CHAPTER II

THE PAINLEVÉ PROPERTY AND DIRECT CALCULATION METHOD

In this chapter, we review a use of the two distinct approaches : ARS method and direct search for the integrals of motion in order to identify the integrability of dynamical systems.

ARS METHOD.

The ARS conjecture was formulated by Ablowitz, Ramani, and Segur (ARS) [7,12] who related integrability to the singularity structure through the Painlevé property of the solutions, i.e., that the only movable singularities on the complex time plane they can have are poles. The ARS algorithm was developed in order to determine whether a nonlinear ODE (or system of ODE's) admits movable branch points.

Let us consider first the case of the nth order ODE

$$\frac{d^n w}{dz^n} = F\left(z,w,w',\ldots,\frac{d^{n-1}w}{dz^{n-1}}\right), \quad j=1,\ldots,n, \qquad (2.1)$$

and assume that the function becomes infinite at the singularity.

The ARS algorithm proceeds in three steps:

A. Finding the Dominant Behavior.

Let us look for a solution of Eq.(2.1) of the form

$$w \sim \alpha(z-z_0)^p , \qquad (2.2)$$

where $Re(p) < 0$ and z_0 is arbitrary. Substituting Eq.(2.2) into Eq.(2.1), one finds all possible values of p. Two or more terms in the equation may balance (depending on α), and the rest can be ignored as $z \to z_0$. For each such choice of p, the terms which can balance are called the leading terms. Requiring that the leading terms do balance determines α.

Example: $w''' + ww'' - 2w^3 + \lambda w^2 + \mu w = 0.$

There are two possible choices:

i) $p = -1$, $\alpha = 3$, the leading terms are w''' and ww'',

ii) $p = -2$, $\alpha = 3$, the leading terms are ww'' and $-2w^3$.

If any of the possible p's is not an integer and if Eq.(2.2) is asymptotic near z_0, then it represents the dominant behavior in the neighborhood of a movable branch point of order p. This means that the equation is not of P-type: here P stands for Painlevé. (The equation is of P-type if all of its solutions have Painlevé property.)

If all possible p's are integer, then for each p, Eq.(2.2) may represent the first term in the Laurent series around a movable pole. In this case, a solution of Eq.(2.1) is

$$w(z) = (z-z_0)^p \sum_{j=0}^{\infty} a_j (z-z_0)^j , \quad 0 < |z-z_0| < R, \qquad (2.3)$$

R is a positive number. z_0 is an arbitrary constant. For an nth order ODE there are still (n-1) arbitrary constants to be sought among the a_j in Eq.(2.3). If they are all found to be present there, Eq.(2.3) will be the general solution. The powers j at which these arbitrary constants enter are called resonances.

B. Finding the Resonances.

We start by keeping only the leading terms in the original equation. Substituting

$$w = \alpha(z-z_0)^p + \beta(z-z_0)^{p+r} , \qquad (2.4)$$

into the leading equation. To leading order in β, this equation reduces to

$$Q(r)\beta(z-z_0)^q = 0, \qquad q \geqslant p + r - n.$$

If the highest derivative of the original equation is a leading term, $q = p + r - n$, and $Q(r)$ is a polynomial of order n. If not, $q > p + r - n$, and the order of the polynomial $Q(r)$ equals the order of highest derivative among the leading terms ($<n$).

The roots of $Q(r)$ determine the resonances.

i) one root is always (-1). It represents the arbitrariness of z_0.

ii) The resonance $r = 0$ corresponds to the coefficient of one of the leading terms being arbitrary.

iii) Any resonance with $Re(r) < 0$ (except $r = -1$) must be ignored, because they violate the hypothesis that $(z-z_0)^p$ is the dominant term in the expansion near z_0.

iv) Any resonance with $Re(r) > 0$ but r not a real integer, indicates a movable branch point at $z = z_0$. The algorithm terminates at this stage.

v) If for every possible (p,α) from step A., all of the roots of $Q(r)$ (except -1 and possibly 0) are positive real integers, then there are no algebraic branch points. Proceed to step C. to check for logarithmic branch points.

vi) The general solution of the nth order ODE in the neighborhood of a movable pole, $Q(r)$, must have $(n-1)$ nonnegative distinct roots, all real integers. If for every (p,α) from step A., $Q(r)$ has fewer than $(n-1)$ roots, then none of the local solutions is general. This means that Eq.(2.2) misses an essential part of the solution.

Example: $w'' + 4ww' + 2w^3 = 0$ (*)

The only dominant behavior is $w \sim \alpha(z-z_0)^{-1}$,

where α satisfies $\alpha^2 - 2\alpha + 1 = 0$.

