Chapter IV

A PATH INTEGRAL APPROACH TO DISORDERED SYSTEMS

This chapter is concerned with the application of the Feynman's path integral formalism introduced in chapter III to disordered systems.

Since the time-dependent Green function can be expressed as

$$G(\underline{r},\underline{r}';t) = \int \mathcal{N}\,\mathcal{D}(\text{path})\exp\left\{\frac{i}{\hbar}S\left[\underline{r}(\tau)\right]\right\},$$

where $\mathcal{N}$ is the normalization constant and $S\left[\underline{r}(\tau)\right]$ is the action, it follows that for disordered systems the averaged time - dependent Green function may be taken as

$$\left\langle G(\underline{r},\underline{r}';t)\right\rangle = \left\langle \int \mathcal{N}\,\mathcal{D}(\text{path})\exp\left[\frac{i}{\hbar}\int_0^t d\tau\left\{\frac{m\dot{\underline{r}}^2(\tau)}{2} - \sum U\left(\underline{r}(\tau)-\underline{R}_\alpha\right)\right\}\right]\right\rangle \quad \text{--------(4.1)}$$

where $\langle\text{----}\rangle$ denotes the average over all the possible configurations of ions.

Edwards and Gulyaev[13] were the first to consider the evaluation of Eq. (4.1) for the case of semiconductors with high density of randomly distributed impurities.

Since the probability distribution $P(\underline{R}_1\ldots\ldots\underline{R}_N)$ for completely disordered systems is

$$P(\underline{R}_1\cdots\cdots\underline{R}_N) = \prod_{\alpha=1}^{N}\frac{d\underline{R}_\alpha}{\int d\underline{R}_\alpha} = \prod_{\alpha=1}^{N}\frac{d\underline{R}_\alpha}{V},$$

where N is the number of the scattering centers and V is the volume of the system, therefore the averaged Green function is

$$\langle G(\underline{r},\underline{r}';t)\rangle = \mathcal{N}\int\int\cdots\int_{\underline{r}(0)=\underline{r}'}^{\underline{r}(t)=\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t\left\{\frac{1}{2}m\dot{\underline{r}}^2-\sum_\alpha U\left(\underline{r}(\tau)-\underline{R}_\alpha\right)\right\}d\tau\right]\mathcal{D}(\text{path})\prod_{\alpha=1}^{N}\left(\frac{d\underline{R}_\alpha}{V}\right)$$

$$= \mathcal{N}\int_{\underline{r}'}^{\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t\frac{1}{2}m\dot{\underline{r}}^2\,d\tau\right]\int\int\cdots\int \exp\left[-\frac{i}{\hbar}\int_0^t\sum_\alpha U\left(\underline{r}(\tau)-\underline{R}_\alpha\right)d\tau\right]\prod_{\alpha=1}^{N}\left(\frac{d\underline{R}_\alpha}{V}\right)\mathcal{D}(\text{path})$$

$$= \mathcal{N}\int_{\underline{r}'}^{\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t\frac{1}{2}m\dot{\underline{r}}^2\,d\tau\right]\left\{\int \exp\left[-\frac{i}{\hbar}\int_0^t U\left(\underline{r}(\tau)-\underline{R}\right)d\tau\,\frac{d\underline{R}}{V}\right]\right\}^N\mathcal{D}(\text{path})$$

$$= \mathcal{N}\int_{\underline{r}'}^{\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t\frac{1}{2}m\dot{\underline{r}}^2\,d\tau\right]\left\{1+\frac{\rho\int\exp\left[-\frac{i}{\hbar}\int_0^t U\left(\underline{r}(\tau)-\underline{R}\right)d\tau\right]d\underline{R}}{N}-1\right\}^N\mathcal{D}(\text{path})$$

$$= \mathcal{N}\int_{\underline{r}'}^{\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t\frac{1}{2}m\dot{\underline{r}}^2\,d\tau\right]\left\{1+\frac{\left[\rho\int\exp\left\{-\frac{i}{\hbar}\int_0^t U\left(\underline{r}(\tau)-\underline{R}\right)d\tau\right\}d\underline{R}\right]-N}{N}\right\}^N\mathcal{D}(\text{path}),$$

where ρ is the density of the scatterers.

