CHAPTER V

CONCLUSION AND OUTLOOK

In the domain of integrability we have always followed a two steps procedure: integrability prognostication through the ARS algorithm, and integrability verification by direct construction of integrals of moion.

In the previous chapters we have reviewed the results of the two methods in the cases of the Toda system and the Henon-Heiles hamiltonian.

1) The Henon-Heiles Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + ax^2 + by^2) + dx^2y - \frac{1}{3}ey^3.$$

There are three cases of integrability:

a) $e = -d, \quad a = b,$

b) $e = -6d, \quad a \neq b,$

c) $e = -16d, \; b = 16a.$

Using the ARS method, cases a) and b) have been given by Bountis et al.[14] and Chang et al.[16] have found the integrable case: case c). We present a direct calculation method of the two cases a) and c). For the case b), Green has calculated the integral of motion by direct calculation method.

2) The Toda system

We have studied a lattice with nearest-neighbor exponential interactions for different boundary conditions:

a) The free-end lattice with three masses:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2} + e^{\epsilon(q_1-q_2)} + e^{q_2-q_3}.$$

There are three cases of integrability:

a) $m_1 = \frac{\epsilon(2\epsilon-1)}{2-\epsilon}$, $m_2 = 2\epsilon-1$, $\frac{1}{2} < \epsilon < 2$,

b) $m_1 = \frac{\epsilon(\epsilon-1)}{2-\epsilon}$, $m_2 = \epsilon-1$, $1 < \epsilon < 2$,

c) $m_1 = \frac{3\epsilon(2\epsilon-1)}{2-3\epsilon}$, $m_2 = 2\epsilon-1$, $1 < \epsilon < 2$.

Using the ARS method, Bountis et al.[14] have found three cases for which the system satisfy sufficient condition for possessing the Painlevé property. Dorizzi et al.[17] presented a direct calculation of the constants of motion for all the three cases above.

b) The fixed-end lattice with two masses:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + e^{-\delta q_1} + e^{\epsilon(q_1-q_2)} + e^{q_2}.$$

There are five cases of integrability

a) $m_1/m_2 = 1$, $\delta = \epsilon = 1$,

b) $m_1/m_2 = 1$, $\delta = 1$, $\epsilon = 1/2$,

c) $m_1/m_2 = 1/3$, $\delta = 1$, $\epsilon = 1/2$,

d) $m_1/m_2 = 1/3$, $\delta = 1/3$, $\epsilon = 1/2$,

e) $m_1/m_2 = 1$, $\delta = 1/2$, $\epsilon = 1/2$.

The integrable cases have been shown by Ramani [18] and Dorizzi et al. [17]. The first has found the integrable cases by the ARS method and the second has calculated the integrals of motion by the direct calculation method.

As yet there are no general methods to check for integrability of dynamical system. The ARS method, although gives only sufficient condition, provides a convenient way for studying the question of integrability. Supplemented by direct calculation of the constants of motion using various methods we find an effective tool for the investigation of integrability of a dynamical system.

The drawbacks of the direct method are obvious. It cannot be applied to find nonpolynomial integrals of motion. Moreover, in the polynomial case, the method becomes cumbersome for polynomials of order higher than 2 in the momenta. More significantly, there is no guarantee that we will be able to solve the differential equations involved in the construction of the coefficients in the polynomial.

Several works have been devoted to the study of dynamical systems using the Painlevé property. New integrable systems have been discovered by direct construction of integral of motion [19] and confirmed the usefulness of the Painlevé criterion. But Grammaticos, Dorizzi, and Ramani [4] have discovered that some systems are integrable and yet they do not possess the full Painlevé property. These systems exhibit a simple singularity expansion in powers of $(t-t_o)^{1/r}$, with r an integer. This is the weak Painlevé concept . The weak Painlevé case has been examined in detail in ref.[20,21], in connection with the existence of fixed singularities.

One of the drawbacks of the ARS approach is the fact that one must first reduced the partial differential equation before applying the algorithm. Weiss, Tabor, and CarnevaLe [22] have done away with this problem by introducing the Painlevé property for the partial differential equations themselves. According to Weiss et al. [21], a partial differential equation will possess the Painlevé property if its solutions are single valued about a movable singularity manifold. The application of the Painlevé method to partial differential equations has been examined in detail in ref [12].