# INTRODUCTION

Let S be a semigroup. An element a of S is called an idempotent of S if $a^2 = a$. For a semigroup S, let E(S) denote the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

An element z of a semigroup S is called a zero of S if $xz = zx = z$ for all $x \in S$. An element e of a semigroup S is called an identity of S if $ex = xe = x$ for all $x \in S$. A zero and an identity of a semigroup are unique if exist and they are denoted by 0 and 1, respectively.

A nonempty subset G of a semigroup S is a subgroup of S if it is a group under the same operation of S.

Let S be a semigroup with identity 1. An element a of S is called a unit of S if there exists $a' \in S$ such that $aa' = a'a = 1$. Let G be the set of all units of S, that is,

$$G = \{a \in S \mid aa' = a'a = 1 \text{ for some } a' \in S\}.$$

Then G is the greatest subgroup of S which has 1 as its identity, and it is called the group of units of S or the unit group of the semigroup S.

An element a of a semigroup S is regular if $a = axa$ for some $x \in S$. A semigroup S is regular if every element of S is regular.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if a = axa and x = xax. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a in S is denoted by $a^{-1}$. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [2, Theorem 1.17]. Then a regular subsemigroup of an inverse semigroup is an inverse semigroup. For any elements a, b of an inverse semigroup S and $e \in E(S)$, the following hold :

$$e^{-1} = e, \ (a^{-1})^{-1} = a \text{ and } (ab)^{-1} = b^{-1}a^{-1}$$

[2, Lemma 1.18].

Every group is an inverse semigroup and the identity of a group is its only idempotent.

Let S be a semigroup and 1 be a symbol not representing any element of S. Let $S \cup 1$ be the semigroup obtaining by extending the operation of S to 1 by $a.1 = 1.a = a$ for all $a \in S$ and $1.1 = 1$. Then $S \cup 1$ is a semigroup having 1 as its identity. Let $S^1$ denote the following semigroup :

$$S^1 = \begin{cases} S & \text{if S has an identity,} \\ S \cup 1 & \text{if S has no identity.} \end{cases}$$

Then for $a \in S$, $S^1a = Sa \cup \{a\}$, $aS^1 = aS \cup \{a\}$.

Let S be a semigroup. Define the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on S as follow :

$$a \mathcal{L} b \iff S^1a = S^1b.$$

$$a \mathcal{R} b \iff aS^1 = bS^1.$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

The relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are called <u>Green's relations</u> on S and they are clearly equivalence relations on S.

Let S be a semigroup. If $a \in S$ is a regular element of S, then $S^1a = Sa$. Hence if the semigroup S is regular, then the following hold : For a, b $\in$ S,

$$a\mathcal{L}b \Longleftrightarrow a = xb \text{ and } b = ya \text{ for some } x, y \in S,$$

and $$a\mathcal{R}b \Longleftrightarrow a = bx \text{ and } b = ay \text{ for some } x, y \in S.$$

In a semigroup S, an $\mathcal{H}$-class of S containing an idempotent e of S is a subgroup of S [2, Theorem 2.16], and it is the greatest subgroup of S having e as its identity. Hence, every subgroup of a semigroup S is contained in $H_e$ for some idempotent e of S where for $a \in S$, $H_a$ denotes the $\mathcal{H}$-class of S containing a. If a semigroup S has an identity 1, then $H_1$ is the group of units of S.

Let X be a set. A <u>partial transformation</u> of X is a map which its domain and its range are subsets of X. If $\alpha$ is a partial transformation of X, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range of $\alpha$, respectively. The <u>empty transformation</u> of X is referred as a map with empty domain, and it is denoted by 0. Let $T_X$ denote the set of all partial transformations of X including the empty transformation 0. For $\alpha, \beta \in T_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \phi$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, let $\alpha\beta : (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \to (\nabla\alpha \cap \Delta\beta)\beta$ be the composition map. Then $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta$. Thus $T_X$ is a semigroup and it is called the <u>partial transformation semigroup</u> on the set X. The empty transformation of X, 0, is the zero of $T_X$. The identity map on X which is denoted by 1 is the identity of the semigroup $T_X$. For any set X, the semigroup $T_X$ is a regular semigroup. For $\alpha \in T_X$, $\alpha$ is an idempotent of $T_X$ if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$. Hence

$$E(T_X) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } x\alpha = x \text{ for all } x \in \nabla\alpha\}.$$

An element $\alpha \in T_X$ is called a <u>1-1 partial transformation</u> of X if $\alpha$ is a one-to-one map. Let $I_X$ denote the set of all 1-1 partial transformations of X, that is,

$$I_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one}\}.$$

Then under the composition of maps, $I_X$ is an inverse subsemigroup of $T_X$ with identity 1 and zero 0, and it is called the <u>symmetric inverse semigroup</u> on the set X and for $\alpha \in I_X$, the inverse map $\alpha^{-1}$, is the inverse of $\alpha$ in $I_X$, so $\Delta\alpha^{-1} = \nabla\alpha$, $\nabla\alpha^{-1} = \Delta\alpha$. For $\alpha \in I_X$, $\alpha$ is an idempotent of $I_X$ if $\alpha$ is the identity map on $\Delta\alpha$. Then

$$E(I_X) = \{\alpha \in I_X \mid \alpha \text{ is the identity map on } \Delta\alpha\}.$$

