CHAPTER III

GRAPHS AND EMBEDDING OF GRAPHS INTO SURFACES

3.1 Graphs

A graph G is an ordered pair (V, E), where V is a finite nonempty set and E is a set of 2-subset of V. Elements of V and E are called vertices and edges of (V, E), respectively.

If {u, v} is an edge of a graph G, we say that v and u are adjacent and vertex v and edge {u, v} are incident.

If v is any vertex in a graph G, we define the degree of vertex v to be the number of edges which are incident to v and denoted by deg v.

Theorem 3.1.1 If G is a graph with n vertices and m edges and d is the average degree of vertices of G, i.e., $d = (\sum_{i=1}^{n} d_i)/n$, then $dn = 2m$.

Proof. Let d_i be the degree of a vertex v_i of G. Since every edge is incident with two vertices, it contributes two to the sum of the degree of vertices, that is,

$$\sum_{i=1}^{n} d_i = 2m.$$

But

$$d = \frac{\sum_{i=1}^{n} d_i}{n}.$$

Hence

$$dn = 2m.$$

Q.E.D.

Let (V_1, E_1) be a graph, then (V_1, E_1) is a subgraph of (V,E) if $V_1 \subseteq V$ and $E_1 \subseteq E$. Observe that the relation "being subgraph" is a partial order on any set of graphs.

Let G be a graph having more than one vertex. By the removal of a vertex v from a graph G, we mean the removal of a vertex v and edges incident with v. The resulting graph is a subgraph of G and will be denoted by G - v. Thus G - v is the maximal subgraph of G not containing v. By the removal of an edge e from a graph G, we mean the removal of that edge only. The resulting graph is a subgraph, denoted by G - e, which is the maximal subgraph of G not containing e.

If every pair of vertices of a graph G are adjacent, then G is called a complete graph. The complete graph with n vertices is denoted by K_n. We see that K_n has $\binom{n}{2}$ edges.

By a path of G, we mean the alternating sequence of vertices and edges,

$$v_0, e_1, v_1, \ldots, e_{r-1}, v_{r-1}, e_r, v_r,$$

in which each edge incident with the two vertices immediately preceding and the following it provided all of its vertices and all of it edges are distinct.

If G consists of a single vertex or every pair of its vertices is joined by a path, we say that G is a connected graph.

Let $G = (V, E)$ be a graph and $C = \{c_1, \ldots, c_n\}$ be a set of n elements, called colors. An n-coloring of G is a function c from V into C such that $c(u) \neq c(v)$ whenever u and v are adjacent vertices of G. If $c(v) = c_i$, we say that we assign the color c_i to v.

By the chromatic number of a graph G, we mean the smallest n for which G has an n-coloring, and denote by $\chi(G)$.

Note that $\chi(K_n) = n$.

3.2 A Realization of a Graph

If P is a set of points in $\mathbb{R}^3$ such that no three of its points lie on the same line and no four of its points lie on the same plane, we say that P is in general position.

Let $G = (V, E)$ be a graph. Let P be a set of points in $\mathbb{R}^3$ and L be a set of line segments such that the endpoints of any element q in L belong to P. We say that (P, L) is a realization of (V, E) if and only if P is in general position and there exist 1 - 1 correspondences $f : V \to P$, $g : E \to L$ such that for any $u, v \in V$, $\{u, v\} \in E$ if and only if $f(u)$, $f(v)$ are endpoints of $g(\{u, v\})$.

