CHAPTER II

FIELD PATTERN AND GAIN OF A HOLLOW CYLINDRICAL ANTENNA

The hollow cylindrical antenna is oriented as shown in fig. 2. The center of the antenna is placed at the origin of the coordinate system with the axis of the antenna along the Z direction. The point $P(r,\theta,\phi)$ is the far point where the field pattern and gain are to be found.

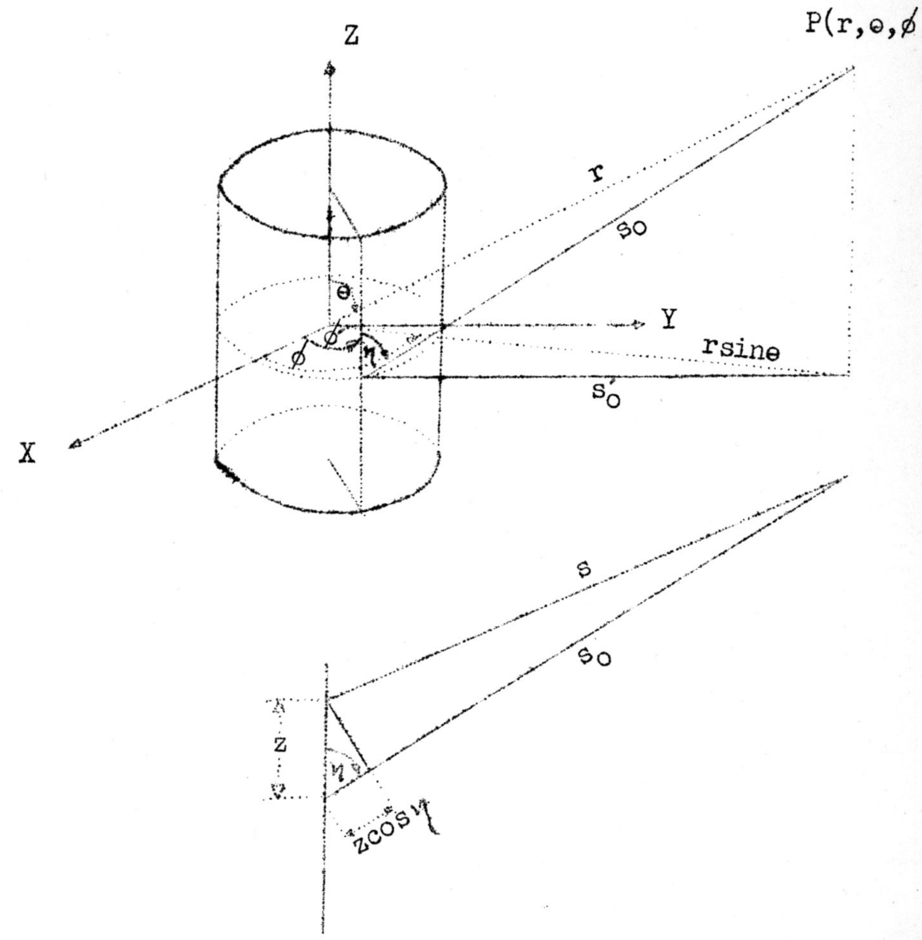


Fig. 2 The Hollow Cylindrical Antenna and Its Orientation

A. Field Pattern Analysis

From fig. 2

$$s_o' = r\sin\theta - a\cos(\phi' - \phi) \tag{1}$$

$$s_o = s_o'\operatorname{cosec}\eta \approx s_o'\operatorname{cosec}\theta \tag{2}$$

$$s = s_o - z\cos\eta \approx s_o - z\cos\theta \tag{3}$$

From equations (2) and (3)

$$s = s_o'\operatorname{cosec}\eta - z\cos\eta \tag{4}$$

$$= s_o'\operatorname{cosec}\theta - z\cos\theta \tag{5}$$

at $\eta > \theta$

$\operatorname{cosec}\eta < \operatorname{cosec}\theta$

$\cos\eta < \cos\theta$

at $\eta < \theta$

$\operatorname{cosec}\eta > \operatorname{cosec}\theta$

$\cos\eta > \cos\theta$

Hence, the value of s obtained from equation (5) is approximately equal to that obtained from equation (4), therefore

$$s = s_o'\operatorname{cosec}\theta - z\cos\theta \tag{5}$$

From equation (1) and equation (5), gives

$$s = \left[r\sin\theta - a\cos(\phi' - \phi)\right]\operatorname{cosec}\theta - z\cos\theta$$

$$= r - a\operatorname{cosec}\theta\cos(\phi' - \phi) - z\cos\theta \tag{6}$$

The retarded vector potential is given by

$$A_z = \frac{\mu}{4\pi}\int_0^{2\pi}\int_{-l}^{l}\frac{[I_z]}{s}\,dz\,d\phi' \tag{7}$$

1. Uniform Current Distribution ($2\ell \ll \lambda$)

$$A_z = \frac{\mu}{4\pi}\int_0^{2\pi}\int_{-\ell}^{\ell}\frac{I_o e^{j(wt-\beta s)}}{s}\,dz\,d\phi' \qquad (8)$$

where I_o = Peak value in time of current

At a large distance the difference between s and r can be neglected in its effect on the amplitude but its effect on the phase must be considered, then

$$A_z = \frac{I_o e^{jwt}}{4\pi r}\int_0^{2\pi}\int_{-\ell}^{\ell} e^{-j\beta s}\,dz\,d\phi' \qquad (9)$$

Substituting equation (6) into equation (9), gives

$$A_z = \frac{I_o e^{jwt}}{4\pi r}\int_0^{2\pi}\int_{-\ell}^{\ell} e^{-j\beta\{r - a\cos\theta\cos(\phi'-\phi) - z\cos\theta\}}\,dz\,d\phi'$$

$$= \frac{I_o e^{jwt}}{4\pi r}\int_0^{2\pi} e^{-j\beta\{r - a\cos\theta\cos(\phi'-\phi)\}}\int_{-\ell}^{\ell} e^{j\beta z\cos\theta}\,dz\,d\phi'$$

$$= \frac{\mu I_o \sin(\beta\ell\cos\theta)\,e^{j(wt-\beta r)}}{2\pi\beta r\cos\theta}\int_0^{2\pi} e^{j\beta a\cos\theta\cos(\phi'-\phi)}\,d\phi' \qquad (10)$$

Assuming that $\beta a \ll 1$

$$e^{j\beta a\cos\theta\cos(\phi'-\phi)}\,d\phi' = 1 + \frac{j\beta a\cos\theta\cos(\phi'-\phi)}{1!} + \frac{\left(j\beta a\cos\theta\cos(\phi'-\phi)\right)^2}{2!} + \frac{\left(j\beta a\cos\theta\cos(\phi'-\phi)\right)^3}{3!} + \frac{\left(j\beta a\cos\theta\cos(\phi'-\phi)\right)^4}{4!} + \frac{\left(j\beta a\cos\theta\cos(\phi'-\phi)\right)^5}{5!} + \frac{\left(j\beta a\cos\theta\cos(\phi'-\phi)\right)^6}{6!} + \frac{\left(j\beta a\cos\theta\cos(\phi'-\phi)\right)^7}{7!} \qquad (11)$$

$$d\phi' = 2\pi \quad (12)$$

$$\cos(\phi'-\phi)d\phi' = \sin(\phi'-\phi)\Big|_0^{2\pi} = 0 \quad (13)$$

$$\cos^2(\phi'-\phi)d\phi' = \frac{(\phi'-\phi)}{2} + \frac{\sin(\phi'-\phi)\cos(\phi'-\phi)}{2}\Big|_0^{2\pi} = \pi \quad (14)$$

$$\cos^3(\phi'-\phi)d\phi' = \sin(\phi'-\phi) - \frac{\sin^3(\phi'-\phi)}{3}\Big|_0^{2\pi} = 0 \quad (15)$$

$$\cos^4(\phi'-\phi)d\phi' = \frac{3(\phi'-\phi)}{8} + \frac{\sin2(\phi'-\phi)}{4} + \frac{\sin4(\phi'-\phi)}{32}\Big|_0^{2\pi} = \frac{3\pi}{4} \quad (16)$$

$$\cos^5(\phi'-\phi)d\phi' = \frac{5\sin(\phi'-\phi)}{8} + \frac{5\sin3(\phi'-\phi)}{48} + \frac{\sin5(\phi'-\phi)}{80}\Big|_0^{2\pi} = 0 \quad (17)$$

$$\cos^6(\phi'-\phi)d\phi' = \frac{5(\phi'-\phi)}{16} + \frac{15\sin2(\phi'-\phi)}{64} + \frac{3\sin4(\phi'-\phi)}{64} + \frac{\sin6(\phi'-\phi)}{192}\Big|_0^{2\pi} = \frac{5\pi}{8} \quad (18)$$

$$\cos^7(\phi'-\phi)d\phi' = \frac{35\sin(\phi'-\phi)}{64} + \frac{7\sin3(\phi'-\phi)}{64} + \frac{7\sin5(\phi'-\phi)}{320} + \frac{\sin7(\phi'-\phi)}{448}\Big|_0^{2\pi} = 0 \quad (19)$$

