

$$g(x) = 2b + 6cx.$$

Conversely, we suppose f and g satisfy (3.2) and (3.3). It is not hard to verify that functions f and g indeed satisfy equation (3.1). □

Next ,we will give some extensions of the Haruki's Lemma as in the following Theorems.

Theorem 3.2. *Given a positive integer n, functions $f_1, \ldots, f_n, g : \mathbb{R} \to \mathbb{R}$ which satisfy equation*

$$\sum_{i=1}^{n} \left(\frac{1}{h_i^2} \Delta_{h_i}^2 f_i(x) \right) = g(x) \tag{3.18}$$

for all $x \in \mathbb{R}$ and $h_1, \ldots, h_n \in \mathbb{R} \backslash \{0\}$ are given by

$$f_i(x) = a_i + \alpha_i(x) + b_i x^2 + c_i x^3 \tag{3.19}$$

and

$$g(x) = 2 \sum_{j=1}^{n} (b_j + 3c_j x), \tag{3.20}$$

where a_i, b_i and c_i are constants and $\alpha_i : \mathbb{R} \to \mathbb{R}$ is an additive function for each $i = 1, 2, 3, \ldots, n$.

Proof. For $n \geq 3$, we suppose $f_i, g : \mathbb{R} \to \mathbb{R}$ satisfy equation (3.18). So, we can write (3.18) in the form

$$\frac{1}{h_1^2}\Delta_{h_1}^2 f_1(x) = g(x) - \sum_{j=2}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right). \tag{3.21}$$

By Lemma 3.1, there exists an additive function $\alpha_1 : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) = a_1 + \alpha_1(x) + b_1x^2 + c_1x^3 \tag{3.22}$$

and

$$g(x) - \sum_{j=2}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right) = 2b_1 + 6c_1x, \tag{3.23}$$

where a_1, b_1 and c_1 are constants. So, we write equation (3.23) in the form

$$\frac{1}{h_2^2}\Delta_{h_2}^2 f_2(x) = g(x) - 2(b_1 + 3c_1x) - \sum_{j=3}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right). \tag{3.24}$$

By Lemma 3.1, there exists an additive function $\alpha_2 : \mathbb{R} \to \mathbb{R}$ such that

$$f_2(x) = a_2 + \alpha_2(x) + b_2x^2 + c_2x^3 \tag{3.25}$$

and

$$g(x) - 2(b_1 + 3c_1x) - \sum_{j=3}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right) = 2b_2 + 6c_2x, \tag{3.26}$$

where a_2, b_2 and c_2 are constants. So, by (3.26), we will get

$$g(x) = 2\sum_{j=1}^{2}(b_j + 3c_jx) + \sum_{j=3}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right). \tag{3.27}$$

For any positive $1 < k < n$, we assume that

$$f_{k-1}(x) = a_{k-1} + \alpha_{k-1}(x) + b_{k-1}x^2 + c_{k-1}x^3 \tag{3.28}$$

and

$$g(x) = 2\sum_{j=1}^{k-1}(b_j + 3c_jx) + \sum_{j=k}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right), \tag{3.29}$$

where a_{k-1}, b_{k-1} and c_{k-1} are constants and α_{k-1} is an additive function. Now, we can write equation (3.29) in the form

$$\frac{1}{h_k^2}\Delta_{h_k}^2 f_k(x) = g(x) - 2\sum_{j=1}^{k-1}(b_j + 3c_j x) - \sum_{j=k+1}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right).$$

By Lemma 3.1, there exists an additive function $\alpha_k : \mathbb{R} \to \mathbb{R}$ such that

$$f_k(x) = a_k + \alpha_k(x) + b_k x^2 + c_k x^3 \tag{3.30}$$

and

$$g(x) - 2\sum_{j=1}^{k-1}(b_j + 3c_j x) - \sum_{j=k+1}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right) = 2b_k + 6c_k x, \tag{3.31}$$

that is,

$$g(x) = 2\sum_{j=1}^{k}(b_j + 3c_j x) + \sum_{j=k+1}^{n}\left(\frac{1}{h_j^2}\Delta_{h_j}^2 f_j(x)\right), \tag{3.32}$$

where a_k, b_k and c_k are constants. Now, we can conclude that (3.28) and (3.29) hold for all $k = 1, 2, 3, \ldots, n-1$. So, by equation (3.32), we have

$$g(x) = 2\sum_{j=1}^{n-1}(b_j + 3c_j x) + \frac{1}{h_n^2}\Delta_{h_n}^2 f_n(x)$$

which can be written in the form

$$\frac{1}{h_n^2}\Delta_{h_n}^2 f_n(x) = g(x) - 2\sum_{j=1}^{n-1}(b_j + 3c_j x).$$

Again, by Lemma 3.1, there exists an additive function $\alpha_n : \mathbb{R} \to \mathbb{R}$ such that

$$f_n(x) = a_n + \alpha_n(x) + b_n x^2 + c_n x^3$$

and

$$g(x) - 2\sum_{j=1}^{n-1}(b_j + 3c_j x) = 2b_n + 6c_n x,$$

that is,

$$g(x) = 2\sum_{j=1}^{n}(b_j + 3c_j x),$$

where a_n, b_n and c_n are constants.

It should be remarked that if $n = 1$, then the solution of equation (3.18) is already given by (3.20) and if $n = 2$, then (3.24) to (3.32) can be omitted. Thus, (3.20) holds for any positive integer n.

Conversely, we suppose $f_1, \ldots, f_n, g : \mathbb{R} \to \mathbb{R}$ are given by (3.19) and (3.20). Using Lemma 3.1, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{1}{h_i^2}\Delta_{h_i}^2 f_i(x)\right) &= \sum_{i=1}^{n}\left(\frac{1}{h_i^2}\Delta_{h_i}^2\left(a_i + \alpha_i(x) + b_i x^2 + c_i x^2\right)\right)\\ &= \sum_{i=1}^{n}\left(2b_i + c_i x\right)\\ &= g(x).\end{aligned}$$

□

Theorem 3.3. *Let A be a non-zero constant, B and C be arbitrary constants. Functions $f, g : \mathbb{R} \to \mathbb{R}$ which satisfy the functional equation*

$$\left(A\frac{1}{t^2}\Delta_t^2 + B\frac{1}{s}\Delta_s + C\right)f(x) = g(x) \tag{3.33}$$

for all $x \in \mathbb{R}$, $t, s \in \mathbb{R}\backslash\{0\}$ are given by

$$\begin{cases} f(x) &= a_0 + \alpha(x) + a_2 x^2 \\ g(x) &= 2a_2 A + a_0 C + C\alpha(x) + a_2 C x^2 \end{cases} \quad \text{if } B = 0 \tag{3.34}$$

and

$$\begin{cases} f(x) &= a_0 + a_1 x + a_2 x^2 \\ g(x) &= 2a_2 A + a_1 B + a_0 C + (2a_2 B + a_1 C)x + a_2 C x^2 \end{cases} \quad \text{if } B \neq 0, \tag{3.35}$$

where a_0, a_1 and a_2 and are constants, and $\alpha : \mathbb{R} \to \mathbb{R}$ is an additive function.