For example, let (V, E) be a graph, where $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$. Let

$P = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and

$L = \{q_1, q_2, q_3, q_4\}$, where

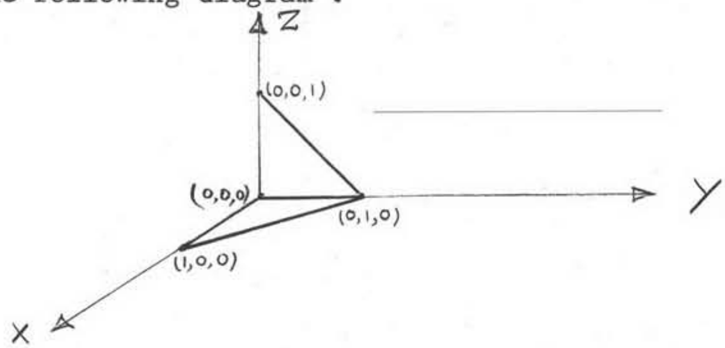
$$q_1 = \{\theta_1(0, 0, 0) + \theta_2(1, 0, 0) : \theta_1, \theta_2 \in \mathbb{R}, 0 < \theta_1 < 1, 0 < \theta_2 < 1 \text{ and } \theta_1 + \theta_2 = 1\},$$

$$q_2 = \{\theta_1(0, 0, 0) + \theta_2(0, 1, 0) : \theta_1, \theta_2 \in \mathbb{R}, 0 < \theta_1 < 1, 0 < \theta_2 < 1 \text{ and } \theta_1 + \theta_2 = 1\},$$

$$q_3 = \{\theta_1(1, 0, 0) + \theta_2(0, 1, 0) : \theta_1, \theta_2 \in \mathbb{R}, 0 < \theta_1 < 1, 0 < \theta_2 < 1 \text{ and } \theta_1 + \theta_2 = 1\},$$

$$q_4 = \{\theta_1(0, 1, 0) + \theta_2(0, 0, 1) : \theta_1, \theta_2 \in \mathbb{R}, 0 < \theta_1 < 1, 0 < \theta_2 < 1 \text{ and } \theta_1 + \theta_2 = 1\}.$$

Then (P, L) is a realization of (V, E) and it can be represented by the following diagram :



We observe that $P \cup (\bigcup_{q \in L} q)$ can be considered as a topological subspace of $\mathbb{R}^3$. We shall denote this topological subspace by $|G|$. Any homeomorphic image of $|G|$ in a topological space (X, τ) will be called a <u>topological realization</u> of (V, E) in (X, τ).

In this section we shall show that every finite graph can be realized in $\mathbb{R}^3$.

Lemma 3.2.1 Given any positive integer n, we can find n points in $\mathbb{R}^3$ such that these n points are in general position.

Proof. Let a_1 be a point in $\mathbb{R}^3$. Choose a point, say a_2, from the complement of the set $\{a_1\}$. Choose a point, say a_3, from the complement of the set $l(a_1, a_2)$. Hence the points a_1, a_2 and a_3 do not lie on the same line. Choose a point, say a_4, from the complement of the set $P(a_1, a_2, a_3)$. Hence the points a_1, a_2, a_3 and a_4 do not lie on the same plane and any three of them do not lie on the same line. Choose a point, say a_5, from the complement of the set $\bigcup_{1 \leq i < j < k \leq 4} P(a_i, a_j, a_k)$. Then the points a_1, a_2, a_3, a_4 and a_5 have no three points lie on the same line and no four points lie on the same plane.

After $a_1, \ldots, a_m$ have been chosen so that no three points lie on the same line and no four points lie on the same plane, we choose a_{m+1} from the complement of the set $\bigcup_{1 \leq i < j < k \leq m} P(a_i, a_j, a_k)$. Hence we can find the points $a_1, \ldots, a_n$ such that these n points are in general position.

Q.E.D.

Theorem 3.2.2 Any graph G = (V, E) can be realized in $\mathbb{R}^3$.

Proof. Let P be a set of points in $\mathbb{R}^3$ such that P is in general position and such that P and V are in 1 - 1 correspondence.

Let $f : V \rightarrow P$ be a 1 - 1 correspondence. We define g on E by setting

$$g(\{v, w\}) = q(f(v), f(w)).$$

Let L be the image of E under g. It can be seen that (P, L) is a realization of (V, E).

Q.E.D.

Note From the assumption that the points in P are in general position, it can be shown that no line segments in L intersect.

