CHAPTER I

PRELIMINARIES

This chapter will give some definitions and theorems which will be used in chapters II and III.

The materials of this chapter are drawn from references [1] - [8]

When we write a vector $\vec{V}$ we shall write its components using superscripts $v^1, \dots, v^n$.

1-1 Existence and Uniqueness theorem

The system of ordinary differential equations

$$\text{(1-1.1)} \qquad \begin{aligned} \frac{dy^1}{dt} &= f^1(y^1, \dots, y^n, t) \\ \frac{dy^2}{dt} &= f^2(y^1, \dots, y^n, t) \\ &\vdots \\ \frac{dy^n}{dt} &= f^n(y^1, \dots, y^n, t) \end{aligned}$$

is equivalent to the vector ordinary differential equation :

$$\text{(1-1.2)} \qquad \frac{d\vec{Y}}{dt} = \vec{f}(\vec{Y}, t)$$

where $\vec{f}$ is a continuous n-vector valued function whose components are functions of the real variable t and the n-vector $\vec{Y}$. The domain of $\vec{f}$ is an open set D in the product space R^{n+1}. We will use these two notations interchangably.

<u>Definition 1-1.3</u> Let $\vec{f} : D \to R^n$ where D is an open subset of R^{n+1}. The vector-valued function $\vec{f}$ is said to be <u>continuous</u> at a point $(\vec{Y}_0, t_0) \in D$ if for any real number $\varepsilon > 0$, there are two real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|\vec{f}(\vec{Y},t) - \vec{f}(\vec{Y}_0,t_0)| < \varepsilon$, whenever $(\vec{Y},t) \in D$, $|t-t_0| < \delta_1$ and $|\vec{Y} - \vec{Y}_0| < \delta_2$.

The vector-valued function $\vec{f}$ is continuous in D if it is continuous at each point of D. Also, it can be easily shown that $\vec{f}$ is continuous on D if and only if each of its components is continuous on D.

<u>Definition 1-1.4</u> Let $\vec{f} : U \to R^n$ where U is a non-empty open subset of R^m. If $\vec{f}$ has continuous partial derivatives of the first order on U then we say that $\vec{f}$ is <u>continuously differentiable</u> on U and we will denote this property by saying that $\vec{f}$ is c^1 on U. Also, for all $k \geq 1$, $\vec{f}$ is c^k on U if $\vec{f}$ has continuous partial derivatives up to and including order k.

<u>Definition 1-1.5</u> Let g be a differentiable function defined on an open subset U of R^n into R. Let $\vec{P}_0 = (x_0^1, \ldots, x_0^n)$ be any point in U. Then the <u>gradient vector of g at $\vec{P}_0$</u> is denoted by $\nabla g(\vec{P}_0)$ and defined by the formula

$$\nabla g = \left(\frac{\partial g}{\partial x^1}, \frac{\partial g}{\partial x^2}, \ldots, \frac{\partial g}{\partial x^n}\right) = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) g$$

where each of the partials is evaluated at $\vec{P}_0$.

<u>Definition 1-1.6</u> Let g be an infinitely differentiable function on open subset U of R^n into R. Let $\vec{Q}_0$ be a fixed point in U. Then the Taylor series expansion of g at the point $\vec{Q}_0$ is the following power series :

$$\begin{aligned} g(\vec{P}) \quad = \quad & g(\vec{Q}_0) + ([(\vec{P} - \vec{Q}_0)\cdot \nabla\ ]\ g)(\vec{Q}_0) \\ & + \frac{1}{2!}([(\vec{P} - \vec{Q}_0)\cdot \nabla\ ]^2\ g)(\vec{Q}_0) \\ & + \ \vdots \\ & + \frac{1}{n!}([(\vec{P} - \vec{Q}_0)\cdot \nabla\ ]^n\ g)(\vec{Q}_0) \ + \ \ldots \end{aligned}$$

In order to fully understand the meaning of this expansion we shall write it out for different cases.