Proof. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ satisfy equation (3.33) with a non-zero constant a. Then, we can write (3.33) in the form

$$\frac{1}{t^2}\Delta_t^2 f(x) = \frac{1}{A}\Big(g(x) - B\frac{1}{s}\Delta_s f(x) - Cf(x)\Big).$$

By Lemma 3.1, we get

$$f(x) = a_0 + \alpha(x) + a_2x^2 + bx^3 \qquad (3.36)$$

and

$$\frac{1}{A}\Big(g(x) - B\frac{1}{s}\Delta_s f(x) - Cf(x)\Big) = 2a_2 + 6bx, \qquad (3.37)$$

where a_0, a_2 and b are constants and $\alpha : \mathbb{R} \to \mathbb{R}$ is an additive function. From the expression of $f(x)$ in equation (3.36), we will have

$$\begin{aligned}\frac{1}{s}\Delta_s f(x) &= \frac{1}{s}\Big(f(x+\frac{1}{2}s) - f(x-\frac{1}{2}s)\Big)\\ &= \frac{1}{s}\Big(\alpha(x+\frac{1}{2}s) - \alpha(x-\frac{1}{2}s) + a_2(x+\frac{1}{2}s)^2 - a_2(x-\frac{1}{2}s)^2\\ &\quad + b(x+\frac{1}{2}s)^3 - b(x-\frac{1}{2}s)^3\Big)\\ &= \frac{1}{s}\alpha(s) + 2a_2x + b(3x^2 + \frac{1}{4}s^2). \end{aligned} \qquad (3.38)$$

So, by (3.36), (3.37) and (3.38), we have

$$\begin{aligned} g(x) &= 2a_2A + 6bAx + B\big(\frac{1}{s}\alpha(s) + 2a_2x + 3bx^2 + b\frac{1}{4}s^2\big)\\ &\quad + C\big(a_0 + \alpha(x) + a_2x^2 + bx^3\big). \end{aligned} \qquad (3.39)$$

Replacing s by rs in equation (3.39), where r denotes any non-zero rational number, we have

$$\begin{aligned} &g(x) - 2a_2A - a_0C - 2a_2Bx - 6bAx - C\alpha(x)\\ &- 3bBx^2 - a_2Cx^2 - bCx^3 - B\frac{1}{s}\alpha(s) - \frac{1}{4}bs^2r^2 = 0. \end{aligned} \qquad (3.40)$$

If equation (3.40) is regarded as a polynomial of r, and noting that the polynomial vanishes for any non-zero rational number r, then

$$\begin{cases} g(x) &= 2a_2A + a_0C + 6bAx + B\frac{1}{s}\alpha(s) + 2a_2Bx + C\alpha(x) + 3bBx^2\\ &\quad +a_2Cx^2 + bCx^3,\\ b &= 0. \end{cases} \qquad (3.41)$$

So, by equation (3.36) and the second equation in (3.41), we get

$$f(x) = a_0 + \alpha(x) + a_2x^2. \tag{3.42}$$

By the first equation in (3.41), we have

$$g(x) = 2a_2A + a_0C + B\frac{1}{s}\alpha(s) + 2a_2Bx + C\alpha(x) + a_2Cx^2. \tag{3.43}$$

If $B = 0$, then by equation (3.43), we obtain

$$g(x) = 2a_2A + a_0C + C\alpha(x) + a_2Cx^2.$$

If $B \neq 0$, then setting $x = 0$ in equation (3.43), we will get

$$\alpha(s) = \frac{1}{B}\big(g(0) - 2a_2A - a_0C\big)s \tag{3.44}$$

for all $s \in \mathbb{R}\backslash\{0\}$. In addition, (3.44) also holds for $s = 0$. Thus, we obtain

$$\alpha(x) = a_1x \tag{3.45}$$

for all $x \in \mathbb{R}\backslash\{0\}$, where $a_1 = \frac{1}{B}\big(g(0) - 2a_2A - a_0C\big)$ is a constant. Hence, (3.42) and (3.43) yield

$$f(x) = a_0 + a_1x + a_2x^2$$

and

$$g(x) = 2a_2A + a_1B + a_0C + (2a_2B + a_1C)x + a_2Cx^2,$$

respectively.

Conversely, if $B = 0$ and f, g are given by (3.34) then

$$\begin{aligned}\left(A\frac{1}{t^2}\Delta_t^2 + B\frac{1}{s}\Delta_s + C\right)f(x) &= A\frac{1}{t^2}\Delta_t^2 f(x) + Cf(x)\\ &= A\frac{1}{t^2}\Delta_t^2\big(a_0 + \alpha(x) + a_2x^2\big) + C\big(a_0 + \alpha(x) + a_2x^2\big)\\ &= 2a_2A + C\big(a_0 + \alpha(x) + a_2x^2\big)\\ &= g(x).\end{aligned}$$

If $B \neq 0$ and f, g are given by equation (3.35), then

$$
\begin{aligned}
\left(A\frac{1}{t^2}\Delta_t^2 + B\frac{1}{s}\Delta_s + C\right)f(x) &= A\frac{1}{t^2}\Delta_t^2 f(x) + B\frac{1}{s}\Delta_s f(x) + Cf(x) \\
&= A\frac{1}{t^2}\Delta_t^2\big(a_0 + a_1x + a_2x^2\big) + B\frac{1}{s}\Delta_s\big(a_0 + a_1x + a_2x^2\big) \\
&\quad + C\big(a_0 + a_1x + a_2x^2\big) \\
&= 2a_2A + a_1B + 2a_2Bx + C\big(a_0 + a_1x + a_2x^2\big) \\
&= g(x).
\end{aligned}
$$

□

CHAPTER IV
THE 2-DIMENSIONAL POISSON FUNCTIONAL EQUATION

In this chapter, we will determine the general solution of the functional equation

$$\Big(\frac{1}{t^2}\Delta^2_{1,t} + \frac{1}{s^2}\Delta^2_{2,s}\Big)f(x,y) = g(x,y) \tag{4.1}$$

analogous to the 2-dimensional Poisson equation of f

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f(x,y) = g(x,y).$$

Theorem 4.1. *Functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfy the functional equation (4.1) for all $x, y \in \mathbb{R}$ and all $t, s \in \mathbb{R}\backslash\{0\}$ if and only if*

$$\begin{aligned} f(x,y) = a_0 &+ \Big(\alpha_1(x) + \alpha_2(y)\Big) + \Big(\beta(x,y) + a_1x^2 + a_2y^2\Big) + \Big(a_3x^3 + a_4y^3 \\ &+ \alpha_3(x)y^2 + \alpha_4(y)x^2\Big) + \Big(\alpha_5(x)y^3 + a_5x^2y^2 + \alpha_6(y)x^3\Big) \\ &+ \Big(a_6x^3y^2 + a_7x^2y^3\Big) + a_8x^3y^3 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} g(x,y) = 2\Big(a_1 + a_2\Big) &+ 2\Big(3a_3x + 3a_4y + \alpha_3(x) + \alpha_4(y)\Big) + 2\Big(a_5(x^2+y^2) \\ &+ 3\alpha_5(x)y + 3\alpha_6(y)x\Big) + 2\Big(a_6(x^3 + 3xy^2) + a_7(y^3 + 3x^2y)\Big) \\ &+ 6a_8(x^3y + xy^3), \end{aligned} \tag{4.3}$$

where $a_1, \ldots, a_8$ are constants, $\alpha_1, \ldots, \alpha_6$ are additive functions and β is a bi-additive function.

