CHAPTER II

SEMIGROUPS OF NUMBERS

In this chapter, we are concerned with semigroups of numbers. The purpose is to determine whether some semigroups of numbers admit a ring structure.

Throughout this chapter, we adopt the following notation :

$\mathbb{N}$	=	the set of all positive integers,
$\mathbb{Z}$	=	the set of all integers,
$\mathbb{Z}^-$	=	the set of all negative integers,
$\mathbb{Q}$	=	the set of all rational numbers,
$\mathbb{Q}^+$	=	the set of all positive rational numbers,
$\mathbb{Q}^-$	=	the set of all negative rational numbers,
$\mathbb{R}$	=	the set of all real numbers,
$\mathbb{R}^+$	=	the set of all positive real numbers,
$\mathbb{R}^-$	=	the set of all negative real numbers,
$[0,1]$	=	$\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$,
$[0,1)$	=	$\{x \in \mathbb{R} \mid 0 \leq x < 1\}$,
$(0,1]$	=	$\{x \in \mathbb{R} \mid 0 < x \leq 1\}$,
$(0,1)$	=	$\{x \in \mathbb{R} \mid 0 < x < 1\}$,
$[1,\infty)$	=	$\{x \in \mathbb{R} \mid x \geq 1\}$,
$(1,\infty)$	=	$\{x \in \mathbb{R} \mid x > 1\}$.

Observe that the semigroups $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ under usual multiplication admit ring structure.

From the definition of admitting a ring structure of a semigroup, it follows that a semigroup S admits a ring structure if and only if the semigroup S^o admits a ring structure.

It has been shown by Chu and Shyr in [1] that the semigroup of all nonnegative integers, $\mathbb{N} \cup \{0\}$, with usual multiplication admits a ring structure. Hence, the semigroup of all positive integers, $\mathbb{N}$, with usual multiplication admits a ring structure since $\mathbb{N}^o \cong \mathbb{N} \cup \{0\}$ under usual multiplication.

Let S be an infinite cyclic semigroup. Suppose S admits a ring structure under an addition +. Let a be a generator of S ; that is ,

$$S = \{a, a^2, a^3, \ldots\}$$

and $a^i \neq a^j$ if $i \neq j$ in $\mathbb{N}$. Then there exists a positive integer k such that $a + a^k = 0$. If $k > 1$, then

$$a + a^k = 0 = a^{k-1}(a + a^k) = a^k + a^{2k-1}$$

and thus $a = a^{2k-1}$ and $2k - 1 > 1$ which is a contradiction. Then $k = 1$ and therefore $a + a = 0$ which implies $a^i + a^i = 0$ for all $i \in \mathbb{N}$. Since $a^i \neq 0$ for all $i \in \mathbb{N}$ and $x + x = 0$ for all $x \in S$, it follows that $a + a^2 = a^j$ for some $j \in \mathbb{N}$ such that $j > 2$. Then $a = a^2 + a^j = a(a + a^{j-1})$. But $a + a^{j-1} = a^m$ for some $m \in \mathbb{N}$. Thus $a = a^{m+1}$, a contradiction.

This shows that any infinite cyclic semigroup does not admit a ring structure.

The semigroup of all positive integers under usual addition and the semigroup of all negative integers under usual addition are clearly infinite cyclic semigroups.

Hence, we have

2.1 Theorem. (1) The semigroup of all positive integers under usual multiplication admits a ring structure.

(2) The semigroup of all positive integers under usual addition does not admit a ring structure.

(3) The semigroup of all negative integers under usual addition does not admit a ring structure.

Let S be a semigroup. Define a relation P on S by $(a,b) \in P$ if and only if $S^1a \subseteq S^1b$. Clearly, P is reflexive and transitive. The relation P is antisymmetric if and only if $S^1a = S^1b$ implies $a = b$. Hence, the relation P on S is a total order on S if and only if

(a) for $a, b \in S$, $S^1a = S^1b$ implies $a = b$, and

(b) for all $a, b \in S$, either $S^1a \subseteq S^1b$ or $S^1b \subsetneq S^1a$.

It was shown by L.J.M. Lawson in [3] that if any semigroup containing more than one element in which the relation P is a total order, then S does not admit a ring structure.

Let S be an infinite cyclic semigroup. Then there exists an element $a \in S$ such that

$$S = \{a^n \mid n = 1,2,3,\ldots\}$$

and $a^i \neq a^j$ if $i \neq j$ in $\mathbb{N}$. Then S has no identity and the semigroup S^1 is not an (infinite) cyclic semigroup. We define $a^0 = 1$. Then

$$S^1 = \{a^n \mid n = 0,1,2,\ldots\},$$

and $a^i \neq a^j$ if $i \neq j$ in $\mathbb{N} \cup \{0\}$. We will show that the relation P define on the semigroup S^1 is a total order. Let m, n be in $\mathbb{N} \cup \{0\}$.

(a) If $S^1a^m = S^1a^n$, then there exist r, s in $\mathbb{N} \cup \{0\}$ such that $a^m = a^r a^n$ and $a^n = a^s a^m$, so $m = r + n$ and $n = s + m$ which imply $m = n$.

(b) Since $m, n \in \mathbb{N} \cup \{0\}$, $m \geq n$ or $n \geq m$. If $m \geq n$, then $S^1a^m = \{a^k \mid k = m, m + 1,\ldots\} \subseteq \{a^k \mid k = n, n + 1,\ldots\} = S^1a^n$. If $n \geq m$, then $S^1a^n \subseteq S^1a^m$.

