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APPENDIX.

DIRECT SEARCH OF THE FREE-END TODA LATTICE

In this appendix we use a direct calculation method at order 6 in the velocities for the cases of the free-end Toda lattice.

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + e^{\epsilon(q_1-q_2)} + e^{q_2-q_3} . \qquad \text{(A1)}$$

We introduce the following change of variables:

$$x = \epsilon(q_1 - q_2),\ y = q_2 - q_3\ ,\ z = m_1q_1 + m_2q_2 + m_3q_3. \qquad \text{(A2)}$$

The equations of motion are derived from the Lagrange equations:

$$\frac{\partial}{\partial t}(m_1\dot q_1) = -\epsilon X\ ,\quad \frac{\partial}{\partial t}(m_2\dot q_2) = \epsilon X - Y\ ,$$

$$\frac{\partial}{\partial t}(m_3\dot q_3) = Y, \qquad \text{(A3)}$$

with $X = e^{\epsilon(q_1-q_2)} = e^{x}$, $Y = e^{q_2-q_3} = e^{y}$.

From Eq.(A2), it is obvious that

$$\ddot x = \epsilon(\ddot q_1 - \ddot q_2) = \frac{\epsilon}{m_2}\left[Y - \frac{\epsilon(m_1 + m_2)X}{m_1}\right] ,$$

$$\ddot y = \ddot q_2 - \ddot q_3 = \frac{\epsilon}{m_2}\left[X - \frac{(m_2 - m_3)Y}{\epsilon m_3}\right] ,$$

$$\ddot z = m_1\ddot q_1 + m_2\ddot q_2 + m_3\ddot q_3 = 0, \qquad \text{(A4)}$$

which, after a scaling in time, read

$$\ddot x = Y - \alpha X,\quad \ddot y = X - \beta Y, \qquad \text{(A5)}$$

with $\alpha = \dfrac{\epsilon(m_1 + m_2)}{m_1}$, $\beta = \dfrac{(m_2 + m_3)}{\epsilon m_3}$.

Let us consider the constant of motion of order 6 in the velocities.

$$
\begin{aligned}
C = {} & e_0\dot x^6 + e_1\dot x^5\dot y + e_2\dot x^4\dot y^2 + e_3\dot x^3\dot y^3 + e_4\dot x^2\dot y^4 + e_5\dot x\dot y^5 \\
& + e_6\dot y^6 + f_0\dot x^4 + f_1\dot x^3\dot y + f_2\dot x^2\dot y^2 + f_3\dot x\dot y^3 + f_4\dot y^4 + g_0\dot x^2 \\
& + g_1\dot x\dot y + g_2\dot y^2 + h.
\end{aligned}
$$

