# CHAPTER III

## GENERALIZED TRANSFORMATION SEMIGROUPS

In this chapter, we characterize generalized transformation semigroups admitting ring structure, in particular, well-known transformation semigroups admitting ring structure. This is the main purpose of this thesis.

Recall that if S is a transformation semigroup on a set X and $\theta \in S$, then the semigroup S under the operation $*$ defined by

$$\alpha * \beta = \alpha\theta\beta$$

for all $\alpha, \beta$ in S is called a <u>generalized transformation semigroup</u> on X and it is denoted by $(S, \theta)$.

Observe that if a transformation semigroup S has a zero 0, then for $\theta \in S$, 0 is also the zero of the generalized transformation semigroup $(S, \theta)$.

Throughout this chapter, the following notation will be used. For any set A, let $1_A$ denote the identity map on A. Let X be a set. For any nonempty subset A of X and for $a \in X$, let $A_a$ denote the partial transformation of X such that $\Delta A_a = A$ and $\nabla A_a = \{a\}$. For $a,b,c \in X$, let $(a,b)$ and $(a,b,c)$ be the permutations on X defined by

$$x(a,b) = \begin{cases} b & \text{if } x = a, \\ a & \text{if } x = b, \\ x & \text{otherwise,} \end{cases}$$

and

$$x(a,b,c) = \begin{cases} b & \text{if } x = a, \\ c & \text{if } x = b, \\ a & \text{if } x = c, \\ x & \text{otherwise.} \end{cases}$$

Recall for the notation of transformation semigroups. For any set X , let

$T_X$ = the partial transformation semigroup on X ,

$\mathcal{T}_X$ = the full transformation semigroup on X ,

$I_X$ = the 1-1 partial transformation semigroup on X ( the symmetric inverse semigroup on X ),

$G_X$ = the permutation group on X ,

$U_X$ = the semigroup of all almost identical partial transformations of X ,

$V_X$ = the semigroup of all almost identical transformations of X ,

$W_X$ = the semigroup of all almost identical 1-1 partial transformations of X ,

$M_X$ = the semigroup of all one-to-one transformations of X ,

$E_X$ = the semigroup of all onto transformations of X , and $C_X$ = the semigroup of all constant partial transformations of X , thus,

$$\mathcal{T}_X = \{ \alpha \in T_X \mid \Delta\alpha = X\} ,$$

$$I_X = \{ \alpha \in T_X \mid \alpha \text{ is one-to-one}\},$$

$$U_X = \{ \alpha \in T_X \mid |S(\alpha)| < \infty\} (\text{where } S(\alpha) = \{x \in \Delta\alpha \mid x\alpha \neq x\}),$$

$$V_X = \{ \alpha \in \mathcal{T}_X \mid |S(\alpha)| < \infty \},$$

$$W_X = \{ \alpha \in I_X \mid |S(\alpha)| < \infty \},$$

$$M_X = \{ \alpha \in \mathcal{T}_X \mid \alpha \text{ is one-to-one}\} = \{ \alpha \in I_X \mid \Delta\alpha = X\},$$

$$E_X = \{ \alpha \in \mathcal{T}_X \mid \nabla\alpha = X\} = \{ \alpha \in T_X \mid \Delta\alpha = \nabla\alpha = X\}$$

and $$C_X = \{ \alpha \in T_X \mid |\nabla\alpha| \leq 1\}.$$

Note that we consider the empty transformation as a constant partial transformation.

The first theorem gives a characterization of a generalized permutation group which admits a ring structure.

3.1 Theorem. For a set X , for $\theta \in G_X$ , the generalized permutation group $(G_X, \theta)$ admits a ring structure if and only if $|X| \leq 2$.

Proof : Assume that the semigroup $(G_X, \theta)$ admits a ring structure under an addition +. Suppose on the contrary that $|X| \geq 3$. Let a, b and c be three distinct elements in X . Then

$$(a,b,c) + (a,c) = \alpha$$

for some $\alpha$ in $(G_X, \theta)^{\circ}$.

