CHAPTER III

WEAKLY FACTORIZABLE TRANSFORMATION SEMIGROUPS

It has been proved in [5] that the symmetric inverse semigroup on a set X is weakly factorizable if and only if X is finite. The main purpose of this chapter is to show that the same thing is true for partial transformation semigroups and full transformation semigroups.

We recall that a semigroup S is said to be weakly factorizable if there exist a subsemigroup T of S which is a union of groups and a set E of idempotents of S such that S = TE. Thus, every factorizable semigroup is weakly factorizable. The following example shows that weakly factorizable semigroups give a generalization of factorizable semigroups.

Example. Let S be the set $\{1, 2, 3, \ldots\}$. Define the operation o on S by $m \circ n = m$. Then S is a (left zero) semigroup, and every element of S is an idempotent, that is, $E(S) = S$. Therefore, for $G \subseteq S$, G is a subgroup of S if and only if $G = \{m\}$ for some $m \in S$. Since for each $m \in S$, $\{m\} \circ E(S) = \{m\} \neq S$, it follows that S is not factorizable. From $S = \bigcup_{m \in S}\{m\}$ and $S = S \circ S = S \circ E(S)$, we have that S is weakly factorizable. #

Let X be a set and let $\alpha, \beta \in T_X$ be such that $\alpha \mathcal{H} \beta$. Then $\alpha \mathcal{L} \beta$ and $\alpha \mathcal{R} \beta$. Since $\alpha \mathcal{L} \beta$, $\alpha = \gamma\beta$ and $\lambda\alpha = \beta$ for some $\gamma, \lambda \in T_X$.

Then $\nabla\alpha = \nabla\gamma\beta \subseteq \nabla\beta$ and $\nabla\beta = \nabla\lambda\alpha \subseteq \nabla\alpha$, so $\nabla\alpha = \nabla\beta$. Since $\alpha \mathcal{R} \beta$, $\alpha = \beta\gamma'$ and $\alpha\lambda' = \beta$ from some $\gamma', \lambda' \in T_X$. Then $\Delta\alpha = \Delta\beta\gamma' \subseteq \Delta\beta$ and $\Delta\beta = \Delta\alpha\lambda' \subseteq \Delta\alpha$, so $\Delta\alpha = \Delta\beta$. Therefore, $\Delta\alpha = \Delta\beta$ and $\nabla\alpha = \nabla\beta$. If β is an idempotent of T_X, then $\nabla\beta \subseteq \Delta\beta$ and hence $\nabla\alpha = \nabla\beta \subseteq \Delta\beta = \Delta\alpha$.

Therefore, for any set X, if $\alpha \in T_X$ belongs to a subgroup of T_X, then $\alpha \mathcal{H} \beta$ for some $\beta \in E(T_X)$ and hence $\nabla\alpha \subseteq \Delta\alpha$.

A main result of this chapter is

3.1 Theorem. For any set X, the partial transformation semigroup on X is weakly factorizable if and only if X is finite.

Proof : Let X be a set. Assume X is finite. Then T_X is factorizable [4, Theorem 3.1], so T_X is weakly factorizable.

Conversely, assume the partial transformation semigroup T_X is weakly factorizable. Then there exists a subsemigroup T of T_X which is a union of groups such that $T_X = TE(T_X)$. To show X is finite, suppose X is infinite. Let $a \in X$. Then $|X \setminus \{a\}| = |X|$, so there exists a one-to-one map α with $\Delta\alpha = X \setminus \{a\}$ and $\nabla\alpha = X$. Thus $\alpha \in T_X$, so $\alpha = \beta\gamma$ for some $\beta \in T$ and $\gamma \in E(T_X)$. It then follows that $\nabla\alpha = \nabla\beta\gamma \subseteq \nabla\gamma$. But $\nabla\alpha = X$, then $\nabla\gamma = X$. Since $\gamma \in E(T_X)$, $\nabla\gamma \subseteq \Delta\gamma$ and $x\gamma = x$ for all $x \in \nabla\gamma$. Therefore γ is the identity map on X. Thus $\alpha = \beta \in T$ which is a union of subgroups of S, so $\alpha \mathcal{H} \delta$ for some $\delta \in E(T_X)$ which implies $\nabla\alpha \subseteq \Delta\alpha$. It is a contradiction because $\nabla\alpha = X$ but $\Delta\alpha = X \setminus \{a\}$. Therefore X is finite. #

By Theorem 3.1 [4] and Theorem 3.1, the following corollary is directly obtained :

3.2 <u>Corollary</u>. Let X be any set. Then the following conditions are equivalent :

(i) X is a finite set.