The condition of the constantcy of C can be written as

$$
\begin{aligned}
0 = dC = {} & 6e_0\dot x^5\ddot x + 5e_1\dot x^4\ddot x\dot y + e_1\dot x^5\ddot y + 4e_2\dot x^3\ddot x\dot y^2 + 2e_2\dot x^4\dot y\ddot y \\
& + 3e_3\dot x^2\ddot x\dot y^3 + 3e_3\dot x^3\dot y^2\ddot y + 2e_4\dot x\ddot x\dot y^4 + 4e_4\dot x^2\dot y^3 + e_5\ddot x\dot y^5 + 5e_5\dot x\dot y\ \ddot y \\
& + 6e_6\dot y^5\ddot y + \frac{\partial f_0}{\partial x}\dot x^5 + \frac{\partial f_0}{\partial y}\dot x^4\dot y + 4f_0\dot x^3\ddot x + \frac{\partial f_1}{\partial x}\dot x^4\dot y + \frac{\partial f_1}{\partial y}\dot x^3\dot y^2 \\
& + 3f_1\dot x^2\dot y\ddot x + f_1\dot x^3\ddot y + \frac{\partial f_2}{\partial x}\dot x^3\dot y^2 + \frac{\partial f_2}{\partial y}\dot x^2\dot y^3 + 2f_2\dot x\ddot x\dot y \\
& + 2f_2\dot x^2\dot y\ddot y + f_3\ddot x\dot y^3 + 3f_3\dot x\dot y^2\ddot y + \frac{\partial f_3}{\partial x}\dot x^2\dot y^3 + \frac{\partial f_3}{\partial y}\dot x\dot y^4 + \frac{\partial f_4}{\partial x}\dot x\dot y^4 \\
& + \frac{\partial f_4}{\partial y}\dot y^5 + 4f_4\dot y^3\ddot y + \frac{\partial g_0}{\partial x}\dot x^3 + \frac{\partial g_0}{\partial y}\dot x^2\dot y + 2g_0\dot x\ddot x + g_1\ddot x\dot y \\
& + g_1\dot x\ddot y + \frac{\partial g_1}{\partial x}\dot x^2\dot y + \frac{\partial g_1}{\partial y}\dot x\dot y^2 + \frac{\partial g_2}{\partial x}\dot x\dot y^2 + \frac{\partial g_2}{\partial y}\dot y^3 + 2g_2\dot y\ddot y + \frac{\partial h}{\partial x}\dot x \\
& + \frac{\partial h}{\partial y}\dot y\,.
\end{aligned}
\tag{A6}
$$

We can restrict ourselves to constant e_i's

Regrouping and equating to zero the coefficients of each monomial in the velocities, we obtain at order 5.

$$
\begin{aligned}
& 6e_0\ddot x + e_1\ddot y + \frac{\partial f_0}{\partial x} = 0, && 5e_1\ddot x + 2e_2\ddot y + \frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial x} = 0, \\
& 4e_2\ddot x + 3e_3\ddot y + \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0, && 3e_3\ddot x + 4e_4\ddot y + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial x} = 0, \\
& 2e_4\ddot x + 5e_5\ddot y + \frac{\partial f_3}{\partial y} + \frac{\partial f_4}{\partial x} = 0, && e_5\ddot x + 6e_6\ddot y + \frac{\partial f_4}{\partial y} = 0.
\end{aligned}
\tag{A7}
$$

The integrability condition for f_i reads

$$\frac{\partial^5}{\partial y^5}(6e_0\ddot{x} + e_1\ddot{y}) - \frac{\partial^5}{\partial x\partial y^4}(5e_1\ddot{x} + 2e_2\ddot{y}) + \frac{\partial^5}{\partial x^2\partial y^3}(4e_2\ddot{x} + 3e_3\ddot{y})$$
$$- \frac{\partial^5}{\partial x^3\partial y^2}(3e_3\ddot{x} + 4e_4\ddot{y}) + \frac{\partial^5}{\partial x^4\partial y}(2e_4\ddot{x} + 5e_5\ddot{y}) - \frac{\partial^5}{\partial x^5}(e_5\ddot{x} + 6e_6\ddot{y}) = 0. \tag{A8}$$

In the particular case where x and y are given by Eq.(A5), Eq.(A8) reduces to

$$\frac{\partial^5}{\partial y^5}(6e_0\ddot{x} + e_1\ddot{y}) - \frac{\partial^5}{\partial x^5}(e_5\ddot{x} + 6e_6\ddot{y}) = 0;$$

Thus
$$6e_0 = e_1\beta, \tag{A9.1}$$
$$\alpha e_5 = 6e_6. \tag{A9.2}$$

The integration for the f_i is

$$\begin{aligned}
f_0 &= e_1(\alpha\beta-1)X + (2e_2\beta - 5e_1)Y,\\
&\equiv A_0X + B_0Y,\\
f_1 &= (5e_1\alpha - 2e_2)X + (3e_3\beta - 4e_2)Y,\\
&\equiv A_1X + B_1Y,\\
f_2 &= (4e_2\alpha - 3e_3)X + (4e_4\beta - 3e_3)Y,\\
&\equiv A_2X + B_2Y,\\
f_3 &= (3e_3\alpha - 4e_4)X - 2e_4Y,\\
&\equiv A_3X + B_3Y,\\
f_4 &= 2e_4\alpha X + (6e_6\beta - e_5)Y,\\
&\equiv A_4X + B_4Y.
\end{aligned} \tag{A10}$$