The resonances are $r = -1$, $r = 0$.

The root (-1) corresponds to the arbitrariness of z_0.

The root (0) does not correspond to the arbitrariness of α, because $\alpha = 1$ is a double root of the leading equation. So, $w = \alpha(z-z_0)^{-1}$ cannot be the first term of a general solution of Eq.(*).

C. Finding the constants of integration

For a given (p,α) from step A., let $r_1 \leqslant r_2 \leqslant, \ldots, \leqslant r_s$ denote the positive integer roots of $Q(r)$. Substituting

$$w = \alpha(z-z_0)^p + \sum_{j=1}^{r_s} a_j (z-z_0)^{p+j}, \quad (s \leqslant n-1), \tag{2.5}$$

into the full equation (2.1)

The coefficient of $(z-z_0)^{p+j-n}$ is

$$Q(j)a_j - R_j(z_0,\alpha,a_1,\ldots,a_{j-1}) = 0, \quad R \equiv (R_1,\ldots,R_n)^T. \tag{2.6}$$

Then:

i) For $j < r_1$, Eq.(2.6) determines a_j

ii) For $j = r_1$, Eq.(2.6) becomes

$$0 \cdot a_{r_1} - R_{r_1}(z_0,\alpha,a_1,\ldots,a_{r_1-1}) = 0.$$

$$\text{If} \quad R_{r_1}(z_0,\alpha,a_1,\ldots,a_{r_1-1}) \neq 0, \tag{2.7}$$

then Eq.(2.6) cannot be satisfied. There is no solution of the Eq.(2.5), and we must introduce logarithm terms into the expansion. Replacing Eq.(2.5) with

$$w = \alpha(z-z_0)^p + \sum_{j=1}^{r-1} a_j(z-z_0)^{p+j} + [a_{r_1} + b_{r_1}\ln(z-z_0)](z-z_0)^{p+r_1} + \ldots$$

The coefficient of $[(z-z_0)^{p+r_1-n}\ln(z-z_0)]$ is $Q(r_1)b_{r_1} = 0$.

iii) If Eq.(2.7) is false ($R_{r_1} = 0$) then a_{r_1} is an arbitrary constant of integration. Proceed to the next coefficient.

For a system of first order ODE's, the basic steps of the algorithm are not essentially different.

The nth order ODE has the form

$$\frac{dw}{dz} = F_j(z;w_1,w_2,\ldots,w_n) \quad j = 1,\ldots,n. \tag{2.8}$$

1. Finding the dominant behavior of the system.

Substituting $w_j \sim \alpha_j(z-z_0)^{p_j}$, $j = 1,\ldots,n$, (2.9)

into Eq.(2.8) and one finds all possible p_j for which there is a balance of leading terms and what the leading terms are. The algorithm stops unless the only possible p_j's are integers.

2. Finding the resonances of the system.

For each p, construct a simplified equation from Eq.(2.8) that retains only the leading terms.

Substituting into the simplified equation

$$w_j = \alpha_j (z-z_o)^{p_j} + \beta_j (z-z_o)^{p_j+r}, \quad j = 1,\dots,n, \tag{2.10}$$

with the same r for every w_j. To leading order in β, this becomes

$$[Q(r)]\beta = 0,$$

where [Q] is an nxn matrix, whose elements depend on r. The resonances are the nonnegative roots of

$$\det[Q(r)] = 0, \quad \text{a polynomial of order} \leqslant n.$$

One root is always (-1), (0) may also be a root. The algorithm stops unless all of the resonances are integers.

3. Finding the constants of integration.

Substituting into Eq.(2.8)

$$w_j \sim \alpha_j (z-z_o)^{p_j} + \sum_{k=1}^{r_s} a_{j_k} (z-z_o)^{p_j+k}, \tag{2.11}$$

where r_s is the largest resonance. The coefficient of each power of $(z-z_o)$ has the form of a matrix generalization of Eq.(2.6). Its treatment is identical to the previous case.

In summary, we say that the ODE (or a system of ODEs) satisfies the necessary condition for the Painlevé property (i.e. for having no movable critical points other than poles), if its solutions can be expanded in Laurent series, near movable singularities at $z-z_0$. The ARS algorithm provides only an indication that the system actually does posses the Painlevé property.

Direct calculation for integrals of motion

In this section we review a direct calculation method for the investigation of the existence of integrals of motion polynomial in the velocities, following Bertrand's approach [2]. This method makes the assumption that the constants of motion are polynomials in the velocities (or momenta). This method is powerful in the simple cases.

