INTRODUCTION

A nonempty subset G of a semigroup S is a *subgroup* of S if it is a group under the same operation of S.

An element a of a semigroup S is an *idempotent* of S if $a^2 = a$. For a semigroup S, let E(S) denote the set of all idempotents of S ; that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

A semigroup S is a *band* if each element of S is an idempotent of S. Hence, a semigroup S is a band if and only if S = E(S). A commutative band is a *semilattice*.

For a semigroup S, the *center* of S, C(S), is the set of all elements of S which commute with every element of S, so

$$C(S) = \{a \in S \mid ax = xa \text{ for all } x \in S\}.$$

A semigroup S is *left cancellative* if for $a, b, x \in S$, $xa = xb$ implies $a = b$. A *right cancellative semigroup* is defined dually. A *cancellative semigroup* is a semigroup which is both left cancellative and right cancellative.

An element z of a semigroup S is called a *zero* of S if $zx = xz = z$ for all $x \in S$. An element e of a semigroup S is called an *identity* of S if $ex = xe = x$ for all $x \in S$. A zero and an identity of a semigroup are unique if exist and they are usually denoted by 0 and 1, respectively.

Let S be a semigroup, and let 0 be a symbol not representing any element of S. The notation $S \cup 0$ denotes the semigroup obtained by extending the binary operation on S to 0 by defining $0.0 = 0$ and $0.a = a.0 = 0$ for all $a \in S$, and the notation S^0 denotes the following semigroup :

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero,} \\ S \cup 0 & \text{if } S \text{ has no zero.} \end{cases}$$

Similarly, let S be a semigroup and 1 a symbol not representing any element of S. The notation $S \cup 1$ denotes the semigroup obtained by extending the binary operation on S to 1 by defining $1.1 = 1$ and $1.a = a.1 = a$ for all $a \in S$, and the notation S^1 denotes the following semigroup :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{if } S \text{ has no identity.} \end{cases}$$

Then for any element a of a semigroup S, $S^1a = Sa \cup \{a\}$, $aS^1 = aS \cup \{a\}$ and $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$.

An element a of a semigroup S is <u>regular</u> if $a = axa$ for some $x \in S$. A semigroup S is <u>regular</u> if every element of S is regular.

In any semigroup S, if $a, x \in S$ such that $a = axa$, then ax and xa are idempotents of S, and $S^1a = Sa$, $aS^1 = aS$, $Sa = Sxa$ and $aS = axS$.

A semigroup S is regular and contains exactly one idempotent if and only if S is a group.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if a = axa and x = xax. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a in S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [2, Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice. If S is an inverse semigroup, then for a, b ε S, e ε E(S), we have that

$$e^{-1} = e, \quad (a^{-1})^{-1} = a \quad \text{and} \quad (ab)^{-1} = b^{-1}a^{-1}$$

[2, Lemma 1.18].

Let S be a semigroup, A a nonempty subset of S. Then A is called a <u>left</u> [right] <u>ideal</u> of S if $SA \subseteq A$ $[AS \subseteq A]$, or equivalently, xa ε A [ax ε A] for all x ε S, a ε A .We call A an <u>ideal</u> of S if A is both a left ideal and a right ideal of S. For a ε S, S^1a $[aS^1, S^1aS^1]$ is the smallest left ideal [right ideal, ideal] of S containing a and it is called the <u>principal left ideal</u> [the principal right ideal, the principal ideal] of S <u>generated by</u> a.

A semigroup S is <u>left simple</u> [right simple, simple] if S is the only left ideal [right ideal, ideal] of S. Hence, a semigroup S is left simple [right simple, simple] if and only if Sa = S [aS = S, SaS = S] for all a ε S.

Let S be a semigroup.Define the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on S as follow :

$$a \mathscr{L} b \quad \text{if and only if} \quad S^1a = S^1b,$$
$$a \mathscr{R} b \quad \text{if and only if} \quad aS^1 = bS^1$$

and

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R} .$$

The relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ are called Green's relations on S and they are clearly equivalence relations on S. For $a \in S$, let H_a $[L_a, R_a]$ denote the $\mathscr{H}$- class $[\mathscr{L}$- class, $\mathscr{R}$- class$]$ of S containing a.

For a semigroup S, $a, x \in S$ such that $a = axa$, we have that $a \mathscr{L} xa$ and $a \mathscr{R} ax$.

In a semigroup S, any $\mathscr{H}$- class of S contains at most one idempotent [2, Lemma 2.15] , an $\mathscr{H}$- class of S containing an idempotent e of S is a subgroup of S [2, Theorem 2.16], and it is the greatest subgroup of S having e as its identity. Hence, every subgroup of a semigroup S is contained in H_e for some idempotent e of S.