At third order we obtain

$$\begin{aligned}
&4f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0,\\
&3f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0,\\
&2f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} = 0,\\
&f_3\ddot{x} + 4f_4\ddot{y} + \frac{\partial g_2}{\partial y} = 0.
\end{aligned} \tag{A11}$$

The integrability condition for g_i reads

$$- \frac{\partial^3}{\partial x^3}(f_3\ddot{x} + 4f_4\ddot{y}) + \frac{\partial^3}{\partial x^2\partial y}(2f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^3}{\partial x\partial y^2}(3f_1\ddot{x} + 2f_2\ddot{y})$$
$$+ \frac{\partial^3}{\partial y^3}(4f_0\ddot{x} + f_1\ddot{y}) = 0. \qquad \text{(A12)}$$

Eq.(A12) is an identity in terms of the independent functions X^2, XY, Y^2, in this case. We obtain a system in terms of A_i, B_i, α, β:

$$4B_0 - \beta B_1 = 0, \qquad \text{(A13.1)}$$

$$\alpha A_3 - 4A_4 = 0, \qquad \text{(A13.2)}$$

$$(\beta A_1 - B_1) + 2(\alpha B_2 - A_2) + 3(\beta A_3 - B_3) + 4(\alpha B_0 - A_0)$$
$$= (\alpha B_3 - A_3) + 2(\beta A_2 - B_2) + 3(\alpha B_1 - A_1) + 4(\beta A_4 - B_4). \qquad \text{(A13.3)}$$

It is possible to calculate g :

$$g_0 = \tfrac{1}{2}(4A_0\alpha - A_1)X^2 + [4(B_0\alpha - A_0) + (A_1\beta - B_1)]XY$$
$$+ \tfrac{1}{2}[2\beta B_2 - 3B_1]Y^2,$$
$$\equiv C_0X^2 + D_0XY + E_0Y^2,$$

$$g_1 = \tfrac{1}{2}(3\alpha A_1 - 2A_2)X^2 + [3(\alpha B_1 - A_1) + 2(\beta A_2 - B_2)$$
$$+ 4(A_0 - \alpha B_0) - (A_1\beta - B_1)]XY + (3\beta B_3 - 2B_2)Y^2,$$
$$\equiv C_1X^2 + D_1XY + E_1Y^2,$$

$$g_2 = (\alpha A_2 - \tfrac{3}{2}A_3)X^2 + (4\beta A_4 + \alpha B_3 - A_3 - 4B_4)XY$$
$$+ (2\beta B_4 - 4B_4)Y^2,$$
$$\equiv C_2X^2 + D_2XY + E_2Y^2. \qquad \text{(A14)}$$

At first order we obtain

$$2g_0\ddot{x} + g_1\ddot{y} + \frac{\partial h}{\partial x} = 0,$$
$$g_1\ddot{x} + 2g_2\ddot{y} + \frac{\partial h}{\partial y} = 0, \qquad (A15)$$

The compatibility condition for the last equation reads

$$\frac{\partial}{\partial y}(2g_0\ddot{x} + g_1\ddot{y}) = \frac{\partial}{\partial x}(g_1\ddot{x} + 2g_2\ddot{y}). \qquad (A16)$$