(ii) T_X is a factorizable semigroup.

(iii) T_X is a weakly factorizable semigroup.

Let X be a set. Then $\mathcal{J}_X = \{\alpha \in T_X \mid \Delta\alpha = X\}$. For $\alpha \in \mathcal{J}_X$, define a relation π_α on X by

$$x\pi_\alpha y \iff x\alpha = y\alpha \qquad (x, y \in X).$$

It is clearly seen that for each $\alpha \in \mathcal{J}_X$, π_α is an equivalence relation on X, and for $x \in X$, the π_α-class containing x is

$$\begin{aligned} x\pi_\alpha &= \{y \in X \mid x\pi_\alpha y\} \\ &= (x\alpha)\alpha^{-1} \quad (= \{y \in X \mid y\alpha = x\alpha\}). \end{aligned}$$

Observe that $|\{x\pi_\alpha \mid x \in X\}| = |\nabla\alpha|$. Let $\alpha, \beta \in \mathcal{J}_X$. If $\alpha \mathcal{L} \beta$, then $\nabla\alpha = \nabla\beta$, and if $\alpha \mathcal{R} \beta$, then $\pi_\alpha = \pi_\beta$ [2, Lemma 2.5 and Lemma 2.6], so if $\alpha \mathcal{H} \beta$, then $\nabla\alpha = \nabla\beta$ and $\pi_\alpha = \pi_\beta$ since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Let $\alpha, \beta \in \mathcal{J}_X$ such that $\alpha \mathcal{R} \beta$. Then $\pi_\alpha = \pi_\beta$. Thus for each $x \in X$, $(x\alpha)\alpha^{-1} = x\pi_\alpha = x\pi_\beta = (x\beta)\beta^{-1}$ and $x\alpha \in \nabla\alpha$, $x\beta \in \nabla\beta$. Hence, the following lemma follows clearly.

3.3 <u>Lemma</u>. Let X be any set and $\alpha, \beta \in \mathcal{J}_X$. If $\alpha \mathcal{R} \beta$, then for each $a \in \nabla\alpha$, $a\alpha^{-1} = b\beta^{-1}$ for some $b \in \nabla\beta$, and for each $c \in \nabla\beta$, $c\beta^{-1} = d\alpha^{-1}$ for some $d \in \nabla\alpha$.

Hence, if $\alpha \mathcal{H} \beta$ in $\mathcal{T}_X$, then $\nabla\alpha = \nabla\beta$ and for each $a \in \nabla\alpha$ there exist $b \in \nabla\alpha$ and $c \in \nabla\beta$ such that $a\beta^{-1} = b\alpha^{-1}$ and $a\alpha^{-1} = c\beta^{-1}$.

The next theorem shows that for any set X, $\mathcal{T}_X$ is weakly factorizable if and only if X is finite.

3.4 Theorem. For any set X, the full transformation semigroup on X is weakly factorizable if and only if X is finite.

Proof : Let X be a set. Assume X is finite. Then $\mathcal{T}_X$ is factorizable [4, Theorem 3.2], so $\mathcal{T}_X$ is weakly factorizable.