Substituting equations (12), (13), (14), (15), (16), (17), (18), and (19) into equation (11), gives

$$\int_0^{2\pi} e^{j\beta a \operatorname{cosec}\theta \cos(\phi'-\phi)} d\phi' = 2\pi + \frac{(j\beta a \operatorname{cosec}\theta)^2 \pi}{2} + \frac{(j\beta a \operatorname{cosec}\theta)^4}{4}\,\frac{3\pi}{4} + \frac{(j\beta a \operatorname{cosec}\theta)^6}{6}\,\frac{5\pi}{8}$$

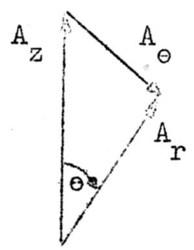
$$= 2\pi\left[1 - \frac{\beta^2 a^2 \operatorname{cosec}^2\theta}{4}\right] \quad (20)$$

Equation (20) change equation (10) to

$$A_z = \frac{\mu I_0 \sin(\beta \ell \cos\theta)}{2\pi\beta r \cos\theta} \cdot \frac{2\pi(4-\beta^2 a^2 \operatorname{cosec}^2\theta)}{4}\, e^{j(wt-\beta r)}$$

$$A_z = \frac{\mu I_o \sin(\beta l \cos\theta)(4-\beta^2 a^2 \operatorname{cosec}^2\theta)}{4\beta r \cos\theta} e^{j(wt-\beta r)} \qquad (21)$$

Since the hollow cylindrical antenna shown in Fig. 2 has only Z component, $A_\phi = 0$, and A_r and A_θ are given by



$$A_r = A_z \cos\theta \qquad (22)$$

$$A_\theta = -A_z \sin\theta \qquad (23)$$

$$A_\phi = 0 \qquad (24)$$

Fig. 3 Resolution of vector potential into A_r and A_θ components,

Substituting equation (21) into equation (22), gives

$$A_r = \frac{\mu I_o \sin(\beta l \cos\theta)(4-\beta^2 a^2 \operatorname{cosec}^2\theta)}{4\beta r} e^{j(wt-\beta r)} \qquad (25)$$

Substituting equation (21) into equation (23), gives

$$A_\theta = \frac{-\mu I_o \sin(\beta l \cos\theta)(4-\beta^2 a^2 \operatorname{cosec}^2\theta)\tan\theta}{4\beta r} e^{j(wt-\beta r)} \qquad (26)$$

From the relation

$$\bar{E} \doteq -j\frac{c^2}{w}\nabla(\nabla\cdot\bar{A}) - jw\bar{A} \qquad (27)$$

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial(r^2 Ar)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi \qquad (28)$$

Since $A_\phi = 0$

$$\therefore\ \nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Ar) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) \qquad (29)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Ar) = \frac{1}{r^2}\frac{\partial}{\partial r}\, r^2\frac{\mu I_0 \sin(\beta l\cos\theta)(4-\beta^2 a^2 \operatorname{cosec}^2\theta)e^{j(wt-\beta r)}}{4\beta r}$$

$$= \frac{\mu I_0 \sin(\beta l\cos\theta)(4-\beta^2 a^2\operatorname{cosec}^2\theta)(1-j\beta r)e^{j(wt-\beta r)}}{4\beta r^2} \qquad (30)$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{-\mu I_0 \sin(\beta l\cos\theta)(4-\beta^2 a^2\operatorname{cosec}^2\theta)\tan\theta\, e^{j(wt-\beta r)}\sin\theta}{4\beta r^2}\right.$$

$$= \frac{-\mu I_0 e^{j(wt-\beta r)}}{4\beta r^2}\left[\frac{8\cos^2\theta\sin(\beta l\cos\theta)-\beta l\cos(\beta l\cos\theta)(4r\sin^2\theta-\beta^2 a^2)\cos\theta}{\cos^2\theta}\right.$$

$$\left. + \frac{\sin(\beta l\cos\theta)(4\sin^2\theta-\beta^2 a^2)}{\cos^2\theta}\right] \qquad (31)$$

Substituting equations (30) and (31) into equation (29), gives

$$\nabla\cdot\bar{A} = \frac{\mu I_0 e^{j(wt-\beta r)}}{4\beta r^2}\left[\sin(\beta l\cos\theta)(4-\beta^2 a^2\operatorname{cosec}^2\theta)(1-j\beta r)-8\sin(\beta l\cos\theta)\right.$$

$$\left. + \frac{\beta l\cos(\beta l\cos\theta)(4\sin\theta-\beta^2 a^2)}{\cos\theta} - \frac{\sin(\beta l\cos\theta)(4\sin^2\theta-\beta^2 a^2)}{\cos^2\theta}\right] \qquad (32)$$

From $$\nabla(\nabla\cdot\bar{A}) = \bar{a}_r\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) + \bar{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\cdot\bar{A}) + a_\phi\frac{1}{r\sin\theta}\frac{\partial(\nabla\cdot\bar{A})}{\partial\phi} \qquad (33)$$

Since $\nabla\cdot\bar{A}$ is independent of ϕ

$$\therefore \quad \nabla(\nabla\cdot\bar{A}) = \bar{a}_r\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) + \bar{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\cdot\bar{A}) \tag{34}$$

$$jw\bar{A} = \bar{a}_r jwA_r + \bar{a}_\theta jwA_\theta + \bar{a}_\phi jwA_\phi$$

Since $A_\phi = 0$ (35)

$$\therefore \quad jw\bar{A} = \bar{a}_r jwA_r + \bar{a}_\theta jwA_\theta \tag{36}$$

Substituting equations (34) and (36) into equation (27), gives

$$\bar{E} = -j\frac{c^2}{w}\left[\bar{a}_r\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) + \bar{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}(\nabla\cdot\bar{A})\right] - \bar{a}_r jwA_r - \bar{a}_\theta jwA_\theta$$

$$= \bar{a}_r\left[\frac{-jc^2}{w}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) - jwA_r\right] + \bar{a}_\theta\left[\frac{-jc^2}{wr}\frac{\partial}{\partial\theta}(\nabla\cdot\bar{A}) - jwA_\theta\right] \tag{37}$$

Then

$$E_r = -j\frac{c^2}{w}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) - jwA_r \tag{38}$$

$$E_\theta = -j\frac{c^2}{wr}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) - jwA_\theta \tag{39}$$

$$E_\phi = 0 \tag{40}$$

$$-j\frac{c^2}{w}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) = -j\frac{c^2}{w}\frac{\partial}{\partial r}\left\{\frac{\mu I_o e^{j(wt-\beta r)}}{4\beta r^2}\left[\sin(\beta l\cos\theta)(4-\beta^2a^2\operatorname{cosec}^2\theta)(1-j\beta r) - 8\sin(\beta l\cos\theta) + \frac{\beta l\cos(\beta l\cos\theta)(4\sin^2\theta-\beta^2a^2)}{\cos\theta} - \frac{\sin(\beta l\cos\theta)(4\sin^2\theta-\beta^2a^2)}{\cos^2\theta}\right]\right\} \tag{41}$$

Let $D = \sin(\beta l\cos\theta)(4-\beta^2a^2\mathrm{cosec}^2\theta)$ (42)

$$F = -8\sin(\beta l\cos\theta) + \frac{\beta l\cos(\beta l\cos\theta)(4\sin^2\theta-\beta^2a^2)}{\cos\theta} - \frac{\sin(\beta l\cos\theta)(4\sin^2\theta-\beta^2a^2)}{\cos^2\theta} \quad (43)$$

Substituting equations (42) and (43) into (41), gives

$$-j\frac{c^2}{w}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) = \frac{-jc^2\mu I_o}{w4\beta}\frac{\partial}{\partial r}\left\{\frac{e^{j(wt-\beta r)}}{r^2}\left[D(1-j\beta r)+F\right]\right\}$$

$$= \frac{j\mu c^2I_oe^{j(wt-\beta r)}}{4w\beta}\left[\frac{\beta^2D}{r}+\frac{j\beta F}{r^2}+\frac{2D}{r^3}+\frac{2F}{r^3}\right] \quad (44)$$

$$-jwA_r = \frac{-jw\mu I_o\sin(\beta l\cos\theta)(4-\beta^2a^2\mathrm{cosec}^2\theta)e^{j(wt-\beta r)}}{4\beta r} \quad (45)$$

Substiting equation (42) into equation (45), gives

$$-jwA_r = \frac{-jw\mu I_oDe^{j(wt-\beta r)}}{4\beta r}$$

$$= \frac{-j\mu\beta^2c^2DI_oe^{j(wt-\beta r)}}{4w\beta r} \quad (46)$$

Substituting equations (44) and (46) into equation (38), gives

$$E_r = \frac{j\mu c^2I_oe^{j(wt-\beta r)}}{4w\beta}\left[\frac{j\beta F}{r^2}+\frac{2D}{r^3}+\frac{2F}{r^3}\right]$$

$$= \frac{j\mu c^2I_oe^{j(wt-\beta r)}}{4w\beta}\left[\frac{j\beta F}{r^2}+\frac{2(D+F)}{r^3}\right] \quad (47)$$