For example, in two dimensions, if we set $\vec{Q}_0 = (a,b)$ and $\vec{P} = (x,y)$, we get

$$\begin{aligned} g(x,y) \quad = \quad & g(a,b) + [(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}]\ g(a,b) \\ & + \frac{1}{2!}([(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\ ]^2\ g)(a,b) + \ldots. \\ & + \frac{1}{n!}([(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\ ]^n\ g)(a,b) + \ldots. \end{aligned}$$

Expanding we get,

$$\begin{aligned} g(x,y) \quad = \quad & g(a,b) + (x-a)\frac{\partial g}{\partial x}(a,b) + (y-b)\frac{\partial g}{\partial y}\ (a,b) \\ & + \frac{1}{2!}[(x-a)^2\ \frac{\partial^2}{\partial x^2}g(a,b) + 2(x-a)(y-b)\frac{\partial^2 g}{\partial x\,\partial y}(a,b) + (y-b)^2\ \frac{\partial^2 g}{\partial y^2}(a,b)] \\ & + \ .\ .\ .\ . \end{aligned}$$

$$+ \frac{1}{n!} \left[(x-a)^n \frac{\partial^n}{\partial x^n} g(a,b) + \binom{n}{n-1} (x-a)^{n-1} (y-b) \frac{\partial^n}{\partial x^{n-1} \partial y} g(a,b) \right.$$

$$\left. + \ldots \binom{n}{1} (x-a)(y-b)^{n-1} \frac{\partial^n g}{\partial x \partial y^{n-1}}(a,b) + (y-b)^n \frac{\partial^n g}{\partial y^n}(a,b) \right]$$

$$+ \ . \ . \ . \ .$$

Note that $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

<u>Definition 1-1.7</u> A real-valued infinitely differentiable function g defined on an open connected set D of R^n is said to be <u>analytic in D</u>, if, for any point $\vec{Y}_0 \varepsilon$ D, the Taylor series expansion of the function at the point $\vec{Y}_0$ converges to the function g in some neighborhood of $\vec{Y}_0$.

<u>Definition 1-1.8</u> Let $\vec{f}$ be a vector-valued function defined on open subset D of R^m into R^n, $\vec{f}$ is said to be an analytic function on D if each of its component function is analytic on D.

The main problem in the theory of ordinary differential equations of the first order is to find solutions to equations of the form 1-1.1. By a solution we mean n continuously differentiable (c^1) functions $y^1, \ldots, y^n$, defined on a real interval I such that

(1) $(y^1(t), \ldots, y^n(t), t)$ is in D when t is in I.

(2) $\frac{d}{dt} y^i(t) = f^i(y^1(t), \ldots, y^n(t), t)$ when t is in I, $i = 1, 2, \ldots, n$.

By an initial value problem for a system of ordinary differential equations we mean the following problem. Let D be an open subset of R^{n+1}, and let $f^1(\vec{Y}, t), \ldots, f^n(\vec{Y}, t)$ be real valued functions which are continuous on D. Let $\vec{Y}_0 = (y_0^1, \ldots, y_0^n)$, and let $(\vec{Y}_0, t_0)$ be a point in D

The problem is to find n continuously differentiable (c^1) functions $y^1, \ldots, y^n$ defined on some open interval containing t_0 such that $\vec{Y} = (y^1, \ldots, y^n)$ is a solution, and which also satisfies $\vec{Y}(t_0) = \vec{Y}_0$.

We use the following notation to describe the initial value problem for the first order system of ordinary differential equations :

$$\text{IVS} \quad : \qquad \frac{d\vec{Y}}{dt} = \vec{f}(\vec{Y}, t), \qquad \vec{Y}(t_0) = \vec{Y}_0$$

If continuously differentiable functions $y^1, \ldots, y^n$ defined on an interval I, containing t_0 can be found such that $\vec{Y}(t_0) = \vec{Y}_0$, then $\vec{Y}$ is said to be a solution of IVS.

<u>Theorem 1-1.9</u> If the n functions $f^i(\vec{Y},t)$ are continuous in a closed and bounded region $\bar{G}$ of R^{n+1}, then given any interior point $(\vec{Y}_0, t_0)$ of this region there exists at least one continuously differentiable curve $\vec{Y} = \vec{Y}(t)$ which is defined in an interval $|t-t_0| \leq a$ and satisfies the system of differentiable equations

$$\frac{d\vec{Y}}{dt} = \vec{f}(\vec{Y}, t) \quad ; \quad \vec{Y}(t_0) = \vec{Y}_0 .$$

The proof can be found in reference [2] pages 13 - 18 .

<u>Theorem 1-1.10</u> Let Ω be an open connected subset of R^{n+1}. For each $i = 1,2,\ldots,n$, let $f^i(\vec{Y},t)$ be c^k (analytic) $k \geq 1$ on Ω . Then for any point $(\vec{Y}_0,t_0) \in \Omega$, there exists neighbourhoods U of $\vec{Y}_0$ in R^n and I of t_0 in R such that for any $\vec{Y}_1 \in U$ and all $t \in I$ there are unique functions defined on I, $\psi^1_{\vec{Y}_1}(t), \ldots, \psi^n_{\vec{Y}_1}(t)$ such that

$$\frac{d\psi^i_{\vec{Y}_1}}{dt} = f^i(\psi^i_{\vec{Y}_1}, t)$$

and

$$\psi^i_{\vec{Y}_1}(t_0) = y^i_1$$

Writing $\psi^i_{\vec{Y}_1}(t) = \psi^i(\vec{Y}_1, t)$ we can furthermore conclude that the function ψ^i are of class c^{k+1} (analytic) in t and of class c^k (analytic) in $\vec{Y}_1$.

