Chapter 5
Excitation of Vortex State

In this chapter we study the response of the vortex state in Bose-Einstein condensation to the excitation made by changing the trap frequency by using the solution of path integral. Firstly, we briefly discuss the physics of this system.

When a quantum system is rotating, particles in the system move in a circle with the circumference having the value of an integer multiple of the de Broglie wavelength. This quantization rule leads to the Bohr model and the discrete energy levels of the hydrogen atom. For a rotating superfluid, it leads to quantized vortices. If we rotate a normal liquid in a bucket, the fluid will finally rotate as if it were a rigid body where the velocity of the particles in the fluid smoothly increases from the center to the edge. But this smooth variation is impossible for particles in a single quantum state. To fulfill the above mentioned quantization rule, the flow field must have some regions where the number of de Broglie wavelengths on a closed path increase by one when going from one path to another. One possibility is a radially symmetric flow field with concentric rings. Between adjacent rings, the number of de Broglie wavelengths on a circumference changes by one.

The vortices in a Bose-Einstein condensate are quantized: when an atom goes around the vortex core, its quantum-mechanical phase changes by exactly 2π. Such quantized vortices play a key role in superfluidity and superconductivity. In superconductors, magnetic flux lines arrange themselves in regular lattices that have been directly imaged. The vortex in rotating gaseous Bose-Einstein condensate was created by a group leaded by Dalibards [27] and Ketterle's group

[28]; they observed the formation of vortex lattices in a rotating Bose-condensed gas produced by spinning laser beams around the condensate, setting it into rotation. Before that the creation of the single vortex was done by Cornell-Wieman group [29]. This is due to the fact that the creation of the vortex, it is not easy to place the vortex core at the center of the condensate. Generally the vortex created is off center. By studying an off-center vortex, they found that the core of the single vortex is not stable but it moves around the center of the condensate. Tempere and Devreese [30] numerically simulated this phenomenon by using the path integral technique with the harmonic model mentioned in Chapter 2 and found the precession of the vortex. The precession is due to the Magnus force which originates from the pressure difference between the inner and the outer sides of the vortex.

In this chapter, we consider the response of the off-center vortex in the non-interacting Bose condensate when the frequency of the trap is changed abruptly. As we learn from Chapter 4 that our time-dependent density matrix has no semi-group property so it is difficult to use the harmonic model which contains the harmonic interaction between particles. As pointed out before, one cannot be sure that using the harmonic model leads to the dissipation of the system since it introduces only the center-of-mass to the calculation. However, our model is not very far from the real situation. If we consider the system in a very short period of time (with respect to the life time of the condensate) we can say that our system is quasi-non-dissipative and the inclusion of the interaction between particles can be neglected.

Consider the vortex state in thermal equilibrium as a prepared state and then physically change the frequency of the trapping potential to another value immediately and see how the vortex state and the condensate itself change with time. The system we consider is two-dimensional. In our system, we excite the anisotropic-trapped condensate in the horizontal plane only and assume that there is no effect on the vertical direction. To write explicitly, we firstly prepare the state with the trap potential

$$V_{trap} = \frac{1}{2}\Omega_{\perp}^{2}\left(x^{2}+y^{2}\right)+\frac{1}{2}\Omega_{z}^{2},$$

and then change the potential to

$$V_{trap}^{excite} = \frac{1}{2}w_{\perp}^{2}\left(x^{2}+y^{2}\right)+\frac{1}{2}\Omega_{z}^{2}.$$

In fact, we do not know the path integral solution for the condensate with vortex so we take the vortex wave function from the Gross-Pitaevskii equation to construct the density matrix. Note that this wave function represents the vortex state at zero temperature. The wave function or the order parameter of the Bose-Einstein condensation with vortex in two dimensions [30] is

$$\psi_{\nu}\left(r\right) = \sqrt{\Omega}re^{-\Omega r^{2}/2+i\theta}, \tag{5.1}$$

where the vortex core is along the z-axis, Ω is a trap frequency and $r = \sqrt{x^{2}+y^{2}}$. The phase factor θ is defined from the circulation around the center of the vortex. The single-particle density matrix is

$$\begin{aligned} \rho\left(r',r\right) &= \psi_{\nu}^{*}\left(r'\right)\psi_{\nu}\left(r\right) \\ &= \Omega rr'e^{-\Omega\frac{\left(r'^{2}+r^{2}\right)}{2}+i\left(\theta-\theta'\right)}. \end{aligned} \tag{5.2}$$