Substituting equations (42) and (43) into equation (32), then differential with respect to θ, gives

$$-\frac{jc^2}{wr}\frac{\partial}{\partial\theta}(\nabla\cdot\bar{A}) = -\frac{jc^2}{wr}\frac{\partial}{\partial\theta}\left\{\frac{\mu I_o e^{j(wt-\beta r)}}{4\beta r^2}\left[D(1-j\beta r)+F\right]\right\}$$

$$= \frac{-j\mu c^2 I_o e^{j(wt-\beta r)}}{4w\beta}\left(\frac{1}{r^3}\frac{\partial D}{\partial\theta} - \frac{j\beta}{r^2}\frac{\partial D}{\partial\theta} + \frac{1}{r^3}\frac{\partial F}{\partial\theta}\right) \qquad (48)$$

$$-jwA_\theta = -jw\,\frac{-\mu I_o \sin(\beta l\cos\theta)(4-\beta^2a^2\operatorname{cosec}^2\theta)\tan\theta e^{j(wt-\beta r)}}{4\beta r} \qquad (49)$$

Substituting equation (42) into equation (49)

$$-jwA_\theta = \frac{j\mu I_o wD\tan\theta e^{j(wt-\beta r)}}{4\beta r}$$

$$= \frac{j\mu\beta c^2 I_o D\tan\theta e^{j(wt-\beta r)}}{4w} \qquad (50)$$

Substituting equations (48) and (50) into equation (39), gives

$$E_\theta = \frac{-j\mu c^2 I_o e^{j(wt-\beta r)}}{4w\beta}\left(\frac{1}{r^3}\frac{\partial D}{\partial\theta} - \frac{j\beta}{r^2}\frac{\partial D}{\partial\theta} + \frac{1}{r^3}\frac{\partial F}{\partial\theta} - \frac{\beta^2 D\tan\theta}{r}\right)$$

Since D and F are independent of r, and $r \gg 1$

$$E_\theta = \frac{j\mu\beta c^2 I_o D\tan\theta e^{j(wt-\beta r)}}{4wr} \qquad (51)$$

Comparing equations (47) and (51). It is found that for $r \gg 1$, E_r can be neglected compare with E_θ

$$\therefore\ E_\theta = \frac{j\mu\beta c^2 I_o D\tan\theta e^{j(wt-\beta r)}}{2wr}$$

$$= \frac{j30\pi I_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\tan\theta\sin(\beta l\cos\theta)e^{j(wt-\beta r)}}{r} \qquad (52)$$

$$E_r = 0 \tag{53}$$

$$E_\phi = 0 \tag{54}$$

From the Relation

$$\bar{H} = \frac{1}{\mu}(\nabla \times \bar{A}) \tag{55}$$

$$\nabla \times \bar{A} = \frac{\bar{a}_r}{r^2 \sin\theta}\left[\frac{\partial(r\sin\theta)A_\phi}{\partial\theta} - \frac{\partial rA_\theta}{\partial\phi}\right] + \frac{\bar{a}_\theta}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \frac{\partial(r\sin\theta)A_\phi}{\partial r}\right] + \frac{\bar{a}_\phi}{r}\left[\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right] \tag{56}$$

Since $A_\phi = 0$, and A_r and A_θ are independent of ϕ

$$\therefore \nabla \times \bar{A} = \frac{\bar{a}_\phi}{r}\left[\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{-a_\phi \mu I_o e^{j(wt-\beta r)}}{4\beta r}\left[-j\beta\tan\theta(4-\beta^2a^2\mathrm{cosec}^2\theta)\sin(\beta l\cos\theta) + \frac{2\beta^2a^2\mathrm{cosec}^2\theta\cot\theta\sin(\beta l\cos\theta)}{r} - \frac{\beta l\sin\theta\cos(\beta l\cos\theta(4-\beta^2a^2\mathrm{cosec}^2\theta)}{r}\right] \tag{57}$$

Substituting equation (57) into (55)

$$\bar{H} = \frac{\bar{a}_\phi I_o e^{j(wt-\beta r)}}{4\beta}\left[\frac{j\beta\tan\theta(4-\beta^2a^2\mathrm{cosec}^2\theta)\sin(\beta l\cos\theta)}{r} - \frac{2\beta^2a^2\mathrm{cosec}^2\theta\cot\theta\sin(\beta l\cos\theta)}{r^2} + \frac{\beta l\sin\theta\cos(\beta l\cos\theta)(4-\beta^2a^2\mathrm{cosec}^2\theta)}{r^2}\right] \tag{58}$$

For $r \gg 1$

$$\bar{H} = \frac{\bar{a}_\phi I_o}{4\beta}\,\frac{j\beta\tan\theta(4-\beta^2a^2\mathrm{cosec}^2\theta)\sin(\beta l\cos\theta)e^{j(wt-\beta r)}}{r} \tag{59}$$

$$H_\phi = \frac{jI_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\tan\theta\sin(\beta l\cos\theta)e^{j(wt-\beta r)}}{4r} \tag{60}$$

$$\text{Or } H_{\phi} = \frac{E_{\theta}}{Z_o} \tag{61}$$

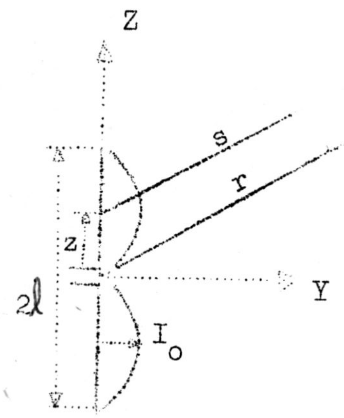
Where Z_o : Intrinsic impedance in free space = 120π

$$H_{\phi} = \frac{E_{\theta}}{120\pi} \tag{62}$$

$$= \frac{jI_o(4-\beta^2 a^2 \mathrm{cosec}^2\theta)\tan\theta \sin(\beta\ell\cos\theta)e^{j(wt-\beta r)}}{4r} \tag{63}$$

2. <u>Sinusoidal Current Distribution</u>

In this case, the current distribution can be given as



$$I_z = I_o \sin\left[\frac{2\pi}{\lambda}(\ell \pm z)\right] e^{j(wt-\beta s)} \qquad (64)$$

where I_o = peak value in time of current

$\ell + z$ is used when $z < 0$

$\ell - z$ is used when $z > 0$

Fig. 4 Relations for symmetrical, hollow cylindrical, center-fed antenna of length 2ℓ

At a large distance, the difference between s and r can be neglected in its effect on the amplitude, but its effect on the phase must be considered.

Substituting equations (6) and (64) into equation (7), gives

$$A_z = \frac{\mu I_o}{4\pi r}\int_0^{2\pi}\left\{\int_{-\ell}^{0}\sin\left[\frac{2\pi}{\lambda}(\ell+z)\right] e^{j\{wt-\beta r+a\beta \cos e c o \cos(\phi'-\phi)+z\cos\theta\}}dz \right.$$

$$\left. + \int_0^{\ell}\sin\left[\frac{2\pi}{\lambda}(\ell-z)\right] e^{j\{wt-\beta r+a\beta \cos e c o \cos(\phi'-\phi)+z\cos\theta\}}dz\right\} d\phi'$$

$$A_z = \frac{\mu I_o}{4\pi r}\int_0^{2\pi} e^{j[wt-\beta r+a\beta\cosec\theta\cos(\phi'-\phi)]}\left\{\int_{-\ell}^{0}\sin\left[\frac{2\pi}{\lambda}(\ell+z)\right]e^{j\beta z\cos\theta}dz + \sin\left[\frac{2\pi}{\lambda}(\ell-z)\right]e^{j\beta z\cos\theta}dz\right\}d\phi' \qquad (65)$$

From tables of integrals

$$\int\sin(c+bx)e^{ax}dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(c+bx)-b\cos(c+bx)\right] \qquad (66)$$

Then

$$\int_{-\ell}^{0}\sin\left(\frac{2\pi\ell}{\lambda}+\frac{2\pi z}{\lambda}\right)e^{j\beta z\cos\theta}dz = \frac{e^{j\beta z\cos\theta}}{(j\beta\cos\theta)^2+\left(\frac{2\pi}{\lambda}\right)^2}\left[j\beta\cos\theta\sin\frac{2\pi}{\lambda}(\ell+z) - \frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(\ell+z)\right]_{-\ell}^{0}$$

$$= \frac{1}{\beta^2\sin^2\theta}\left[j\beta\cos\theta\sin\beta\ell-\beta\cos\beta\ell+e^{-j\beta\ell\cos\theta}\right] \qquad (67)$$

$$\int_0^{\ell}\sin\left(\frac{2\pi\ell}{\lambda}-\frac{2\pi z}{\lambda}\right)e^{j\beta z\cos\theta}dz = \frac{e^{j\beta z\cos\theta}}{\beta^2\sin^2\theta}\left[j\beta\cos\theta\sin\frac{2\pi}{\lambda}(\ell-z)+\beta\cos\frac{2\pi}{\lambda}(\ell-z)\right]_0^{\ell}$$

$$= \frac{1}{\beta^2\sin^2\theta}\left[\beta e^{j\beta\ell\cos\theta} - j\beta\cos\theta\sin\beta\ell-\beta\cos\beta\ell\right] \qquad (68)$$