Case $\alpha = 0$. That is, $(a,b,c) + (a,c) = 0$. Then we have

$$(a,b,c)\theta\theta^{-1}(a,c) + (a,c)\theta\theta^{-1}(a,c) = 0$$

and

$$(a,c)\theta^{-1}\theta(a,b,c) + (a,c)\theta^{-1}\theta(a,c) = 0,$$

so we have $(a,b) + 1_X = 0$ and $(b,c) + 1_X = 0$, respectively. Hence we have $(a,b) = (b,c)$, which is a contradiction.

Case $\alpha \neq 0$. Then we have

$$(a,b,c)\theta\theta^{-1}(a,c) + (a,c)\theta\theta^{-1}(a,c) = \alpha\theta\theta^{-1}(a,c)$$

which implies

$$(a,b) + 1_X = \alpha(a,c).$$

Then we have

$$(a,b)\theta^{-1}\theta(a,b) + (a,b)\theta^{-1}\theta 1_X = (a,b)\theta^{-1}\theta\alpha(a,c)$$

which implies

$$1_X + (a,b) = (a,b)\alpha(a,c).$$

Thus $\alpha(a,c) = (a,b)\alpha(a,c)$. Hence $a\alpha(a,c) = a(a,b)\alpha(a,c) = b\alpha(a,c)$. Since $\alpha(a,c)$ is a one-to-one map, we have $a = b$, which is a contradiction.

This proves that $|X| \leq 2$.

Conversely, assume that $|X| \leq 2$.

If $X = \emptyset$, then $G_X = \{0\}$ and $\theta = 0$, and clearly, $(G_X, \theta)$ admits a ring structure.

<u>Case $|X| = 1$</u> . Say $X = \{a\}$. Then $G_X = \{\{a\}_a\}$ and $\theta = \{a\}_a$. Thus $(G_X, \theta)^\circ$ is isomorphic to the multiplicative semigroup $\mathbb{Z}_2$ and hence $(G_X, \theta)$ admits a ring structure.

<u>Case $|X| = 2$</u>. Say $X = \{a,b\}$, $a \neq b$. Then $G_X = \{1_X, (a,b)\}$. If $\theta = 1_X$, then $(G_X, \theta)^\circ$ is isomorphic to the multiplicative semigroup $\mathbb{Z}_3$, so $(G_X, \theta)$ admits a ring structure.

Assume $\theta = (a,b)$. Define the operation + on $G_X \cup \{0\}$ by

$$1_X + 1_X = (a,b),\ (a,b) + (a,b) = 1_X\ ,\ 0 + 0 = 0,$$

$$1_X + 0 = 0 + 1_X = 1_X,\ (a,b) + 0 = 0 + (a,b) = (a,b)$$

and

$$1_X + (a,b) = (a,b) + 1_X = 0.$$

It is easy to see that the generalized permutation group $(G_X, \theta)$ admits a ring structure under this addition. □

The following corollary follows from Theorem 3.1 when $\theta = 1_X$.

3.2 <u>Corollary</u>. For a set X, the permutation group on X admits a ring structure if and only if $|X| \leq 2$.

3.3 <u>Theorem</u>. For a set X, for $\theta \in I_X$, the generalized 1-1 partial transformation semigroup $(I_X, \theta)$ admits a ring structure if and only if either $\theta = 0$ or $|X| \leq 1$.

Proof : Assume that the semigroup $(I_X, \theta)$ admits a ring structure under an addition + and $\theta \neq 0$. First, we show that $|\Delta\theta| = 1$ and $\Delta\theta = \nabla\theta$. Suppose $|\Delta\theta| > 1$. Let a and b be two distinct elements of $\Delta\theta$. Then $a\theta \neq b\theta$ and



$$\{a\}_a + \{a\}_b = \alpha$$

for some $\alpha \in I_X$, and thus

$$\{a\}_a\theta\{a\theta\}_{a\theta} + \{a\}_b\theta\{a\theta\}_{a\theta} = \alpha\theta\{a\theta\}_{a\theta}$$

and

$$\{a\}_a\theta\{b\theta\}_{b\theta} + \{a\}_b\theta\{b\theta\}_{b\theta} = \alpha\theta\{b\theta\}_{b\theta} ,$$