This theorem is called the existence and uniqueness theorem for the first order ordinary differential equations or the fundamental theorem of ordinary differential equation and the proof for the c^k case is shown in references [2] pages 18-22 and [7] pages 372-373. For the case where $f^i(\vec{Y},t)$ are analytic for all $i = 1,2,\dots,n$ one can find the proof in reference [3] pages 210-215.

Notation : We shall use the notation $\frac{d\vec{\psi}}{dt} = \dot{\vec{\psi}}$ interchangeably and also for $\frac{d^2\vec{\psi}}{dt^2} = \ddot{\vec{\psi}}$.

Now, we want to extend the fundamental theorem of ordinary differential equation of the first order to ordinary differential equations of the second order. Since these are the objects of study of this thesis. We want to show that whenever $\vec{H}$ is c^k (analytic) $k \geq 1$ in $2n + 1$ real variables then the differential equations $\ddot{\psi}^i = H^i(\vec{\psi}, \dot{\vec{\psi}}, t)$, $i = 1,2,\dots,n$ with initial conditions $\psi^i(t_0) = p^i_0$, $\dot{\psi}^i(t_0) = v^i_0$ has a unique solution.

<u>Theorem 1-1.11</u> Let $\vec{H}$ be c^k (analytic) $k \geq 1$ on an open connected subset Ω of R^{2n+1} then for all $(\vec{P}_0,\vec{V}_0,t_0) \in \Omega$, there exist neighbourhoods U of $\vec{P}_0$, V of $\vec{V}_0$ in R^n and an interval I of t_0 in R such that for any $\vec{P}_1 \in U$, and $\vec{V}_1 \in V$ there are unique functions $\psi^1_{\vec{P}_1,\vec{V}_1}(t),\ldots,\psi^n_{\vec{P}_1,\vec{V}_1}(t)$ defined on I such that $\ddot{\vec{\psi}}_{\vec{P}_1,\vec{V}_1} = \vec{H}(\vec{\psi}_{\vec{P}_1,\vec{V}_1},\dot{\vec{\psi}}_{\vec{P}_1,\vec{V}_1},t)$

and

$$\vec{\psi}_{\vec{P}_1,\vec{V}_1}(t_0) = \vec{P}_1 , \quad \dot{\vec{\psi}}_{\vec{P}_1,\vec{V}_1}(t_0) = \vec{V}_1$$

Writing $\vec{\psi}_{\vec{P}_1,\vec{V}_1}(t) = \vec{\psi}(\vec{P}_1,\vec{V}_1,t)$ we can furthermore conclude that the function $\vec{\psi}$ is of class c^{k+2} (analytic) in t and of class c^k(analytic) with respect to $(\vec{P}_1,\vec{V}_1)$.

<u>Proof</u> Let $\vec{H}$ be c^k (analytic) $k \geq 1$ on a subset Ω of R^{2n+1}. Let $(\vec{P}_0,\vec{V}_0,t_0)$ be any point in Ω.

We shall prove that there exists a unique solution $\vec{\phi} = (\phi^1,\ldots,\phi^n)$ which satisfies the system of second order ordinary differential equations

(1) $$\ddot{\vec{\psi}} = \vec{H}(\vec{\psi},\dot{\vec{\psi}}, t)$$

$$\vec{\psi}(\vec{P},\vec{V},t_0) = \vec{P}$$ (where $\vec{P}$ is a point in some neighbourhood of $\vec{P}_0$)

$$\dot{\vec{\psi}}(\vec{P},\vec{V},t_0) = \vec{V}$$ (where $\vec{V}$ is a point in some neighbourhood of $\vec{V}_0$).