Substituting equations (67) and (68) into (65), gives

$$A_z = \frac{\mu I_o}{4\pi r}\int_0^{2\pi} e^{j[wt-\beta r+a\beta\cosec\theta\cos(\phi'-\phi)]}\left\{\frac{1}{\beta^2\sin^2\theta}\left[j\beta\cos\theta\sin\beta\ell-\beta\cos(\beta\ell) + \beta e^{j\beta\ell\cos\theta} - j\beta\cos\theta\sin(\beta\ell) - \beta\cos(\beta\ell)\right]\right\}d\phi'$$

$$A_z = \frac{\mu I_o}{4\pi r}\int_0^{2\pi} e^{j[wt-\beta r+a\beta\cosec\theta\cos(\phi'-\phi)]}\left[\frac{\beta(e^{j\beta\ell\cos\theta}+e^{-j\beta\ell\cos\theta}) - 2\beta\cos\beta\ell}{\beta^2\sin^2\theta}\right]d\phi'$$

$$= \frac{\mu I_o\left[\cos(\beta\ell\cos\theta)-\cos\beta\ell\right]e^{j(wt-\beta r)}}{2\pi\beta r\sin^2\theta}\int_0^{2\pi}e^{ja\beta\cosec\theta\cos(\phi'-\phi)}d\phi' \qquad (69)$$

Substituting equation (20) into equation (69), gives

$$A_z = \frac{\mu I_o}{2\pi\beta r}\frac{\left[\cos(\beta l\cos\theta)-\cos\beta l\right]}{\sin^2\theta}e^{j(wt-\beta r)}\cdot 2\pi\left[1-\frac{\beta^2a^2\mathrm{cosec}^2\theta}{4}\right]$$

$$= \frac{\mu I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{4\beta r\sin^2\theta} \tag{70}$$

From fig. 3

$$A_r = A_z\cos\theta \tag{71}$$

$$A_\theta = -A_z\sin\theta \tag{72}$$

Substituting equation (70) into equation (71), gives

$$A_r = \frac{\mu I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]\cos\theta e^{j(wt-\beta r)}}{4\beta r\sin^2\theta}$$

$$= \frac{\mu I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\cos\theta\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{4\beta r\sin^2\theta} \tag{73}$$

Substituting equation (70) into equation (72), gives

$$A_\theta = \frac{-\mu I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]\sin\theta e^{j(wt-\beta r)}}{4\beta r\sin^2\theta}$$

$$= \frac{-\mu I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{4\beta r\sin\theta} \tag{74}$$

From $\bar{E} = \frac{-jc^2}{w}\nabla(\nabla\cdot\bar{A})-jw\bar{A}$

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

Since $A_\phi = 0$

$$\therefore\ \nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) \tag{75}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\, r^2 \frac{\mu I_o(4-\beta^2 a^2 \mathrm{cosec}^2\theta)\cos\theta\,[\cos(\beta l\cos\theta)-\cos\beta l]\, e^{j(wt-\beta r)}}{4\beta r\sin^2\theta}$$

$$= \frac{\mu I_o(4-\beta^2 a^2 \mathrm{cosec}^2\theta)\cos\theta\,[\cos(\beta l\cos\theta)-\cos\beta l]}{4\beta r^2\sin^2\theta}\frac{\partial}{\partial r} r e^{j(wt-\beta r)}$$

$$= \frac{\mu I_o(4-\beta^2 a^2 \mathrm{cosec}^2\theta)\cos\theta\,[\cos(\beta l\cos\theta)-\cos\beta l]}{4\beta r^2\sin^2\theta}\,[e^{j(wt-\beta r)}$$

$$-j\beta r e^{j(wt-\beta r)}]$$

$$= \frac{\mu I_o(4-\beta^2 a^2 \mathrm{cosec}^2\theta)\cos\theta\,[\cos(\beta l\cos\theta)-\cos\beta l]\,(1-j\beta r)e^{j(wt-\beta r)}}{4\beta r^2\sin^2\theta} \qquad (76)$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{-\mu I_o(4-\beta^2 a^2 \mathrm{cosec}^2\theta)\,[\cos(\beta l\cos\theta)-\cos\beta l]\, e^{j(wt-\beta r)}\sin\theta}{4\beta r\sin\theta}$$

$$= \frac{-\mu I_o e^{j(wt-\beta r)}}{4\beta r^2\sin\theta}\Big\{[(4-\beta^2 a^2 \mathrm{cosec}^2\theta)][\beta l\sin\theta\sin(\beta l\cos\theta)]$$

$$+ [\cos(\beta l\cos\theta)-\cos\beta l]\,(2\beta^2 a^2 \mathrm{cosec}^2\theta\cot\theta)\Big\}$$

$$= \frac{-\mu I_o e^{j(wt-\beta r)}}{4\beta r^2\sin\theta}\Big\{(4-\beta^2 a^2 \mathrm{cosec}^2\theta)[\beta l\sin\theta\sin(\beta l\cos\theta)]$$

$$+ 2\beta^2 a^2 \mathrm{cosec}^2\theta\cot\theta\,[\cos(\beta l\cos\theta)-\cos\beta l]\Big\} \qquad (77)$$

$$\nabla\cdot\bar{A} = \frac{\mu I_o e^{j(wt-\beta r)}}{4\beta r^2 \sin\theta}\Big\{(4-\beta^2a^2\operatorname{cosec}^2\theta)\cot\theta\,[\cos(\beta l\cos\theta)-\cos\beta l]\,(1-j\beta r)$$

$$-(4-\beta^2a^2\operatorname{cosec}^2\theta)\,[\beta l\sin\theta\sin(\beta l\cos\theta)]$$

$$+2\beta^2a^2\operatorname{cosec}^2\theta\cot\theta\,[\cos(\beta l\cos\theta)-\cos\beta l]\Big\} \qquad (78)$$

From $\nabla(\nabla\cdot\bar{A}) = \bar{a}_r\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) + \bar{a}_r\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\cdot\bar{A}) + \bar{a}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\nabla\cdot\bar{A})$

Since $\nabla\cdot\bar{A}$ is independent of ϕ

$$\therefore\quad \nabla(\nabla\cdot\bar{A}) = \bar{a}_r\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) + \bar{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\cdot\bar{A}) \qquad (79)$$

From $jw\bar{A} = \bar{a}_r jwA_r + \bar{a}_\theta jwA_\theta + \bar{a}_\phi jwA_\phi$

Since $A_\phi : 0$

$\therefore\quad jw\bar{A} = \bar{a}_r jwA_r + \bar{a}_\theta jwA_\theta$

Substituting equations (78) and (79) into equation (27)

$$\bar{E} = \bar{a}_r\, -\frac{jc^2}{w}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) - jwA_r + \bar{a}_\theta\, -\frac{jc^2}{w}\frac{1}{r}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) - jwA_\theta \qquad (80)$$

$$\therefore\quad E_r = \frac{-jc^2}{w}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) - jwA_r \qquad (81)$$

$$E_o = \frac{-jc^2}{wr}\frac{\partial}{\partial\theta}(\nabla\cdot\bar{A}) - jwA_\theta \qquad (82)$$

$$E_\phi = 0 \qquad (83)$$

Let $$M = (4-\beta^2a^2\operatorname{cosec}^2\theta)\cot\theta\left[\cos(\beta l\cos\theta)-\cos\beta l\right] \tag{84}$$

$$N = -(4-\beta^2a^2\operatorname{cosec}^2\theta)\left[\beta l\sin\theta\sin(\beta l\cos\theta)\right]$$

$$+2\beta^2a^2\operatorname{cosec}^2\theta\cot\theta\left[\cos(\beta l\cos\theta)-\cos\beta l\right] \tag{85}$$

Substituting equations (84) and (85) into equation (78), gives

$$\nabla\cdot\bar{A} = \frac{\mu I_o e^{j(wt-\beta r)}}{4\beta r^2\sin\theta}\left[M(1-j\beta r)+N\right] \tag{86}$$

$$-j\frac{c^2}{w}\frac{\partial}{\partial r}(\nabla\cdot\bar{A}) = -j\frac{c^2}{w}\frac{\partial}{\partial r}\,\frac{I_o\left[M+N-jM\beta r\right]e^{j(wt-\beta r)}}{4\beta r^2\sin\theta}$$

$$= \frac{-j30\pi I_o}{\beta\sin\theta}\left[-\frac{j\beta N}{r^2}-\frac{M\beta^2}{r}-\frac{2(M+N)}{r^3}\right]e^{j(wt-\beta r)} \tag{87}$$

$$-jwA_r = \frac{-jw\, I_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\cos\theta\left[\cos(\beta l\cos\theta-\cos\beta l\right]e^{j(wt-\beta r)}}{4\beta r\sin^2\theta} \tag{88}$$

Substituting equation (84) into equation (88)

$$-jwA_r = \frac{-jw\mu I_o M e^{j(wt-\beta r)}}{4\beta r\sin\theta}$$

$$= \frac{-j30\pi I_o\beta^2 M e^{j(wt-\beta r)}}{\beta^2 r\sin\theta} \tag{89}$$