which imply $\{a\}_{a\theta} = \alpha\theta\{a\theta\}_{a\theta}$ and $\{a\}_{b\theta} = \alpha\theta\{b\theta\}_{b\theta}$, respectively. From $a\theta = a\{a\}_{a\theta} = a\alpha\theta\{a\theta\}_{a\theta}$, we have $a\alpha\theta = a\theta$. From $b\theta = a\{a\}_{b\theta} = a\alpha\theta\{b\theta\}_{b\theta}$, we have $a\alpha\theta = b\theta$. Hence $a\theta = b\theta$ which is a contradiction because $\theta$ is one-to-one. This shows that $|\Delta\theta| = 1$, say $\Delta\theta = \{x\}$. Suppose $x\theta \neq x$. Let $\beta \in I_X$ be such that

$$(x, x\theta) + \{x\}_x = \beta .$$

Then

$$(x, x\theta)\theta\{x\theta\}_{x\theta} + \{x\}_x\theta\{x\theta\}_{x\theta} = \beta\theta\{x\theta\}_{x\theta}$$

which implies

$$\{x\theta\}_{x\theta} + \{x\}_{x\theta} = \beta\theta\{x\theta\}_{x\theta} \qquad (1)$$

and hence

$$\{x\}_x\theta\{x\theta\}_{x\theta} + \{x\}_x\theta\{x\}_{x\theta} = \{x\}_x\theta\beta\{x\theta\}_{x\theta}$$

which implies $\{x\}_{x\theta} = \{x\}_x\theta\beta\theta\{x\theta\}_{x\theta}$, and thus $x\theta = x\theta\beta\theta$. Since $\theta$ is a one-to-one map, $x\theta\beta = x$, and hence $x\theta \in \Delta\beta$. It then follows that $\beta\theta\{x\theta\}_{x\theta} = \{x\theta\}_{x\theta}$. From (1), we have

$$\{x\theta\}_{x\theta} + \{x\}_{x\theta} = \{x\theta\}_{x\theta}.$$

This implies $\{x\}_{x\theta} = 0$, a contradiction. This shows that $x\theta = x$.

Next, we will prove that $|X| \leqslant 1$. To prove this, suppose that $X \smallsetminus \{x\} \neq \emptyset$. Let $y \in X \smallsetminus \{x\}$. Then there is an element $\gamma \in I_X$ such that

$$\{x\}_x + \{x\}_y = \gamma .$$

Then

$$\{x\}_x\theta\{x\}_x + \{x\}_y\theta\{x\}_x = \gamma\theta\{x\}_x$$

and

$$\{x\}_x\theta\{x\}_x + \{x\}_x\theta\{x\}_y = \{x\}_x\theta\gamma ,$$

which imply that $\{x\}_x = \gamma\theta\{x\}_x$ and $\{x\}_x + \{x\}_y = \{x\}_x\theta\gamma$, respectively. If $\gamma = 0$, then $\{x\}_x = 0$, a contradiction. Hence

$$\{x\}_x + \{x\}_y = \{x\}_x\theta\gamma = \{x\}_{x\gamma} .$$

From $\{x\}_x = \gamma\theta\{x\}_x$, we have $x\gamma \in \Delta\theta = \{x\}$, so $x\gamma = x$. Hence $\{x\}_x + \{x\}_y = \{x\}_x$, so $\{x\}_y = 0$, a contradiction.

This proves that $|X| \leqslant 1$.

Conversely, assume that $\theta = 0$ or $|X| \leqslant 1$. If $\theta = 0$, the semigroup $(I_X, \theta)$ is a zero semigroup, so it admits a ring structure. If $|X| \leqslant 1$, then $I_X$ is $\{0\}$ or $\{0, 1_X\}$, so $(I_X, \theta)$ is either a zero semigroup or a Kronecker semigroup of order $\leq 2$, and hence the semigroup $(I_X, \theta)$ admits a ring structure. □

3.4 Corollary. For a set X, the 1-1 partial transformation semigroup on X admits a ring structure if and only if $|X| \leq 1$.

