CHAPTER II

METHOD OF ANALYSIS

2.1 Introduction

In the method proposed herein, the framed panels are first substituted by equivalent orthotropic plates to form a substitute closed-tube structure (Fig. 1). The elastic properties of the equivalent plates are determined by the method presented by Moselhi, et al. (9). By means of certain simplifying assumptions and the infinitesimal strain displacement relations, the strain energy is then expressed in terms of the assumed axial displacement in the corner column and twisting angle, both being functions of height. Applying the principle of minimum total potential energy and the Ritz technique, the warping displacement and twisting angle can be solved. The stresses in the substitute tube are obtained from the warping displacement and twisting angle and the internal forces in the actual structure are evaluated by integration.

The detial of the method will now be described.

2.2 Assumptions

The following simplifying assumptions are made.

a) The material is homogenous, isotropic and linear elastic.

b) The out-of-plane effects are negligible so that only in-plane deformations are considered.

c) The floor slabs are assumed to be rigid in their own planes, so that each cross section undergoes rigid body displacements, and axial deformations in the girders can be neglected.

d) Each panel consists of closely spaced columns and spandrel beams so that it may be replaced by an equivalent orthotropic plate.

f) The points of contraflexure may be assumed to occur at midspans of columns and spandrel beams.

g) Deformations in the joints can be neglected.

2.3 Formulation of the Method

2.3.1 Replacement of Framed Panels by Equivalent Orthotropic Plates

The method of evaluation of the equivalent plate properties used in this study follows closely that proposed by Moselhi, el at. (9). In replacing a panel of the framed-tube by an equivalent orthotropic plate, we stipulate that both the individual unit of the framed panels (Figs. 2,3) and the orthotropic plate should undergo the same displacements when subjected to statically equivalent applied forces.

(a) Modulus of Elasticity E_z

Fig. 2 depicts a typical unit of the framed panel under the action of an axial force P. The axial displacement Δ_{af} (neglecting deformation in the joint in region 2) is

$$\Delta_{af} = P(h - d_b) / A_c E \qquad (1)$$

in which h = story height, d_b = beam depth, A_c = cross sectional area of column, and E = modulus of elasticity of the members of the actual structure. For the equivalent plate shown by the dotted line in Fig. 2, the axial displacement, Δ_{ap}, is

$$\Delta_{ap} = Ph / tdE_z \qquad (2)$$

where d = bay width, t = plate thickness obtained by equating the area of the plate to the cross sectional area of the column for each typical unit. Equating the axial displacement in Eqs. (1) and (2) yields the equivalent elastic modulus, E_z :

$$E_z = EA_c / (1 - \frac{d_b}{h}) \, td \qquad (3)$$

For framed-tubes having corner columns larger than the interior ones, we will assume that part of the cross sectional area A_{cp} of each corner column is required to form the equivalent closed-tube. This area, which is assumed to be fully effective in supporting both direct and shear stresses, is

$$A_{cp} = 0.5 \, (t_1 d_1 + t_2 d_2) \qquad (4)$$

where t_1 and t_2 = equivalent plate thicknesses of panels 1 and 2, respectively, and d_1 and d_2 = bay widths of panels 1 and 2, respectively. The remaining area, A^*_{cc}, is assumed to be concentrated at the corner and act as an axial boom which can take only axial forces (4). This area is simply

$$A^*_{cc} = A_{cc} - A_{cp} \tag{5}$$

in which A_{cc} = cross sectional area of the corner column in the actual structure.

For the general case, the equivalent elastic moduli E_z for panel 1 and panel 2 may be different. In such a case the equivalent elastic modulus of the axial boom, A^*_{cc}, may be assumed to be given by the average value of those for the two panels.

(b) Shear Modulus G_{zs}

Consider the frame segment under a lateral force Q as shown in Fig. 3. The shear modulus of the equivalent plate will be chosen so that the horizontal displacement of both the typical frame unit and the equivalent plate are the same when both are subjected to the same shearing force.