An element $\alpha \in T_X$ is called a <u>full transformation</u> of X if $\Delta\alpha = X$. Let $\mathcal{J}_X$ denote the set of all full transformations of X, that is,

$$\mathcal{J}_X = \{\alpha \in T_X \mid \Delta\alpha = X\}.$$

Then under the composition of maps, $\mathcal{J}_X$ is a regular subsemigroup of $T_X$ with identity 1 and it is called the <u>full transformation semigroup</u> on the set X. Therefore

$$E(\mathcal{J}_X) = \{\alpha \in \mathcal{J}_X \mid \alpha \text{ is the identity map on } \nabla\alpha\}.$$

For any set X, let $G_X$ denote the permutation group on X, that is,

$$G_X = \{\alpha : X \to X \mid \alpha \text{ is one-to-one and onto}\}.$$

Then $G_X$ is the group of units of $T_X$, also of $I_X$ and of $\mathcal{J}_X$.

For any set A, let $|A|$ denote the cardinality of A.

A semigroup S is said to be factorizable if there exist a subgroup G of S and a set E of idempotents of S such that S = GE ( = $\{ge \mid g \in G, e \in E\}$). Observe that if a semigroup S is factorizable as S = GE, then S = GE(S). Every factorizable semigroup is regular [4, Proposition 2.2]. If a semigroup S has an identity and S is factorizable as GE, then G is the group of units of S [4, Theorem 2.4].

A semigroup S is called a weakly factorizable semigroup if there exist a subsemigroup T of S which T is a union of groups and a set E of idempotents of S such that S = TE. Then factorizable semigroups are weakly factorizable semigroups. But the converse is not true.

A transformation semigroup on a set X is a semigroup of maps from subsets of X onto subsets of X and the operation is the composition of maps. Let S be a transformation semigroup and let $\theta \in S$. The semigroup S under the operation * defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S$ is called a generalized transformation semigroup on the set X, and it is denoted by $(S, \theta)$. Note that for any set X and for $\theta \in T_X$, the generalized transformation semigroup $(T_X, \theta)$ is referred as a generalized partial transformation semigroup on X. A generalized full transformation semigroup on a set and a generalized 1-1 partial transformation semigroup on a set are referred similarly. Observe that for any set X, if $\theta$ is the identity map on X, then the generalized transformation semigroups $(T_X, \theta)$, $(\mathcal{T}_X, \theta)$ and $(I_X, \theta)$ are the partial transformation semigroup, the full transformation semigroup and the 1-1 partial transformation semigroup (the symmetric inverse semigroup) on the set X, respectively.

Let X be a set. A partial transformation $\alpha$ on X is said to be <u>almost identical</u> if there exists at most a finite number of elements x in the domain of $\alpha$ such that $x\alpha \neq x$. Therefore, a partial transformation $\alpha$ on X is almost identical if and only if the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$ is finite.

Let X be a set, $U_X = \{\alpha \in T_X \mid \alpha$ is almost identical$\}$, $V_X = \{\alpha \in \mathcal{T}_X \mid \alpha$ is almost identical$\}$ and $W_X = \{\alpha \in I_X \mid \alpha$ is almost identical$\}$. It is shown in the first chapter that under the composition of maps $U_X$, $V_X$ and $W_X$ are regular semigroups which are called the <u>semigroup of almost identical partial transformations</u> on X, the <u>semigroup of almost identical full transformations</u> on X and the <u>semigroup of almost identical 1-1 partial transformations</u> on X, respectively. We prove in Chapter I that for any set X, the transformation semigroups $U_X$, $V_X$ and $W_X$ are factorizable.

Generalized partial transformation semigroups, generalized full transformation semigroups and generalized 1-1 partial transformation semigroups are studied in the second chapter. The following are shown. Let X be a set and S be $T_X$, $\mathcal{T}_X$ or $I_X$. If $\theta \in S$, then the generalized transformation semigroup $(S, \theta)$ is regular if and only if $\theta$ is a permutation on X. The main result of this chapter is to show that the semigroup $(S, \theta)$ is factorizable if and only if $\theta$ is a permutation on X and X is a finite set. Moreover, it is shown that for any set X, if S is $U_X$, $V_X$ or $W_X$ and $\theta \in S$, then the semigroup $(S, \theta)$ is factorizable if and only if $\theta$ is a permutation on X.

In the last chapter, weakly factorizable transformation semigroups are studied. We show that the partial transformation semigroup on a set X is weakly factorizable if and only if X is finite, and it is also showed that the full transformation semigroup on a set X is weakly factorizable if and only if X is finite.