Proof : This follows from Theorem 3.3 when $\theta = 1_X$. □

3.5 Theorem. For a set X, for $\theta \in \mathcal{T}_X$, the generalized full transformation semigroup $(\mathcal{T}_X, \theta)$ admits a ring structure if and only if $|X| \leq 1$.

Proof : Assume that the semigroup $(\mathcal{T}_X, \theta)$ admits a ring structure under an addition +. Suppose $|X| \geq 2$. Let a and b be two distinct elements in X. Then we have

$$X_a + X_b = \alpha$$

for some $\alpha \in (\mathcal{T}_X, \theta)^{\circ}$.

Case $\alpha = 0$. That is, $X_a + X_b = 0$. Then we have

$$X_a\theta X_b + X_b\theta X_b = 0$$

which implies $X_b + X_b = 0$. It then follows that $X_a = X_b$, which is impossible because $a \neq b$.

Case $\alpha \neq 0$. Then we have

$$X_a\theta X_b + X_b\theta X_b = \alpha\theta X_b$$

which implies $\quad X_b + X_b = X_b$ ,

so $X_b = 0$ , a contradiction.

This proves that $|X| \leq 1$.

The converse is obvious. □

3.6 Corollary. For a set X , the full transformation semigroup on X admits a ring structure if and only if $|X| \leq 1$.

Proof : This follows from Theorem 3.5 when $\theta = 1_X$. □

If X is a set such that $|X| \leq 1$, it is easy to see that for any $\theta \in T_X$, the semigroup $(T_X, \theta)$ admits a ring structure. For any set X, if $\theta = 0$, then the semigroup $(T_X, \theta)$ is a zero semigroup, so it admits a ring structure.

3.7 Theorem. Let X be a nonempty set and let θ be a nonzero element in $T_X$ such that $\Delta\theta = X$, $\nabla\theta = X$ or θ is one-to-one. If the generalized partial transformation semigroup $(T_X, \theta)$ admits a ring structure, then $|X| = 1$.

Proof : Assume that the semigroup $(T_X, \theta)$ admits a ring structure under an addition +.

(1) Let $\Delta\theta = X$ .

Suppose $|X| > 1$. Let a and b be two distinct elements of X . Then there exists $\alpha \in T_X$ such that

$$X_a + X_b = \alpha .$$

Case $\alpha = 0$. Then we have

$$X_a\theta X_b + X_b\theta X_b = 0$$

which implies $X_b + X_b = 0$. It then follows that $X_a = X_b$, which is a contradiction.

Case $\alpha \neq 0$. Then we have

$$X_a\theta X_a + X_a\theta X_b = X_a\theta\alpha$$

and thus

$$X_a + X_b = X_a\theta\alpha.$$

But $X_a\theta\alpha = X_c$ for some $c \in X$, so

$$X_a + X_b = X_c.$$

Therefore we have

$$X_a\theta X_c + X_b\theta X_c = X_c\theta X_c$$

which implies $X_c + X_c = X_c$. Thus $X_c = 0$, a contradiction.

This proves that $|X| = 1$.

(2) Let $\nabla\Theta = X$.

Let a be an element of $\nabla\Theta$. Then $x\theta = a$ for some $x \in X$. Let $y \in \Delta\Theta$. Then

$$X_x + X_y = \alpha$$

for some $\alpha \in T_X$.

Case $\alpha = 0$. That is, $X_x + X_y = 0$. Then we have

$$X_x\theta X_x + X_y\theta X_x = 0$$

which implies $X_x + X_x = 0$, and so $X_x = X_y$. This proves that $x = y$ and hence $y\theta = a$.

Case $\alpha \neq 0$. Then we have

$$X_x\theta X_x + X_x\theta X_y = X_x\theta\alpha ,$$

and so

$$X_x + X_y = X_{x\theta\alpha}.$$



Let $z = x\theta\alpha$. Then

$$X_x + X_y = X_z$$

and hence

$$X_x\theta X_x + X_y\theta X_x = X_z\theta X_x.$$

If $z \in \Delta\Theta$, then $X_x + X_x = X_x$, and so $X_x = 0$, a contradiction.

