CHAPTER I

SEMIGROUPS ADMITTING A RING STRUCTURE

The multiplicative structure of any ring is by definition a semigroup with zero. If $(S, \cdot)$ is a semigroup with zero, then the following are clearly equivalent :

(1) $(S, \cdot)$ is isomorphic to the multiplicative structure of some ring.

(2) There exists a binary operation + on S such that $(S,+,\cdot)$ is a ring.

A semigroup S (which is not necessary to have a zero) is said to <u>admit a ring structure</u> if the semigroup S^0 satisfies (1) (which is equivalent to (2)). Observe that if the semigroups S and T are isomorphic, then S admits a ring structure if and only if T admits a ring structure.

Not every semigroup admits a ring structure. It is shown by the following examples :

<u>Example</u>. A semigroup S is called a <u>left</u> [right] <u>zero semigroup</u> if $ab = a$ [$ab = b$] for all $a, b \in S$.

Observe that a left [right] zero semigroup S has a zero if and only if $|S| = 1$.

A left [right] zero semigroup S admits a ring structure if and only if $|S| = 1$. To prove this for left zero semigroups, assume that a left zero semigroup S admits a ring structure by an additive operation +. Suppose $|S| > 1$. Let a, b be two distinct elements in S. Then $a + b = c$ for some $c \in S^o$, so

$$a + a = a(a + b) = ac.$$

If $c \neq 0$, then $a + a = a$, so $a = 0$, a contradiction. If $c = 0$, then $a + b = 0 = a + a$, so $a = b$ which is also a contradiction. This proves that if S admits a ring structure, then $|S| = 1$. The converse is obvious. Hence, if $|S| > 1$, S does not admit a ring structure.

<u>Example.</u> A semigroup S with zero 0 is called a <u>Kronecker semigroup</u> if

$$ab = \begin{cases} a & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

Let S be a Kronecker semigroup. It is obvious that S admits a ring structure if $|S| \leqslant 2$. Claim that S does not admit a ring structure if $|S| > 2$. Assume that $|S| > 2$ and S admits a ring structure with an additive operation +. Let a and b be two distinct nonzero elements in S. Then $a + b = c$ for some $c \in S \smallsetminus \{a, b\}$. Thus

$$0 = ac = a(a + b) = a,$$

a contradiction.

This shows that a Kronecker semigroup S admits a ring structure if and only if $|S| \leqslant 2$.

A semigroup S with zero 0 is said to be a <u>zero semigroup</u> if $ab = 0$ for all $a,b \in S$.

Let (S, .) be a zero semigroup with zero 0. If there is a binary operation + on S such that (S, +) is a commutative group having 0 as its identity, then $(S, +, \cdot)$ is clearly a ring. We shall show that every zero semigroup admits a ring structure. A proof is given as follows : Let S be a zero semigroup with zero 0.

Case : S is finite. Let $|S| = n$. Let C_n be a cyclic group of order n with identity e. Then there is a one - to - one map Ψ from S onto C_n with $\Psi(0) = e$. Define an operation + on S by $a + b = c$ if and only if $\Psi(a)\,\Psi(b) = \Psi(c)$ in C_n. Then (S, +) is a commutative group having 0 as its identity.

(*)
Case : S is infinite. Let F(S) be the set of all finite subsets of S. Then $|S| = |F(S)|$ [4, Theorem 22.17]. Define the operation * on F(S) by

$$A * B = (A \setminus B) \cup (B \setminus A)$$

for all A, B $\in$ F(S). It is clearly seen that (F(S), *) is a commutative group having $\emptyset$ as its identity. Since $|S| = |F(S)|$, there exists a one - to - one map ψ from S onto F(S) with $\psi(0) = \emptyset$. Define the operation + on S by $x + y = z$ if and only if $\psi(x) * \psi(y) = \psi(z)$. Then (S,+) is a commutative group having 0 as its identity.

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Dr. Sidney S.Mitchell kindly provided the proof for the case that S is infinite. The author would like to thank him for his helpful assistance.

Next, we study semigroups which are unions of groups and admit ring structure.

Let a semigroup S be a union of subgroups. Then $S = \bigcup_{i\in I} G_i$ for some index set I and for some subgroups $G_i (i \in I)$ of S. For each $i \in I$, let e_i be the identity of the group G_i. Then $G_i \subseteq H_{e_i}$ for all $i \in I$. Thus $S = \bigcup_{i\in I} G_i \subseteq \bigcup_{i\in I} H_{e_i} \subseteq \bigcup_{e\in E(S)} H_e \subseteq S$, and hence $S = \bigcup_{e\in E(S)} H_e$. This shows that a semigroup S is a union of groups if and only if $S = \bigcup_{e\in E(S)} H_e$.