The lateral deflection for the frame segment, Δ_{1f}, may be computed as the sum of that due to bending, Δ_b, and that due to shear deformation in the members, Δ_v:

$$\Delta_{1f} = \Delta_b + \Delta_v \tag{6}$$

By applying The Virtual Force Method, and neglecting the deformation in the joint, one can easily obtain the lateral displacement as

$$\Delta_{1f} = \frac{Qh}{E}\left[\frac{(h - d_b)^3}{12h\ I_c} + \frac{h(d - d_c)^3}{12d^2\ I_b}\right]$$

$$+ \frac{Qh}{G}\left[\frac{h(d - d_c)}{d^2\ A_{rb}} + \frac{h - d_b}{h\ A_{rc}}\right] \qquad (7)$$

in which I_b and I_c = moments of inertia of the beams and columns, respectively, A_{rb} and A_{rc} = reduced or effective shear area of the beams and columns, respectively, d_c = width of column, and G = shear modulus of the material of the actual structure.

The shear deformation of the equivalent plate, Δ_{1p}, is simply given by

$$\Delta_{1p} = Qh \,/\, G_{zs}\ td \qquad (8)$$

where G_{zs} = equivalent shear modulus. By equating the two deflections in Eqs. (7) and (8) one obtains

$$G_{zs} = E \,/\, td\ C_{zs} \qquad (9)$$

$$\text{in which } C_{zs} = \frac{(h - d_b)^3}{12h\ I_c} + \frac{h(d - d_c)^3}{12\ d^2\ I_c}$$

$$+ \frac{E}{G}\left[\frac{h(d - d_c)}{d^2\ A_{rb}} + \frac{h - d_b}{h\ A_{rc}}\right] \qquad (10)$$

2.3.2 Total Potential Energy

Due to symmetry, the total strain energy, U , stored in the structure (Fig. 4) may be expressed as

$$U = 2U_1 + 2U_2 + 4U_c \tag{11}$$

in which U_1 and U_2 = strain energy stored in panels 1 and 2, respectively, and U_c = strain energy stored in the concentrated area A^*_{cc} at the corner of equivalent tube.

For the state of plane stress in each panel, the strain energy, U_1, U_2 and U_c are given, respectively, by

$$U_1 = \frac{1}{2}\int_{o}^{H}\int_{-c}^{c} (\sigma_x \epsilon_x + \sigma_z \epsilon_z + \tau_{xz}\gamma_{xz})\, t_1 \, dx\, dz \tag{12}$$

$$U_2 = \frac{1}{2}\int_{o}^{H}\int_{-b}^{b} (\sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{yz}\gamma_{yz})\, t_2 \, dy\, dz \tag{13}$$

$$U_c = \frac{1}{2}\int_{o}^{H} A^*_{cc}\, \sigma_c \epsilon_c \, dz \tag{14}$$

in which x, y, z = Catesian coordinates, 2c = length of the shorter panel (panel 1), 2b = length of the longer panel (panel 2), H = total height of the structure, ϵ_x, ϵ_y, ϵ_z = unit normal strains in x, y and z directions, σ_x, σ_y, σ_z = normal stresses in x,y and z directions, γ_{xz} and γ_{yz} = unit shearing strains in panels 1 and 2, respectively, τ_{xz} and τ_{yz} = corresponding shear stresses,

ϵ_c and σ_c = axial strain and axial stress at the corner of equivalent tube.

The strain energy due to the normal strains in the horizontal direction is negligible because of the high in-plane stiffness of the floor slabs. Consequently Eqs. (12) and (13) reduce to

$$U_1 = \frac{1}{2} \int_0^H \int_{-c}^{c} (\sigma_z \epsilon_z + \tau_{xz} \gamma_{xz}) \, t_1 \, dx \, dz \qquad (15)$$

$$U_2 = \frac{1}{2} \int_0^H \int_{-b}^{b} (\sigma_z \epsilon_z + \tau_{yz} \gamma_{yz}) \, t_2 \, dy \, dz \qquad (16)$$

Furthermore, due to restraint provided by the rigid floor slabs, the cross section of equivalent tube moves as a rigid body in their own plane. Thus, for a small angle of twist θ, a point whose coordinates are (X, Y, Z) will experience the following displacements

$$u\,(X, Y, Z) = -\,Y \cdot \theta\,(Z) \qquad (17)$$

$$v\,(X, Y, Z) = X \cdot \theta\,(Z) \qquad (18)$$

in which u and v = displacements in the x and y directions, respectively. In addition, this point will also be displaced longitudinally. Therefore, the displacement of each cross section of the tube is completely defined by the warping displacement, w (x, y, z), and by the twisting angle, θ (z). In view of the strain - displacement relationships

$$\epsilon_z = \frac{\partial w}{\partial z} \tag{19}$$

$$\gamma_{xz} = \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{\partial w}{\partial s} + b\,\frac{\partial \theta}{\partial z} \tag{20}$$

$$\gamma_{yz} = -\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{\partial w}{\partial s} + c\,\frac{\partial \theta}{\partial z} \tag{21}$$

in which s = peripheral line along the equivalent tube, with the positive direction shown in Fig. 4.