Let $\psi^1 = y^1 \;, \quad \psi^2 = y^2 \;, \; \dots \;, \psi^n = y^n$

$\mathring{\psi}^1 = y^{n+1} \;, \quad \mathring{\psi}^2 = y^{n+2} \;, \; \dots \;, \mathring{\psi}^n = y^{2n} \;.$

Hence

$$\frac{dy^1}{dt} = \mathring{\psi}^1 = y^{n+1}$$

$$\frac{dy^2}{dt} = \dot{\psi}^2 = y^{n+2}$$

$$\vdots$$

$$\frac{dy^n}{dt} = \mathring{\psi}^n = y^{2n}$$

$$\frac{dy^{n+1}}{dt} = \ddot{\psi}^1 = H^1(\psi^1, \dots, \psi^n, \mathring{\psi}^1, \dots, \dot{\psi}^n, t)$$

$$= H^1(y^1, \dots, y^n, y^{n+1}, \dots, y^{2n}, t)$$

$$\vdots$$

$$\frac{dy^{2n}}{dt} = H^n(y^1, \dots, y^n, y^{n+1}, \dots, y^{2n}, t)$$

This is a system of ordinary differential equation of the first order which we shall denote by

(2) $$\frac{dy^i}{dt} = f^i(y^1, \dots, y^{2n}, t)$$

where for each $i = 1,2,\dots,n$, $f^i(y^1,\dots, y^{2n},t) = y^{n+i}$ and for each $i = n+1,\dots, 2n$, $f^i(y^1,\dots, y^{2n}, t) = H^{i-n}(y^1,\dots,y^{2n},t)$.

This system of differential equations has initial conditions $\vec{Y}(t_0) = (p^1,\dots,p^n, v^1,\dots, v^n)$.

The n-vector valued function $\vec{f}$ defined above is continuous, hence theorem 1-1.9 implies that there exists at least one solution of (2).

Lemma 1-1.12 If $\vec{\phi} = (\phi^1, \ldots, \phi^n)$ is a solution of (1), then $\vec{\phi}^* = (\phi^1, \ldots, \phi^n, \mathring{\phi}^1, \ldots, \mathring{\phi}^n)$ is a solution of (2). Conversely, if $\vec{\phi}^* = (\phi^1, \ldots, \phi^{2n})$ is a solution of (2), then the first n component $\phi^1, \ldots, \phi^n$ are solutions of (1).

Proof Assume that $\vec{\phi}$ is a solution of

(1) $$\text{DS} : \ddot{\psi}^i = H^i(\psi^1, \ldots, \psi^n, \mathring{\psi}^1, \ldots, \mathring{\psi}^n, t)$$

$$\text{IC} : \mathring{\psi}^i(t_0) = v^i, \quad \psi^i(t_0) = p^i$$

Hence for each $i = 1,2,\ldots, n$, we have

$$\ddot{\phi}^i = H^i(\phi^1, \ldots, \phi^n, \mathring{\phi}^1, \ldots, \mathring{\phi}^n, t)$$

and $$\phi^i(t_0) = p^i \quad , \quad \mathring{\phi}^i(t_0) = v^i$$

We will prove that $\vec{\phi}^* = (\phi^1, \ldots, \phi^n, \mathring{\phi}^1, \ldots, \mathring{\phi}^n)$ satisfies

(2) $$\text{DS} : \frac{dy^i}{dt} = f^i(y^1, \ldots, y^{2n}, t)$$

$$\text{IC} : \vec{Y}(t_0) = (p^1, \ldots, p^n, v^1, \ldots, v^n)$$

Since $$\frac{d\phi^1}{dt} = \mathring{\phi}^1, \ldots, \frac{d\phi^n}{dt} = \mathring{\phi}^n$$

and by the assumption, we get

$$\frac{d\mathring{\phi}^1}{dt} = \ddot{\phi}^1 = H^1(\phi^1, \ldots, \phi^n, \mathring{\phi}^1, \ldots, \mathring{\phi}^n, t)$$

$$\vdots$$

$$\frac{d\mathring{\phi}^n}{dt} = \ddot{\phi}^n = H^n(\phi^1, \ldots, \phi^n, \mathring{\phi}^1, \ldots, \mathring{\phi}^n, t)$$

Since $\phi^i(t_0) = p^i$, $\mathring{\phi}^i(t_0) = v^i$

Hence $\vec{\phi}^*(t_0) = (\phi^1(t_0),\ldots,\phi^n(t_0),\mathring{\phi}^1(t_0),\ldots,\mathring{\phi}^n(t_0))$

$\vec{\phi}^*(t_0) = (p^1,\ldots, p^n, v^1,\ldots, v^n)$

Thus, $\vec{\phi}^* = (\phi^1,\ldots, \phi^n, \mathring{\phi}^1,\ldots, \mathring{\phi}^n)$ is a solution of (2).

<u>Uniqueness</u>

We will prove that $\vec{\phi}^* = (\phi^1,\ldots,\phi^n,\mathring{\phi}^1,\ldots,\mathring{\phi}^n)$ is the unique solution of (2).