Under usual addition, $\mathbb{N}$ and $\mathbb{Z}^-$ are infinite cyclic semigroups. Clearly, under usual addition $\mathbb{N}^1 \cong \mathbb{N} \cup \{0\}$ and $(\mathbb{Z}^-)^1 \cong \mathbb{Z}^- \cup \{0\}$. Hence, we have

2.2 <u>Theorem</u>. (a) The semigroup of all nonnegative integers under usual addition does not admit a ring structure.

(b) The semigroup of all nonpositive integers under usual addition does not admit a ring structure.

The next theorem shows that each of the semigroups $\mathbb{Q}^+$, $\mathbb{Q}^+ \cup \{0\}$, $\mathbb{Q}^-$, $\mathbb{Q}^- \cup \{0\}$, $\mathbb{R}^+$, $\mathbb{R}^+ \cup \{0\}$, $\mathbb{R}^-$, $\mathbb{R}^- \cup \{0\}$ under usual addition does not admit a ring structure.

2.3 Theorem. Let S be the semigroup $\mathbb{Q}^+$, $\mathbb{Q}^+ \cup \{0\}$, $\mathbb{Q}^-$, $\mathbb{Q}^- \cup \{0\}$, $\mathbb{R}^+$, $\mathbb{R}^+ \cup \{0\}$, $\mathbb{R}^-$ or $\mathbb{R}^- \cup \{0\}$ under usual addition +. Then S does not admit a ring structure.

Proof : Recall that for any semigroup T, for $a \in T$, $T^1a = Ta \cup \{a\}$. To prove the theorem, let x,y be elements of S.

(a) Assume $(S + x) \cup \{x\} = (S + y) \cup \{y\}$. Then $x \in (S + y) \cup \{y\}$ and $y \in (S + x) \cup \{x\}$. Thus ($x = y$ or $x = s + y$ for some $s \in S$) and ($y = x$ or $y = t + x$ for some $t \in S$). If S is $\mathbb{Q}^+$, $\mathbb{Q}^+ \cup \{0\}$, $\mathbb{R}^+$ or $\mathbb{R}^+ \cup \{0\}$, then ($x = y$ or $x = s + y$) implies $x \geqslant y$, and ($x = y$ or $y = t + x$) implies $y \geqslant x$. If S is $\mathbb{Q}^-$, $\mathbb{Q}^- \cup \{0\}$, $\mathbb{R}^-$ or $\mathbb{R}^- \cup \{0\}$, then ($x = y$ or $x = s + y$) implies $x \leqslant y$, and ($x = y$ or $y = t + x$) implies $y \leqslant x$. Hence $x = y$.

(b) If $x = y$, then $(S + x) \cup \{x\} = (S + y) \cup \{y\}$.

Case $x < y$. If S is $\mathbb{Q}^+$, $\mathbb{Q}^+ \cup \{0\}$, $\mathbb{R}^+$ or $\mathbb{R}^+ \cup \{0\}$, then $y = r + x$ for some $r \in S$, and hence $(S + y) \cup \{y\} \subsetneq (S + x) \cup \{x\}$. If S is $\mathbb{Q}^-$, $\mathbb{Q}^- \cup \{0\}$, $\mathbb{R}^-$ or $\mathbb{R}^- \cup \{0\}$, then $x = s + y$ for some $s \in S$, and thus $(S + x) \cup \{x\} \subsetneq (S + y) \cup \{y\}$.

Case $y < x$. If S is $\mathbb{Q}^+$, $\mathbb{Q}^+ \cup \{0\}$, $\mathbb{R}^+$ or $\mathbb{R}^+ \cup \{0\}$, then $(S + x) \cup \{x\} \subsetneq (S + y) \cup \{y\}$. If S is $\mathbb{Q}^-$, $\mathbb{Q}^- \cup \{0\}$, $\mathbb{R}^-$ or $\mathbb{R}^- \cup \{0\}$, then $(S + y) \cup \{y\} \subsetneq (S + x) \cup \{x\}$.

This shows that the relation P defined as before is a total order on S. Thus S does not admit a ring structure. □

Next, we study the semigroups $(0,1)$, $(0,1]$, $(1,\infty)$ and $[1,\infty)$ under usual multiplication .

2.4 Theorem If S is the semigroup $(0,1)$, $(0,1]$, $(1,\infty)$ or $[1,\infty)$ under usual multiplication, then S does not admit a ring structure.

Proof : (a) Let x, y be elements of S such that $S^1x = S^1y$. Then $x = sy$ and $y = tx$ for some $s,t \in S^1$. Then $x = stx$, so $st = 1$ which implies $s = t = 1$. Thus $x = y$.

(b) Let x, y be elements of S. Then $x = y$, $x > y$ or $x < y$. If $x = y$, then $S^1x = S^1y$.

Case $S = (0, 1)$ or $(0, 1]$. If $x > y$, then $\frac{y}{x} \in S$ and so $S^1y = (S^1\frac{y}{x})x \subseteq S^1x$. Similarly, $x < y$ implies $S^1x \subseteq S^1y$.

Case $S = (1,\infty)$ or $[1,\infty)$. If $x > y$, then $\frac{x}{y} \in S$, and thus $S^1x = (S^1\frac{x}{y})y \subseteq S^1y$. Similarly, $x < y$ implies $S^1y \subseteq S^1x$.

Hence the semigroup S does not admit a ring structure. □

Under usual multiplication, $(0,1)^0 \cong [0,1)$, $(0,1]^0 \ncong [0,1]$, $(1,\infty)^0 \cong (1,\infty) \cup \{0\}$ and $[1,\infty)^0 \cong [1,\infty) \cup \{0\}$. Hence we have the following corollary :

2.5 Corollary. If S is the semigroup $[0,1)$, $[0,1]$, $(1,\infty) \cup \{0\}$ or $[1,\infty) \cup \{0\}$ under usual multiplication, S does not admit a ring structure.