Substituting equations (87) and (89) into equation (81), gives

$$E_r = \frac{-j30\pi I_o e^{j(wt-\beta r)}}{\beta^2\sin\theta}\left[-\frac{j\beta N}{r^2}-\frac{M\beta^2}{r}-\frac{2(M+N)}{r^3}+\frac{M\beta^2}{r}\right]$$

$$= \frac{-j30\pi I_o e^{j(wt-\beta r)}}{\beta^2\sin\theta}\left[-\frac{j\beta N}{r^2}-\frac{2(M+N)}{r^3}\right] \tag{90}$$

From equations (82) and (86), gives

$$-\frac{jc^2}{wr}\frac{\partial}{\partial\theta}(\nabla\cdot A) = \frac{jc^2}{wr}\frac{\partial}{\partial\theta}\frac{\mu I_o\left[M(1-j\beta r)+N\right]}{4\beta r^2\sin\theta}$$

$$= \frac{-jc\mu I_o e^{j(wt-\beta r)}}{4\beta^2 r^3}\left\{\frac{\left(\frac{\partial}{\partial\theta}(M+N) - j\beta r\frac{\partial M}{\partial\theta}\right)\sin\theta - (M+N-j\beta Mr)\cos\theta}{\sin^2\theta}\right\} \quad (91)$$

$$= \frac{-j30\pi I_o e^{j(wt-\beta r)}}{\beta^2\sin\theta}\left[\frac{1}{r^3}\frac{\partial}{\partial\theta}(M+N) - \frac{j\beta}{r^2}\frac{\partial M}{\partial\theta} - \frac{(M+N)\cot\theta}{r^3} + \frac{j\beta M\cot\theta}{r^2}\right]$$

$$-jwA_\theta = \frac{jw\mu I_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]\cot\theta e^{j(wt-\beta r)}}{4\beta r\sin\theta\cot\theta} \quad (92)$$

Substituting equation (84) into equation (92), gives

$$-jwA_\theta = \frac{j\beta c\mu I_o M e^{j(wt-\beta r)}}{4\beta r\sin\theta\cot\theta}$$

$$= \frac{j30\pi I_o M\beta^2 e^{j(wt-\beta r)}}{r\beta^2\sin\theta\cot\theta} \quad (93)$$

Substituting equations (91) and (93) into equation (82), gives

$$E_\theta = \frac{-j30\pi I_o e^{j(wt-\beta r)}}{\beta^2 r\sin\theta}\left[\frac{1}{r^3}\frac{\partial(M+N)}{\partial\theta} - \frac{j\beta\partial M}{r^2\partial\theta} - \frac{(M+N)\cot\theta}{r^3} + \frac{j\beta M\cot\theta}{r^2} - \frac{M\beta^2}{\cot\theta}\right] \quad (94)$$

For $r \gg 1$

$$E_\theta = \frac{-j30\pi I_o}{r\beta^2\sin\theta}\left(-\frac{M\beta^2}{\cot\theta}\right)e^{j(wt-\beta r)}$$

$$= \frac{j30\pi I_o M e^{j(wt-\beta r)}}{r\cos\theta} \quad (95)$$

For $r \gg 1$, $E_\theta \gg E_r$

$\therefore$ E_r can be neglected when compared with E_θ

$$E_\theta = \frac{j30\pi I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\cot\theta\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{r\cos\theta}$$

$$= \frac{j30\pi I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{r\sin\theta} \quad (96)$$

$$E_r = 0 \quad (97)$$

$$E_\phi = 0 \quad 998)$$

From $$H_\phi = \frac{E_\theta}{Z_o} \quad (99)$$

$$Z_o = 120\pi$$

$$\therefore \quad H_\phi = \frac{j30\pi I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{120\pi r\sin\theta}$$

$$= \frac{jI_o(4-\beta^2a^2\mathrm{cosec}^2e)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{4r\sin\theta} \quad (100)$$

$$H_r = 0 \quad (101)$$

$$H_\theta = 0 \quad (102)$$

B. Power Gain Analysis

The Power Gain (G) of an antenna is defined as

$$G = \frac{\text{Maximum radiation intensity}}{\text{Maximum radiation intensity from a reference antenna with same power input}}$$

Or

$$G = \frac{\text{Maximum Poynting vector}}{\text{Maximum Poynting vector from a reference antenna with same power input}}$$

Often the reference is a linear $\frac{1}{2}$ wave length antenna. In this analysis the reference is a $\frac{1}{2}$ wave length dipole antenna.

The average Poynting vector (P) is defined as

$$P = \frac{1}{2} \operatorname{Re} (\bar{E} \times \bar{H}^*) \tag{103}$$

$$\bar{E} \times \bar{H}^* = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ E_r & E_\theta & E_\phi \\ H_r^* & H_\theta^* & H_\phi^* \end{vmatrix}$$

$$= \bar{a}_r (E_\theta H_\phi^* - E_\phi H_\theta^*) - \bar{a}_\theta (E_r H_\phi^* - E_\phi H_r^*) + \bar{a}_\phi (E_r H_\theta^* - E_\theta H_r^*) \tag{104}$$

Since $E_\phi = E_r = H_\theta^* = H_r^* = 0$

$\therefore$ Equation (104) becomes

$$\bar{E} \times \bar{H}^* = \bar{a}_r (E_\theta H_\phi^*) \tag{105}$$

Substituting equation (105) into equation (103), gives

$$P = \frac{1}{2} \operatorname{Re} (E_\theta H_\phi^*) \tag{106}$$

1. Uniform Current Distribution

From equation (52)

$$E_\theta = \frac{j30\pi I_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\tan\theta\sin(\beta l\cos\theta)e^{j(wt-\beta r)}}{r}$$

$$= \frac{30\pi I_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\tan\theta\sin(\beta l\cos\theta)\left[-\sin(wt-\beta r)+j\cos(wt-\beta r)\right]}{r} \quad (107)$$

From equation (60)

$$H_\phi = \frac{jI_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\tan\theta\sin(\beta l\cos\theta)e^{j(wt-\beta r)}}{4r}$$

$$= \frac{I_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\tan\theta\sin(\beta l\cos\theta)\left[-\sin(wt-\beta r)+j\cos(wt-\beta r)\right]}{4r} \quad (108)$$

Then

$$H_\phi^* = \frac{I_o(4-\beta^2a^2\operatorname{cosec}^2\theta)\tan\theta\sin(\beta l\cos\theta)\left[-\sin(wt-\beta r)-j\cos(wt-\beta r)\right]}{4r} \quad (109)$$

Substituting equations (107) and (109) into (106)

$$P = \frac{1}{2}\operatorname{Re}\left\{\frac{30\pi I_o^2(4-\beta^2a^2\operatorname{cosec}^2\theta)^2\tan^2\theta\sin^2(\beta l\cos\theta)\left[\sin^2(wt-\beta r)+\cos^2(wt-\beta r)\right]}{4r^2}\right\}$$

$$= \frac{15\pi I_o^2(4-\beta^2a^2\operatorname{cosec}^2\theta)^2\tan^2\theta\sin^2(\beta l\cos\theta)}{4r^2} \quad (110)$$

Since $\beta^2a^2 \ll 1$

Equation (110) becomes

$$P = \frac{60\pi I_o^2\tan^2\theta\sin^2(\beta l\cos\theta)}{r^2} \quad (111)$$

Maximizing P, gives

$$\text{Pmax.} \quad = \quad \frac{60\pi I_o^{\,2}\tan^2\frac{\pi}{2}\sin^2(\beta l\cos\frac{\pi}{2})}{r^2} \tag{112}$$

By using l'Hospital's rule, equation (112) becomes

$$\text{Pmax.} \quad = \quad \frac{60\pi I_o^{\,2}\beta^2 l^2}{r^2} \tag{113}$$

From equation (5-80) and equation (5-81) 'Antennas' by Kraus (6)

For Thin Linear Antenna, assuming sinusoidal current distribution

$$H_\phi = \frac{jI_o}{2\pi r}\left[\frac{\cos(\beta\frac{2l}{2}\cos\theta)-\cos(\beta\frac{2l}{2})}{\sin\theta}\right]e^{j(wt-\beta r)}$$

$$= \frac{jI_o}{2\pi r}\left[\frac{\cos(\beta l\cos\theta)-\cos\beta l}{\sin\theta}\right]\left[\cos(wt-\beta r)+j\sin(wt-\beta r)\right]$$

$$= \frac{I_o}{2\pi r}\left[\frac{\cos(\beta l\cos\theta)-\cos\beta l}{\sin\theta}\right]\left[-\sin(wt-\beta r)+j\cos(wt-\beta r)\right] \tag{114}$$