Hence $z \notin \Delta\Theta$. From $X_x + X_y = X_z$, we also have

$$\{x\}_x\theta X_x + \{x\}_x\theta X_y = \{x\}_x\theta X_z$$

which implies

$$\{x\}_x + \{x\}_y = \{x\}_z ,$$

and so

$$\{x\}_x\theta\{a\}_a + \{x\}_y\theta\{a\}_a = \{x\}_z\theta\{a\}_a = 0.$$

If $y\theta \neq a$, then $\{x\}_a = 0$, which is a contradiction. Thus $y\theta = a$.

This proves that $\nabla\theta = \{a\}$ . Hence $|\nabla\theta| = 1$ and so $|X| = 1$.

(3) Let $\theta$ be a one-to-one map.

Let a and b be elements of X such that $a\theta = b$. Then we have

$$X_a + X_b = \alpha$$

for some $\alpha \in T_X$.

<u>Case $\alpha = 0$</u>. Then $X_a + X_b = 0$, and thus

$$X_a\theta X_b + X_b\theta X_b = 0$$

If $b \notin \Delta\theta$, then $X_b\theta X_b = 0$ and hence $X_b = X_a\theta X_b = 0$, which is a contradiction. Therefore we have $b \in \Delta\theta$, and so from $X_a\theta X_b + X_b\theta X_b = 0$, we have $X_b + X_b = 0$. It then follows that $X_a = X_b$, and hence $a = b$.

<u>Case $\alpha \neq 0$</u>. Then we have

$$X_a\theta X_a + X_a\theta X_b = X_a\theta\alpha.$$

It then follows that

$$X_a + X_b = X_a\theta\alpha.$$

Hence $X_a\theta\alpha = X_c$ for some $c \in X \setminus \{a,b\}$. Therefore $X_a + X_b = X_c$ and so

$$\{a\}_a \theta X_a + \{a\}_a \theta X_b = \{a\}_a \theta X_c$$

which implies

$$\{a\}_a + \{a\}_b = \{a\}_c$$

and thus

$$\{a\}_a \theta \{b\}_b + \{a\}_b \theta \{b\}_b = \{a\}_c \theta \{b\}_b .$$

Since $a\theta = b$ and $c \neq a$, we have that $c\theta \neq b$, and therefore $\{a\}_c \theta \{b\}_b = 0$. Thus

$$\{a\}_b + \{a\}_b \theta \{b\}_b = 0.$$

If $a \neq b$, then $\{a\}_b \theta \{b\}_b = 0$ which implies $\{a\}_b = 0$, a contradiction. Hence $a = b$.

This proves that $a\theta = a$ for all $a \in \Delta\Theta$.

Next, claim that $|\Delta\Theta| = 1$. Suppose $|\Delta\Theta| > 1$. Let x and y be two distinct elements in $\Delta\Theta$. Then we have

$$X_x + X_y = \beta$$

for some $\beta \in T_X$.

<u>Case $\beta = 0$</u>. Then we have

$$X_x \theta X_x + X_y \theta X_x = 0$$

and therefore $X_x + X_x = 0$. It then follows that $X_x = X_y$, a contradiction.

Case $\beta \neq 0$. Then we have

$$X_x \theta X_x + X_x \theta X_y = X_x \theta \beta.$$

Thus

$$X_x + X_y = X_z$$

for some $z \in X \setminus \{x,y\}$ and hence

$$\{x\}_x \theta X_x + \{x\}_x \theta X_y = \{x\}_x \theta X_z.$$

It then follows that

$$\{x\}_x + \{x\}_y = \{x\}_z$$

and so

$$\{x\}_x \theta \{y\}_y + \{x\}_y \theta \{y\}_y = \{x\}_z \theta \{y\}_y$$

which implies $\{x\}_y = 0$, a contradiction.

This proves that $|\Delta\Theta| = 1$, say $\Delta\Theta = \{a\}$. Then $\Delta\Theta = \nabla\Theta = \{a\}$.