Let X be a set. A partial transformation of X is a map which its domain and its range are subsets of X . If α is a partial transformation of X, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range of α, respectively. The empty transformation of X is referred as a map with empty domain, and it is denoted by 0. Let T_X denote the set of all partial transformations of X including the empty transformation 0.For $\alpha, \beta \in T_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \emptyset$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \emptyset$, let $\alpha\beta$ be the composition map of $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ (α restricted to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$) and $\beta|_{(\nabla\alpha \cap \Delta\beta)}$.

Then for $\alpha,\beta \in T_X$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$. Thus T_X is a semigroup under the operation defined above and it is called the <u>partial transformation semigroup</u> on the set X. Hence the empty transformation of X is the zero of X and the identity map on X which is denoted by 1_X is the identity of the semigroup T_X. A partial transformation α of X is called a <u>1-1 partial transformation</u> of X if α is a one-to-one map. Let I_X denote the set of all 1-1 partial transformations of X; that is,

$$I_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one}\}.$$

Then I_X is an inverse subsemigroup of T_X with identity 1_X and zero 0, and it is called the <u>1-1 partial transformation semigroup</u> or the <u>symmetric inverse semigroup</u> on the set X. By a <u>tranformation</u> of a set X we mean a mapping of X into itself. Then an element $\alpha \in T_X$ is a transformation of X if and only if $\Delta\alpha = X$. Let $\mathcal{T}_X$ denote the set of all transformations of X ; that is

$$\mathcal{T}_X = \{\alpha \in T_X \mid \Delta\alpha = X\}.$$

Then $\mathcal{T}_X$ is a regular subsemigroup of T_X with identity 1_X and it is called the <u>full transformation semigroup</u> on the set X. The permutation group on X is denoted by G_X. Then $G_X = \{\alpha \in T_X \mid \Delta\alpha = \nabla\alpha = X$ and α is one-to-one$\}$. Observe that $G_X \subseteq I_X \subsetneq T_X$ and $G_X \subseteq \mathcal{T}_X \subseteq T_X$. The semigroup of all one-to-one transformations of X and the semigroup of all onto transformations of X are denoted by M_X and E_X, respectively. Hence

$$M_X = \{\alpha : X \to X \mid \alpha \text{ is one-to-one}\}$$

$$= \{\alpha \in I_X \mid \Delta\alpha = X\}$$

and

$$E_X = \{\alpha : X \to X \mid \alpha \text{ is onto}\}$$

$$= \{\alpha \in \mathcal{T}_X \mid \nabla\alpha = X\} .$$

We denote the semigroup of all constant partial transformations of X by C_X ; that is ,

$$C_X = \{\alpha \in T_X \mid |\nabla\alpha| \leqslant 1\} .$$

Let X be a set. The <u>shift</u> of a partial transformation α of X, $S(\alpha)$, is defined to be the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$. A partial transformation α of X is said to be <u>almost identical</u> if the shift of α is finite; that is, $|S(\alpha)| < \infty$, where for any set A, $|A|$ denotes the cardinality of A. Let

$$U_X = \{\alpha \in T_X \mid \alpha \text{ is almost identical}\},$$

$$V_X = \{\alpha \in \mathcal{T}_X \mid \alpha \text{ is almost identical }\}$$

and

$$W_X = \{\alpha \in I_X \mid \alpha \text{ is almost identical }\}.$$

If $\alpha,\beta \in T_X$, then $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$. Hence, U_X , V_X and W_X are subsemigroups of T_X, $\mathcal{T}_X$ and I_X, respectively, and U_X, V_X and W_X are referred respectively as the <u>semigroup of all almost identical partial transformations</u> of X , the <u>semigroup of all almost identical transformations</u> of X and the <u>semigroup of all almost identical 1-1 partial transformations</u> of X .

By a <u>transformation semigroup</u> on a set X, we mean a subsemigroup of the partial transformation semigroup on X. Let S be a transformation semigroup and let $\theta \in S$. The semigroup S under the operation * defined by $\alpha * \beta = \alpha\theta\beta$ for all α, $\beta \in S$ is called a <u>generalized transformation semigroup</u> on the set X , and it is denoted by (S,θ).

Let S and T be semigroups and ψ a map from S into T. The map ψ is a homomorphism from S into T if

$$(ab)\psi = (a\psi)(b\psi)$$

for all a, b ε S. A homomorphism ψ from S into T is an isomorphism if ψ is a one-to-one map. If there exists an isomorphism from S onto T, we say that the semigroups S and T are isomorphic, and we write $S \cong T$.

Let S be a semigroup, we say that S admits a ring structure if the semigroup S^{0} is isomorphic to the multiplicative structure of some ring.

In the first chapter of this thesis, we study admitting ring structure of semigroups which are unions of groups. To determine whether some semigroups of numbers admit ring structure is the purpose of Chapter II. The main study of this thesis is in the last chapter. In this chapter, we characterize well-known generalized transformation semigroups admitting ring structure and well-known transformation semigroups admitting ring structure.