CHAPTER II

METHOD OF SOLUTION

1. Proposed Deflection Function

The energy method to be used in finding an approximated solution depends on good deflection functions compatible with the nature of the problem. The deflection function is assumed in the form of polynomials of fourth degree as follows.

$$w = C_1 + C_2 x + C_3 y + C_4 x^2 + C_5 xy + C_6 y^2 + C_7 x^3 + C_8 x^2 y + C_9 xy^2 + C_{10} y^3 + C_{11} x^4 + C_{12} x^3 y + C_{13} x^2 y^2 + C_{14} xy^3 + C_{15} y^4 \qquad (1)$$

The co-ordinate axes are located as shown in Fig. 1. Because of symmetry, the deflection w must be an even function of y. Therefore

$$C_3 = C_5 = C_8 = C_{10} = C_{12} = C_{14} = 0 \qquad (2)$$

Thus eq. (1) becomes

$$w = C_1 + C_2 x + C_4 x^2 + C_6 y^2 + C_7 x^3 + C_9 xy^2 + C_{11} x^4 + C_{13} x^2 y^2 + C_{15} y^4 \qquad (3)$$

There are nine more unknowns in eq. (3) to be determined. Some of them can be found by rotating the co-ordinates axes from $x - y$ to $\xi - \eta$ by an angle of 120°. The relationship between the two is

$$
\begin{aligned}
x &= -\frac{1}{2}(\xi + \sqrt{3}\,\eta) \\
y &= \frac{1}{2}(\sqrt{3}\,\xi - \eta)
\end{aligned}
\qquad (4)
$$

Substituting eq. (4) into eq. (3) one has

$$
\begin{aligned}
w = {} & c_1 - \frac{1}{2}c_2\xi - \frac{\sqrt{3}}{2}c_2\eta + \left(\frac{1}{4}c_4 + \frac{3}{4}c_6\right)\xi^2 \\
& + \frac{\sqrt{3}}{2}(c_4 - c_6)\xi\eta + \left(\frac{3}{4}c_4 + \frac{1}{4}c_6\right)\eta^2 \\
& + \left(-\frac{1}{8}c_7 - \frac{3}{8}c_9\right)\xi^3 + \left(-\frac{3\sqrt{3}}{8}c_7 - \frac{\sqrt{3}}{8}c_9\right)\xi^2\eta \\
& + \left(-\frac{9}{8}c_7 + \frac{5}{8}c_9\right)\xi\eta^2 + \left(-\frac{3\sqrt{3}}{8}c_7 - \frac{\sqrt{3}}{8}c_9\right)\eta^3 \\
& + \left(\frac{3}{16}c_{13} + \frac{1}{16}c_{11} + \frac{9}{16}c_{15}\right)\xi^4 \\
& + \left(-\frac{1}{8}c_{13} + \frac{9}{8}c_{11} + \frac{9}{8}c_{15}\right)\xi^2\eta^2 \\
& + \left(\frac{\sqrt{3}}{4}c_{13} + \frac{\sqrt{3}}{4}c_{11} - \frac{3\sqrt{3}}{4}c_{15}\right)\xi^3\eta \\
& + \left(-\frac{\sqrt{3}}{4}c_{13} + \frac{3\sqrt{3}}{4}c_{11} - \frac{\sqrt{3}}{4}c_{15}\right)\xi\eta^3 \\
& + \left(\frac{3}{16}c_{13} + \frac{9}{16}c_{11} + \frac{1}{16}c_{15}\right)\eta^4
\end{aligned}
\qquad (5)
$$

From Fig. 1, it is seen that the deflection function must be an even function of η. Therefore, from eq. (5) the following results can be obtained.

$$C_2 = 0,\ C_4 = C_6,\ C_7 = -\frac{C_9}{3},\ C_{11} = \frac{C_{13}}{2} = C_{15} \quad (6)$$

Substituting eq. (6) into eq. (3) and rearrange, one has

$$w = C_1 + C_4(x^2 + y^2) + C_7(x^3 - 3xy^2) + C_{11}(x^4 + 2x^2y^2 + y^4) \quad (7)$$