Let us consider the motion of a particle in a two dimensional potential V(x,y). The Hamiltonian governing the system reads

$$H = \frac{1}{2}(m_1\dot{x}^2 + m_2\dot{y}^2) + V(x,y). \qquad (2.12)$$

The equations of motion associated with the system are simply

$$\ddot{x} = -\frac{\partial V}{\partial x} \equiv V_x \quad , \quad \ddot{y} = -\frac{\partial V}{\partial y} \equiv V_y. \qquad (2.13)$$

For the complete integrability of the system, one needs the existence of a second constant of motion, besides the Hamiltonian.

Let us look for the case of a constant of motion quadratic in the velocities. The general form of an integral is

$$C = g_o\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h \quad , \tag{2.14}$$

The conditions of the constancy of C can be written as

$$0 \equiv \frac{dC}{dt} = g_{ox}\dot{x}^2 + g_{oy}\dot{x}^2\dot{y} + g_{1x}\dot{x}^2\dot{y} + g_{1y}\dot{x}\dot{y}^2 + g_{2x}\dot{x}\dot{y}^2 + g_{2y}\dot{y}^3 + 2g_o\ddot{x}\dot{x} + g_1\ddot{y}\dot{x} + g_1\dot{y}\ddot{x} + 2g_2\ddot{y}\dot{y} + h_x\dot{x} + h_y\dot{y}. \tag{2.15}$$

Regrouping and equating to zero the coefficients of each monomial in the velocities, we obtain at order three

$$g_{ox} = 0 \ , \qquad g_{oy} + g_{1x} = 0 \ ,$$
$$g_{1y} + g_{2x} = 0 \ , \quad g_{2y} = 0. \tag{2.16}$$

The solution of this system of equations is

$$g_o = \alpha y^2 + \beta y + \gamma \ ,$$
$$g_1 = -2\alpha xy - \beta x + \delta y + k \ , \tag{2.17}$$
$$g_2 = \alpha x^2 - \delta x + \zeta .$$

At first order we obtain

$$h_x + 2g_o\ddot{x} + g_1\ddot{y} = 0 \ ,$$
$$h_y + g_1\ddot{x} + 2g_2\ddot{y} = 0 \ . \tag{2.18}$$

The integrability condition for h reads

$$\frac{\partial}{\partial y}[2g_oV_x + g_1V_y] = \frac{\partial}{\partial x}[g_1V_x + 2g_2V_y], \tag{2.19}$$

or, equivalently,

$$2(g_0 - g_2)V_{xy} + (2g_{0y} - g_{1x})V_x - (2g_{2x} - g_{1y})V_y$$
$$-g_1(V_{xx} - V_{yy}) = 0. \qquad (2.20)$$

For the system to possess an integral of motion quadratic in velocities, the potential must satisfy Eqs. (2.18) and (2.19).

Let us look for the case of a constant of motion cubic in velocities of the form

$$C = f_0\dot{x}^3 + f_1\dot{x}^2\dot{y} + f_2\dot{x}\dot{y}^2 + f_3\dot{y}^3 + g_0\dot{x} + g_1\dot{y}. \qquad (2.21)$$

The condition $\frac{dC}{dt} = 0$ leads to a system of partial differential equations obtained by equating to zero the coefficients of each monomial $\dot{x}^m\dot{y}^n$. We obtain

$$\frac{\partial f_0}{\partial x} = 0 \quad , \quad \frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial x} = 0 \quad , \quad \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0 \quad ,$$
$$\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial x} = 0 \quad , \quad \frac{\partial f_3}{\partial y} = 0. \qquad (2.22)$$

The first set of PDEs for f_i can be solved in a straight-forward way giving:

$$f_0 = \alpha y^3 + \beta y^2 + \gamma y + \delta,$$
$$f_1 = -(3\alpha y^2 + 2\beta y + \gamma)x + \epsilon y^2 + \xi y + \eta,$$
$$f_2 = (3\alpha y + \beta)x - (2\epsilon y + \xi)x + \theta y + \kappa, \qquad (2.23)$$
$$f_3 = -\alpha x^3 + \epsilon x^2 - \theta x + \lambda.$$

The next set of equations read

$$3f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0,$$

$$2f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0, \qquad (2.24)$$

$$f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} = 0.$$

One replaces f_i from Eq.(2.23) and $\ddot{x}$, $\ddot{y}$ from the equations of motion, $\ddot{x} = -\partial V/\partial x$, $\ddot{y} = -\partial V/\partial y$, and integrates for g_i. The compatibility condition for the integration reads

$$\frac{\partial^2}{\partial x^2}(f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^2}{\partial x\partial y}(2f_1\ddot{x} + 2f_2\ddot{y}) + \frac{\partial^2}{\partial y^2}(3f_0\ddot{x} + f_1\ddot{y}) = 0. \qquad (2.25)$$

The last equation reads:

$$g_0\ddot{x} + g_1\ddot{y} = 0. \qquad (2.26)$$

For the system to possess an integral of motion cubic in velocities, the potential must satisfy Eqs.(2.25) and(2.26).