Obviously, $\vec{f} = (y^{n+1},\ldots, y^{2n}, H^1(\vec{Y},t),\ldots, H^n(\vec{Y},t))$ is c^k (analytic) $k \geq 1$. By theorem 1-1.10, we conclude that $\vec{\phi}^* = (\phi^1,\ldots,\phi^n,\mathring{\phi}^1,\ldots,\mathring{\phi}^n)$ is the unique solution of (2).

Conversely, we assume that $\vec{\phi}^* = (\phi^1,\ldots, \phi^{2n})$ is a solution of (2). We will show that $\vec{\phi} = (\phi^1,\ldots, \phi^n)$ is a solution of (1). By the assumption we have that

$$\mathring{\phi}^1 = \phi^{n+1} ,\ldots, \mathring{\phi}^n = \phi^{2n} .$$

Then differentiate the above equation with respect to t and using the fact that ϕ^i, $i = 1,2,\ldots, 2n$ satisfy (2), we obtain

$$\ddot{\phi}^1 = \mathring{\phi}^{n+1} = H^1(\phi^1,\ldots,\phi^{2n},t) = H^1(\vec{\phi},\dot{\vec{\phi}},t)$$

$$\ddot{\phi}^2 = \mathring{\phi}^{n+2} = H^2(\phi^1,\ldots,\phi^{2n},t) = H^2(\vec{\phi},\dot{\vec{\phi}},t)$$

$$\vdots$$

$$\ddot{\phi}^n = \mathring{\phi}^{2n} = H^n(\phi^1,\ldots,\phi^{2n},t) = H^n(\vec{\phi},\dot{\vec{\phi}},t) .$$

Therefore, for each $i = 1,2,\ldots, n$

$$\ddot{\phi}^i = H^i(\vec{\phi}, \dot{\vec{\phi}}, t)$$

Since $\vec{\phi}^*$ satisfies the initial condition

$$\vec{\phi}^*(t_0) = (p^1,\ldots,p^n, v^1,\ldots,v^n)$$

Hence $\phi^i(t_0) = p^i , \quad i = 1,2,\ldots, n$

$$\phi^{n+j}(t_0) = v^j , \quad j = 1,2,\ldots, n .$$

Since

$$\dot{\phi}^j = \phi^{n+j}, \quad j = 1,2,\ldots,n$$

Then

$$\dot{\phi}^j(t_0) = \phi^{n+j}(t_0) = v^j , \quad j = 1,2,\ldots, n.$$

Hence $\vec{\phi} = (\phi^1,\ldots,\phi^n)$ satisfies the 2^{nd} order ordinary differential equation (1).

Therefore, $\vec{\phi} = (\phi^1,\ldots,\phi^n)$ is a solution of (1).

Uniqueness

Suppose $\vec{\Phi}$ is another solution of (1) such that for some $i = 1,2,\ldots,n$, $\phi^i \neq \Phi^i$.

Similarly, we can prove that $\vec{\Phi}^* = (\Phi^1,\ldots,\Phi^n,\dot{\Phi}^1,\ldots,\dot{\Phi}^n)$ is a solution of (2) which satisfies the given initial condition. Since there exists some i such that $\phi^i \neq \Phi^i$, hence $\vec{\Phi}^* \neq \vec{\phi}^*$ which contradicts the existence and uniqueness theorem for the ordinary differential equation of first order (Theorem 1-1.10) . Hence for all $i = 1,2,\ldots,n$, we have $\phi^i = \Phi^i$. That is $\vec{\phi} = \vec{\Phi}$. Thus, $\vec{\phi}$ is the unique solution of (1).

This completes the proof of lemma 1-1.12.

Since $(\vec{P}_0,\vec{V}_0,t_0)$ be any point in $\Omega \subseteq R^{2n+1}$. By theorem 1-1.10, there exist neighbourhoods U of $\vec{P}_0$, V of $\vec{V}_0$ and an interval I of t_0 in R such that for all $\vec{P}_1 \in U$ and $\vec{V}_1 \in V$, there exists a unique function $\vec{\phi}^* = (\phi^1,\ldots,\phi^{2n})$ defined on I such that for each $i = 1,2,\ldots,2n$, $\phi^i(\vec{P}_1,\vec{V}_1,t)$ satisfies

$$\frac{d\phi^i}{dt} = \overset{i}{f}(\vec{\phi}^*, \dot{\vec{\phi}}^*, t)$$

$$\phi^i(\vec{P}_1,\vec{V}_1,t_0) = p_1^i \quad , \quad i = 1,2,\ldots,n$$

$$\phi^i(\vec{P}_1,\vec{V}_1,t_0) = v_1^i \quad \text{where} \quad i = n+1,\ n+2,\ldots,2n.$$

The functions ϕ^i are of class c^{k+1} (analytic) in t and of class c^k (analytic) in $(\vec{P}_1,\vec{V}_1)$ for all $i = 1,2,\ldots, n$.