The remaining four constants in eq. (7) must be chosen such that the deflection function satisfy the boundary conditions and the equilibrium equation. Consider one of the edge of the triangular plate, the exact boundary conditions are

$$w \Big|_{\frac{2a}{3},\ 0} = 0 \quad (8)$$

$$M_x \Big|_{-a/3,\ y} = 0 \quad (9)$$

$$V_x \Big|_{-a/3,\ y} = 0 \quad (10)$$

To facilitate the solution, the boundary condition (9) is changed to the vanish of the total bending moment effect. That is,

$$\int_0^{a/\sqrt{3}} M_x \Big|_{-\frac{a}{3},\ y} dy = \int_0^{a/\sqrt{3}} -D\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]_{-\frac{a}{3},\ y} dy = 0 \quad (11)$$

Now, three of the remaining constants can be found by forcing them to satisfy the boundary conditions (8), (10) and

(11). Substitute eq.(7) into eqs.(8), (10) and (11) yields three simultaneous algebraic eqs. as follows :

$$81\,C_1 + 36\,a^2C_3 + 24a^3C_5 + 16\,a^4C_7 = 0 \qquad (12)$$

$$9a\,(1 + \nu)C_3 - 9a^2(1 - \nu)C_5 + 8a^3\,(1 + \nu)C_7 = 0 \qquad (13)$$

$$9(1 - \nu)C_5 + 4a\,(5 - \nu)C_7 = 0 \qquad (14)$$

Solving for C_3, C_5 and C_7 in term of C_1 one gets.

$$C_3 = \frac{-27(1 - \nu)}{8a^2(2 - \nu)}\,C_1 \qquad (15)$$

$$C_5 = \frac{-27(5 - \nu)(1 + \nu)}{8a^3(7 + \nu)(2 - \nu)}\,C_1 \qquad (16)$$

$$C_7 = \frac{243\,(1 - \nu^2)}{32a^4(7 + \nu)(2 - \nu)}\,C_1 \qquad (17)$$

Rewriting eqs.(15), (16) and (17) as

$$C_3 = kC_1 \qquad (18)$$

$$C_5 = mC_1 \qquad (19)$$

$$C_7 = r.C_1 \qquad (20)$$

where $k = -\dfrac{27\,(1 - \nu)}{8a^2(2 - \nu)}$

$$m = -\frac{27\,(5 - \nu)(1 + \nu)}{8a^3(7 + \nu)(2 - \nu)}$$

$$n = \frac{243\,(1 - \nu^2)}{32a^4(7 + \nu)(2 - \nu)}$$

Then the proposed deflection function becomes

$$w = C_1 \left[1 + k(x^2 + y^2) + m(x^3 - 3xy^2) + n(x^4 + 2x^2y^2 + y^4) \right] \tag{21}$$

2. Solution for Concentrated Load at Centroid

The proposed approximated method of solution is the stationary value of the total potential energy. The strain energy in pure bending of the plate is [4]

$$V = \frac{D}{2} \iint \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \times \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right\} \right] dA \tag{22}$$

The potential energy of the concentrated load P applied at the centroid $x = 0$, $y = 0$ is

$$-Pw \Big|_{x=0,\, y=0} = -PC_1 \tag{23}$$

Therefore the total potential energy of the plate is

$$I = D \int_{-\frac{a}{3}}^{\frac{2a}{3}} \int_0^{y} \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \times \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right\} \right] dy \, dx - PC_1 \tag{24}$$

Substituting eq. (21) into eq. (24) and integrating yields

$$I = \frac{4\ DC_1^2}{\sqrt{3}} \left[(1 + \nu)k^2a^2 + \frac{32}{405}(5 + 3\nu)\ n^2a^6 \right.$$

$$+ \frac{8}{9}(1 + \nu)\ kna^4 + (1 - \nu)m^2a^4$$

$$\left. + \frac{16}{45}(1 - \nu)mna^5 \right] - PC_1 \qquad (25)$$

The coefficient C_1 can now be determined by the principle of stationary value of the total potential energy, that is

$$\frac{\partial I}{\partial C_1} = 0$$

from which one obtains

$$C_1 = \frac{4\ \sqrt{3}\ a^2\ (2 - \nu)^2\ (7 + \nu)^2 P}{729(1 - \nu^2)(29 - 10\nu - 3\nu^2)D} \qquad (26)$$

Make a substitution of C_1, k, m and n into eq. (21) one has the solution of the equilateral triangular plate loaded by a concentrated force at the centroid.