Since $\left(1+\frac{X}{N}\right)^N = \exp X$, therefore $\left\{1+\frac{\left[\rho\int\exp\left\{-\frac{i}{\hbar}\int_0^t U\left(\underline{r}(\tau)-\underline{R}\right)d\tau\right\}d\underline{R}\right]-N}{N}\right\}^N$

$$= \exp\left\{\left[\rho\int\exp\left\{-\frac{i}{\hbar}\int_0^t U\left(\underline{r}(\tau)-\underline{R}\right)d\tau\right\}d\underline{R}\right]-N\right\}.$$

Thus

$$\langle G(\underline{r},\underline{r}';t)\rangle = \mathcal{N}\int_{\underline{r}'}^{\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t \frac{1}{2} m \dot{\underline{r}}^2 d\tau + \left\{\rho\int \exp\left[-\frac{i}{\hbar}\int_0^t U(\underline{r}(\tau)-\underline{R})\, d\tau\right]\right.\right.$$

$$\left. - N\right] \mathcal{D}(\text{path})$$

$$= \mathcal{N}\int_{\underline{r}'}^{\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t \frac{1}{2} m \dot{\underline{r}}^2 d\tau + \left\{\rho\int \exp\left[-\frac{i}{\hbar}\int_0^t U(\underline{r}(\tau)-\underline{R})\, d\tau\right] d\underline{R}\right\}\right.$$

$$\left. -\rho\int d\underline{R}\right] \mathcal{D}(\text{path})$$

$$= \mathcal{N}\int_{\underline{r}'}^{\underline{r}} \exp\left[\frac{i}{\hbar}\int_0^t \frac{1}{2} m \dot{\underline{r}}^2 d\tau + \rho\int d\underline{R}\left\{\exp\left[-\frac{i}{\hbar}\int_0^t U(\underline{r}(\tau)-\underline{R})\, d\tau\right]\right.\right.$$

$$\left.\left. - 1\right\}\right] \mathcal{D}(\text{path}) \,.$$

$$\text{------- } (4.2)$$

This is an exact expression for the average time-dependent Green function for completely disordered systems. By writing $t \rightarrow i\hbar\beta$, Eq. (4.2) becomes

$$\langle G(r,r';\beta)\rangle = \int \mathcal{N}\mathcal{D}(\text{path}) \exp\left[-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \frac{1}{2} m \dot{\underline{r}}^2 + \rho\int d\underline{R}\right.$$

$$\left.\left\{\exp\left[-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\, U\left(\underline{r}(\tau)-\underline{R}\right)\right] - 1\right\}\right].$$

$$\text{---- } (4.3)$$

By considering $U(\underline{r}(\tau)-\underline{R})$ weak, the exponential of $U(\underline{r}(\tau)-\underline{R})$ in Eq. (4.3) can be expanded and only the linear and quadratic terms survive. Thus Eq. (4.3) becomes

$$\langle G(\underline{r},\underline{r}';\beta)\rangle = \int \mathcal{N}\,\mathcal{D}(\text{path}) \exp\Bigg[-\frac{1}{\hbar}\int_0^{\hbar\beta}\frac{m\dot{\underline{r}}^2}{2}\,d\tau - \frac{\rho}{\hbar}\int d\underline{R}\int_0^{\hbar\beta} d\tau\, U(\underline{r}$$

$$+\frac{\rho}{2\hbar^2}\int d\underline{R}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, U(\underline{r}(\tau)-\underline{R})\, U(\underline{r}(\tau')-\underline{R})$$

------------ (4.4)