Proof. Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfy equation (4.1). So, we write (4.1) in the form

$$\frac{1}{s^2}\Delta^2_{2,s}f(x,y) = g(x,y) - \frac{1}{t^2}\Delta^2_{1,t}f(x,y).$$

For a fixed $x \in \mathbb{R}$, by Lemma 3.1, we have

$$f(x,y) = A^0(x) + A^1(x,y) + B(x)y^2 + C(x)y^3 \tag{4.4}$$

and

$$g(x,y) - \frac{1}{t^2}\Delta_{1,t}^2 f(x,y) = 2B(x) + 6C(x)y, \tag{4.5}$$

where $A^0, B, C : \mathbb{R} \to \mathbb{R}$ are unknown functions and $A^1 : \mathbb{R}^2 \to \mathbb{R}$ is additive in the second variable. Substituting (4.4) into (4.5), we get

$$g(x,y) - \frac{1}{t^2}\Delta_{1,t}^2\big(A^0(x) + A^1(x,y) + B(x)y^2 + C(x)y^3\big) = 2B(x) + 6C(x)y. \tag{4.6}$$

Setting $y = 0$ in (4.6) and noting that $A^1(x,0) = 0$ will give

$$\frac{1}{t^2}\Delta_{1,t}^2 A^0(x) = g(x,0) - 2B(x). \tag{4.7}$$

By Lemma 3.1, we have

$$A^0(x) = a_0 + \alpha_1(x) + a_1x^2 + a_3x^3 \tag{4.8}$$

and

$$g(x,0) - 2B(x) = 2a_1 + 6a_3x, \tag{4.9}$$

where a_0, a_1, a_3 are constants and α_1 is an additive function. Substituting (4.7) into (4.6), we obtain

$$g(x,y) = \frac{1}{t^2}\Delta_{1,t}^2\big(A^1(x,y) + B(x)y^2 + C(x)y^3\big) + g(x,0) + 6C(x)y. \tag{4.10}$$

Next, we apply the operator $\frac{1}{s^2}\Delta_{2,s}^2$ to both sides of (4.10) to get

$$\frac{1}{s^2}\Delta_{2,s}^2 g(x,y) = \frac{1}{t^2}\Delta_{1,t}^2\big(2B(x) + 6C(x)y\big). \tag{4.11}$$

For a fixed $x \in \mathbb{R}$ in (4.11), by Lemma 2.3, we have

$$g(x,y) = D^0(x) + D^1(x,y) + E(x)y^2 + F(x)y^3 \tag{4.12}$$

and

$$\frac{1}{t^2}\Delta_{1,t}^2\big(2B(x)+6C(x)y\big)=2E(x)+6F(x)y, \tag{4.13}$$

where $D^0, E, F : \mathbb{R} \to \mathbb{R}$ are unknown functions and $D^1 : \mathbb{R}^2 \to \mathbb{R}$ is additive in the second variable. Replacing y by ry in (4.13), where r denotes any rational number, we get

$$\frac{1}{t^2}\Delta_{1,t}^2 B(x)-E(x)+\big(\frac{1}{t^2}\Delta_{1,t}^2 3C(x)y-3F(x)y\big)r=0. \tag{4.14}$$

If equation (4.14) is regarded as a polynomial of r, and noting that the polynomial vanishes for any rational number r, then we have

$$\begin{cases} \frac{1}{t^2}\Delta_{1,t}^2 B(x)=E(x), \\ \\ \frac{1}{t^2}\Delta_{1,t}^2 C(x)y=F(x)y. \end{cases} \tag{4.15}$$

We will start with solving the first equation in (4.15). By Lemma 3.1, we will get

$$B(x)=a_2+\alpha_3(x)+a_5x^2+a_6x^3, \tag{4.16}$$

and

$$E(x)=2a_5+6a_6x, \tag{4.17}$$

where a_2, a_5 and a_6 are constants and $\alpha_3 : \mathbb{R} \to \mathbb{R}$ is an additive function. Next, setting $y=1$ in the second equation in (4.15) will give

$$\frac{1}{t^2}\Delta_{1,t}^2 C(x)=F(x).$$

Again ,by Lemma 3.1, we have

$$C(x)=a_4+\alpha_5(x)+a_7x^2+a_8x^3 \tag{4.18}$$

and

$$F(x)=2a_7+6a_8x, \tag{4.19}$$

where a_4, a_7, a_8 are constants and $\alpha_5 : \mathbb{R} \to \mathbb{R}$ is an additive function. So, by (4.12), (4.17) and (4.19), we get

$$g(x,y) = D^0(x) + D^1(x,y) + \big(2a_5 + 6a_6x\big)y^2 + \big(2a_7 + 6a_8x\big)y^3, \tag{4.20}$$

and by (4.10), (4.16) and (4.18), we obtain

$$\frac{1}{t^2}\Delta^2_{1,t}A^1(x,y) = D^1(x,y) - 6a_4y - 6\alpha_5(x)y - 6a_7x^2y - 6a_8x^3y. \tag{4.21}$$

For fixed $y \in \mathbb{R}$ in (4.21), by Lemma 3.1, we get

$$A^1(x,y) = \alpha_2(y) + \beta(x,y) + \alpha_4(y)x^2 + \alpha_6(y)x^3 \tag{4.22}$$

and

$$2\alpha_4(y) + 6\alpha_6(y)x = D^1(x,y) - 6a_4y - 6\alpha_5(x)y - 6a_7x^2y - 6a_8x^3y, \tag{4.23}$$

where $\alpha_2, \alpha_4, \alpha_6 : \mathbb{R} \to \mathbb{R}$ are unknown functions and $\beta : \mathbb{R}^2 \to \mathbb{R}$ is additive in the first variable.

Setting $x = 0$ in (4.23) to obtain

$$\alpha_4(y) = \frac{1}{2}D^1(0,y) - 3a_4y.$$

Since D^1 is additive in the second variable, we infer that α_4 is an additive function. Next, setting $x = 0$ in (4.22) to obtain

$$\alpha_2(y) = A^1(0,y),$$

Since A^1 is additive in the second variable, we infer that α_2 is an additive function. Combining results from (4.4), (4.8), (4.16), (4.18) and (4.22), we get

$$\begin{aligned} f(x,y) &= a_0 + \alpha_1(x) + a_1x^2 + a_3x^3 + \alpha_2(y) + \beta(x,y) + \alpha_4(y)x^2 + \alpha_6(y)x^3 \\ &\quad + \big(a_2 + \alpha_3(x) + a_5x^2 + a_6x^3\big)y^2 + \big(a_4 + \alpha_5(x) + a_7x^2 + a_8x^3\big)y^3. \end{aligned} \tag{4.24}$$

Next, substituting (4.24) and (4.20) in (4.1) will give

$$\begin{aligned} &2a_1 + 6a_3x + 2\alpha_4(y) + 6\alpha_6(y)x + 2a_2 + 2\alpha_3(x) + 6a_4y + 6\alpha_5(x)y \\ &+ \big(2a_5 + 6a_7y\big)x^2 + \big(2a_6 + 6a_8y\big)x^3 = D^0(x) + D^1(x,y). \end{aligned} \tag{4.25}$$

Replacing r by ry in (4.25), where r denotes any rational number, we have

$$D^0(x) - 2a_1 - 2a_2 - 6a_3x - 2\alpha_3(x) - 2a_5x^2 - 2a_6x^3 + \big(D^1(x,y) - 6a_4y - 2\alpha_4(y) - 6\alpha_5(x)y - 6\alpha_6(y)x - 6a_7x^2y - 6a_8x^3y\big)r = 0. \quad (4.26)$$

If equation (4.26) is regarded as a polynomial of r, and noting that the polynomial vanishes for any rational number r, then we get

$$\begin{cases} D^0(x) = 2a_1 + 2a_2 + 6a_3x + 2\alpha_3(x) + 2a_5x^2 + 2a_6x^3 \\ \\ D^1(x,y) = 6a_4y + 2\alpha_4(y) + 6\alpha_5(x)y + 6\alpha_6(y)x + 6a_7x^2y + 6a_8x^3y. \end{cases} \quad (4.27)$$

Therefore, by (4.20) and (4.27), we have

$$g(x,y) = 2a_1 + 2a_2 + 6a_3x + 6a_4y + 2\alpha_3(x) + 2\alpha_4(y) + 2a_5(x^2+y^2) + 6\alpha_5(x)y + 6\alpha_6(y)x + 2a_6(x^3+3xy^2) + 2a_7(y^3+3x^2y) + 6a_8(x^3y+xy^3). \quad (4.28)$$

Thus, we infer that f satisfies (4.2) and g satisfies (4.28).