Our next step is to show that $|X| = 1$. We suppose that there exists an element b in $X \setminus \{a\}$. Then there exists $\gamma \in T_X$ such that $X_a + X_b = \gamma$. If $\gamma = 0$, then $0 = X_a \theta X_a + X_b \theta X_a = X_a$, a contradiction. Therefore $\gamma \neq 0$. From $X_a + X_b = \gamma$, we have

$$X_a \theta X_a + X_a \theta X_b = X_a \theta \gamma$$

which implies

$$X_a + X_b = X_c$$

for some $c \in X \smallsetminus \{a,b\}$. Hence

$$\{a\}_a\theta X_a + \{a\}_a\theta X_b = \{a\}_a\theta X_c.$$

Thus

$$\{a\}_a + \{a\}_b = \{a\}_c,$$

and hence

$$\{a\}_a\theta\{a\}_a + \{a\}_b\theta\{a\}_a = \{a\}_c\theta\{a\}_a$$

which implies $\{a\}_a = 0$ since $b,c \notin \Delta\theta = \{a\}$. This is a contradiction. □

3.8 Corollary. For a set X, the partial transformation semigroup on X admits a ring structure if and only if $|X| \leq 1$.

We characterize almost identical transformation semigroups admitting ring structure in the three following theorems.

3.9 Theorem. For a set X, the semigroup of all almost identical partial transformations of X, $U_X$, admits a ring structure if and only if $|X| \leq 1$.

Proof : Assume that the semigroup $U_X$ admits a ring structure under an addition +. Suppose on the contrary that $|X| \geq 2$. Let a and b be two distinct elements in X. Then we have

$$\{a\}_a + \{a\}_b = \alpha$$

for some $\alpha \in U_X$. Therefore

$$\{a\}_a\{a\}_a + \{a\}_b\{a\}_a = \alpha\{a\}_a$$

and

$$\{a\}_a\{a\}_a + \{a\}_a\{a\}_b = \{a\}_a\alpha ,$$

which imply $\{a\}_a = \alpha\{a\}_a$ and $\{a\}_a + \{a\}_b = \{a\}_a\alpha$ , respectively. From $\{a\}_a = \alpha\{a\}_a$. We have that $a \in \Delta\alpha$ and $a\alpha = a$. It then follows that $\{a\}_a\alpha = \{a\}_a$. Hence from $\{a\}_a + \{a\}_b = \{a\}_a\alpha$, we have $\{a\}_b = 0$, a contradiction.

The converse is obvious . □

3.10 Theorem. For a set X, the semigroup of all almost identical transformations of X, $V_X$, admits a ring structure if and only if $|X| \leqslant 1$.

Proof : Assume that the semigroup $V_X$ admits a ring structure under an addition +. Suppose $|X| \geqslant 2$. Let a and b be two distinct elements in X. Define the maps $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} b & \text{if } x = a, \\ x & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} a & \text{if } x = b, \\ x & \text{otherwise.} \end{cases}$$

Then $\alpha$ , $\beta \in V_X$, and so

$$\alpha + \beta = \gamma$$

for some $\gamma \in V_X^0$. Thus

$$\alpha(a,b) + \beta(a, b) = \gamma(a, b)$$

which implies

$$\beta + \alpha = \gamma(a, b) .$$

Hence $\gamma = \gamma(a, b)$. Next, we claim that $b \notin \nabla\gamma$ . To prove this, we suppose that there is an element x in X such that $x\gamma = b$. Since $\gamma = \gamma(a,b)$, we have

$$b = x\gamma = x\gamma(a,b) = b(a,b) = a,$$

a contradiction. Hence $b \notin \nabla\gamma$ . From $\alpha + \beta = \gamma$ , we also have

$$\alpha\beta + \beta\beta = \gamma\beta$$

and thus

$$\beta + \beta = \gamma\beta$$

If $\gamma = 0$, then $0 = \gamma = \gamma\beta$ . If $\gamma \neq 0$, then for each $x \in X$ , $x\gamma \neq b$ since $b \notin \nabla\gamma$, and thus $x\gamma\beta = x\gamma$ . Hence $\gamma\beta = \gamma$ . Therefore

$$\alpha + \beta = \gamma = \gamma\beta = \beta + \beta ,$$

so $\alpha = \beta$ , a contradiction.