Since the material is linear elastic, the corresponding normal and shearing stresses at any point are

$$\sigma_z = E_z\left(\frac{\partial w}{\partial z}\right) \tag{22}$$

$$\tau_{xz} = G_1\left(\frac{\partial w}{\partial s} + b\,\frac{\partial \theta}{\partial z}\right) \tag{23}$$

$$\tau_{yz} = G_2\left(\frac{\partial w}{\partial s} + c\,\frac{\partial \theta}{\partial z}\right) \tag{24}$$

in which G_1 and G_2 = equivalent shear modulus of the panel 1 and panel 2, respectively.

Substituting Eqs. (19, 20, 21) and (22, 23, 24) into Eqs. (15, 16, 14) yields

$$U_1 = \frac{1}{2}\int_{o}^{H}\int_{-c}^{c} \left[E_1\left(\frac{\partial w}{\partial z}\right)^2 + G_1\left(\frac{\partial w}{\partial s} + b\frac{\partial \theta}{\partial z}\right)^2 \right] t_1 ds\, dz \tag{25}$$

$$U_2 = \frac{1}{2}\int_0^H\int_{-b}^{b}\left[E_2\left(\frac{\partial w}{\partial z}\right)^2 + G_2\left(\frac{\partial w}{\partial s} + c\,\frac{\partial \theta}{\partial z}\right)^2 \right] t_2 ds\, dz \qquad (26)$$

$$U_c = \frac{1}{2}\int_0^H A^*_{cc}\, E^*_c \left(\frac{\partial w_c}{\partial z}\right)^2 dz \qquad (27)$$

where E_1 and E_2 = elastic modulus in panel 1 and panel 2, respectively, and w_c and E^*_c = axial displacement and elactic modulus at the corner of the equivalent tube, respectively.

In view of Eqs. (25, 26, 27) and Eq. (11), the total strain energy, U, can be written in the more compact form as

$$U = \frac{1}{2}\int_0^H\left[\oint \left\{ E_z\left(\frac{\partial w}{\partial z}\right)^2 + G_{zs}\left(\frac{\partial w}{\partial s} + \rho\,\frac{\partial \theta}{\partial z}\right)^2 \right\} t\, ds\right] dz + (4)\cdot\frac{1}{2}\int_0^H A^*_{cc}\, E^*_c \left(\frac{\partial w_c}{\partial z}\right)^2 dz \qquad (28)$$

in which ρ = the perpendicular distance from origin, 0, which is the twist center of the cross section, to the tangential segment ds.

The potent al energy of external load, V, is given simply as

$$V = -\int_0^H T\cdot\theta\cdot dz \qquad (29)$$

in which T = torque per unit length along the height.

The total potential energy of the structure, π_p, which is the sum of the strain energy and potential energy of external load, takes on a minimum value at stable equilibrium, i.e.

$$\pi_p = 2\,U_1 + 2\,U_2 + 4\,U_c + V = \text{Minimum} \qquad (30)$$

2.3.3 Approximate Solutions by Ritz Method

In order to make the proposed method simple and general enough so that framed-tubes with any variation of story properties along the height of the structures can be analysed, we will employ the Ritz method to solve for the approximate displacement functions. The displacement fields that are employed in this study are based on the observations of the solutions obtained by using the computer program (ETABS, Ref. 11) and the results presented by Coull and Bose (8).

With the equivalent tube regarded as a thin walled structure, a linear variation of the axial displacement along the cross section is a good approximation. Thus

$$w(x, y, z) = \left[\frac{xy}{bc} \right] w_c(z) \qquad (31)$$

in which w_c = axial displacement at the corner of the equivalent tube, whose variation along the height is taken as

$$w_c(z) = A \sin \frac{\pi}{2}\xi + B(\cos \pi\xi - 1) + C(\cos 2\pi\xi - 1) \qquad (32)$$

where ξ = height parameter = $\frac{z}{H}$, and A, B, C = undetermined constants.

Two types of functions have been tried for the twisting angle. For the general case, a polynomial approximation may be employed.