$$w = \frac{(2 - \nu)(7 + \nu)a^2 P}{\sqrt{3}\ D(29 - 10\nu - 3\nu^2)} \left[\frac{4(2 - \nu)(7 + \nu)}{243\ (1 - \nu^2)} \right.$$

$$- \frac{1\ (7 + \nu)}{18\ (1 + \nu)} \left(\frac{x^2}{a^2} + \frac{y^2}{a^2} \right) - \frac{1\ (5 - \nu)}{18(1 - \nu)} \left(\frac{x^3}{a^3} - \frac{3xy^2}{a^3} \right)$$

$$\left. + \frac{1}{8} \left(\frac{x^4}{a^4} + \frac{2x^2y^2}{a^4} + \frac{y^4}{a^4} \right) \right] \qquad (27)$$

The expression for the bending moments, twisting moments and shearing force can be found by appropriate diffrentiation as follows.

$$M_x = \frac{(2-\nu)(7+\nu)P}{\sqrt{3}\,(29-10\nu-3\nu^2)}\left[\frac{1}{9}(7+\nu)+\frac{1}{3}(5-\nu)\frac{x}{a}-\frac{1}{2}(3+\nu)\frac{x^2}{a^2}-\frac{1}{2}(1+3\nu)\frac{y^2}{a^2}\right] \qquad (28)$$

$$M_y = \frac{(2-\nu)(7+\nu)P}{\sqrt{3}\,(29-10\nu-3\nu^2)}\left[\frac{1}{9}(7+\nu)-\frac{1}{3}(5-\nu)\frac{x}{a}-\frac{1}{2}(1+3\nu)\frac{x^2}{a^2}-\frac{1}{2}(3+\nu)\frac{y^2}{a^2}\right] \qquad (29)$$

$$M_{xy} = -M_{yx} = \frac{(2-\nu)(7+\nu)P}{\sqrt{3}\,(29-10\nu-3\nu^2)}\left[\frac{1}{3}(5-\nu)\frac{y}{a}+(1-\nu)\frac{xy}{a^2}\right] \qquad (30)$$

$$Q_x = \frac{(2-\nu)(7+\nu)P}{\sqrt{3}\,(29-10\nu-3\nu^2)}\left[-\frac{4x}{a^2}\right] \qquad (31)$$

$$Q_y = \frac{(2-\nu)(7+\nu)P}{\sqrt{3}\,(29-10\nu-3\nu^2)}\left[-\frac{4y}{a^2}\right] \qquad (32)$$

and the corner force is

$$R = 2\left[M_{xy}\right]_{-\frac{a}{3},\,-\frac{a}{\sqrt{3}}}$$

$$= -\frac{8}{9}\,\frac{(2-\nu)(7+\nu)P}{(29-10\nu-3\nu^2)} \qquad (33)$$

Note that for $\nu = 0.3$ the corner reaction is

$$R = -0.429 \ P$$

which does not agree with the equilibrium requirement $R = \frac{P}{3}$. This is because of the approximated boundary condition (11) and the deflection function selected may not be very good. However, it will be shown later that the proposed solution yields reasonable agreement with the experimental results, eventhough the proposed approximated method of solution does not satisfy equilibrium exactly.

3. Solution for Partial Triangular Load

The solution of the equilateral triangular plate supported at the corners and loaded by partial uniformly distributed triangular load as shown in Fig. 2. will now be solved.

The potential energy of the load is

$$\iint q \, w \, d \, A$$

$$= 2q \int_{-\frac{e}{3}}^{\frac{2e}{3}} \int_{0}^{y} C_1 \left[1 + k(x^2 + y^2) + m\,(x^3 - 3xy^2) + n\,(x^4 + 2x^2y^2 + y^4) \right] dy \, dx$$

$$= \frac{2C_1 q}{\sqrt{3}} \left[\frac{1}{2} e^2 + \frac{1}{18} k e^4 + \frac{2}{5x27} m e^5 + \frac{4}{5x81} n e^6 \right] \quad (34)$$

Combine eq. (34) with eq. (22) and after appropriate integration gives the total potential energy of the plate as

$$I = \frac{4DC^2}{\sqrt{3}} \Big[(1 + \nu) k^2 a^2 + \frac{32}{9x45} (5 + 3\nu) n^2 a^6 + \frac{8}{9} (1 + \nu) kna^4 + (1 - \nu) m^2 a^4 + \frac{16}{45} (1 - \nu) mna^5 \Big] - \iint q \, w \, dA \quad (35)$$