Considering energy E of the electron approaches $-\infty$, therefore t must approach zero and hence $\underline{r}(\tau)$ is small. Therefore an expansion of $U(\underline{r}(\tau)-\underline{R})$ in $\underline{r}(\tau)$ can be made and we need only keep the first two terms of the expansion. If we define

$$\overline{U} = \int d\underline{R}\, U ,$$

$$\overline{U^2} = \int d\underline{R}\, U^2 ,$$

$$W = \int d\underline{R}\, (\nabla U)^2 .$$

Eq. (4.4) becomes

$$\langle G(\underline{r},\underline{r}';\beta)\rangle = \int \mathcal{N}\,\mathcal{D}(\text{path}) \exp\Bigg[-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{r}}^2(\tau) - \rho\,\overline{U}\,\beta + \rho\,\overline{U^2}\,\beta^2$$

$$-\frac{\rho W}{\hbar^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \frac{1}{3}\left\{\underline{r}(\tau)-\underline{r}(\tau')\right\}^2\Bigg] . \quad \text{-----} ($$

The term $\rho \bar{U} \beta$ in the exponent of Eq.(4.5) becomes infinite in the limit $\rho \to \infty$; however, we are free to choose our energy origin wherever we please; if therefore we choose this origin as $\bar{U}$, the possibility of $\rho \bar{U} \beta$ reaching infinity is removed. Hence Eq.(4.5) becomes

$$\langle G(\underline{r}, \underline{r}'; \beta) \rangle = \int \mathcal{N} \mathcal{D}(\text{path}) \exp\left[-\frac{m}{2\hbar} \int_0^{\hbar\beta} d\tau \, \dot{\underline{r}}^2(\tau) + \rho \overline{U}^2 \beta^2 - \frac{\rho \omega}{\hbar^2} \int_0^{\hbar\beta} \int_0^{\hbar\beta} d\tau \, d\tau' \frac{1}{3} \left\{ \underline{r}(\tau) - \underline{r}(\tau') \right\}^2 \right] \quad \text{----- (4.6)}$$

Eq.(4.4) can also be written as

$$\langle G(\underline{r}, \underline{r}'; \beta) \rangle = \int \mathcal{N} \mathcal{D}(\text{path}) \exp\left[-\frac{m}{2\hbar} \int_0^{\hbar\beta} d\tau \, \dot{\underline{r}}^2(\tau) + \frac{\rho}{2\hbar^2} \int_0^{\hbar\beta} \int_0^{\hbar\beta} d\tau \, d\tau' \, W\left[\underline{r}(\tau) - \underline{r}(\tau') \right] \right]$$

where $W\left[r(\tau) - r(\tau')\right]$ is the correlation function of the potential and is equal to $\int d\underline{R} \, U\left(\underline{r}(\tau) - \underline{R}\right) U\left(\underline{r}(\tau') - \underline{R}\right)$.

Taking the Fourier transform with respect to $\underline{k}$, the correlation function of the potential $W\left[\underline{r}(\tau) - \underline{r}(\tau')\right]$ becomes

$$W\left[\underline{r}(\tau) - \underline{r}(\tau')\right] = \frac{1}{(2\pi)^3} \int d\underline{k} \, \left| u(\underline{k}) \right|^2 \exp\left[i\underline{k} \cdot \left\{ \underline{r}(\tau) - \underline{r}(\tau') \right\} \right].$$

It can be seen that the correlation function $W\left[r(\tau) - r(\tau')\right]$ depends on the form of the scattering potential. If the scattering potential is a delta function, $u(\underline{k})$ is a constant, therefore the correlation function is again a delta function. For a screened-Coulomb potential,

$u(\underline{k}) = \dfrac{4\pi}{k^2 + \lambda^2}$, where λ is the screening length, the correlation function turns out to be an exponential having a characteristic length. As a model for theoretical study it is, however, preferable to choose the correlation function as Gaussian function, $W[\underline{r}(\tau)-\underline{r}(\tau')] = \exp\left[-\frac{(\underline{r}(\tau)-\underline{r}(\tau'))^2}{L^2}\right]$ where L is the correlation length. The reason for choosing this function is that it can represent the characteristic length of the exponential function as well as the characteristic length of the delta function.