Thus

$$\theta = K_1\xi + K_2\xi^2 + K_3\xi^3 \tag{33}$$

in which θ = angle of twist, and K_1, K_2, K_3 = undetermined constants.

For the special case of a framed-tube of uniform properties along the height and subjected to a uniformly distributed torque, the following simpler approximation for the twisting angle has been found to be satisfactory:

$$\theta = K \sin \frac{\pi}{2}\xi \tag{34}$$

with K being the undetermined constant.

It should be observed that the assumed displacement functions (w, θ) satisfy the geometric boundary conditions, i.e. they both vanish at the base. Furthermore, the assumed displacement fields give rise to zero axial stress at the free end.

For the special case of a framed-tube of uniform properties along the height and subjected to a uniformly distributed torque, a simple solution can be derived. Substituting the assumed displacement functions from Eqs. (31), (32) and (34) into Eqs. (25), (26), (27) and (29) and integrating leads to

$$2U_1 = \left(\frac{\pi^2}{12}\frac{E_1t_1c}{H} + \frac{G_1t_1H}{c}\right)A^2 + \left(\frac{\pi^2}{3}\frac{E_1t_1c}{H} + \frac{3G_1t_1H}{c}\right)B^2$$

$$+ \left(\frac{4\pi^2}{3}\frac{E_1t_1c}{H} + \frac{3G_1t_1H}{c}\right)C^2 - \left(\frac{8\pi}{9}\frac{E_1t_1c}{H} + \frac{32}{3\pi}\frac{G_1t_1H}{c}\right)AB$$

$$+ \left(4\frac{G_1t_1H}{c}\right)BC - \left(\frac{32\pi}{45}\frac{E_1t_1c}{H} + \frac{128}{15\pi}\frac{G_1t_1H}{c}\right)AC$$

$$- G_1t_1b\left(2A - \frac{8}{3}B - \frac{64}{15}C\right)K + \frac{\pi^2}{4}\frac{G_1t_1b^2c}{H}K^2 \quad (35)$$

$$2U_2 = \left(\frac{\pi^2}{12}\frac{E_2t_2b}{H} + \frac{G_2t_2H}{b}\right)A^2 + \left(\frac{\pi^2}{3}\frac{E_2t_2b}{H} + \frac{3G_2t_2H}{b}\right)B^2$$

$$+ \left(\frac{4\pi^2}{3}\frac{E_2t_2b}{H} + \frac{3G_2t_2H}{b}\right)C^2 - \left(\frac{8\pi}{9}\frac{E_2t_2b}{H} + \frac{32}{3\pi}\frac{G_2t_2H}{b}\right)AB$$

$$+ \left(\frac{4G_2t_2H}{b}\right)BC - \left(\frac{32\pi}{45}\frac{E_2t_2b}{H} + \frac{128}{15\pi}\frac{G_2t_2H}{b}\right)AC$$

$$+ G_2t_2c\left(2A - \frac{8}{3}B - \frac{64}{15}C\right)K + \frac{\pi^2}{4}\frac{G_2t_2c^2b}{H}K^2 \quad (36)$$

$$4\,U_c = \frac{A^*_{cc}\,E^*_c}{H}\left[\frac{\pi^2}{4}A^2 + \pi^2 B^2 + 4\pi^2 C^2 - \frac{8\pi}{3}AB - \frac{32\pi}{15}AC\right] \quad (37)$$

$$V = -\frac{2}{\pi}\,ToHK \quad (38)$$

where To = uniformly distributed torque per unit length along the height. By introducing Eqs. (35), (36), (37) and (38) into Eq. (30), and setting to zero the partial derivative of π_p with respect to each undertermined constant we obtain

$$K = \frac{4To\,H^2}{\pi^3 bc\,(G_1t_1b + G_2t_2c)} + \frac{4}{\pi^2}\left(\frac{H}{bc}\right)^2\left(A - \frac{4}{3}B - \frac{32}{15}C\right)F_2 \quad (39)$$

$$\left(\frac{\pi^2}{24}F_1 + \frac{1}{2}\right)A - \left(\frac{2\pi}{9}F_1 + \frac{8}{3\pi}\right)B - \left(\frac{8\pi}{45}F_1 + \frac{32}{15\pi}\right)C = \left(\frac{F_2}{2}\right)K \quad (40)$$