The constant C_1 can be determined from the condition $\frac{\partial I}{\partial C_1} = 0$ from which gives

$$\frac{\frac{q}{4D} \left[\frac{1}{2} e^2 + \frac{1}{18} k e^4 + \frac{2}{5x27} m e^5 + \frac{4}{5x81} n e^6 \right]}{(1+\nu) k^2 a^6 + \frac{32}{9x45} (5+3\nu) n^2 a^6 + \frac{8}{9} (1+\nu) kna^4 + (1-\nu) m^2 a^4 + \frac{16}{45} (1-\nu) mna^5} \quad (30$$

Substitution of C_1, k, m and n into eq. (21) one has the solution of the equilateral triangular plate loaded by partial uniformly distributed load.

The deflection function is

$$w = \left[\frac{q(7+\nu)(2-\nu)\left\{40(7+\nu)(2-\nu)a^4 e^2 - 15(7+\nu)(1-\nu)a^2 e^4\right\}}{7290a^2 D(1-\nu^2)(29-10\nu-3\nu^2)} + \frac{q(7+\nu)(2-\nu)\left\{6(1-\nu)e^6 - 4(5-\nu)(1+\nu)a\, e^5\right\}}{7290a^2 D(1-\nu^2)(29-10\nu-3\nu^2)} \right] \times$$

$$\times \left[1 - \frac{27}{8}\frac{(1-\nu)}{(2-\nu)}\left(\frac{x^2}{a^2}+\frac{y^2}{a^2}\right) - \frac{27}{8}\frac{(5-\nu)(1+\nu)}{(7+\nu)(2-\nu)}\left(\frac{x^3}{a^3}-\frac{3xy^2}{a^3}\right) + \frac{243}{32}\frac{(1-\nu^2)}{(7+\nu)(2-\nu)}\left(\frac{x^4}{a^4}+\frac{2x^2y^2}{a^4}+\frac{y^4}{a^4}\right)\right] \quad (37)$$

The moment resultants, shear forces and the corner force are found by differentiation as the following.

$$M_x = C\left[\frac{1}{9}(7+\nu) + \frac{1}{3}(5-\nu)\frac{x}{a} - \frac{1}{2}(3+\nu)\frac{x^2}{a^2} - \frac{1}{2}(1+3\nu)\frac{y^2}{a^2}\right] \quad (38)$$

$$M_y = C\left[\frac{1}{9}(7+\nu) - \frac{1}{3}(5-\nu)\frac{x}{a} - \frac{1}{2}(1+3\nu)\frac{x^2}{a^2} - \frac{1}{2}(3+\nu)\frac{y^2}{a^2}\right] \quad (39)$$

$$M_{xy} = C\left[\frac{1}{3}(5-\nu)\frac{y}{a} + (1-\nu)\frac{xy}{a^2}\right] \quad (40)$$

$$Q_x = C\left[-\frac{4x}{a^2}\right] \quad (41)$$

$$Q_y = C\left[-\frac{4y}{a^2}\right] \quad (42)$$

$$R = 2C\left[\frac{1}{3}(5-\nu)\frac{y}{a} + (1-\nu)\frac{xy}{a^2}\right] \quad (43)$$

where $C = \text{const}$

$$C = \left[\frac{q\left\{40(7+\nu)(2-\nu)a^4e^2-15(7+\nu)(1-\nu)a^2e^4\right\}}{120a^4(29-10\nu-3\nu^2)} + \right.$$

$$\left. + \frac{q\left\{6(1-\nu)e^6-4(5-\nu)(1+\nu)ae^5\right\}}{120a^4(29-10\nu-3\nu^2)} \right] \qquad (44)$$

For $\nu = 0.3$ the corner force is

$$R = -\left[0.2475-0.03823\left(\frac{e}{a}\right)^2-0.01219\left(\frac{e}{a}\right)^3+0.00209\left(\frac{e}{a}\right)^4\right] \qquad (45)$$

It is seen that R is a function of $\frac{e}{a}$. If e is equal to a

$$R = -0.19917\ qa^2$$

The exact value of R is $-\frac{qa^2}{3\sqrt{3}} = -0.19245qa^2$