Bezak[14, 15] have studied the averaged Green function of disordered systems by using the Gaussian correlation function. In order to simplify the problem, he studied the approximate form of the Gaussian correlation function, i.e. keeping only the first two terms of the expansion. The result is

$$\langle G(\underline{r},\underline{r}';\beta)\rangle = \int \mathcal{N}\mathcal{D}(\text{path}) \exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{r}}^2(\tau) + \frac{\gamma^2}{2\hbar^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[1-\frac{\{\underline{r}(\tau)-\underline{r}(\tau')\}^2}{L^2}\right]\right\}$$

$$= \int \mathcal{N}\mathcal{D}(\text{path}) \exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{r}}^2(\tau) + \frac{\gamma^2\beta^2}{2} - \frac{\gamma^2}{2\hbar^2 L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right\} \quad \text{---- (4.7)}$$

Comparing the two Eqs.(4.7)and (4.6), it can be seen that the problem considered by Bezák is essentially the same problem as that considered by Edwards and Gulyaev. The difference being that, instead of characterizing the potential by the density of scatterers, ρ , as Edwards and Gulyaev, Bezák characterized the potential by two parameters: γ (whose square is the variance of the potential energy) and L (correlation length).

To solve the problem in Eq.(4.6), Edwards and Gulyaev further simplified this equation by replacing the non-local effect, $[\underline{r}(\tau)-\underline{r}(\tau')]^2$, by a local one, $[\underline{r}(\tau)]^2$. Thus Eq. (4.6) becomes

$$\langle G(\underline{x},\underline{x}';\beta)\rangle = \exp\left(\rho\,\overline{U^2}\beta^2\right)\int \mathcal{N}\mathcal{D}(\text{path})\exp\left[-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{x}}^2(\tau) - \frac{\rho W}{3\hbar^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{x}^2(\tau)\right].$$

It can now be seen that the problem of evaluating the averaged Green function by the method of Edwards and Gulyaev is the same as that of the harmonic oscillator which has already been calculated at the end of chapter III.

Bezák, however, retained the non-local effect and set about the problem by using the method described at the end of chapter III, reducing the path integrals to a product of two functions, $\exp\left(\frac{1}{\hbar}S_{cl}\right)$ and $F_y(\hbar\beta)$,

where
$$S_{cl} = -\frac{m}{2}\int_0^{\hbar\beta} d\tau\, \dot{\underline{x}}_c^2(\tau) - \frac{\gamma^2}{2\hbar L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[\underline{x}_c(\tau)-\underline{x}_c(\tau')\right]^2 ,$$

$$F_y(\hbar\beta) = \int \mathcal{N}\mathcal{D}(\text{path})\exp\left[-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{y}}^2(\tau) - \frac{\gamma^2}{2\hbar^2 L^2}\iint d\tau\, d\tau' \left[\underline{y}(\tau)-\underline{y}(\tau')\right.\right.$$

Thus Eq.(4.7) becomes

$$\langle G(\underline{x},\underline{x}';\beta)\rangle = \exp\left(\frac{\gamma^2\beta^2}{2}\right)\exp\left(\frac{1}{\hbar}S_{cl}\right)F_y(\hbar\beta). \qquad \text{-----}(4.8)$$

By using the principle of least action, $\delta S_{cl} = 0$, S_{cl} for disordered systems can be calculated as follows:

$$\delta S_{cl} = -\frac{m}{2}\int_0^{\hbar\beta} d\tau\, 2\dot{\underline{x}}_c(\tau)\,\delta\dot{\underline{x}}_c(\tau) - \frac{\eta^2}{2\hbar L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, 2\left[\underline{x}_c(\tau)-\underline{x}_c(\tau')\right]\delta\left(\underline{x}_c(\tau)-\underline{x}_c(\tau')\right)$$

$$= -\frac{m}{2}\int_0^{\hbar\beta} d\tau\, 2\dot{\underline{x}}_c(\tau)\frac{d}{dt}\,\delta\underline{x}_c(\tau) - \frac{\eta^2}{\hbar L^2}\left\{\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left(\underline{x}_c(\tau)-\underline{x}_c(\tau')\right)\delta\underline{x}_c(\tau)\right.$$