Conversely, suppose $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies (4.2) and $g : \mathbb{R}^2 \to \mathbb{R}$ satisfies (4.3). So, we get

$$\begin{aligned}\Big(\frac{1}{t^2}\Delta^2_{1,t} + \frac{1}{s^2}\Delta^2_{2,s}\Big)f(x,y) &= \Big(\frac{1}{t^2}\Delta^2_{1,t} + \frac{1}{s^2}\Delta^2_{2,s}\Big)\big(a_0 + \alpha_1(x) + \alpha_2(y) + \beta(x,y) + a_1x^2 \\ &\quad + a_2y^2 + a_3x^3 + a_4y^3 + \alpha_3(x)y^2 + \alpha_4(y)x^2 + \alpha_5(x)y^3 \\ &\quad + a_5x^2y^2 + \alpha_6(y)x^3 + a_6x^3y^2 + a_7x^2y^3 + a_8x^3y^3\big) \\ &= 2a_1 + 2a_2 + 6a_3x + 6a_4y + 2\alpha_3(x) + 2\alpha_4(y) \\ &\quad + 2a_5(x^2+y^2) + 6\alpha_5(x)y + 6\alpha_6(y)x + 2a_6(x^3+3xy^2) \\ &\quad + 2a_7(y^3+3x^2y) + 6a_8(x^3y+xy^3) \\ &= g(x,y).\end{aligned}$$

□

In equation (4.1), if $g(x,y) = 0$ for $x, y \in \mathbb{R}$ then we get that equation (4.1) reduces to

$$\Big(\frac{1}{t^2}\Delta^2_{1,t} + \frac{1}{s^2}\Delta^2_{2,s}\Big)f(x,y) = 0 \quad (4.29)$$

which is analogous to 2-dimensional Laplace equation

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})f(x,y)=0.$$

Corollary 4.2. *A function $f:\mathbb{R}^2\to\mathbb{R}$ satisfies the functional equation (4.29) for all $x,y\in\mathbb{R}$ and $s,t\in\mathbb{R}\backslash\{0\}$ if and only if there exist additive functions $\alpha_1,\alpha_2:\mathbb{R}\to\mathbb{R}$ and a bi-additive function $\beta:\mathbb{R}^2\to\mathbb{R}$ such that*

$$\begin{aligned} f(x,y) = a_0+\alpha_1(x)+\alpha_2(y)+\beta(x,y)+a_1(x^2-y^2)+a_3(x^3-3xy^2)\\ +a_4(y^3-3x^2y)+a_9(x^3y-xy^3) \end{aligned} \tag{4.30}$$

for all $x,y\in\mathbb{R}$, where a_0,a_1,a_3,a_4, and a_9 are constants.

Proof. Suppose $f:\mathbb{R}^2\to\mathbb{R}$ satisfies equation (4.29). So, by Theorem 4, we have

$$\begin{aligned} f(x,y) = a_0+\Big(\alpha_1(x)+\alpha_2(y)\Big)+\Big(\beta(x,y)+a_1x^2+a_2y^2\Big)+\Big(a_3x^3+a_4y^3\\ +\alpha_3(x)y^2+\alpha_4(y)x^2\Big)+\Big(\alpha_5(x)y^3+a_5x^2y^2+\alpha_6(y)x^3\Big)\\ +\Big(a_6x^3y^2+a_7x^2y^3\Big)+a_8x^3y^3 \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} 2\Big(a_1+a_2\Big)+2\Big(3a_3x+3a_4y+\alpha_3(x)+\alpha_4(y)\Big)+2\Big(a_5(x^2+y^2)+3\alpha_5(x)y\\ +3\alpha_6(y)x\Big)+\Big(2a_6(x^3+3xy^2)+2a_7(y^3+3x^2y)\Big)+6a_8(x^3y+xy^3)=0, \end{aligned} \tag{4.32}$$

where $a_0,\dots,a_8$ are constants, $\alpha_1,\dots,\alpha_4$ are additive functions and β is a bi-additive function. Replacing y by ry in (4.32), where r denotes any rational number, we get

$$\begin{aligned} 2a_1+2a_2+6a_3x+2\alpha_3(x)+2a_5x^2+2a_6x^3+(6a_4y+2\alpha_4(y)+6\alpha_5(x)y\\ +6\alpha_6(y)x+6a_7x^2y+6a_8xy^3)r+(2a_5y^2+6a_6xy^2)r^2+(2a_7y^3+6a_8xy^3)r^3=0. \end{aligned} \tag{4.33}$$

If equation (4.33) is regarded as a polynomial of r, and noting that the polynomial vanishes for any rational number r, then we obtain

$$\begin{cases} 2a_1 + 2a_2 + 6a_3x + 2\alpha_3(x) + 2a_5x^2 + 2a_6x^3 = 0, \\ 6a_4y + 2\alpha_4(y) + 6\alpha_5(x)y + 6\alpha_6(y)x + 6a_7x^2y + 6a_8xy^3 = 0, \\ 2a_5y^2 + 6a_6xy^2 = 0, \\ 2a_7y^3 + 6a_8xy^3 = 0. \end{cases} \tag{4.34}$$

Next, setting $y = 1$ in the third equation and the fourth equation in (4.34), and replacing x by rx in (4.34), where r denotes any rational number, we have

$$\begin{cases} 2a_1 + 2a_2 + (6a_3x + 2\alpha_3(x))r + 2a_5x^2r^2 + 2a_6x^3r^3 = 0, \\ 6a_4y + 2\alpha_4(y) + (6\alpha_5(x)y + 6\alpha_6(y)x)r + 6a_7x^2yr^2 + 6a_8xy^3r^3 = 0, \\ 2a_5 + 6a_6xr = 0, \\ 2a_7 + 6a_8xr = 0. \end{cases} \tag{4.35}$$

If each equation in (4.35) is regarded as a polynomial of r, and noting that the polynomial vanishes for any rational number r, then we get

$$\begin{cases} a_1 + a_2 = 0, \\ 3a_3x + \alpha_3(x) = 0, \\ 3a_4y + \alpha_4(y) = 0, \\ \alpha_5(x)y + \alpha_6(y)x = 0, \\ a_i = 0, \end{cases} \tag{4.36}$$

where $i = 5, 6, 7, 8$. Next, setting $x = 1$ in the third equation in (4.36) to obtain

$$\alpha_6(y) = a_9y, \tag{4.37}$$

where $a_9 = \alpha_5(1)$ is a constant. Substituting (4.37) in the third equation in (4.36) will give

$$\alpha_5(x)y + a_9xy = 0. \tag{4.38}$$

Setting $y = 1$ in (4.38), we get

$$\alpha_5(x) = -a_9y. \tag{4.39}$$

Combining results from equations (4.36), (4.37) and (4.39), we obtain equation (4.31) becomes

$$f(x,y) = a_0 + \alpha_1(x) + \alpha_2(y) + \beta(x,y) + a_1(x^2 - y^2) + a_3(x^3 - 3xy^2) \\ + a_4(y^3 - 3x^2y) + a_9(x^3y - xy^3)$$

Conversely, suppose a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies equation (4.30). It is not hard to verify that functions f indeed satisfies equation (4.29). $\square$

We can use Theorem 4.1 to solve the 3-dimensional Laplace equation as the next chapter.