$$-\left(\frac{2\pi}{9}F_1 + \frac{8}{3\pi}\right)A + \left(\frac{\pi^2}{6}F_1 + \frac{3}{2}\right)B + C = -\left(\frac{2}{3}F_2\right)K \quad (41)$$

$$-\left(\frac{8\pi}{45}F_1 + \frac{32}{15\pi}\right)A + B + \left(\frac{2\pi^2}{3}F_1 + \frac{3}{2}\right)C = -\left(\frac{16}{15}F_2\right)K \quad (42)$$

in which
$$F_1 = \frac{bc}{H^2}\left(\frac{E_1t_1c + E_2t_2b + 3A^*_{cc}\,E^*_c}{G_1t_1b + G_2t_2c}\right) \quad (43.a)$$

$$F_2 = \frac{bc}{H}\left(\frac{G_1t_1b - G_2t_2c}{G_1t_1b + G_2t_2c}\right) \quad (43.b)$$

Eqs. (39) - (42) can be easily solved for the undetermined constants. Observe that the coefficient matrix for Eqs.(40)-(42) is symmetric.

2.3.5 Internal Forces in the Actual Structure

The distribution of stresses in the equivalent continuous system can be evaluated by using Eqs. (22), (23) and (24) as follows:

$$\sigma_c = E_c^* \left(\frac{\partial w_c}{\partial z}\right) = \frac{E_c^*}{H} \left(\frac{\partial w_c}{\partial \xi}\right) \tag{44}$$

$$\sigma_1 = \left[\frac{xy}{bc}\right]\left[\frac{E_1}{E_c^*}\right]\sigma_c \tag{45}$$

$$\sigma_2 = \left[\frac{xy}{bc}\right]\left[\frac{E_2}{E_c^*}\right]\sigma_c \tag{46}$$

$$\tau_{xz} = G_1\left[-\frac{1}{c} w_c + b\frac{\partial\theta}{\partial z}\right] \tag{47}$$

$$\tau_{yz} = G_2\left[\frac{1}{b} w_c + c\frac{\partial\theta}{\partial z}\right] \tag{48}$$

where σ_c = axial stress in the corner column, σ_1 and σ_2 = axial stresses in panel 1 and panel 2, respectively.

The internal forces in each member of the actual structure can be evaluated by multiplying the stress at the centroid of that member with the equivalent tributary plate area. The results are

Axial force in column at level z_i :

$$P_c = A_{cc} \left[\sigma_c \right]_{\xi = \frac{z_i}{H}} \tag{49}$$

$$P_1(x_i) = t_1 d_1 \left[\sigma_1 (x_i, b, \xi) \right]_{\xi = \frac{z_i}{H}} \tag{50}$$

$$P_2(y_i) = t_2 d_2 \left[\sigma_2 (c, y_i, \xi) \right]_{\xi = \frac{z_i}{H}} \tag{51}$$

where P_c, P_1 and P_2 = column axial force in the corner, panel 1 and Panel 2, respectively, and z_i = height from the base to the point of contraflexure of the columns.

Shear force in column at level z_i :

$$Q_1 = t_1 d_1 \left[\tau_{xz} (\xi) \right]_{\xi = \frac{z_i}{H}} \tag{52}$$

$$Q_2 = t_2 d_2 \left[\tau_{yz} (\xi) \right]_{\xi = \frac{z_i}{H}} \tag{53}$$

$$Q_{c1} = 0.5\, Q_1 \tag{54}$$

$$Q_{c2} = 0.5\, Q_2 \tag{55}$$

where Q_1 and Q_2 = column shearing force in panel 1 and panel 2, respectively, and Q_{c1} and Q_{c2} = corner column shearing force in panel 1 and panel 2, respectively.

Shear force in spandrel beam at level z_j :

$$V_1 = t_1 h \left[\tau_{xz}(\xi) \right]_{\xi = \frac{z_j}{H}} \tag{56}$$

$$V_2 = t_2\, h \left[\tau_{yz}(\xi) \right]_{\xi = \frac{z_j}{H}} \tag{57}$$

where V_1 and V_2 = shear force at points of contraflexure in the spandrel beams in panel 1 and panel 2, respectively, and z_j = the height from the base to the floor level.

Bending moments in the spandrel beams and columns, at the face of a column and a spandrel beam, are approximately obtained by multiplying the corresponding shear force by the distance from the point of contraflexure to the section under consideration.