Eq.(A16) is an identity in terms of the independent functions X^3, Y^3, X^2Y, XY^2, in this case .We obtain a system in terms of C_i, D_i, α, β :

$$4(D_0 - \alpha E_0) - 2(\beta D_1 - E_1) - (D_1 - \alpha E_1) - 2(E_2 - \beta D_2) = 0, \qquad (A17.1)$$
$$2(C_0 - \alpha D_0) + (D_1 - \beta C_1) + 2(\alpha D_1 - C_1) + 4(\beta C_2 - D_2) = 0, \qquad (A17.2)$$
$$2E_0 - \beta E_1 = 0, \qquad (A17.3)$$
$$2C_2 - \alpha C_1 = 0. \qquad (A17.4)$$

Eqs.(A13.1),(A13.2),(A17.3) and (A17.4) reduced to the equation in terms α and β :

$$(3\alpha\beta - 4)(-3\alpha\beta + 6)(\alpha\beta - 4) = 0.$$

a) $\alpha = 2$, $\beta = 2$: This integral is the product of the constant of degree 3 in the velocities.

b) $\alpha = 2$, $\beta = 1$: This integral is the product of the constant of degree 4 in the velocities.

c) $\alpha = \frac{2}{3}$, $\beta = 2$: This corresponds to

$$m_1 = 3\epsilon\frac{(2\epsilon-1)}{2-3\epsilon}, \quad m_2 = 2\epsilon - 1, \quad m_3 = 1, \ 1 < \epsilon < 2.$$

Substitute Eqs.(A13.1),(A13.2),(A17.3) and $\alpha = \frac{2}{3}$, $\beta = 2$ into Eq.(A13.3),which gives

$$-e_5 + 12e_6 = 0, \tag{A18}$$

But Eq.(A9) is $4e_5 - 36e_6 = 0$.

This means that $e_5 = 0$, $e_6 = 0$.

For case $\alpha = \frac{2}{3}$, $\beta = 2$, we find the functions f_i 's , g_i's and h:

$$f_0 = 4X - 8Y, \quad f_1 = 14X - 16Y, \quad f_2 = \frac{50}{3}X - 10Y,$$

$$f_3 = 8X - 2Y, \quad f_4 = \frac{4}{3}X,$$

$$g_0 = -\frac{5}{3}X^2 + \frac{20}{3}XY + 4Y^2,$$

$$g_1 = -\frac{8}{3}X^2 + 6XY + 4Y^2,$$

$$g_2 = -\frac{8}{9}X^2 + \frac{4}{3}XY + Y^2.$$

$$h = \frac{4}{9}X^2 Y + \frac{4}{27}X^3. \qquad (e_4 = 1)$$

The constant C is given by

$$\begin{aligned} C = {} & 4\dot{x}^6 + 12\dot{x}^5\dot{y} + 13\dot{x}^4\dot{y}^2 + 6\dot{x}^3\dot{y}^3 + \dot{x}^2\dot{y}^4 + 4(e^x - 2e^y)\dot{x}^4 \\ & + (14e^x - 16e^y)\dot{x}^3\dot{y} + 10(\tfrac{5}{3}e^x - e^y)\dot{x}^2\dot{y}^2 + 2(4e^x - e^y)\dot{x}\dot{y}^3 \\ & + \tfrac{4}{3}e^x\dot{y}^4 + (-\tfrac{5}{3}e^{2x} + \tfrac{20}{3}e^{x+y} + 4e^{2y})\dot{y}^2 + \tfrac{4}{27}e^{3x} + \tfrac{4}{9}e^{2x+y} \\ & + (-\tfrac{8}{3}e^{2x} + 6e^{x+y} + 4e^{2y})\dot{x}\dot{y} + (-\tfrac{8}{9}e^{2x} + \tfrac{4}{3}e^{x+y} + e^{2y})\dot{y}^2. \end{aligned}$$

CIRRICULUM VITAE

Miss Kanitta Chantanachai was born on April 5, 1966 in Bangkok. She received her B.Sc. degree in Physics from Mahidol University in 1986. Now she studies for M.Sc. degree in Physics at Chukalongkorn University.