CHAPTER V
THE 3-DIMENSIONAL LAPLACE FUNCTIONAL EQUATION

In this chapter, we will give the general solution of the functional equation

$$\Big(\frac{1}{h_1^2}\Delta^2_{1,h_1}+\frac{1}{h_2^2}\Delta^2_{2,h_2}+\frac{1}{h_3^2}\Delta^2_{3,h_3}\Big)f(x,y,z)=0 \tag{5.1}$$

analogous to the 3-dimensional Laplace equation

$$\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)f(x,y,z)=0$$

when no regularity assumptions are imposed on f.

Theorem 5.1. *The general solution of the functional equation (5.1) for all $x,y,z\in\mathbb{R}$ and $h_1,h_2,h_3\in\mathbb{R}\backslash\{0\}$ is given by*

$$\begin{aligned}f(x,y,z)=\ &a+\alpha_x(x)+\alpha_y(y)+\alpha_z(z)+\beta_{xy}(x,y)+\beta_{xz}(x,z)+\beta_{yz}(y,z)\\&+\gamma(x,y,z)+\big(b_{xy}+\alpha_{xy}(z)\big)(x^2-y^2)+\big(b_{xz}+\alpha_{xz}(y)\big)(x^2-z^2)\\&+\big(b_{yz}+\alpha_{yz}(x)\big)(y^2-z^2)+\big(c_{xy}+\eta_{xy}(z)\big)(x^3-3xy^2)\\&+\big(c_{xz}+\eta_{xz}(y)\big)(x^3-3xz^2)+\big(c_{yx}+\eta_{yx}(z)\big)(y^3-3x^2y)\\&+\big(c_{yz}+\eta_{yz}(x)\big)(y^3-3yz^2)+\big(c_{zx}+\eta_{zx}(y)\big)(z^3-3x^2z)\\&+\big(c_{zy}+\eta_{zy}(x)\big)(z^3-3y^2z)+\varphi_{xy}(z)(x^3y-xy^3)\\&+\varphi_{xz}(y)(x^3z-xz^3)+\varphi_{yz}(z)(y^3z-yz^3)\end{aligned} \tag{5.2}$$

where $a, b_{..}'s, c_{..}'s$ are constants, $\alpha_{..}'s, \eta_{..}'s, \varphi_{..}'s$ are additive functions, $\beta_{..}'s$ are bi-additive functions and γ is a 3-additive function.

Proof. Suppose $f:\mathbb{R}^3\to\mathbb{R}$ satisfies equation (5.1). So, we can write (5.1) in the form

$$\frac{1}{h_3^2}\Delta^2_{3,h_3}f(x,y,z)=-\Big(\frac{1}{h_1^2}\Delta^2_{1,h_1}+\frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)f(x,y,z). \tag{5.3}$$

For fixed $x, y \in \mathbb{R}$ in (5.3), by Lemma 3.1, we have

$$f(x,y,z) = A^0(x,y) + A^1(x,y,z) + B(x,y)z^2 + C(x,y)z^3, \tag{5.4}$$

where $A^0, B, C : \mathbb{R}^2 \to \mathbb{R}$ are unknown functions and $A^1 : \mathbb{R}^3 \to \mathbb{R}$ is additive in the third variable. Substituting (5.4) into (5.3) and simplifying, we get

$$\begin{aligned}&\Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)\Big(A^0(x,y) + A^1(x,y,z) + B(x,y)z^2 + C(x,y)z^3\Big)\\ &+ 2B(x,y) + 6C(x,y)z = 0.\end{aligned} \tag{5.5}$$

Let r be an arbitrary rational number. If we replace z by rz in (5.5) and note that $A^1(x,y,rz) = rA^1(x,y,z)$, the resulting equation can be regarded as a polynomial of r. Thus, all coefficients must vanish; that is

$$\begin{cases}\Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)A^0(x,y) + 2B(x,y) = 0,\\ \Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)A^1(x,y,z) + 6C(x,y)z = 0,\\ \Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)B(x,y)z^2 = 0,\\ \Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)C(x,y)z^3 = 0.\end{cases} \tag{5.6}$$

Firstly, we turn to the first equation in (5.6) and we write it in the form

$$\Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)A^0(x,y) = -2B(x,y). \tag{5.7}$$

By Theorem 4.1, we have

$$\begin{aligned}A^0(x,y) = a_0 &+ \Big(\alpha_1(x) + \alpha_2(y)\Big) + \Big(\beta_1(x,y) + a_1x^2 + a_2y^2\Big) + \Big(a_3x^3 + a_4y^3\\ &+ \alpha_3(x)y^2 + \alpha_4(y)x^2\Big) + \Big(\alpha_6(x)y^3 + k_1x^2y^2 + \alpha_8(y)x^3\Big)\\ &+ \Big(k_2x^3y^2 + k_3x^2y^3\Big) + k_4x^3y^3\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}B(x,y) = &-\Big(a_1 + a_2\Big) - \Big(3a_3x + 3a_4y + \alpha_3(x) + \alpha_4(y)\Big) - \Big(k_1(x^2 + y^2)\\ &+ 3\alpha_6(x)y + 3\alpha_8(y)x\Big) - \Big(k_2(x^3 + 3xy^2) + k_3(y^3 + 3x^2y)\Big)\\ &- 3k_4(x^3y + xy^3),\end{aligned} \tag{5.9}$$

where $a_1, \ldots, a_4, k_1, \ldots, k_4$ are constants, $\alpha_1, \ldots, \alpha_6$ are additive functions and β_1 is a bi-additive function. Next, substituting (5.9) in the third equation in (5.6) and setting $z = 1$ will give

$$-4k_1 - 12k_2 x - 12k_3 y - 36k_4 xy = 0. \tag{5.10}$$

Since (5.10) holds for all $x, y \in \mathbb{R}$, all coefficients must vanish; that is, $k_1 = k_2 = k_3 = k_4 = 0$. Thus, equation (5.8) and (5.9) simplify to

$$\begin{aligned} A^0(x,y) = a_0 + \Big(\alpha_1(x) + \alpha_2(y)\Big) + \Big(\beta_1(x,y) + a_1x^2 + a_2y^2\Big) + \Big(a_3x^3 + a_4y^3 \\ + \alpha_3(x)y^2 + \alpha_4(y)x^2\Big) + \Big(\alpha_6(x)y^3 + \alpha_8(y)x^3\Big) \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} B(x,y) = & -\Big(a_1 + a_2\Big) - \Big(3a_3x + 3a_4y + \alpha_3(x) + \alpha_4(y)\Big) \\ & - 3\Big(\alpha_6(x)y + \alpha_8(y)x\Big). \end{aligned} \tag{5.12}$$

Next, we turn to the second equation in (5.6) and we can write it in the form

$$\Big(\frac{1}{h_1^2}\Delta^2_{1,h_1} + \frac{1}{h_2^2}\Delta^2_{2,h_2}\Big)A^1(x,y,z) = -6C(x,y)z. \tag{5.13}$$