Then

$$H_\phi^* = \frac{I_o}{2\pi r}\left[\frac{\cos(\beta l\cos\theta)-\cos\beta l}{\sin\theta}\right]\left[-\sin(wt-\beta r)-j\cos(wt-\beta r)\right] \tag{115}$$

$$E_\theta = \frac{j60I_o}{r}\left[\frac{\cos(\beta l\cos\theta)-\cos\beta l}{\sin\theta}\right]e^{j(wt-\beta r)}$$

$$= \frac{60I_o}{r}\left[\frac{\cos(\beta l\cos\theta)-\cos\beta l}{\sin\theta}\right]\left[-\sin(wt-\beta r)+j\cos(wt-\beta r)\right] \tag{116}$$

$$P = \frac{1}{2}\,\mathrm{Re}\,(\bar{E}\times\bar{H}^*)$$

Since $E_\phi = H_\theta^* = H_r^* = E_r = 0$

$$P = \frac{1}{2}\,\mathrm{Re}(E_\theta H_\phi^*)$$

$$P = \frac{1}{2} \mathrm{Re}\left\{ \frac{60 I_o^2}{2\pi r^2} \left[\frac{\cos(\beta l \cos\theta) - \cos\beta l}{\sin\theta} \right]^2 \left[\sin^2(wt - \beta r) + \cos^2(wt - \beta r) \right] \right\}$$

$$= \frac{60}{4\pi} \frac{I_o^2}{r^2} \left[\frac{\cos(\beta l \cos\theta) - \cos\beta l}{\sin\theta} \right]^2 \qquad (117)$$

$$\therefore \quad P_{\frac{\lambda}{2}\text{dipole}} = \frac{15 I_o^2 \left[\cos(\pi \cos\theta) + 1 \right]^2}{\pi r^2 \sin^2\theta} \qquad (118)$$

Maximizing equation (118), gives

$$P_{\frac{\lambda}{2}\text{dipole (max.)}} = \frac{15 I_o^2 \left[1 + \cos(\pi \cos\frac{\pi}{2}) \right]^2}{\pi r^2 \sin^2\frac{\pi}{2}} \qquad (119)$$

$$= \frac{60 I_o^2}{\pi r^2} \qquad (120)$$

$$\text{Power Gain} = G = \frac{P_{max.}}{P_{\frac{\lambda}{2}\text{dipole(max.)}}} \qquad (121)$$

Substituting equations (113) and (120) into equation (121), gives

$$G = \frac{\dfrac{60\pi I_o^2 \beta^2 l^2}{r^2}}{\dfrac{60 I_o^2}{\pi r^2}}$$

$$= \pi^2 \beta^2 l^2 \qquad (122)$$

2. Sinusoidal Current Distribution

From equation (96)

$$E_\theta = \frac{j30\pi I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{r\sin\theta}$$

$$= \frac{30\pi I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]\left[-\sin(wt-\beta r)+j\cos(wt-\beta r)\right]}{r\sin\theta} \quad (123)$$

From equation (100)

$$H_\phi = \frac{jI_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]e^{j(wt-\beta r)}}{4r\sin\theta}$$

$$= \frac{I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]\left[-\sin(wt-\beta r)+j\cos(wt-\beta r)\right]}{4r\sin\theta}$$

$$H_\phi^* = \frac{I_o(4-\beta^2a^2\mathrm{cosec}^2\theta)\left[\cos(\beta l\cos\theta)-\cos\beta l\right]\left[-\sin(wt-\beta r)-j\cos(wt-\beta r)\right]}{4r\sin\theta} \quad (124)$$

Substituting equations (123) and (124) into (106), gives

$$P = \frac{1}{2}\,\mathrm{Re}\left\{\frac{30\pi I_o^2(4-\beta^2a^2\mathrm{cosec}^2\theta)^2\left[\cos(\beta l\cos\theta)-\cos\beta l\right]^2}{4r^2\sin^2\theta}\right\}$$

$$= \frac{15\pi I_o^2(4-\beta^2a^2\mathrm{cosec}^2\theta)^2\left[\cos(\beta l\cos\theta)-\cos\beta l\right]^2}{2r^2\sin^2\theta} \quad (125)$$

Maximizing equation (125), gives

$$P_{max.} = \frac{15\pi I_o^2(4-\beta^2a^2\mathrm{cosec}^2\frac{\pi}{2})^2\left[\cos(\beta l\cos\frac{\pi}{2})-\cos\beta l\right]^2}{2r^2\sin^2\frac{\pi}{2}}$$

$$= \frac{15\pi I_o^2(4-\beta^2a^2)^2(1-\cos\beta l)^2}{2r^2} \quad (126)$$

Substituting equations (120) and (126) into equation (116), gives

$$G = \frac{\dfrac{15\pi I_o^2(4-\beta^2a^2)^2(1-\cos\beta l)^2}{2r^2}}{\dfrac{60I_o^2}{\pi r^2}}$$

$$= \frac{\pi^2(1-\cos\beta l)^2(4-\beta^2a^2)^2}{8} \qquad (127)$$

Since $\beta a \ll 1$

$$\therefore \quad G = 2\pi^2(1-\cos\beta l)^2 \qquad (128)$$

C. Data Precalculated for the Experiment Set Up

The antenna used in the experiment: $l = 1$ m.

$a = 8$ cm.

(1) At frequency = 150 MHz.

$$\beta a = \frac{2\pi f \times 0.08}{c} = \frac{0.16\pi f}{3\times10^8} = 0.08\pi$$

$$\beta^2 a^2 = 0.0064\pi^2 = 0.0634 \qquad (129)$$

$$\beta l = \frac{2\pi f l}{c} = \frac{2\pi\times1.5\times10^8}{3\times10^8} = \pi \qquad (130)$$

Substituting equations (129) and (130) into equations (96) and (100), give

$$E_\theta = \frac{j30\pi I_o(4-0.0634\operatorname{cosec}^2\theta)\left[\cos(\pi\cos\theta)-\cos\pi\right]}{r\sin\theta}$$

$$= \frac{j30\pi I_o(4-0.0634\operatorname{cosec}^2\theta)\left[1+\cos(\pi\cos\theta)\right]}{r\sin\theta} \qquad (131)$$

$$H_\phi = \frac{jI_o(4-0.0634\operatorname{cosec}^2\theta)\left[1+\cos(\pi\cos\theta)\right]}{4r\sin\theta} \qquad (132)$$

(2) At frequency = 126 MHz.

$$\beta a = \frac{2\pi f\times0.08}{c} \doteq \frac{2\pi\times1.26\times10^8\times0.08}{3\times10^8} = 0.2102$$

$$\beta^2 a^2 = 0.0445 \qquad (133)$$

$$\beta l = \frac{2\pi \times 1.26\times10^{8}}{3\times10^{8}} = 0.838\pi \qquad (134)$$

Substituting equations (133) and (134) into equations (96) and (100), give

$$E_{\theta} = \frac{j30\pi I_{o}(4-0.0445\mathrm{cosec}^{2}\theta)\left[\cos(0.838\pi\cos\theta)-\cos 0.838\pi\right]}{r\sin\theta}$$

$$= \frac{j30\pi I_{o}(4-0.0445\mathrm{cosec}^{2}\theta)\left[\cos(0.838\pi\cos\theta)+0.97235\right]}{r\sin\theta} \qquad (135)$$

$$H_{\phi} = \frac{jI_{o}(4-0.0445\mathrm{cosec}^{2}\theta)\left[\cos(0.838\pi\cos\theta)+0.97235\right]}{4r\sin\theta} \qquad (136)$$

D. Computer Program for Theoretical Analysis

This theoretical analysis is run by the computer NEAC-SERIES 2200 which was installed at the Computer Science Center, Chulalongkorn University.

```
C       PROGRAM SOLVING THE FIELDS OF A HOLLOW CYLINDRICAL ANTENNA
C       EAI = FIELD STRENGTH OF THE ANTENNA
C       Y  = MAXIMUM VALUE OF THE FIELD
C       XY = NORMALIZE VALUE OF THE FIELD
        DIMENSION EA1(181),XY(181)
        ID=1
      5 WRITE(3,15)
     15 FORMAT(1H1,10X,6HDEGREE,10X,5HVALUE, 8X,10HNORMALIZE ,/)
        A=0
        I=0
        GO TO (10,20,70),ID
     10 X=A*3.1416/180.
        I=I+1
        AE01=3.1416*COS(X)
         E0A1=4.-(0.0634/(SIN(X)*SIN(X)))
        E02=(1.+COS(AE01))/SIN(X)
         EA1(I)=E0A1*E02
        IF(A.EQ.360.)GO TO 40
        A=A+2.
        GO TO 10
     20 X=A*3.1416/180.
        I=I+1
        AE02 = 0.838*3.1416*COS(X)
        E04 =(0.97235+COS(AE02))/SIN(X)
        E0A2=4.-(0.0445/(SIN(X)*SIN(X)))
        EA1(I)=E0A2*E04
        IF(A.EQ.360.)GO TO 40
        A=A+2.
        GO TO 20
     40 Y=EA1(46)
        NN=0
        DO45N=1,181
        XY(N)=EA1(N)/Y
        WRITE(3,35)   NN,EA1(N),XY(N)
     35 FORMAT(I14,2F17.6)
        NN=NN+2
     45 CONTINUE
         ID=ID+1
        GO TO 5
     70 STOP
        END
```