By lemma 1-1.12, $\vec{\phi} = (\phi^1,\ldots, \phi^n)$ is the unique solution of (1) such that for each $i = 1,2,\ldots,n$, $\ddot{\phi}^i = H^i(\vec{\phi},\dot{\vec{\phi}},t)$ and $\phi^i(\vec{P}_1,\vec{V}_1,t_0) = p_1^i$, $\phi^{n+i}(\vec{P}_1,\vec{V}_1,t_0) = v_1^i$.

Since

$$\phi^{n+i} = \dot{\phi}^i \text{ , for } i = 1,2,\ldots,n$$

hence

$$\phi^{n+i}(\vec{P}_1,\vec{V}_1,t_0) = \dot{\phi}^i(\vec{P}_1,\vec{V}_1,t_0) \text{ .}$$

Thus

$$v_1^i = \dot{\phi}^i(\vec{P}_1,\vec{V}_1,t_0) \text{ for } i = 1,2,\ldots,n$$

For each $i = 1,2,\ldots,n$, $\dot{\phi}^i = \phi^{n+i}$ which is of class c^{k+1} (analytic) in t (by theorem 1-1.10). Hence $\dot{\phi}^i$ is of class c^{k+1} (analytic) in t. Therefore, ϕ^i is of class c^{k+2} (analytic) in t for each $i = 1,2,\ldots,n$. The proof is complete.

1-2 Implicit function theorem

Definition 1-2.1 Let $f^1, \ldots, f^n$ be n real-valued functions defined on an open set U in R^n and let $\vec{f} = (f^1, \ldots, f^n)$. By the Jacobian of $\vec{f}$ we mean the real-valued function $J_{\vec{f}}$ whose values are given by the determinant

$$J_{\vec{f}}(\vec{X}_0) = \det \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(\vec{X}_0) & \frac{\partial f^1}{\partial x^2}(\vec{X}_0) & \cdots & \frac{\partial f^1}{\partial x^n}(\vec{X}_0) \\ \vdots & & & \\ \frac{\partial f^n}{\partial x^1}(\vec{X}_0) & \frac{\partial f^n}{\partial x^2}(\vec{X}_0) & \cdots & \frac{\partial f^n}{\partial x^n}(\vec{X}_0) \end{bmatrix}$$

at those points $\vec{X}_0$ in U where all partial $\frac{\partial f^i}{\partial x^j}(\vec{X}_0)$ exist.

The notation $\frac{\partial(f^1, \ldots, f^n)}{\partial(x^1, \ldots, x^n)}$ is also used for $J_{\vec{f}}(\vec{X})$ where $\vec{X} = (x^1, \ldots, x^n)$.

Theorem 1-2.2 (The inverse function theorem)

Let $\vec{f} : U \to R^n$, $\vec{X}_0 \in U$ where U is an open subset of R^n. If $\vec{f}$ is c^1 function on U and $J_{\vec{f}}(\vec{X}_0) \neq 0$, then there exists open sets V of $\vec{X}_0$ and W of $\vec{f}(\vec{X}_0)$ which are subsets of U and $\vec{f}[U]$ respectively and a unique function $g : W \to V$ such that

(1) $\vec{f}[V] = W$

(2) $\vec{f}$ is a one to one function on V

(3) $\vec{g}[W] = V$ and $\vec{g}(\vec{f}(\vec{X})) = \vec{X} \quad \forall \vec{X} \in V$

(4) $\vec{g}$ is c^1 function on W.

For the proof of this theorem, we refer the reader to reference [1] pages 144-146.

Given a curve in the xy-plane we can express it in implicit form i.e. $f(x,y) = 0$ and sometimes we can express it in explicit form i.e. $y = f(x)$.

Example given $F(x,y) = x^3 + y^3 - 1$, then the relation defined by $x^3 + y^3 - 1 = 0$ is a function. That is $x^3 + y^3 - 1 = 0$ can be solved explicitly for y in terms of x ($y = \sqrt[3]{1 - x^3}$), yielding a unique solution.