$$\left. - \int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left(-\underline{x}_c(\tau')+\underline{x}_c(\tau)\right)\delta\underline{x}_c(\tau')\right\}$$

$$= -\frac{m}{2}\left\{2\dot{\underline{x}}_c(\tau)\,\delta\underline{x}_c(\tau)\Big|_0^{\hbar\beta} - \int_0^{\hbar\beta} d\tau\, 2\ddot{\underline{x}}_c(\tau)\,\delta\underline{x}_c(\tau)\right\} - \frac{\eta^2}{\hbar L^2}$$

$$\left\{\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left(\underline{x}_c(\tau)-\underline{x}_c(\tau')\right)\delta\underline{x}(\tau) + \int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left(\underline{x}_c(\tau)-\underline{x}_c(\tau')\right)\delta\underline{x}_c(\tau)\right\}$$

$$= m\int_0^{\hbar\beta} d\tau\, \ddot{\underline{x}}_c(\tau)\,\delta\underline{x}_c(\tau) - \frac{2\eta^2}{\hbar L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left(\underline{x}_c(\tau)-\underline{x}_c(\tau')\right)\delta\underline{x}_c(\tau) = 0$$

Thus $$m\ddot{\underline{x}}_c(\tau) = \frac{2\eta^2}{\hbar L^2}\int_0^{\hbar\beta} d\tau'\left(\underline{x}_c(\tau)-\underline{x}_c(\tau')\right)$$

$$= \frac{2}{\hbar}\left(\frac{\eta^2}{L^2}\,\frac{2\beta}{m}\right)\frac{m}{2\beta}\int_0^{\hbar\beta} d\tau'\left(\underline{x}_c(\tau)-\underline{x}_c(\tau')\right),$$

$$\ddot{\underline{x}}_c(\tau) = \frac{\omega_G^2}{\hbar\beta}\int_0^{\hbar\beta} d\tau'\left[\underline{x}_c(\tau)-\underline{x}_c(\tau')\right], \quad \text{-------} \quad (4.9)$$

where $\qquad \omega_G = \dfrac{\gamma}{L}\left(\dfrac{2\beta}{m}\right)^{\frac{1}{2}}$.

The characteristic frequency ω_G has been interpreted by Bezák as being due to itinerant oscillators.
Eq.(4.9) can be rewritten as

$$\ddot{\underline{r}}_c(\tau) = \frac{\omega_G^2}{\hbar\beta}\int_0^{\hbar\beta} d\tau' \, \underline{r}_c(\tau) - \frac{\omega_G^2}{\hbar\beta}\int_0^{\hbar\beta} d\tau' \, \underline{r}_c(\tau')$$

$$= \omega_G^2 \, \underline{r}_c(\tau) - C , \qquad \text{--------- (4.10)}$$

where

$$C = \frac{\omega_G^2}{\hbar\beta}\int_0^{\hbar\beta} d\tau' \, \underline{r}_c(\tau') .$$

The solution of Eq.(4.10) is

$$\underline{r}_c(\tau) = \frac{C}{\omega_G^2} + A\exp\left(\omega_G\tau\right) + B\exp\left(-\omega_G\tau\right), \qquad \text{------ (4.11)}$$

where A ,B and C are constants.

By applying the boundary conditions

$$\underline{r}_c(0) = \underline{r}' , \quad \underline{r}_c(\hbar\beta) = \underline{r} \quad \text{and} \quad C = \frac{\omega_G^2}{\hbar\beta}\int_0^{\hbar\beta} d\tau' \, \underline{r}_c(\tau') ,$$

we obtain

$$A = \frac{1}{2}\left(\underline{r} - \underline{r}'\right)\frac{1}{\exp(\omega_G\hbar\beta) - 1} ,$$

$$B = \frac{1}{2}\left(\underline{r} - \underline{r}'\right)\frac{1}{\exp(-\omega_G\hbar\beta) - 1} ,$$

$$C = \frac{\omega_G^2}{2}\left\{\underline{r} + \underline{r}' - A\left[\exp(\omega_G\hbar\beta) + 1\right] - B\left[\exp(-\omega_G\hbar\beta) + 1\right]\right\} .$$