1. Field Pattern of the Hollow Cylindrical Antenna from Theory at
   Frequency = 150 MHz.

DEGREE	VALUE	NORMALIZE
0*	.9999999999E+99*	.1270131586E+99
2	-.002502	-.000318
4	-.003783	-.000480
6	-.002552	-.000324
8	.002439	.000310
10	.012439	.001580
12	.028696	.003645
14	.052451	.006662
16	.084933	.010788
18	.127356	.016176
20	.180908	.022978
22	.246741	.031339
24	.325959	.041401
26	.419604	.053295
28	.528638	.067144
30	.653925	.083057
32	.796213	.101130
34	.956107	.121438
36	1.134049	.144039
38	1.330296	.168965
40	1.544890	.196221
42	1.777645	.225784
44	2.028117	.257597
46	2.295588	.291570
48	2.579052	.327574
50	2.877203	.365443
52	3.188427	.404972
54	3.510803	.445918
56	3.842109	.487998
58	4.179833	.530894
60	4.521199	.574252
62	4.863188	.617689
64	5.202582	.660796
66	5.536001	.703145
68	5.859957	.744292
70	6.170908	.783787
72	6.465320	.821181
74	6.739729	.856034
76	6.990810	.887925
78	7.215437	.916455
80	7.410752	.941263
82	7.574222	.962026
84	7.703693	.978470
86	7.797438	.990377
88	7.854197	.997586
90	7.873200	1.000000
92	7.854189	.997585
94	7.797423	.990375
96	7.703669	.978467
98	7.574191	.962022
100	7.410714	.941258
102	7.215393	.916450
104	6.990760	.887918
106	6.739674	.856027
108	6.465260	.821173
110	6.170844	.783778
112	5.859890	.744283
114	5.535932	.703136
116	5.202511	.660787

118	4.863116	.617680
120	4.521127	.574243
122	4.179762	.530885
124	3.842038	.487989
126	3.510735	.445909
128	3.188361	.404964
130	2.877139	.365434
132	2.578991	.327566
134	2.295529	.291562
136	2.028062	.257591
138	1.777594	.225778
140	1.544843	.196215
142	1.330252	.168960
144	1.134010	.144034
146	.956071	.121434
148	.796181	.101126
150	.653897	.083054
152	.528613	.067141
154	.419582	.053292
156	.325941	.041399
158	.246726	.031337
160	.180896	.022976
162	.127346	.016175
164	.084925	.010787
166	.052445	.006661
168	.028692	.003644
170	.012437	.001580
172	.002438	.000310
174	-.002552	-.000324
176	-.003783	-.000480
178	-.002502	-.000318
180	4797.211713	609.309012
182	.002503	.000318
184	.003783	.000481
186	.002551	.000324
188	-.002441	-.000310
190	-.012442	-.001580
192	-.028700	-.003645
194	-.052456	-.006663
196	-.084941	-.010789
198	-.127366	-.016177
200	-.180921	-.022979
202	-.246756	-.031341
204	-.325977	-.041403
206	-.419625	-.053298
208	-.528662	-.067147
210	-.653953	-.083061
212	-.796245	-.101134
214	-.956143	-.121443
216	-1.134089	-.144044
218	-1.330339	-.168971
220	-1.544937	-.196227
222	-1.777696	-.225791
224	-2.028171	-.257604
226	-2.295646	-.291577
228	-2.579113	-.327581
230	-2.877267	-.365451
232	-3.188494	-.404981
234	-3.510872	-.445927
236	-3.842179	-.488007
238	-4.179905	-.530903

240	-4.521271	-.574261
242	-4.863260	-.617698
244	-5.202653	-.660805
246	-5.536070	-.703154
248	-5.860024	-.744300
250	-6.170972	-.783795
252	-6.465380	-.821188
254	-6.739785	-.856041
256	-6.990860	-.887931
258	-7.215481	-.916461
260	-7.410790	-.941268
262	-7.574253	-.962030
264	-7.703716	-.978473
266	-7.797454	-.990379
268	-7.854205	-.997587
270	-7.873200	-1.000000
272	-7.854181	-.997584
274	-7.797407	-.990373
276	-7.703646	-.978464
278	-7.574160	-.962018
280	-7.410676	-.941253
282	-7.215348	-.916444
284	-6.990709	-.887912
286	-6.739618	-.856020
288	-6.465200	-.821165
290	-6.170780	-.783770
292	-5.859823	-.744275
294	-5.535862	-.703127
296	-5.202440	-.660778
298	-4.863045	-.617671
300	-4.521055	-.574233
302	-4.179690	-.530876
304	-3.841968	-.487980
306	-3.510666	-.445901
308	-3.188294	-.404955
310	-2.877075	-.365426
312	-2.578929	-.327558
314	-2.295471	-.291555
316	-2.028007	-.257584
318	-1.777543	-.225771
320	-1.544796	-.196209
322	-1.330209	-.168954
324	-1.133971	-.144029
326	-.956036	-.121429
328	-.796150	-.101121
330	-.653869	-.083050
332	-.528588	-.067138
334	-.419561	-.053290
336	-.325923	-.041396
338	-.246711	-.031336
340	-.180883	-.022975
342	-.127336	-.016173
344	-.084917	-.010786
346	-.052439	-.006660
348	-.028688	-.003644
350	-.012434	-.001579
352	-.002436	-.000309
354	.002553	.000324
356	.003783	.000480
358	.002501	.000318
360	-599.656074	-76.164212

I16077921

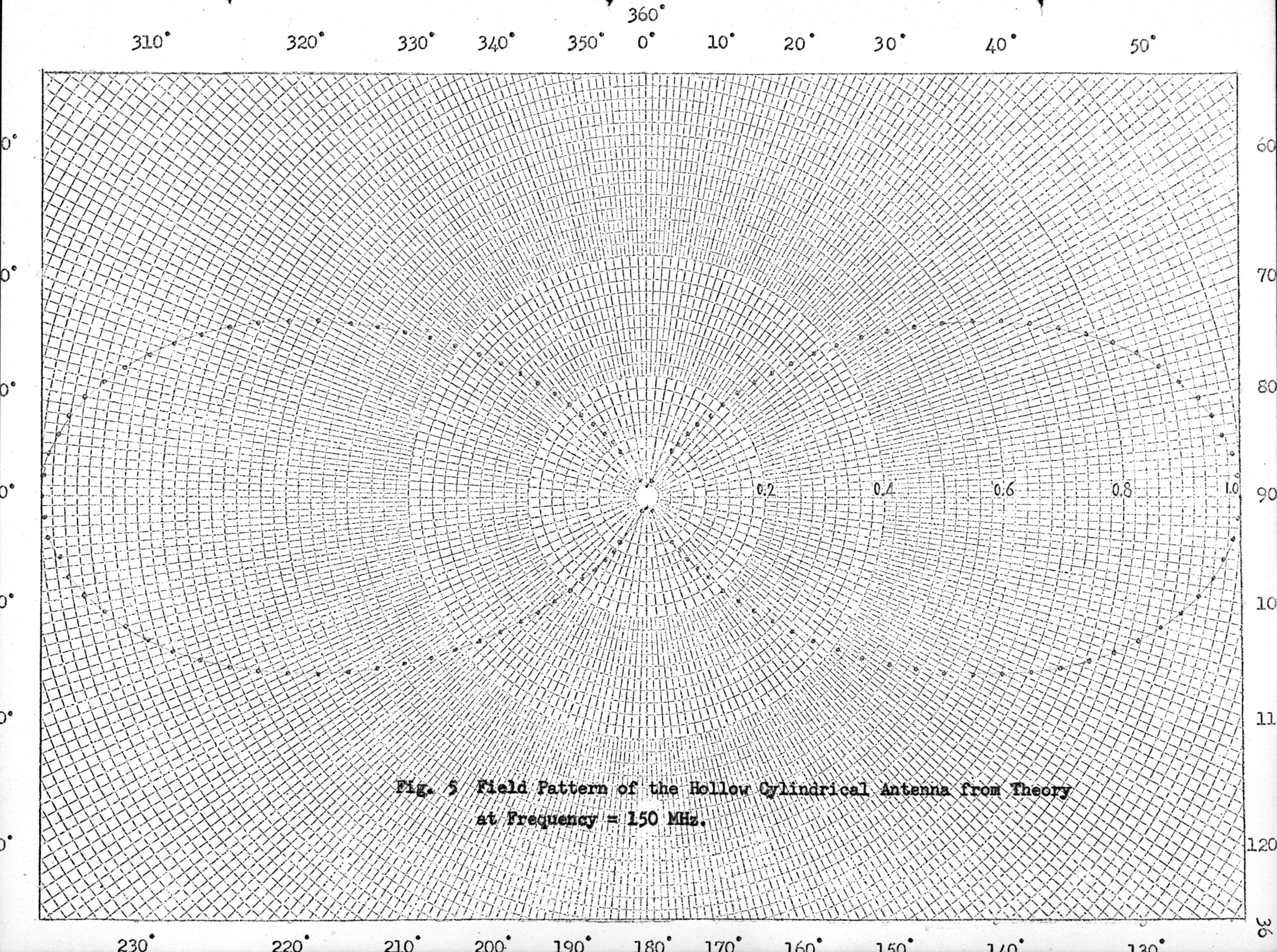


Fig. 5 Field Pattern of the Hollow Cylindrical Antenna from Theory at Frequency = 150 MHz.