However, if we are given an equation of the form $F(x,y) = 0$, this does not necessarily represent an explicit function for example $x^2 + y^2 - 5 = 0$. When is the relation defined by $F(x,y) = 0$ also a function? In other words, when can the equation $F(x,y) = 0$ be solved explicitly for y in terms of x, yielding a unique solution? The implicit function theorem deals with this question locally.

It tells us that, given a point $(x_0, y_0) \in R^2$ such that $F(x_0, y_0) = 0$, under certain conditions there will be a neighbourhood of (x_0, y_0) such that in this neighbourhood the relation defined by $F(x,y) = 0$ is also a function. The conditions are that F and $\frac{\partial F}{\partial y}$ be continuous in some neighbourhood of (x_0, y_0) and that $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$.

<u>Theorem 1-2.3</u> (The implicit function theorem)

Let $f^1, \ldots, f^n$ be c^k $(k \geq 1)$ (or analytic) real-valued functions of $x^1, \ldots, x^m, y^1, \ldots, y^n$ defined in some neighbourhood $U \times V$ of a point $(a^1, \ldots, a^m, b^1, \ldots, b^n)$ in $R^m \times R^n$ such that

$$(1)\quad f^i(a^1,\ldots, a^m, b^1,\ldots, b^n) = 0 \quad \text{for } 1 \le i \le n$$

and $$(2)\quad \det \left(\frac{\partial f^i}{\partial y^j}\right)(a^1,\ldots, a^m, b^1,\ldots, b^n) \neq 0 \quad \text{for } 1 \le i, j \le n.$$

Then there is an open neighbourhood $W_0 \subseteq U$ of $(a^1,\ldots, a^m)$ such that, for any connected open neighbourhood $W \subseteq W_0$ of $(a^1,\ldots, a^m)$, there is a unique system of n real-valued functions $g^1,\ldots, g^n$, defined and C^k $(k \ge 1)$ (or analytic) in W and such that

$$(3)\quad g^i(a^1,\ldots, a^m) = b^i \quad \text{for } 1 \le i \le n, \quad \text{and}$$

$$(4)\quad f^i(x^1,\ldots, x^m, g^1(x^1,\ldots, x^m),\ldots, g^n(x^1,\ldots,x^m)) = 0$$

for $1 \le i \le n$ and any $(x^1,\ldots, x^m) \in W$.

For the proof of this theorem one is referred to [4] pages 270-273.

1-3 Analytic function of several variables

<u>Theorem 1-3.1</u> The set S of all convergent power series in n variables over the field of real numbers is an integral domain.

For the proof one is refered to [8] pages 129-130. This theorem says that if the product fg of two real-valued analytic functions f and g is identically zero in some neighbourhood of $\vec{X}_0 \in R^n$, then at least one of the function f and g is identically zero in a neighbourhood of $\vec{X}_0$.

<u>Theorem 1-3.4</u> Let $\vec{V} = (v^1, \dots, v^n)$. Let h be a function of n variables $v^1, v^2, \dots, v^n$ which is analytic. If h is identically zero, then all of the coefficients of the power series are zero.

<u>Proof</u> Since h is analytic in $v^1, v^2, \dots, v^n$.

By definition 1-1.7, 1-1.6 we get,

$$\begin{aligned} h(v^1, \dots, v^n) &= h(\vec{0}) + [v^1 \frac{\partial}{\partial v^1} + \dots + v^n \frac{\partial}{\partial v^n}] h(\vec{0}) \\ &+ \frac{1}{2!} [v^1 \frac{\partial}{\partial v^1} + \dots + v^n \frac{\partial}{\partial v^n}]^2 h(\vec{0}) \\ &\quad \vdots \\ &+ \frac{1}{k!} [v^1 \frac{\partial}{\partial v^1} + \dots + v^n \frac{\partial}{\partial v^n}]^k h(\vec{0}) \\ &+ \dots \end{aligned}$$

For simplicity, we will use the following notations

(1) $c_{j_1} v^{j_1} = \sum_{j_1=1}^{n} c_{j_1} v^{j_1}$, $c_{j_1 \dots j_k} v^{j_1} \dots v^{j_k} = \sum_{j_1=1}^{n} \dots \sum_{j_k=1}^{n} c_{j_1 \dots j_k} v^{j_1} \dots v^{j_k}$

(2) $D_{j_1} h = \dfrac{\partial h}{\partial v^{j_1}}$; $j_1 = 1, 2, \dots, n$

$$D_{j_1 \dots j_k} h = \frac{\partial^k h}{\partial v^{j_1} \dots \partial v^{j_k}} \; ; \; j_k = 1, 2, \dots, n \; ; \; k = 1, 2, \dots$$