For each fixed $z \in \mathbb{R}$ in (5.13), by Theorem 4.1, we get

$$\begin{aligned} A^1(x,y,z) = \alpha_3(z) &+ \Big(\Phi_1(x,z) + \Phi_2(y,z)\Big) + \Big(\kappa(x,y,z) + \zeta_1(z)x^2 + \zeta_2(z)y^2\Big) \\ &+ \Big(\zeta_3(z)x^3 + \zeta_4(z)y^3 + \Phi_3(x,z)y^2 + \Phi_4(y,z)x^2\Big) \\ &+ \Big(\Phi_5(x,z)y^3 + \zeta_5(z)x^2y^2 + \Phi_6(y,z)x^3\Big) \\ &+ \Big(\zeta_6(z)x^3y^2 + \zeta_7(z)x^2y^3 + \zeta_8(z)x^3y^3\Big) \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} C(x,y)z = & -\frac{1}{3}\Big(\zeta_1(z) + \zeta_2(z)\Big) - \Big(\zeta_3(z)x + \zeta_4(z)y + \frac{1}{3}\Phi_3(x,z) + \frac{1}{3}\Phi_4(y,z)\Big) \\ & - \Big(\frac{1}{3}\zeta_5(z)(x^2+y^2) + \Phi_5(x,z)y + \Phi_6(y,z)x\Big) - \Big(\zeta_6(z)(\frac{1}{3}x^3 + xy^2) \\ & + \zeta_7(z)(\frac{1}{3}y^3 + x^2y) + \zeta_8(z)(x^3y + xy^3)\Big), \end{aligned} \tag{5.15}$$

where $\alpha_3, \zeta_1, \ldots, \zeta_8 : \mathbb{R} \to \mathbb{R}$ are unknown functions, $\Phi_1, \ldots, \Phi_6 : \mathbb{R}^2 \to \mathbb{R}$ is additive in the first variable and $\kappa : \mathbb{R}^3 \to \mathbb{R}$ is additive in the first variable and the second variable. Setting $x = y = 0$ in (5.14), we have

$$A^1(0,0,z) = \alpha_3(z).$$

Since A^1 is additive in the third variable, we infer that α_3 is an additive function. We take Δ^2_{3,h_3} to both sides of (5.14) to obtain

$$\begin{aligned}\Delta^2_{3,h_3}\Big(&\Phi_1(x,z) + \Phi_2(y,z) + \kappa(x,y,z) + \zeta_1(z)x^2 + \zeta_2(z)y^2 + \zeta_3(z)x^3 + \zeta_4(z)y^3 \\ &+ \Phi_3(x,z)y^2 + \Phi_4(y,z)x^2 + \Phi_5(x,z)y^3 + \zeta_5(z)x^2y^2 + \Phi_6(y,z)x^3 + \zeta_6(z)x^3y^2 \\ &+ \zeta_7(z)x^2y^3 + \zeta_8(z)x^3y^3\Big) = 0. \end{aligned} \tag{5.16}$$

Let s be an arbitrary rational number. If we replace x by rx and y by sy in (5.16), and note that $\Phi_i(rx,z) = r\Phi_i(x,z), \Phi_j(sy,z) = s\Phi_j(y,z)$ for all $i = 1,3,5$ and $j = 2,4,6$ and $\kappa(rx,sy,z) = rs\kappa(x,y,z)$, the resulting equation can be be regarded as a polynomial of r and s. Thus, all coefficients must vanish; that is

$$\begin{cases} \Delta^2_{3,h_3}\zeta_i(z) = 0, \\ \Delta^2_{3,h_3}\Phi_j(x,z) = 0, \\ \Delta^2_{3,h_3}\kappa(x,y,z) = 0, \end{cases} \tag{5.17}$$

for all $i = 1,2,\ldots,8$ and $j = 1,2,\ldots,6$. Next, replacing z by $z + h_3$ in equation (5.17) to obtain

$$\begin{cases} \sum_{i=0}^{2}(-1)^i\binom{2}{i}\zeta_i\big(z + (2-i)h_3\big) = 0, \\ \sum_{i=0}^{2}(-1)^i\binom{2}{i}\Phi_j\big(x, z + (2-i)h_3\big) = 0, \\ \sum_{i=0}^{2}(-1)^i\binom{2}{i}\kappa\big(x,y,z + (2-i)h_3\big) = 0. \end{cases} \tag{5.18}$$

So, $\zeta_1, \ldots, \zeta_8, \Phi_1, \ldots, \Phi_6$ and κ are polynomial functions of order 1. By Theorem 2.3, there exist additive functions $\xi_1, \ldots, \xi_8 : \mathbb{R} \to \mathbb{R}$, $\beta_2, \ldots, \beta_7 : \mathbb{R}^2 \to \mathbb{R}$ are additive in the second variable and $\gamma : \mathbb{R}^3 \to \mathbb{R}$ is additive in the third variable such that

$$\begin{cases} \zeta_i(z) = r_i + \xi_i(z), \\ \Phi_j(x,z) = \phi_j(x) + \beta_{j+1}(x,z), \\ \kappa(x,y,z) = \beta_8(x,y) + \gamma(x,y,z), \end{cases} \tag{5.19}$$

where $r_1 \ldots, r_8$ are constants, $\phi_1, \ldots, \phi_6 : \mathbb{R} \to \mathbb{R}$ and $\beta_8 : \mathbb{R}^2 \to \mathbb{R}$ are unknown functions. Setting $z = 0$ in the second equation and the third equation in (5.19) will give

$$\Phi_j(x, 0) = \phi_j(x) \tag{5.20}$$

and

$$\kappa(x, y, 0) = \beta_8(x, y). \tag{5.21}$$

In equation (5.20), since Φ_j is additive in the first variable , we infer that $\phi_1, \ldots, \phi_6$ are an additive functions. Consequently, we can see from the second equation in (5.19) that $\beta_2, \ldots, \beta_7$ must also be bi-additive functions. In equation (5.21), κ is additive in the first variable and the second variable, we conclude that β_8 is a bi-additive function. Therefore, we can see from the third equation in (5.19) that γ must also be a 3-additive function. So, equation (5.14) becomes

$$\begin{aligned} A^1(x, y, z) = \alpha_3(z) &+ \Big(\phi_1(x) + \beta_2(x, z) + \phi_2(y) + \beta_3(y, z)\Big) + \Big(\beta_8(x, y) \\ &+ \gamma(x, y, z) + \big(r_1 + \xi_1(z)\big)x^2 + \big(r_2 + \xi_2(z)\big)y^2\Big) + \Big(\big(r_3 + \xi_3(z)\big)x^3 \\ &+ \big(r_4 + \xi_4(z)\big)y^3 + \big(\phi_3(x) + \beta_4(x, z)\big)y^2 + \big(\phi_4(y) + \beta_5(y, z)\big)x^2\Big) \\ &+ \Big(\big(\phi_5(x) + \beta_6(x, z)\big)y^3 + \big(r_5 + \xi_5(z)\big)x^2y^2 + \big(\phi_6(y) + \beta_7(y, z)\big)x^3\Big) \\ &+ \Big(\big(r_6 + \xi_6(z)\big)x^3y^2 + \big(r_7 + \xi_7(z)\big)x^2y^3 + \big(r_8 + \xi_8(z)\big)x^3y^3\Big) \end{aligned} \tag{5.22}$$

and equation (5.15) becomes

$$\begin{aligned} C(x, y)z = &-\frac{1}{3}\Big(r_1 + \xi_1(z) + r_2 + \xi_2(z)\Big) - \Big(\big(r_3 + \xi_3(z)\big)x + \big(r_4 + \xi_4(z)\big)y \\ &+ \frac{1}{3}\big(\phi_3(x) + \beta_4(x, z)\big) + \frac{1}{3}\big(\phi_4(y) + \beta_5(y, z)\big)\Big) \\ &- \Big(\frac{1}{3}\big(r_5 + \xi_5(z)\big)\big(x^2 + y^2\big) + \big(\phi_5(x) + \beta_6(x, z)\big)y \\ &+ \big(\phi_6(y) + \beta_7(y, z)\big)x\Big) - \Big(\big(r_6 + \xi_6(z)\big)\big(\frac{1}{3}x^3 + xy^2\big) \\ &+ \big(r_7 + \xi_7(z)\big)\big(\frac{1}{3}y^3 + x^2y\big) + \big(r_8 + \xi_8(z)\big)\big(x^3y + xy^3\big)\Big). \end{aligned} \tag{5.23}$$