Let us consider the case of a constant of order 4 in the velocities. The form of a fourth-order constant is

$$C = f_0\dot{x}^4 + f_1\dot{x}^3\dot{y} + f_2\dot{x}^2\dot{y}^2 + f_3\dot{x}\dot{y}^3 + f_4\dot{y}^4 + g_0\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h. \qquad (2.27)$$

The first set of partial differential equations for f_i can be solved in a straightforward way giving

$$\begin{aligned} f_0 &= Ay^3 + By^2 + Cy + D, \\ f_1 &= -3Axy^2 - 2Bxy - Cx + Ey^2 + Fy + G, \\ f_2 &= x^2(3Ay + B) - x(2Ey + F) + Hy + I, \\ f_3 &= -Ax^3 + Ex^2 - Hx + J, \\ f_4 &= K. \end{aligned} \qquad (2.28)$$

The PDE's for g_i read

$$\begin{aligned} &4f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0, \\ &3f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0, \\ &2f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} = 0, \\ &f_3\ddot{x} + 4f_4\ddot{y} + \frac{\partial g_2}{\partial y} = 0. \end{aligned} \qquad (2.29)$$

The compatibility condition for the integration reads

$$- \frac{\partial^3}{\partial x^3}(f_3 \ddot{x} + 4f_4 \ddot{y}) + \frac{\partial^3}{\partial x^2 \partial y}(2f_2 \ddot{x} + 3f_3 \ddot{y}) - \frac{\partial^3}{\partial x \partial y^2}(3f_1 \ddot{x} + 2f_2 \ddot{y})$$
$$+ \frac{\partial^3}{\partial y^3}(4f_0 \ddot{x} + f_1 \ddot{y}) = 0. \qquad (2.30)$$

We obtain the equations for h:

$$2g_0 \ddot{x} + g_1 \ddot{y} + \frac{\partial h}{\partial x} = 0,$$
$$g_1 \ddot{x} + 2\, g_2 \ddot{y} + \frac{\partial h}{\partial y} = 0. \qquad (2.31)$$

The compatibility condition for the last equation reads:

$$\frac{\partial}{\partial y}(2g_0 \ddot{x} + g_1 \ddot{y}) = \frac{\partial}{\partial x}(g_1 \ddot{x} + 2g_2 \ddot{y}). \qquad (2.32)$$

An integral of motion quartic in velocities exits whenever the potential satisfies the Eqs. (2.30) and (2.32).

Let us now consider the case of a constant of order 5 in the velocities. The form of a fifth-order constant is

$$C = e_0 \dot{x}^5 + e_1 \dot{x}^4 \dot{y} + e_2 \dot{x}^3 \dot{y}^2 + e_3 \dot{x}^2 \dot{y}^3 + e_4 \dot{x} \dot{y}^4 + e_5 \dot{y}^5 + f_0 \dot{x}^3$$
$$+ f_1 \dot{x}^2 \dot{y} + f_2 \dot{x} \dot{y}^2 + f_3 \dot{y}^3 + g_1 \dot{x} + g_2 \dot{y}. \qquad (2.33)$$

By equating to zero the coefficients of order 4 in $dC/dt = 0$, we obtain a system of partial differential equations for f_i , which leads to the new compatibility condition (2.34). As soon as this last condition is satisfied, one can calculate the functions f_i. The problem is then reduced to the search of the g_i's from relations that read the same as in the case of constants of order 3 in the velocities.

The compatibility condition gives

$$\frac{\partial^4}{\partial y^4}(5e_0\ddot{x} + e_1\ddot{y}) - \frac{\partial^4}{\partial x\partial y^3}(4e_1\ddot{x} + 2e_2\ddot{y}) + \frac{\partial^4}{\partial x^2\partial y^2}(3e_2\ddot{x} + 3e_3\ddot{y})$$
$$- \frac{\partial^4}{\partial x^3\partial y}(2e_3\ddot{x} + 4e_4\ddot{y}) + \frac{\partial^4}{\partial x^4}(e_4\ddot{x} + 5e_5\ddot{y}) = 0. \qquad (2.34)$$

The PDE's for g_i read

$$3f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_1}{\partial x} = 0,$$
$$2f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} = 0,$$
$$f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_2}{\partial y} = 0. \qquad (2.35)$$

and the last compatibility condition gives

$$g_1\ddot{x} + g_2\ddot{y} = 0. \qquad (2.36)$$

Let us now consider the case of a constant of order 6 in the velocities. The computations are similar but more complicated.