Then the many-body density matrix is simply a product of one-particle density matrices

$$\rho\left(\bar{r}',t|\bar{r},0\right) = \frac{1}{N!}\sum_{P}\prod_{j=1}^{N}\rho\left(Pr_{j}',r_{j}\right). \tag{5.3}$$

The time dependent density matrix can be written as

$$\rho\left(\bar{r}',t|\bar{r},0\right)=\int d\bar{r}\int d\bar{r}'\;K_w\left(\bar{r}',t|\bar{r}'',0\right)\rho_\Omega\left(\bar{r}''|\bar{r}'\right)K_w\left(\bar{r}',0|\bar{r},t\right)$$

$$=\frac{1}{N!}\sum_P\prod_{j=1}^{N}\int dr_j\int dr_j'\;K_w\left(Pr_j',t|r_j'',0\right)\rho_\Omega\left(r_j''|r_j'\right)K_w^*\left(r_j',0|r_j,t\right),\qquad(5.4)$$

where we have used the propagator $K_w(x',t|x)$ for the harmonic oscillator as a time-evolution operator, which is

$$K_w(r',t|r)=\left(\frac{1}{2}\frac{w}{i\pi\sin wt}\right)^{\frac{1}{2}}\exp\left(-\frac{w}{2}\frac{(r'^2+r^2)\cos wt-2r'r}{i\sin wt}\right).\qquad(5.5)$$

However, the statistical calculation of this density matrix is very complicated since we cannot find the closed form of the single-particle density matrix so we must calculate it numerically. In the case of the vortex placed at the distance a away from the center of the condensate we have

$$\psi\left(\vec{r}\right)=\sqrt{\Omega}\sqrt{(x-a)^2+y^2}\exp\left[i\left[\arctan\frac{y}{x-a}\right]-\frac{\Omega\left(x^2+y^2\right)}{2}\right],\qquad(5.6)$$

where the maximum of the condensate is at the origin and the vortex center is situated at a distance a from the center. Express $\psi\left(\vec{r}\right)$ in polar coordinates,

$$\psi\left(\vec{r}\right)=\sqrt{\Omega}\sqrt{a^2-2ar\cos\theta+r^2}\exp\left[i\arcsin\left(\frac{r\sin\theta}{\sqrt{a^2+r^2-2ar\cos\theta}}\right)-\frac{\Omega r^2}{2}\right].\qquad(5.7)$$

then the density matrix of the system can be written as

$$\begin{aligned}\rho_\Omega\left(\vec{r}',\vec{r}\right) &= \Omega\sqrt{\left(a^2-2ar'\cos\theta'+(r')^2\right)\left(a^2-2ar\cos\theta+r^2\right)}\\ &\quad\exp\left[i\left(\theta'-\theta\right)-\frac{\Omega r'^2}{2}-\frac{\Omega r^2}{2}\right].\end{aligned}\qquad(5.8)$$

To see the time-evolution of the system, we let it evolves in time under the influence of the harmonic potential o frequency w. Mathematically speaking, we use the propagator of the harmonic oscillator of frequency w as the time-evolution operator. Thus

$$
\begin{aligned}
\rho(\vec{r},\theta,t) &\equiv \rho\left(\vec{r},t|\vec{r},0\right) \\
&= \int d^2r' \int d^2r'' K_w\left(r,t|r',0\right)\rho_\Omega\left(r',r''\right)K_w^*\left(r'',t|r,0\right) \\
&= \Omega\left(\frac{w}{2\pi\sin wt}\right)^3 \exp\left(-\Omega a^2\right) \\
&\times \int \left(r'\right)^2 dr' e^{-\frac{1}{2}(\Omega+iw\cot wt)(r')^2} \int d\theta' e^{i\theta'+\alpha\rho'(\Omega+iw\cot wt)\cos\theta'+\frac{iwrr'}{\sin wt}\cos(\theta-\theta')} \\
&\times \int \left(r''\right)^2 dr'' e^{-\frac{1}{2}(\Omega-iw\cot wt)(r'')^2} \int d\theta'' e^{-i\theta''+ar''(\Omega-iw\cot wt)\cos\theta''-\frac{iwrr''}{\sin wt}\cos(\theta-\theta'')}
\end{aligned}
\tag{5.9}
$$

This expression is too complicated so we evaluate the density distribution of the particles by the numerical calculation. Fig. 5.1 depicts the snapshot of $\frac{1}{12}$ period of rotation.