Substituting for C in Eq.(4.11), we obtain

$$\underline{r}_c(\tau) = \frac{1}{2}\left\{\underline{r} + \underline{r}' - A\left[\exp(\omega_G\hbar\beta) + 1\right] - B\left[\exp(-\omega_G\hbar\beta) + 1\right]\right\}$$

$$+ A\exp\left(\omega_G\tau\right) + B\exp\left(-\omega_G\tau\right) ,$$

$$\underline{r}_c(\tau) = \frac{1}{2}(\underline{r}+\underline{r}') + A\left\{\exp(\omega_G\tau) - \frac{1}{2}\left(\exp(\omega_G\hbar\beta)+1\right)\right\}$$

$$B\left\{\exp(-\omega_G\tau) - \frac{1}{2}\left(\exp(-\omega_G\hbar\beta)+1\right)\right\} .$$

------ (4.12)

Substituting for A and B in Eq.(4.12), we obtain

$$\underline{r}_c(\tau) = \frac{1}{2}(\underline{r}+\underline{r}') + \frac{1}{2}(\underline{r}-\underline{r}')\frac{\left[\exp\{\omega_G(\tau-\hbar\beta)\} + \exp\{-\omega_G(\tau-\hbar\beta)\} - \left\{\exp(\omega_G\tau)+\exp(-\omega_G\tau)\right\}\right]}{2-\left[\exp(\omega_G\hbar\beta)+\exp(-\omega_G\hbar\beta)\right]}$$

$$= \frac{1}{2}(\underline{r}+\underline{r}') + \frac{1}{2}(\underline{r}-\underline{r}')\left[\frac{\cosh\{\omega_G(\tau-\hbar\beta)\} - \cosh\{\omega_G\tau\}}{1-\cosh\{\omega_G\hbar\beta\}}\right].$$

By using the identity $\cosh(x)-\cosh(y) = 2\sinh\left(\frac{x+y}{2}\right)$

and $2\sinh^2\left(\frac{x}{2}\right) = \cosh(x)-1$,

we obtain

$$\underline{r}_c(\tau) = \frac{1}{2}(\underline{r}+\underline{r}') + \frac{1}{2}(\underline{r}-\underline{r}')\left[\frac{\sinh\omega_G(\tau-\frac{1}{2}\hbar\beta)}{\sinh\frac{1}{2}\omega_G\hbar\beta}\right] . \quad \text{----(4.13)}$$

Since

$$S_{cl} = -\frac{m}{2}\int_0^{\hbar\beta} d\tau\, \dot{\underline{r}}_c^2(\tau) - \frac{\eta^2}{2\hbar L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left[\underline{r}_c(\tau)-\underline{r}_c(\tau')\right]^2$$

$$S_{cl} = -\frac{m}{2}\left\{ \dot{x}_c(\tau)\, x_c(\tau)\Big|_0^{\hbar\beta} - \int_0^{\hbar\beta} d\tau\, x_c(\tau)\, \ddot{x}_c(\tau) \right\} - \frac{\eta^2}{2\hbar L^2}\left\{ \int_0^{\hbar\beta}\!\!\int_0^{\hbar\beta} d\tau\, d\tau' \Big[x_c^2(\tau) + 2x_c(\tau)x_c(\tau') + x_c^2(\tau') \right.$$

$$= -\frac{m}{2}\left\{ \dot{x}_c(\hbar\beta)\, x_c(\hbar\beta) - \dot{x}(0)\, x(0) \right\} + \frac{m}{2}\int_0^{\hbar\beta} d\tau\, x_c(\tau)\, \ddot{x}_c(\tau)$$

$$- \frac{\eta^2}{2\hbar L^2}\left\{ 2\int_0^{\hbar\beta} d\tau' \int_0^{\hbar\beta} d\tau\, x_c^2(\tau) - 2\int_0^{\hbar\beta}\!\!\int_0^{\hbar\beta} d\tau\, d\tau'\, x_c(\tau)\, x_c(\tau') \right\}$$