2. Field Pattern of the Hollow Cylindrical Antenna from Theory at Frequency = 126 MHz.

DEGREE	VALUE	NORMALIZE
0*	.9999999999E+99*	.1281783352E+99
2	-93.103254	-11.933820
4	-7.540075	-.966474
6	-.073925	-.009476
8	1.368331	.175390
10	1.733700	.222223
12	1.836599	.235412
14	1.872346	.239994
16	1.898011	.243284
18	1.931952	.247634
20	1.980101	.253806
22	2.044090	.262008
24	2.124030	.272255
26	2.219518	.284494
28	2.330008	.298657
30	2.454932	.314669
32	2.593733	.332460
34	2.745857	.351959
36	2.910731	.373093
38	3.087743	.395782
40	3.276220	.419940
42	3.475404	.445472
44	3.684441	.472266
46	3.902367	.500199
48	4.128095	.529132
50	4.360418	.558911
52	4.597997	.589364
54	4.839372	.620303
56	5.082963	.651526
58	5.327078	.682816
60	5.569929	.713944
62	5.809649	.744671
64	6.044307	.774749
66	6.271933	.803926
68	6.490544	.831947
70	6.698168	.858560
72	6.892872	.883517
74	7.072797	.906579
76	7.236177	.927521
78	7.381377	.946133
80	7.506915	.962224
82	7.611491	.975628
84	7.694006	.986205
86	7.753584	.993841
88	7.789588	.998456
90	7.801630	1.000000
92	7.789582	.998456
94	7.753574	.993840
96	7.693991	.986203
98	7.611471	.975626
100	7.506891	.962221
102	7.381348	.946129
104	7.236144	.927517
106	7.072760	.906575
108	6.892833	.883512
110	6.698125	.858555
112	6.490499	.831941
114	6.271886	.803920
116	6.044258	.774743

118	5.809599	.744665
120	5.569878	.713938
122	5.327026	.682809
124	5.082911	.651519
126	4.839321	.620296
128	4.597947	.589357
130	4.360368	.558905
132	4.128047	.529126
134	3.902320	.500193
136	3.684396	.472260
138	3.475361	.445466
140	3.276179	.419935
142	3.087705	.395777
144	2.910695	.373088
146	2.745823	.351955
148	2.593702	.332456
150	2.454904	.314666
152	2.329983	.298653
154	2.219496	.284491
156	2.124011	.272252
158	2.044074	.262006
160	1.980089	.253805
162	1.931943	.247633
164	1.898006	.243283
166	1.872341	.239994
168	1.836588	.235411
170	1.733662	.222218
172	1.368193	.175373
174	-.074508	-.009550
176	-7.543635	-.966931
178	-93.166548	-11.941933
180*	.1112101189E+14*	.1425472791E+13
182	93.040013	11.925714
184	7.536517	.966018
186	.073342	.009401
188	-1.368470	-.175408
190	-1.733737	-.222228
192	-1.836610	-.235414
194	-1.872352	-.239995
196	-1.898017	-.243285
198	-1.931961	-.247635
200	-1.980113	-.253808
202	-2.044105	-.262010
204	-2.124048	-.272257
206	-2.219540	-.284497
208	-2.330033	-.298660
210	-2.454960	-.314673
212	-2.593764	-.332464
214	-2.745890	-.351964
216	-2.910767	-.373097
218	-3.087782	-.395787
220	-3.276261	-.419946
222	-3.475447	-.445477
224	-3.684486	-.472271
226	-3.902413	-.500205
228	-4.128144	-.529139
230	-4.360467	-.558917
232	-4.598048	-.589370
234	-4.839423	-.620309
236	-5.083014	-.651532
238	-5.327129	-.682823

240	-5.569980	-.713951
242	-5.809699	-.744678
244	-6.044356	-.774755
246	-6.271980	-.803932
248	-6.490589	-.831953
250	-6.698210	-.858565
252	-6.892912	-.883522
254	-7.072833	-.906584
256	-7.236209	-.927525
258	-7.381405	-.946136
260	-7.506940	-.962227
262	-7.611511	-.975631
264	-7.694021	-.986207
266	-7.753594	-.993843
268	-7.789593	-.998457
270	-7.801630	-1.000000
272	-7.789577	-.998455
274	-7.753563	-.993839
276	-7.693976	-.986201
278	-7.611452	-.975623
280	-7.506867	-.962218
282	-7.381320	-.946125
284	-7.236112	-.927513
286	-7.072724	-.906570
288	-6.892793	-.883507
290	-6.698083	-.858549
292	-6.490454	-.831936
294	-6.271839	-.803914
296	-6.044210	-.774737
298	-5.809549	-.744658
300	-5.569828	-.713931
302	-5.326975	-.682803
304	-5.082860	-.651513
306	-4.839270	-.620290
308	-4.597896	-.589351
310	-4.360319	-.558898
312	-4.127999	-.529120
314	-3.902273	-.500187
316	-3.684351	-.472254
318	-3.475318	-.445460
320	-3.276138	-.419930
322	-3.087666	-.395772
324	-2.910659	-.373083
326	-2.745790	-.351951
328	-2.593672	-.332453
330	-2.454876	-.314662
332	-2.329958	-.298650
334	-2.219475	-.284489
336	-2.123993	-.272250
338	-2.044059	-.262004
340	-1.980077	-.253803
342	-1.931935	-.247632
344	-1.898000	-.243282
346	-1.872335	-.239993
348	-1.836576	-.235409
350	-1.733624	-.222213
352	-1.368055	-.175355
354	.075091	.009625
356	7.547196	.967387
358	93.229897	11.950053

360*-.1390137168E+13*-.1781854680E+12

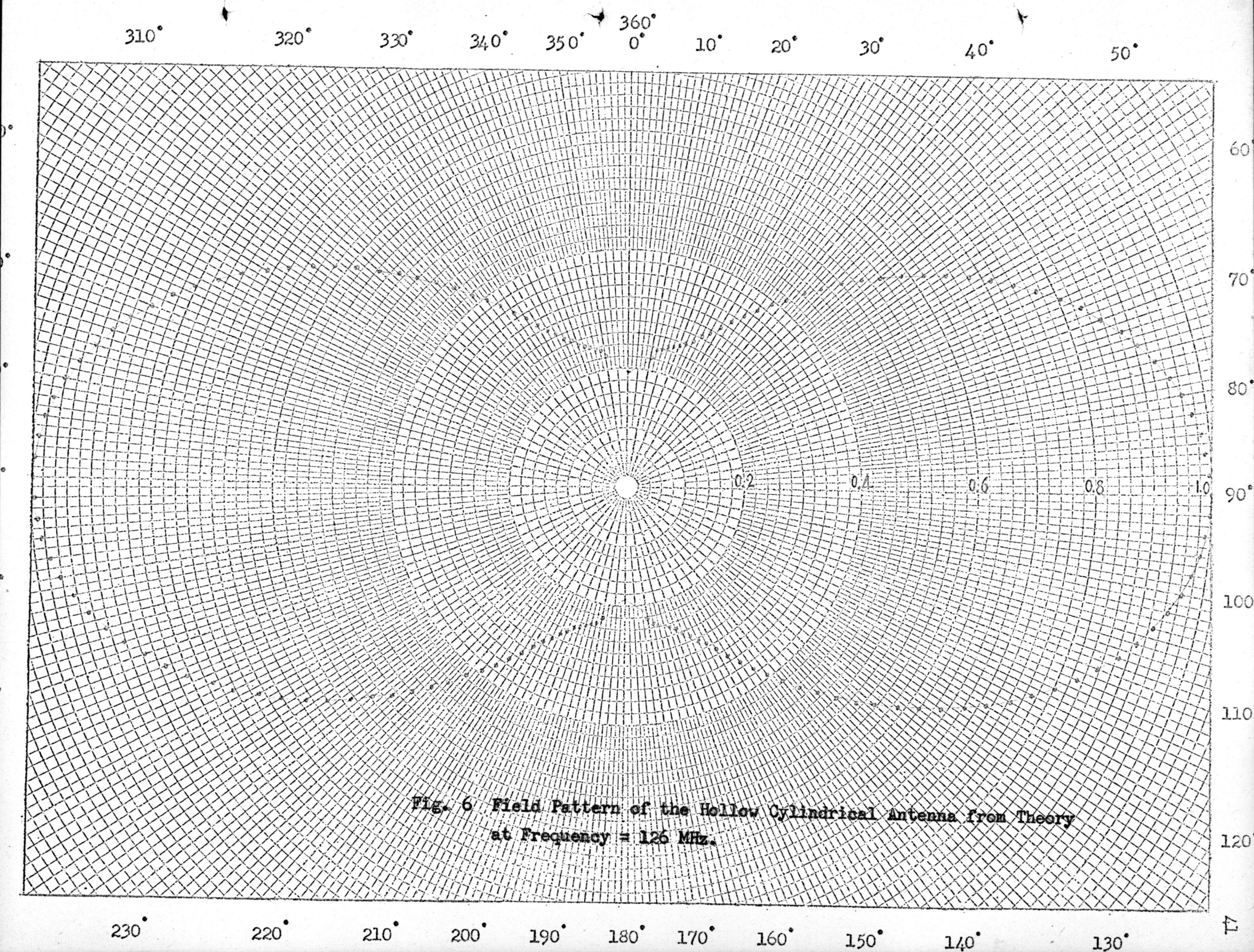


Fig. 6 Field Pattern of the Hollow Cylindrical Antenna from Theory at Frequency = 126 MHz.