Let

$$c_0 = h(\vec{0})$$

$$c_{j_1} = D_{j_1} h(\vec{0})$$

$$c_{j_1 j_2} = \begin{cases} \frac{1}{2!} D_{j_1 j_2} h(\vec{0}) & \text{where } j_1 = j_2 \\ c^*_{j_1 j_2} D_{j_1 j_2} h(\vec{0}) & \text{where } j_1 \neq j_2 \end{cases}$$

$$c_{j_1\cdots j_k} = \begin{cases} \frac{1}{k!} D_{j_1\cdots j_k} h(\vec{0}) & \text{where } j_1 = j_2 = \cdots = j_k \\ c^*_{j_1\cdots j_k} D_{j_1\cdots j_k} h(\vec{0}), & \text{otherwise} \end{cases}$$

Hence, $h(v^1,\ldots,v^n) = c_0 + c_{j_1} v^{j_1} + c_{j_1 j_2} v^{j_1} v^{j_2} + \ldots + c_{j_1\cdots j_k} v^{j_1} \ldots v^{j_k} + \ldots$

By the assumption, $h = 0$

Therefore, we have $h(\vec{0}) = 0$ and $D_{j_1\cdots j_k} h(\vec{0}) = 0$, $k = 1,2,\ldots$

Thus, $c_0 = 0$, $c_{j_1\cdots j_k} = 0$ where $j_k = 1,2,\ldots,n$; $k = 1,2,\ldots$

The proof is complete.

<u>Corollary 1-3.5</u> Let f and g be two analytic functions in the n variables $v^1, v^2, \ldots, v^n$ in the same region such that f and g are identically equal. Let $f(v^1,\ldots,v^n) = a_0 + a_{j_1} v^{j_1} + \ldots + a_{j_1\cdots j_k} v^{j_1} \ldots v^{j_k} + \ldots$

and let $g(v^1,\ldots,v^n) = b_0 + b_{j_1} v^{j_1} + \ldots + b_{j_1\cdots j_k} v^{j_1} \ldots v^{j_k} + \ldots$

Then $a_0 = b_0$, for each $k = 1,2,3,\ldots$ $a_{j_1\cdots j_k} = b_{j_1\cdots j_k}$ for $1 \le j_k \le n$

<u>Proof</u> Let $h \equiv f-g$. Hence $h \equiv 0$ (by the assumption). Since f,g are analytic functions of $v^1,\ldots, v^n$

Hence f-g is analytic in $v^1,\ldots, v^n$, so is h.

Thus, $h = (a_0 - b_0) + (a_{j_1} - b_{j_1}) v^{j_1} + \ldots + (a_{j_1\cdots j_k} - b_{j_i\cdots j_k}) v^{j_1} \ldots v^{j_k} + \ldots$

By theorem 1-3.4, implies that $a_0 - b_0 = 0$, $a_{j_1} - b_{j_1} = 0, \ldots,$

$a_{j_1\cdots j_k} - b_{j_1\cdots j_k} = 0, \ldots.$

Hence $a_0 = b_0$, $a_{j_1} = b_{j_1}, \ldots, a_{j_1 \ldots j_k} = b_{j_1 \ldots j_k}$ for $k = 1,2,\ldots$ and $j_k = 1,2,\ldots, n$. Then the proof is complete .

Theorem 1-3.6 Let $b = (b^1,\ldots,b^n) \varepsilon R^n$ be such that $b^i \neq 0$ for $1 \leq i \leq n$ and that the series $\sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} c_{m_1 \ldots m_n} (b^1)^{m_1} \ldots (b^n)^{m_n} < \infty$. Then for any r^i, $0 < r^i < |b^i|$ for $1 \leq i \leq n$, the power series $\sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} c_{m_1 \ldots m_n} (z^1)^{m_1} \ldots (z^n)^{m_n}$ is absolutely convergent for all z^i, $|z^i| < r^i$ and it can be rearranged.

For the proof one is refered to [4] pages 199-200.

Theorem 1-3.7 Let x and y be two independent variables. If f is a differentiable function of x and g is a differentiable of y such that $f(x) = g(y)$, then f and g are identically equal constant.

Proof Assume $f(x) = g(y)$

Since f is differentiable.

Hence, $\frac{df}{dx}(x) = 0$ for all x

Therefore $f \equiv k_1$ where k_1 is a constant.

Similarly, we can prove that $g \equiv k_2$ where k_2 is a constant.

By the assumption, we conclude that $k_1 = k_2$.

Hence f and g are identically equal constant.

This completes proof.