The converse is obvious. □

3.11 Theorem. For a set X, the semigroup of all almost identical 1-1 partial transformations of X , $W_X$, admits a ring structure if and only if $|X| \leq 1$.

Proof : A proof of this theorem can be given identically to the proof of Theorem 3.9, only replacing $U_X$ by $W_X$ in every place. □

Recall that for any set **X** ,

$$M_X = \{\alpha : X \longrightarrow X \mid \alpha \text{ is one-to-one}\}$$

and

$$E_X = \{ \alpha : X \rightarrow X \mid \alpha \text{ is onto}\}.$$

It is known that for any set X , $M_X = G_X$ if and only if $|X| < \infty$ and $E_X = G_X$ if and only if $|X| < \infty$ .

3.12 Theorem. For a set X , the semigroup of all one-to-one transformations of X , $M_X$ , admits a ring structure if and only if $|X| \leq 2$.

Proof : Assume that the semigroup $M_X$ admits a ring structure under an addition +. Suppose $|X| \geq 3$. Let a,b and c be three distinct elements in X. Then

$$(a,b,c) + (a,c) = \alpha$$

for some $\alpha \in M_X^0$ .

Case $\alpha = 0$. Then

$$(a,b,c)(a,c) + (a,c)(a,c) = 0$$

and

$$(a,c)(a,b,c) + (a,c)(a,c) = 0$$

which imply $(a,b) + 1_X = 0$ and $(b,c) + 1_X = 0$, respectively. It then follows that $(a,b) = (b,c)$, which is a contradiction.

Case $\alpha \neq 0$. Then we have

$$(a,b)(a,b,c) + (a,b)(a,c) = (a,b)\alpha$$

and thus

$$(a,c) + (a,b,c) = (a,b)\alpha$$

which implies $\alpha = (a,b)\alpha$ , and so $a\alpha = a(a,b)\alpha = b\alpha$ , which is a contradiction because $\alpha$ is a one-to-one map.

Conversely, assume that $|X| \leq 2$. Then $M_X = G_X$ which admits a ring structure by Corollary 3.2. □

3.13 Theorem. For a set X , the semigroup of all onto transformations of X , $E_X$ , admits a ring structure if and only if $|X| \leq 2$.

Proof : Assume that the semigroup $E_X$ admits a ring structure under an addition +. Suppose $|X| \geq 3$. Let a, b and c be three distinct elements in X . Then we have

$$(a,b,c) + (a,c) = \alpha$$

for some $\alpha \in E_X^{\circ}$

Case $\alpha = 0$. That is, $(a,b,c) + (a,c) = 0$. Then we have

$$(a,b,c)(a,c) + (a,c)(a,c) = 0$$

and

$$(a,c)(a,b,c) + (a,c)(a,c) = 0$$

which imply $(a,b) + 1_X = 0$ and $(b,c) + 1_X = 0$, respectively. Thus $(a,b) = (b,c)$, a contradiction.

Case $\alpha \neq 0$. Then we have

$$(a,b,c)(a,c) + (a,c)(a,c) = \alpha(a,c)$$

and thus

$$(a,b) + 1_X = \alpha(a,c) .$$

Therefore

$$(a,b)(a,b) + 1_X(a,b) = \alpha(a,c)(a,b)$$

which implies

$$1_X + (a,b) = \alpha(a,c,b).$$

It then follows that $\alpha(a,c) = \alpha(a,c,b)$. Since $\alpha$ is onto, there is an element $x$ in $X$ such that $x\alpha = c$. Then

$$a = x\alpha(a,c) = x\alpha(a,c,b) = c(a,c,b) = b,$$

a contradiction.

Conversely, assume that $|X| \leq 2$. Then $E_X = G_X$ which admits a ring structure by Corollary 3.2. □

Let $X$ be a set. Recall that the semigroup of all constant partial transformations of $X$ , $C_X = \{\alpha \in T_X \mid |\nabla\alpha| \leq 1\}$. The next two theorems deal with the semigroup $C_X$ and the semigroup $(C_X, \theta)$ for $\theta \in C_X$ , respectively.

3.14 Theorem. For a set $X$ , the semigroup of all constant partial transformations of $X$ , $C_X$ , admits a ring structure if and only if $|X| \leq 1$.