$$= -\frac{m}{2}\left\{ \dot{x}_c(\hbar\beta)\, x_c(\hbar\beta) - \dot{x}(0)\, x(0) \right\} + \frac{1}{2}\int_0^{\hbar\beta} d\tau\, x_c(\tau)\left[m\ddot{x}_c(\tau) - \frac{2\eta^2}{\hbar L^2} \int d\tau' \left(x_c(\tau) - x_c(\tau') \right) \right.$$

and since

$$m\ddot{x}_c(\tau) - \frac{2\eta^2}{\hbar L^2}\int d\tau' \left(x_c(\tau) - x_c(\tau') \right) = 0 ,$$

therefore

$$S_{cl} = -\frac{m}{2}\left\{ \dot{x}_c(\hbar\beta)\, x_c(\hbar\beta) - \dot{x}(0)\, x(0) \right\} . \qquad (4.14)$$

Differentiating Eq.(4.13) with respect to t, we obtain

$$\dot{x}_c(\tau) = \frac{1}{2}\left(x - x' \right) \frac{\omega_G \cosh \omega_G\left(\tau - \frac{1}{2}\hbar\beta\right)}{\sinh \frac{1}{2}\omega_G \hbar\beta} . \qquad (4.15)$$

Substituting Eq.(4.15) when $\tau = 0$ and $\tau = \hbar\beta$ and Eq.(4.13) when $\tau = 0$ and $\tau = \hbar\beta$ in Eq.(4.14), we obtain

$$S_{cl} = -\frac{1}{4} m\omega_G \coth\left(\frac{1}{2}\beta\hbar\omega_G\right) \cdot \left(x - x' \right)^2 .$$

Substituting for S_{cl} in Eq.(4.8), we obtain

$$\langle G(x, x'; \beta) \rangle = \exp\left(\frac{\eta^2\beta^2}{2}\right) \exp\left[-\frac{m\omega_G}{4\hbar} \coth\left(\frac{1}{2}\beta\hbar\omega_G\right) \cdot \left(x - x' \right)^2 \right] F_\eta$$

$$= \exp\left(\frac{\eta^2\beta^2}{2}\right) \exp\left[-\left(\frac{m_G}{2\hbar^2\beta}\right) \cdot \left(x - x' \right)^2 \right] F_\eta(\hbar\beta) .$$

Here the " second effective mass " m_G has been introduced:

$$m_G = m \frac{1}{2}\beta\hbar\omega_G \coth\left(\frac{1}{2}\beta\hbar\omega_G\right) .$$

Therefore the only function now remaining to be calculated in order to obtain the averaged expression for the time-dependent Green function is $F_\gamma(\hbar\beta)$. Bezák used a short-cut by changing the exponent in the expression for $F_\gamma(\hbar\beta)$ to the integro-differential equation. The method is as follows:

Since the exponent in the integral of $F_\gamma(\hbar\beta)$ is

$$-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{y}}^2(\tau) - \frac{\gamma^2}{2\hbar^2 L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[\underline{y}(\tau) - \underline{y}(\tau')\right]^2$$

$$= -\frac{m}{2\hbar}\left[\underline{y}(\tau)\dot{\underline{y}}(\tau)\Big|_0^{\hbar\beta} - \int_0^{\hbar\beta} d\tau\, \underline{y}(\tau)\ddot{\underline{y}}(\tau)\right] - \frac{m\omega_G^2}{4\hbar^2\beta}\left\{\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}^2(\tau)\right.$$

$$+\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}^2(\tau') + 2\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}(\tau)\underline{y}$$

$$= \frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \underline{y}(\tau)\ddot{\underline{y}}(\tau) - \frac{m\omega_G^2}{2\hbar}\int_0^{\hbar\beta} d\tau\, \underline{y}^2(\tau) + \frac{m\omega_G^2}{2\hbar^2\beta}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}(\tau)\underline{y}($$