The form of a sixth-order constant is

$$C = e_0\dot{x}^6 + e_1\dot{x}^5\dot{y} + e_2\dot{x}^4\dot{y}^2 + e_3\dot{x}^3\dot{y}^3 + e_4\dot{x}^2\dot{y}^4 + e_5\dot{x}\dot{y}^5$$
$$+ e_6\dot{y}^6 + f_0\dot{x}^4 + f_1\dot{x}^3\dot{y} + f_2\dot{x}^2\dot{y}^2 + f_3\dot{x}\dot{y}^3 + f_4\dot{y}^4 + g_0\dot{x}^2$$
$$+ g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h.$$

The condition $dC/dt = 0$ leads to a system of partial differential equations obtained by equating to zeo the coefficients of each $\dot{x}^m\dot{y}^n$. We obtain at order 5

$$6e_0\ddot{x} + e_1\ddot{y} + \frac{\partial f}{\partial x} = 0, \qquad 5e_1\ddot{x} + 2e_2\ddot{y} + \frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial x} = 0,$$
$$4e_2\ddot{x} + 3e_3\ddot{y} + \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0, \qquad 3e_3\ddot{x} + 4e_4\ddot{y} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial x} = 0,$$
$$2e_4\ddot{x} + 5e_5\ddot{y} + \frac{\partial f_3}{\partial y} + \frac{\partial f_4}{\partial x} = 0, \qquad e_5\ddot{x} + 6e_6\ddot{y} + \frac{\partial f_4}{\partial y} = 0. \qquad (2.37)$$

The compatibility condition gives

$$\frac{\partial^5}{\partial y^5}(6e_0\ddot{x} + e_1\ddot{y}) - \frac{\partial^5}{\partial x\partial y^4}(5e_1\ddot{x} + 2e_2\ddot{y}) + \frac{\partial^5}{\partial x^2\partial y^3}(4e_2\ddot{x} + 3e_3\ddot{y})$$
$$- \frac{\partial^5}{\partial x^3\partial y^2}(3e_3\ddot{x} + 4e_4\ddot{y}) + \frac{\partial^5}{\partial x^4\partial y}(2e_4\ddot{x} + 5e_5\ddot{y}) - \frac{\partial^5}{\partial x^5}(e_5\ddot{x} + 6e_6\ddot{y}) = 0. \tag{2.38}$$

The problem is then reduced to the search of g_i's and h from relations that read the same as in the case of constants of order 4 in the velocities.

The next set of equations reads:

$$\begin{aligned}
&4f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0,\\
&3f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0,\\
&2f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} = 0,\\
&f_3\ddot{x} + 4f_4\ddot{y} + \frac{\partial g_2}{\partial y} = 0.
\end{aligned} \tag{2.39}$$

The compatibility condition for the integration reads

$$- \frac{\partial^3}{\partial x^3}(f_3\ddot{x} + 4f_4\ddot{y}) + \frac{\partial^3}{\partial x^2\partial y}(2f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^3}{\partial x\partial y^2}(3f_1\ddot{x} + 2f_2\ddot{y})$$
$$+ \frac{\partial^3}{\partial y^3}(4f_0\ddot{x} + f_1\ddot{y}) = 0. \tag{2.40}$$

At first order we obtain

$$\begin{aligned}
&2g_0\ddot{x} + g_1\ddot{y} + \frac{\partial h}{\partial x} = 0,\\
&g_1\ddot{x} + 2g_2\ddot{y} + \frac{\partial h}{\partial y} = 0,
\end{aligned} \tag{2.41}$$

The compatibility condition for the last equation reads

$$\frac{\partial}{\partial y}(2g_0\ddot{x} + g_1\ddot{y}) = \frac{\partial}{\partial x}(g_1\ddot{x} + 2g_2\ddot{y}). \tag{2.42}$$

The constant of order 6 in velocities exists whenever the potential satisfies the PDE's Eq.(2.38), Eq.(2.40) and Eq.(2.42). The search is facilitated by the fact that in nontrivial known cases of integrable potentials the f_i are just constants, independent of x and y.