For a semigroup S, for $a \in S$, it is clearly seen that the $\mathcal{H}$-class of S containing a is the same as the $\mathcal{H}$-class of the semigroup S^0 containing a. Thus a semigroup S is a union of groups if and only if S^0 is a union of groups.

1.1 Theorem. Let a semigroup S be a union of groups. If S admits a ring structure, then for each element a of S, the additive inverse of a is in H_a; that is, a and the additive inverse of a are in the same subgroup of S.

Proof : In this proof, for $e \in E(S)$, $a \in H_e$, we will use the notation a^{-1} to denote the inverse element of a in the group H_e. Assume that the semigroup S admits a ring structure under an additive operation +.

First, we will prove that this theorem is true for any idempotent of S. Let e be an idempotent of S. Let x be the additive inverse of e ; that is, $e + x = 0$. Then $e + xe = 0$. Therefore

$x = xe$. Hence we have

$$xe + x^2 = x + x^2 = 0.$$

It then follows that $x^2 = e$. Since S is a union of groups, $x \in H_f$ for some $f \in E(S)$. Then $e = x^2 \in H_f$. Thus $e = f$. That is, $x \in H_e$. This proves that the additive inverse of e is in H_e.

Next, we will prove the theorem as stated, let a be an element of S, and let b be the additive inverse of a. Then $a + b = 0$. Thus we have $aa^{-1} + ba^{-1} = 0$. Since $aa^{-1} \in E(S)$, we have that $ba^{-1} \in H_{aa^{-1}} = H_a$. Thus $ba^{-1}a \in H_a$. From $aa^{-1} + ba^{-1} = 0$, we also have $aa^{-1}a + ba^{-1}a = a + ba^{-1}a = 0$. It then follows that $b = ba^{-1}a$. But $ba^{-1}a \in H_a$, so $b \in H_a$.

This proves that for each element a of S, a and the additive inverse of a are in the same subgroup of S. □

In general, a semigroup which is a union of groups need not be an inverse semigroup. The next theorem shows that a semigroup which is a union of groups and admits a ring structure must be an inverse semigroup. First, the following lemma is required.

1.2 Lemma. Let R be a ring having the property that for each element a of S, $a^2 = 0$ implies $a = 0$. Then every idempotent of R is in the center of R.

Proof : Let $a \in R$ and $e \in R$ such that $e^2 = e$. Then $(ea - eae)^2 = 0$ and $(ae - eae)^2 = 0$. By assumption, these imply that

$$ea - eae = 0 = ae - eae, \text{ and hence } ae = ea. \quad \square$$

1.3 Theorem. Let a semigroup S be a union of groups. If S admits a ring structure, then S is an inverse semigroup.

Proof : Let a be an element of S^o such that $a^2 = 0$. Since S^o is a union of groups, $a \in H_e$ of S^o for some $e \in E(S^o)$. Then $0 = a^2 \in H_e$, so $H_e = H_o = \{0\}$ and thus $a = 0$. By Lemma 1.2, we have that every idempotent of S^o is in the center of S^o. This implies that any two idempotents of S commute. Because S is a union of groups, S is regular. Hence S is an inverse semigroup. □

A band is clearly a union of trivial groups. By Theorem 1.3, we have

1.4 Corollary. A band which admits a ring structure is a semilattice.

A semigroup S is a right group if S is right simple and left cancellative. A left group is defined dually. If S is a right group, then S is a union of groups and $ef = f$ for all $e, f \in E(S)$ [2, Exercises for § 1.11(2)]. Dually, if S is a left group, then S is a union of groups and $ef = e$ for all $e, f \in E(S)$. If a right [left] group S is an inverse semigroup, then for all $e, f \in E(S)$, $f = ef = fe = e$ [$e = ef = fe = f$] and thus S is a group. Hence, we obtain from Theorem 1.3 that

1.5 Corollary. A right [left] group admitting a ring structure is a group.

1.6 Theorem. Let a semigroup S be a union of groups. If S admits a ring structure, then for all $a, b \in S$, $Sa + Sb = Sc$ for some $c \in S$.