We can see the rotation of the condensate peak around the center of the trap with the fluctuation in the amplitude. The frequency of the rotation is in agreement with the experimental result but in our case the decay of the oscillation is not present since the calculation we made does not include the dissipation term. However we anticipate that such a decay could be obtained in the perturbation framework. It is noted that the calculation has been made without taking the statistics into account but from the previous chapter, we found that the statistics makes change only in the size but not in the physical behavior of the condensate. Hence the result we have is more or less a representative of the dynamical behaviour of the vortex in Bose-Einstein condensation under a small perturbation.

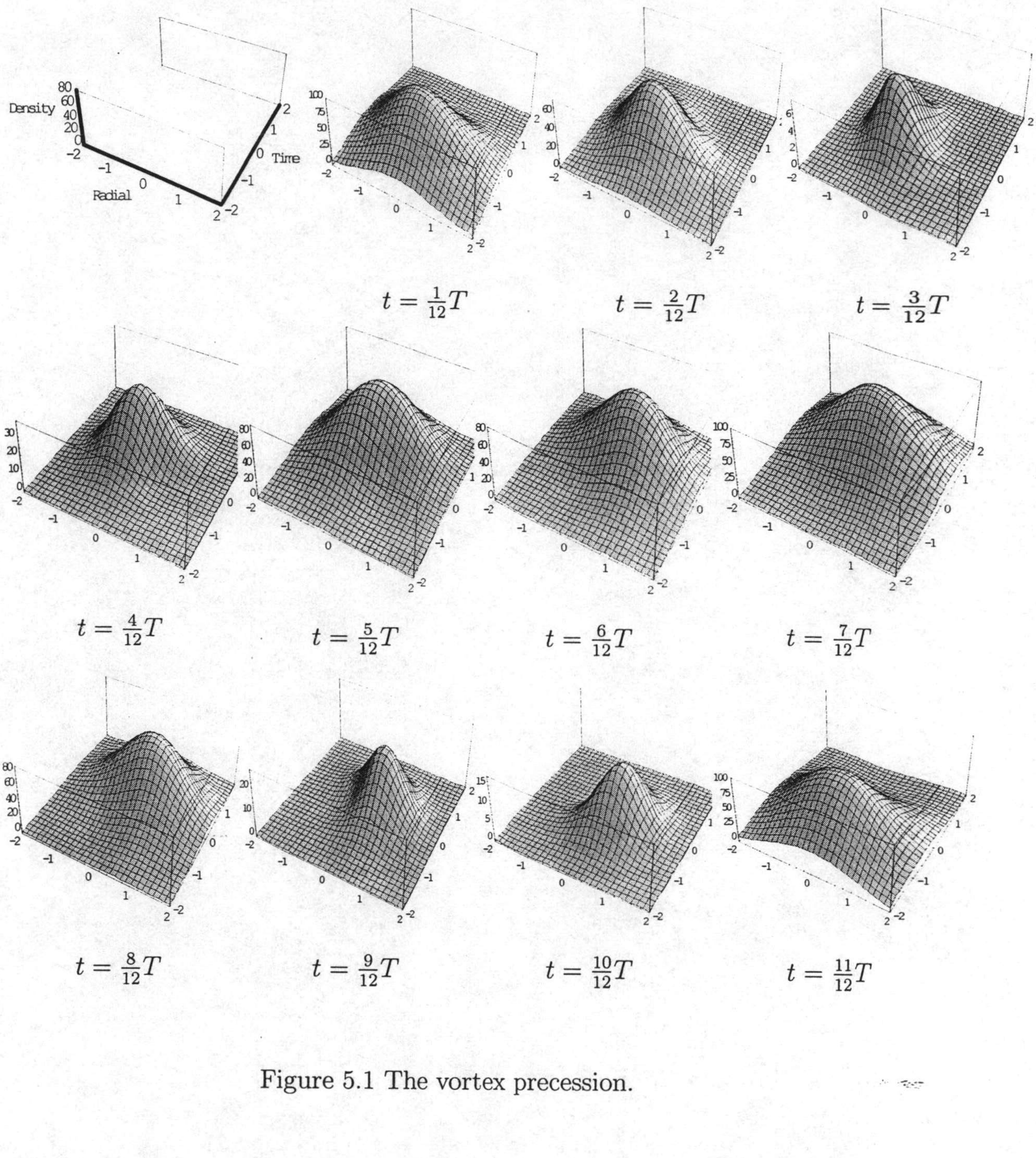

Figure 5.1 The vortex precession.