Conversely, assume that the full transformation semigroup $\mathcal{T}_X$ is weakly factorizable. Then there exists a subsemigroup T of $\mathcal{T}_X$ which is a union of groups such that $\mathcal{T}_X = TE(\mathcal{T}_X)$. To show X is finite, suppose that X is infinite. Let $a \in X$. Then $|X| = |X \setminus \{a\}|$, so there exists a one-to-one map such that $\Delta\alpha = X$ and $\nabla\alpha = X \setminus \{a\}$. Thus $\alpha \in \mathcal{T}_X$, so $\alpha = \beta\gamma$ for some $\beta \in T$ and $\gamma \in E(\mathcal{T}_X)$. Since $\gamma \in E(\mathcal{T}_X)$, $x\gamma = x$ for all $x \in \nabla\gamma$. Because $\nabla\alpha = \nabla\beta\gamma \subseteq \nabla\gamma$ and $\nabla\alpha = X \setminus \{a\}$, then $\nabla\gamma = X$ or $\nabla\gamma = X \setminus \{a\}$.

Case $\nabla\gamma = X$. Then γ is the identity map on X. Thus $\alpha = \beta \in T$ which is a union of groups, so $\alpha \mathcal{H} \delta$ for some $\delta \in E(\mathcal{T}_X)$. By Lemma 3.3, $\nabla\alpha = \nabla\delta$ and for each $x \in \nabla\delta = \nabla\alpha$, there exists $y \in \nabla\alpha$ such that $x\alpha^{-1} = y\delta^{-1}$. Let $b = a\alpha$. Then $b \in \nabla\alpha = X \setminus \{a\}$, so there exists $c \in \nabla\alpha$ such that $b\alpha^{-1} = c\delta^{-1}$. Since $\delta \in E(\mathcal{T}_X)$, $x\delta = x$ for all $x \in \nabla\delta = \nabla\alpha$. Then $c\delta = c$. Thus $c \in c\delta^{-1}$, so $c \in b\alpha^{-1}$ which implies $c\alpha = b$. Since $c\alpha = b = a\alpha$ and α is one-to-one, $c = a$. Hence

$a = c \in \nabla\alpha = X\setminus\{a\}$ which is a contradiction.

Case $\nabla\gamma = X\setminus\{a\}$. Since $\alpha = \beta\gamma$ and α is one-to-one, we have that β is one-to-one. Then for each $x \in X$, $x\pi_\beta = (x\beta)\beta^{-1} = \{x\}$, it implies π_β is the identity equivalence relation on X (that is, $\pi_\beta = \{(x, x) \mid x \in X\}$). Because $\beta \in T$ which is a union of subgroups of $\mathcal{T}_X$, $\beta \mathcal{H} \delta$ for some $\delta \in E(\mathcal{T}_X)$. Therefore $\nabla\beta = \nabla\delta$ and $\pi_\beta = \pi_\delta$. Thus π_δ is the identity equivalence relation on X, that is, for each $x \in X$, $x\pi_\delta = \{x\}$. But since for each $x \in X$, $x\pi_\delta = \{y \in X \mid y\delta = x\delta\}$, it follows that δ is one-to-one. Because $\delta \in E(\mathcal{T}_X)$, $x\delta = x\delta^2 = (x\delta)\delta$ for all $x \in X$ which implies $x\delta = x$ for all $x \in X$. Hence $\nabla\delta = X$, so $\nabla\beta = \nabla\delta = X$. Therefore there exist $b, c \in X$ such that $b\beta = a$ and $c\beta = a\gamma$. Thus $b\alpha = b\beta\gamma = a\gamma = a\gamma\gamma = (a\gamma)\gamma = c\beta\gamma = c\alpha$. Since α is one-to-one, we then have that $b = c$. Hence $a\gamma = c\beta = b\beta = a \notin X\setminus\{a\} = \nabla\gamma$, a contradiction.

This proves that X is finite, as required. #

It has been proved in Theorem 3.2 [4] that for any set X, $\mathcal{T}_X$ is factorizable if and only if X is finite. Combining this with Theorem 3.4, we have the following corollary.

3.5 Corollary. Let X be any set. Then the following conditions are equivalent :

(i) X is a finite set.

(ii) $\mathcal{T}_X$ is a factorizable semigroup.

(iii) $\mathcal{T}_X$ is a weakly factorizable semigroup.