Proof : Assume that S admits a ring structure under an addition $+$. Let $a,b \in S$. Since S is a regular semigroup, $Sa = Se$ and $Sb = Sf$ for some $e, f \in E(S)$. Let $e' = f - fe$. Claim that $Se + Sf = Se + Se'$. To prove this, let $x_1, x_2 \in S$. Then $x_1e + x_2f = (x_1 + x_2f)e + x_2(f - fe) = (x_1 + x_2f)e + x_2e' \in Se + Se'$. Thus $Se + Sf \subsetneq Se + Se'$. Conversely, we have $x_1e + x_2e' = x_1e + x_2(f - fe) = (x_1 - x_2f)e + x_2f \in Se + Sf$. Thus $Se + Se' \subsetneq Se + Sf$. It follows that $Se + Sf = Se + Se'$. Next, claim that $Se + Se' = S(e + e')$. Observe that $e'e = 0$. By Theorem 1.3, S is an inverse semigroup, so $ee' = e'e = 0$. Let $y_1, y_2 \in S$. Then $y_1e + y_2e' = (y_1e + y_2e')(e + e') \in S(e + e')$. Thus $Se + Se' \subsetneq S(e + e')$. Conversely, we have $y_1(e + e') = y_1e + y_1e' = y_1e + y_1(f - fe)$ $= (y_1 - y_1f)e + y_1f \in Se + Sf = Se + Se'$. Thus $S(e + e') \subsetneq Se + Se'$. Hence $S(e + e') = Se + Se'$. It then follows that $Sa + Sb = Se + Sf$ $= Se + Se' = S(e + e')$. □

Let S be a regular semigroup. Assume $E(S) \subsetneq C(S)$. Let $a \in S$. Then $a = axa$ for some $x \in S$, so $ax, xa \in E(S) \subsetneq C(S)$ which implies $ax = axax = a(xa)x = xaax = xa(ax) = xaxa = xa$. Since $a = axa$, $a \mathcal{L} xa$ and $a \mathcal{R} ax$. Thus $a \mathcal{L} xa$ and $a \mathcal{R} xa$ and hence $a \in H_{xa}$ which is a subgroup of S. This shows that if S is a regular semigroup with $E(S) \subsetneq C(S)$, then S is a union of groups.

An inverse semigroup need not be a union of groups. The following theorem shows that an inverse semigroup which admits a ring structure is a union of groups.

1.7 Theorem. Let S be an inverse semigroup. If S admits a ring structure, then S is a union of groups.

Proof : Assume that S admits a ring structure under an additive operation +. Since S is inverse, S^o is inverse. Let $a \in S^o$ such that $a^2 = 0$. Then $a + a^{-1}a = x$ for some $x \in S^o$, and thus

$$x^2 = (a + a^{-1}a)^2 = a + a^{-1}a = x.$$

Therefore $x \in E(S^o)$. Since S^o is an inverse semigroup, any two idempotents of S^o commute. Then we have

$$xa^{-1}a = a^{-1}ax = a^{-1}a(a + a^{-1}a) = a^{-1}a$$

because $a^{-1}a \in E(S)$. It follows that

$$a^{-1}a = xa^{-1}a = (a + a^{-1}a)a^{-1}a = a + a^{-1}a$$

which implies $a = 0$. By Lemma 1.2, $E(S^o) \subseteq C(S^o)$ and hence $E(S) \subseteq C(S)$. But S is regular, so S is a union of groups. □

Let Y be a semilattice. Let a semigroup $S = \bigcup_{\alpha \in Y} G_\alpha$ be a disjoint union of subgroups G_α of S. We call S a semilattice Y of groups G_α if $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

If S is a semilattice of groups, then $E(S) \subsetneq C(S)$ [2, Lemma 4.8] and hence S is an inverse semigroup.

Let S be an inverse semigroup which is a union of groups. Then $S = \bigcup_{e \in E(S)} H_e$ and E(S) is a semilattice. If $e, f \in E(S)$, then for

$a \varepsilon H_e$, $b \varepsilon H_f$, we have

$$abb^{-1}(abb^{-1})^{-1} = abb^{-1}(bb^{-1}a^{-1}) = abb^{-1}a^{-1} = ab(ab)^{-1},$$

and

$$(abb^{-1})^{-1}\ abb^{-1} = (bb^{-1}a^{-1})abb^{-1} = (bb^{-1})(a^{-1}a)(bb^{-1}) = fef = ef.$$

Since $abb^{-1}(abb^{-1})^{-1} = (abb^{-1})^{-1}\ abb^{-1}$, $ab(ab)^{-1} = ef$. Let $g \varepsilon E(S)$ such that $ab \varepsilon H_g$. Then we have $g = ab(ab)^{-1} = ef$, and hence $H_g = H_{ef}$. It then follows that $ab \varepsilon H_{ef}$. Hence $H_eH_f \subsetneq H_{ef}$. This proves that S is a semilattice E(S) of groups H_e.

Hence from Theorem 1.3, we have the following remark : If S is a semigroup which is a union of groups and S admits a ring structure, then S is a semilattice of groups. Also, from Theorem 1.7, we have the following : If an inverse semigroup S admits a ring structure, then S is a semilattice of groups.