and since $\int_0^{\hbar\beta} d\tau\, \underline{y}(\tau)\ddot{\underline{y}}(\tau) = \int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}(\tau)\ddot{\underline{y}}(\tau)\,\delta(\tau-\tau')$

and $\int_0^{\hbar\beta} d\tau\, \underline{y}^2(\tau) = \int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}^2(\tau)\,\delta(\tau-\tau')$,

therefore

$$-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{y}}^2(\tau) - \frac{\gamma^2}{2\hbar^2 L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[\underline{y}(\tau) - \underline{y}(\tau')\right]^2$$

$$= \frac{m}{2\hbar}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}(\tau)\ddot{\underline{y}}(\tau)\,\delta(\tau-\tau') + \frac{m\omega_G^2}{2\hbar}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}^2(\tau)\,\delta(\tau-$$

$$+ \frac{m\omega_G^2}{2\hbar^2\beta}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}(\tau)\underline{y}(\tau') ,$$

$$-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{y}}^2(\tau) - \frac{\eta^2}{2\hbar^2 L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[\underline{y}(\tau) - \underline{y}(\tau')\right]^2$$

$$= -\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}(\tau)\left[\frac{m}{2\hbar}\left\{\left(-\frac{\partial^2}{\partial\tau'^2} + \omega_G^2\right)\delta(\tau-\tau') - \frac{\omega_G^2}{\hbar\beta}\right\}\right]\underline{y}(\tau')$$

$$= -\int\int d\tau\, d\tau'\, \underline{y}(\tau)\, A(\tau,\tau')\, \underline{y}(\tau'),$$

where

$$A(\tau,\tau') = \frac{m}{2\hbar}\left[\left(-\frac{\partial^2}{\partial\tau'^2} + \omega_G^2\right)\delta(\tau-\tau') - \frac{\omega_G^2}{\hbar\beta}\right].$$

Thus

$$F_\eta(\hbar\beta) = \mathcal{N}\int \mathcal{D}\underline{y}(\tau)\, \exp\left[-\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\, \underline{y}(\tau)\, A(\tau,\tau')\, \underline{y}(\tau')\right].$$

By solving the integro-differential equation and doing direct path integration, Bezák obtained

$$F_\eta(\hbar\beta) = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\left(\frac{\gamma}{\sinh\gamma}\right)^{3/2}\prod_{n=1}^{\infty}\left(\frac{\pi(n-\frac{1}{2})}{(\gamma^2+\xi_n^2)^{1/2}}\right)^3, \text{ where } \gamma = \tfrac{1}{2}\beta\hbar\omega_G$$

with the restriction that

$$\left\{1 + \left(\frac{\xi_n}{\gamma}\right)^2\right\}\xi_n = \tan\xi_n \quad .$$

Therefore in this way Bezák was able to obtain the formula for the averaged time-dependent Green function of disordered systems:

$$\langle G(\underline{r},\underline{r}';\beta)\rangle = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\exp\left(\frac{\eta^2\beta^2}{2}\right)\exp\left(-\frac{m_G}{2\hbar^2\beta}(\underline{r}-\underline{r}')^2\right)$$

$$\left(\frac{\gamma}{\sinh\gamma}\right)^{3/2}\prod_{n=1}^{\infty}\left(\frac{\pi(n-\frac{1}{2})}{(\gamma^2+\xi_n^2)^{1/2}}\right)^3,$$

where $\omega_G = \frac{\eta}{L}\left(\frac{2\beta}{m}\right)^{\frac{1}{2}}$, $m_G = m\gamma\coth\gamma$, $\beta = \frac{1}{k_B T}$,

$\gamma = \frac{1}{2}\beta\hbar\omega_G$, and with the restriction that

$$\left\{1+\left(\frac{\xi_n}{\gamma}\right)^2\right\}\xi_n = \tan \xi_n .$$

Although Bezák has obtained the explicit formula for the averaged time-dependent Green function of disordered systems, the restriction which is a transcendental equation cannot be solved exactly. In the next chapter, we will write the averaged time-dependent Green function in terms of cumulant series. In this way the restriction in Bezák's method is removed and the averaged Green function can be evaluated.