Next, substituting (5.23) in the fourth equation in (5.6) and simplifying, we have

$$\big(r_5 + \xi_5(z)\big)z^2 + 3\big(r_6 + \xi_6(z)\big)xz^2 + 3\big(r_7 + \xi_7(z)\big)yz^2 + 9\big(r_8 + \xi_8(z)\big)xyz^2 = 0. \tag{5.24}$$

If we replace z by rz in (5.24), and note that $\xi_i(rz) = r\xi_i(z)$ for all $i = 5, 6, 7, 8$, the resulting equation can be be regarded as a polynomial of r. Thus, all coefficients must vanish; that is

$$\begin{cases} \big(r_5 + 3r_6x + 3r_7y + 9r_8xy\big)z^2 = 0, \\ \big(\xi_5(z) + 3\xi_6(z)x + 3\xi_7(z)y + 9\xi_8(z)xy\big)z^3 = 0. \end{cases} \tag{5.25}$$

For any fixed $z \in \mathbb{R}\backslash\{0\}$ in (5.25), we have

$$\begin{cases} r_5 + 3r_6x + 3r_7y + 9r_8xy = 0, \\ \xi_5(z) + 3\xi_6(z)x + 3\xi_7(z)y + 9\xi_8(z)xy = 0. \end{cases} \tag{5.26}$$

Since each equation in (5.26) holds for all $x, y \in \mathbb{R}$, all coefficients must vanish; that is, $r_5 = r_6 = r_7 = r_8 = 0$ and $\xi_5(z) = \xi_6(z) = \xi_7(z) = \xi_8(z) = 0$ for all $z \in \mathbb{R}$. Hence, equation (5.22) reduces to

$$\begin{aligned} A^1(x, y, z) = \alpha_3(z) &+ \Big(\phi_1(x) + \beta_2(x, z) + \phi_2(y) + \beta_3(y, z)\Big) + \Big(\beta_8(x, y) + \gamma(x, y, z) \\ &+ \big(r_1 + \xi_1(z)\big)x^2 + \big(r_2 + \xi_2(z)\big)y^2\Big) + \Big(\big(r_3 + \xi_3(z)\big)x^3 \\ &+ \big(r_4 + \xi_4(z)\big)y^3 + \big(\phi_3(x) + \beta_4(x, z)\big)y^2 + \big(\phi_4(y) + \beta_5(y, z)\big)x^2\Big) \\ &+ \Big(\big(\phi_5(x) + \beta_6(x, z)\big)y^3 + \big(\phi_6(y) + \beta_7(y, z)\big)x^3\Big) \end{aligned} \tag{5.27}$$

and equation (5.23) reduces to

$$\begin{aligned} C(x, y)z = \; -\frac{1}{3}\Big(r_1 + \xi_1(z) + r_2 + \xi_2(z)\Big) &- \Big(\big(r_3 + \xi_3(z)\big)x + \big(r_4 + \xi_4(z)\big)y \\ + \frac{1}{3}\big(\phi_3(x) + \beta_4(x, z)\big) + \frac{1}{3}\big(\phi_4(y) + \beta_5(y, z)\big)\Big) &- \Big(\big(\phi_5(x) + \beta_6(x, z)\big)y \\ + \big(\phi_6(y) + \beta_7(y, z)\big)x\Big). \end{aligned} \tag{5.28}$$

Setting $z = 0$ in (5.27) will give

$$\begin{aligned} &\phi_1(x) + \phi_2(y) + \beta_8(x, y) + r_1x^2 + r_2y^2 + r_3x^3 + r_4y^3 \\ &+ \phi_3(x)y^2 + \phi_4(y)x^2 + \phi_5(x)y^3 + \phi_6(y)x^3 = 0. \end{aligned} \tag{5.29}$$

If we replace x by rx and y by sy in (5.29), and note that $\phi_i(rx) = r\phi_i(x), \phi_j(sy) = s\phi_j(y)$ for all $i = 1, 3, 5$ and $j = 2, 4, 6$ and $\beta_8(rx, sy) = rs\kappa(x, y)$, the resulting equation can be be regarded as a polynomial of r and s. Thus, all coefficients must vanish; that is $r_1 = r_2 = r_3 = r_4 = 0$, $\phi_1(x) = \phi_2(y) = \phi_3(x) = \phi_4(y) = \phi_5(x) = \phi_6 = 0$ and $\beta_8(x, y) = 0$. So, equation (5.27) reduces to

$$\begin{aligned} A^1(x, y, z) = \alpha_3(z) + \Big(\beta_2(x, z) + \beta_3(y, z)\Big) + \Big(\gamma(x, y, z) + \alpha_5(z)x^2 + \xi_2(z)y^2\Big) \\ + \Big(\alpha_{10}(z)x^3 + \alpha_{11}(z)y^3 + \beta_4(x, z)y^2 + \beta_5(y, z)x^2\Big) \\ + \Big(\beta_6(x, z)y^3 + \beta_7(y, z)x^3\Big) \end{aligned} \tag{5.30}$$

and equation (5.28) reduces to

$$\begin{aligned} C(x, y)z = & -\frac{1}{3}\Big(\alpha_5(z) + \xi_2(z)\Big) - \Big(\alpha_{10}(z)x + \alpha_{11}(z)y + \frac{1}{3}\beta_4(x, z) + \frac{1}{3}\beta_5(y, z)\Big) \\ & - \Big(\beta_6(x, z)y + \beta_7(y, z)x\Big), \end{aligned} \tag{5.31}$$

where $\alpha_5 \equiv \xi_1$, $\alpha_{10} \equiv \xi_3$ and $\alpha_{11} \equiv \xi_4$. Next, setting $z = 1$ in (5.31) will give

$$C(x, y) = a_5 + \alpha_7(x) + \alpha_9(y) - \alpha_{12}(x)y - \alpha_{13}(y)x \tag{5.32}$$

where $a_5 = -\frac{1}{3}\big(\alpha_5(1) + \xi_2(1)\big)$ is a constant, $\alpha_7(x) = -\alpha_{10}(1)x - \frac{1}{3}\beta_4(x, 1)$, $\alpha_9(y) = -\alpha_{11}(1)y - \frac{1}{3}\beta_5(y, 1)$, $\alpha_{12}(x) = \beta_6(x, 1)$ and $\alpha_{13}(y) = \beta_7(y, 1)$ are additive functions for all $x, y \in \mathbb{R}$. Next, substituting (5.30) and (5.32) in (5.13) to obtain

$$\begin{aligned} 2\alpha_5(z) + 2\xi_2(z) + 6\alpha_{10}(z)y + 2\beta_4(x, z) + 2\beta_5(y, z) + 6\beta_6(x, z)y + 6\beta_7(y, z)x \\ + 6a_5z + 6\alpha_7(x)z + 6\alpha_9(y)z - 6\alpha_{12}(x)yz - 6\alpha_{13}(y)xz = 0. \end{aligned} \tag{5.33}$$

Setting $x = y = 0$ in (5.33) will give

$$\xi_2(z) = -3a_5z - \alpha_5(z). \tag{5.34}$$

Setting $y = 0$ in (5.33) and by (5.34), we get

$$\beta_4(x, z) = -3\alpha_7(x)z - 3\alpha_{10}(z)x. \tag{5.35}$$

Setting $x = 0$ in (5.33), by (5.34) and (5.35), we have

$$\beta_5(y, z) = -3\alpha_9(y)z - 3\alpha_{11}(z)y. \tag{5.36}$$

So, by (5.33), (5.34), (5.35) and (5.36), we obtain

$$\beta_6(x,z)y + \beta_7(y,z)x = \alpha_{12}(x)yz + \alpha_{13}(y)xz. \tag{5.37}$$