Proof : Assume that the semigroup $C_X$ admits a ring structure under an addition +. Suppose there are two distinct elements in X, say a,b. Let $A = \{a,b\}$. Then

$$A_a + \{b\}_b = \alpha$$

for some $\alpha \in C_X$.

Case $\alpha = 0$. Then we have

$$A_a\{b\}_b + \{b\}_b\{b\}_b = 0$$

which implies $\{b\}_b = 0$, a contradiction

Case $\alpha \neq 0$. Then $\alpha = B_c$ for some nonempty subset B of X and for some $c \in X$. Thus we have

$$A_aA_a + A_a\{b\}_b = A_aB_c$$

and

$$A_aA_a + \{b\}_bA_a = B_cA_a$$

which imply that $A_a = A_aB_c$ and $A_a + \{b\}_a = B_cA_a$, respectively. From $A_a = A_aB_c$, we have $a = c$, and thus $B_cA_a = B_a = B_c$. Hence $A_a + \{b\}_a = B_c = A_a + \{b\}_b$ and therefore $a = b$, a contradiction.

This proves that $|X| \leq 1$.

The converse is obvious. □

3.15 Theorem. For a set X, for $\theta \in C_X$, the generalized transformation semigroup $(C_X, \theta)$ admits a ring structure if and only if either $\theta = 0$ or $|X| \leq 1$.

Proof : Assume that the semigroup $(C_X, \theta)$ admits a ring structure under an addition +. Suppose $\theta \neq 0$. Let A be a nonempty subset of X and $x \in X$ such that $\theta = A_x$. Let $y \in A$. Suppose that $x \neq y$. Let $B = \{x,y\}$. Then

$$B_x + \{y\}_y = \alpha$$

for some $\alpha \in C_X$.

Case $x \notin A$. Then we have that

$$B_x A_x B_x + \{y\}_y A_x B_x = \alpha A_x B_x$$

and

$$B_y A_x B_x + B_y A_x \{y\}_y = B_y A_x \alpha,$$

which imply $\{y\}_x = \alpha A_x$ and $B_x = B_x\alpha$, respectively. From $0 \neq \{y\}_x = \alpha A_x$, we have $\Delta\alpha = \{y\}$ since $|\nabla\alpha| = 1$ and $\nabla\alpha \subseteq A$. Thus $B_x = B_x\alpha = 0$, a contradiction.

Case $x \in A$. Because

$$B_x A_x B_x + B_x A_x \{y\}_y = B_x A_x \alpha,$$

we have that $B_x = B_x\alpha$, and hence $\nabla\alpha = \{x\}$.

Let $\Delta\alpha = C$. Then $B_x + \{y\}_y = C_x$, so

$$B_x A_x B_x + \{y\}_y A_x B_x = C_x A_x B_x$$

which implies

$$B_x + \{y\}_x = C_x.$$

It then follows that $B_x + \{y\}_y = B_x + \{y\}_x$ which implies $y = x$, a contradiction.

Hence this proves that $A = \{x\}$ and thus $\theta = \{x\}_x$.

The next step is to prove that $|X| \leq 1$. Suppose that there exists an element y in $X \setminus \{x\}$. Let $D = \{x,y\}$. Then we have

$$D_x + D_y = \beta$$

for some $\beta \in C_X$. Thus

$$D_x\{x\}_x D_x + D_y\{x\}_x D_x = \beta\{x\}_x D_x$$

which implies $D_x = \beta\{x\}_x$, and hence $\Delta\beta = D$ and $\nabla\beta = \{x\}$. It then follows that $D_x + D_y = D_x$, which implies $D_y = 0$, a contradiction.

Conversely, assume $\theta = 0$ or $|X| \leq 1$. If $\theta = 0$, then $(C_X, \theta)$ is a zero semigroup, so it admits a ring structure. If $|X| \leq 1$, then $C_X = \{0\}$ or $\{0, 1_X\}$, so $(C_X, \theta)$ is either a zero semigroup or a Kronecker semigroup of order $\leq 2$, and hence it admits a ring structure. □