3.3 Embedding of a Graph into a Surface

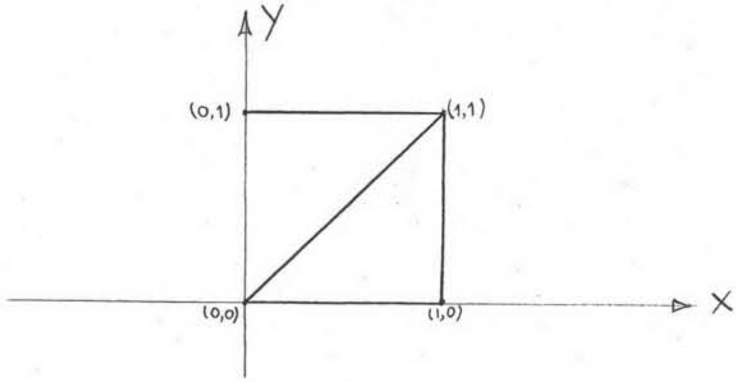
Let S be a surface. An embedding h of any realization $|G|$ of a graph G to S will be called an embedding of G into S. The images of points and line segments under any embedding will be also refered to as points and lines, respectively.

Roughly speaking an embedding of a graph G into a surface S is a drawing of G on S. Observe that the lines in the drawing do not intersect, this followsfrom fact that the embedding is an injection.

To illustrate this, let G = (V, E) be the graph given in the example of Section 3.2. Then $h : |G| \to \mathbb{R}^2$, defined by

$$h(x, y, z) = (x + y, y + z)$$

is an embedding of graph G = (V, E) into $\mathbb{R}^2$. The image of this embedding can be represented by the following diagram.



If (P, L) is a realization of graph G = (V, E) and $h : |G| \to S$ is an embedding from $|G|$ into a surface S, then the components of $S-h(|G|)$ are called the regions of the embedding. If each region of an embedding h is homeomorphic to $\mathbb{R}^2$, then h will be called a proper embedding. When the proper embedding $h : |G| \to S$ exists, we say that G can be properly embedded into S. We shall call a region whose boundary consists of three points and three lines, a triangle.

If G has an embedding into a surface S, we say that G can be embedded into S or G is S-embeddable. A maximal S-embeddable graph is an S-embeddable graph to which no edge can be added without losing S-embeddability. From this we can show that a graph G is maximal S-embeddable if and only if each region of the embedding is a triangle.

Let G and G' be any two connected graphs with n vertices m edges and n' vertices, m' edges, respectively, and G, G' can be properly embedded into a surface S_t. It can be shown that

$n - m + r = n' - m' + r'$, where r and r' are numbers of regions of the embeddings. Proof of this fact can be found in [4] or [8].

We denote the number $n - m + r$ by $\chi(S_t)$ and call <u>the Euler characteristic of S_t</u>.

By using special graph for each surface S_t, it can be shown that

$$\chi(S_t) = 2 - 2t.$$

From this we have the following theorem.

<u>Theorem</u> 3.3.1 If G is a connected graph with n vertices, m edges and G can be properly embedded into a surface S_t, then $n - m + r = 2 - 2t$, where r is the number of regions of the embedding.

<u>Corollary</u> 3.3.2 If G is a connected graph with n-vertices, m edges and G is maximal S_t-embeddable, then $m = 3(n - (2 - 2t))$.

<u>Proof</u>. Since G is maximal S_t-embeddable, hence each region of the embedding of G is a triangle.

Let r be the number of these regions.

By counting argument, we see that $3r = 2m$.

By Theorem 3.3.1, we have

$$n - m + r = 2 - 2t ,$$

so

$$n - m + \frac{2m}{3} = 2 - 2t .$$

Hence

$$m = 3(n - (2 - 2t)).$$

Q.E.D.

Corollary 3.3.3 If G is a connected graph with n vertices m edges and G can be properly embedded into a surface S_t, then $m \leq 3(n - (2 - 2t))$.

Proof. This follows immediately from Corollary 3.3.2.

Q.E.D.