Setting $y = 1$ in (5.37) will give

$$\beta_6(x,z) = \alpha_{12}(x)z - \alpha_{14}(z)x, \tag{5.38}$$

where $\alpha_{14}(z) = \beta_7(1,z) - \alpha_{13}(1)z$ is an additive function for all $z \in \mathbb{R}$. By (5.37) and (5.38), we have

$$\beta_7(y,z)x = \alpha_{13}(y)xz - \alpha_{14}(z)xy. \tag{5.39}$$

Setting $x = 1$ in (5.39) to obtain

$$\beta_7(y,z) = \alpha_{13}(y)z - \alpha_{14}(z)y. \tag{5.40}$$

Thus, by (5.35), (5.36), (5.38) and (5.40), we get that equation (5.30) becomes

$$\begin{aligned} A^1(x,y,z) = \alpha_3(z) &+ \Big(\beta_2(x,z) + \beta_3(y,z)\Big) + \Big(\gamma(x,y,z) + \alpha_5(z)x^2 \\ &- 3\big(a_5 z + \alpha_5(z)\big) + \alpha_{10}(z)x^3 + \alpha_{11}(z)y^3 \\ &- 3\big(\alpha_7(x) + \alpha_{10}(z)x\big)y^2 - 3\big(\alpha_9(y) + \alpha_{11}(z)y\big)x^2 \\ &+ \big(\alpha_{12}(x)z - \alpha_{14}(z)x\big)y^3 + \big(\alpha_{13}(y)z + \alpha_{14}(z)y\big)x^3. \end{aligned} \tag{5.41}$$

Combining results from equations (5.4), (5.11), (5.12), (5.32) and (5.41), and writing it in the form

$$\begin{aligned} f(x,y,z) = a_0 &+ \alpha_1(x) + \alpha_2(y) + \alpha_3(z) + \beta_1(x,y) + \beta_2(x,z) + \beta_3(y,z) \\ &+ \gamma(x,y,z) + \big(a_6 - 3a_9 z + \alpha_5(z)\big)(x^2 - y^2) + \big(a_1 - a_6 \\ &- 3a_8 y + \alpha_4(y)\big)(x^2 - z^2) + \big(a_2 + a_6 - 3a_7 x + \alpha_3(x)\big)(y^2 - z^2) \\ &+ \big(a_7 + \alpha_{10}(z)\big)(x^3 - 3xy^2) + \big(a_3 - a_7 + \alpha_8(y)\big)(x^3 - 3xz^2) \\ &+ \big(a_8 + \alpha_{11}(z)\big)(y^3 - 3x^2y) + \big(a_4 - a_8 + \alpha_6(x)\big)(y^3 - 3yz^2) \\ &+ \big(a_9 + \alpha_9(y)\big)(z^3 - 3x^2z) + \big(a_5 - a_9 + \alpha_7(x)\big)(z^3 - 3y^2z) \\ &+ \alpha_{12}(x)(y^3z - yz^3) + \alpha_{13}(y)(x^3z - xz^3) + \alpha_{14}(z)(x^3y - xy^3), \end{aligned}$$

where $a_6, \ldots, a_9$ are arbitrary constants. Next, let $a = a_0, b_{xy} = a_6, b_{xz} = a_1 - a_6, b_{yz} = a_2 + a_6, c_{xy} = a_7, c_{xz} = a_3 - a_7, c_{yx} = a_8, c_{yz} = a_4 - a_8, c_{zx} = a_9$ and $c_{zy} = a_5 - a_9$ be constants, let $\alpha_{xy}, \alpha_{xz}, \alpha_{yz} : \mathbb{R} \to \mathbb{R}$ be functions defined by $\alpha_{xy}(z) = -3a_9 z + \alpha_5(z), \alpha_{xz}(y) = -3a_8 y + \alpha_4(y)$ and $\alpha_{yz}(x) = -3a_7 x + \alpha_3(x)$ for all $x, y, z \in \mathbb{R}$, and we define $\eta_{xy} \equiv \alpha_{10}, \eta_{xz} \equiv \alpha_8, \eta_{yx} \equiv \alpha_{11}, \eta_{yz} \equiv \alpha_6, \eta_{zx} \equiv \alpha_9, \eta_{zy} \equiv \alpha_7, \varphi_{xy} \equiv \alpha_{14}, \varphi_{xz} \equiv \alpha_{13}, \varphi_{yz} \equiv \alpha_{12}, \beta_{xy} \equiv \beta_1, \beta_{xz} \equiv \beta_2$ and $\beta_{yz} \equiv \beta_3$. Therefore, we conclude that the general solution of (5.1) is

$$\begin{aligned}
f(x,y,z) = {} & a + \alpha_x(x) + \alpha_y(y) + \alpha_z(z) + \beta_{xy}(x,y) + \beta_{xz}(x,z) + \beta_{yz}(y,z) \\
& + \gamma(x,y,z) + \big(b_{xy} + \alpha_{xy}(z)\big)(x^2 - y^2) + \big(b_{xz} + \alpha_{xz}(y)\big)(x^2 - z^2) \\
& + \big(b_{yz} + \alpha_{yz}(x)\big)(y^2 - z^2) + \big(c_{xy} + \eta_{xy}(z)\big)(x^3 - 3xy^2) \\
& + \big(c_{xz} + \eta_{xz}(y)\big)(x^3 - 3xz^2) + \big(c_{yx} + \eta_{yx}(z)\big)(y^3 - 3x^2 y) \\
& + \big(c_{yz} + \eta_{yz}(x)\big)(y^3 - 3yz^2) + \big(c_{zx} + \eta_{zx}(y)\big)(z^3 - 3x^2 z) \\
& + \big(c_{zy} + \eta_{zy}(x)\big)(z^3 - 3y^2 z) + \varphi_{xy}(z)(x^3 y - xy^3) \\
& + \varphi_{xz}(y)(x^3 z - xz^3) + \varphi_{yz}(z)(y^3 z - yz^3).
\end{aligned}$$

Conversely, suppose a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies equation (5.2). It is not hard to verify that functions f indeed satisfies equation (5.1). □

REFERENCES

[1] J. Aczél, H. Haruki, M.A. McKiernan, G. N. Sakovič. *General and regular solutions of functional equations characterizing harmonic polynomials*, Aequationes Math. 1 (1968), 37-53.

[2] J.A. Bake. *An analogue of the wave equation and certain related functional equations*, Canad. Math. Bull. 12 (1969), 837-846.

[3] S.Czerwik. *Functional Equations and Inequalities in Several variable*, World Scientific, 2001.

[4] D.P. Flemming. *Generalizations of the wave equation which allow weak and strong discontinuities*, SIAM J. Appl. Math. 23 (1972), 221-224.

[5] D. Girod. *On the functional equation* $\Delta_{T_1y}\Delta_{T_2y}f = 0$, Aequationes Math. 9 (1973), 157-164.

[6] H. Haruki. *On the functional equation* $f(x+t,y)+f(x-t,y) = f(x,y+t)+f(x,y-t)$, Aequationes Math. 5 (1970), 118-119.

[7] H. Haruki. *On the general solution of a nonsymmetric partial difference functional equation analogous to the wave equation*, Aequationes Math. 36 (1988), 20-31.

[8] Kucharzewski. *Some remarks on a difference-functional equation*, Aequationes Math. 4 (1970), 399-400.

[9] M. A. McKiernan. *The general solution of some finite difference equations analogous to the wave equation*, Aequatione Math. 8 (1972), 263-266.

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