CHAPTER II

EXAMPLE 1 : THE SENTENTIAL CALCULUS (SC)

In this chapter we formulate a theory of sentential logic and we prove a completeness theorem for it by using the theory of Boolean Algebras. The knowledge about Boolean Algebra used in this chapter is in Appendix A. At the end of this chapter we give some subtheories of SC which have SC as their only complete and consistent extension.

We first introduce the symbols of SC which are the following:

(i) a denumerable set of sentence variables :

$p, q, r, s, p_1, q_1, r_1, s_1, \ldots\ldots\ldots\ldots\ldots,$

(ii) logical connectives : $\sim, \rightarrow, \vee, \wedge,$

(iii) parentheses : (,).

We define sentences of SC as in Definition 1.1, and rules of inference are as 1.2.

The axioms of SC are the following :

A1. $p \rightarrow (q \rightarrow p)$

A2. $(s \rightarrow (p \rightarrow q)) \rightarrow ((s \rightarrow p) \rightarrow (s \rightarrow q))$

A3. $p \wedge q \rightarrow p$

A4. $p \wedge q \rightarrow q$

A5. $p \rightarrow (q \rightarrow p \wedge q)$

A6. $p \rightarrow p \vee q$

A7. $q \rightarrow p \vee q$

A8. $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow (p \vee q \rightarrow r))$

A9. $\sim p \rightarrow (p \rightarrow q)$

A10. $(p \rightarrow q) \rightarrow ((p \rightarrow \sim q) \rightarrow \sim p)$

A11. $p \vee \sim p$.

From now on we say that the sentence ϕ is a theorem of SC instead of saying ϕ is a theorem of the set of all axioms of SC, and instead of saying the set of all theorems of SC, we say the theory SC, or SC.

Note that if $\vdash_{sc} \phi$, then $\vdash_{sc} S^{b_1, \ldots, b_n}_{\phi_1, \ldots, \phi_n} \phi \mid$ where $b_1, \ldots, b_n$ are sentence variables in ϕ and $\phi_1, \ldots, \phi_n$ are some sentences.

2.1 Definition. A variant of a sentence ϕ of SC is a sentence obtained from ϕ by alphabetic changes of the variables of such a sort that two occurrences of the same variable in ϕ remain occurrences of the same variable, and two occurrences of distinct variables in ϕ remain occurrences of distinct variables.

Thus if $a_1, \ldots, a_n$ are distinct variables, and $b_1, \ldots, b_n$ are distinct variables, and there is no variable among $b_1, \ldots, b_n$ which occurs in ϕ and does not occur among $a_1, \ldots, a_n$, then $S^{a_1, \ldots, a_n}_{b_1, \ldots, b_n} \phi \mid$ is a variant of ϕ.

Note that if ϕ is a variant of ψ, then ψ is a variant of ϕ; and any variant of a variant of ϕ is a variant of ϕ; and any sentence ϕ is a variant of itself.

2.2 Definition. A finite sequence of sentences $\psi_1, \ldots, \psi_n$ is called a proof from a set of sentences Σ in SC if and only if each ψ_i, $1 \le i \le n$, is

(i) a sentence in Σ, or

(ii) a variant of an axiom, or

(iii) a conclusion from ψ_j $(j < i)$ by Subs. where the variable substituted for does not occur in the sentences in Σ, or

(iv) a conclusion from ψ_j, ψ_k $(j,k < i)$ by MP.

Such a finite sequence of sentences, ψ_n being the final sentence of the sequence, is called more explicitly a proof of ψ_n from Σ and we use the notation $\Sigma \vdash_{sc} \psi_n$.

2.3 Lemma. For any sentence ϕ of SC, $\vdash_{sc} \phi \rightarrow \phi$.

Proof. 1. $p \rightarrow (q \rightarrow p)$ by A1.

2. $\phi \rightarrow (\phi \rightarrow \phi)$ by (1) and Subs.

3. $\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)$ by (1) and Subs.

4. $(s \rightarrow (p \rightarrow q)) \rightarrow ((s \rightarrow p) \rightarrow (s \rightarrow q))$ by A2.

5. $(\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)) \rightarrow ((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi))$ by (4) and Subs.

6. $(\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)$ by (3), (5) and MP.

7. $\phi \rightarrow \phi$ by (2), (6) and MP.

Hence $\phi \rightarrow \phi$ is a theorem of SC.

2.4 Theorem. (Deduction Theorem.) Let Σ be a set of sentences in SC, ϕ, ψ be sentences in SC. If $\Sigma \cup \{\phi\} \vdash_{sc} \psi$, then $\Sigma \vdash_{sc} \phi \rightarrow \psi$.

Proof. Assume $\Sigma \cup \{\phi\} \vdash_{sc} \psi$. Then there is a finite sequence of sentences $\theta_1, \ldots, \theta_n$ such that $\theta_n = \psi$ and for each i, $1 \leq i \leq n$, θ_i is a variant of an axiom, or $\theta_i \in \Sigma$, or $\theta_i = \phi$, or θ_i is a conclusion from θ_j $(j < i)$ by Subs. where the variable substituted for does not occur in the sentences in $\Sigma \cup \{\phi\}$, or θ_i is a conclusion from θ_j, θ_k $(j, k < i)$ by MP.

Claim that $\Sigma \vdash_{sc} \phi \rightarrow \theta_i$, $1 \leq i \leq n$. We will show this by induction on i. For $i = 1$, θ_1 is a variant of an axiom or $\theta_1 \in \Sigma$ or $\theta_1 = \phi$. Suppose θ_1 is a variant of an axiom or $\theta_1 \in \Sigma$. Since $\vdash_{sc} \theta_1 \rightarrow (\phi \rightarrow \theta_1)$, $\Sigma \vdash_{sc} \phi \rightarrow \theta_1$. Suppose $\theta_1 = \phi$. Since from Lemma 2.3 we have $\vdash_{sc} \phi \rightarrow \phi$, $\Sigma \vdash_{sc} \phi \rightarrow \theta_1$.

Now assume $\Sigma \vdash_{sc} \phi \rightarrow \theta_j$ for all $j < k \leq n$. If θ_k is a variant of an axiom, or $\theta_k \in \Sigma$, or $\theta_k = \phi$, then we can prove similarly to the case $i = 1$ that $\Sigma \vdash_{sc} \phi \rightarrow \theta_k$. If θ_k is a conclusion from θ_j $(j < k)$ by Subs., then by induction hypothesis we have $\Sigma \vdash_{sc} \phi \rightarrow \theta_j$, and we get $\phi \rightarrow \theta_k$ from $\phi \rightarrow \theta_j$ by Subs. because the variable substituted for does not occur in ϕ, so $\Sigma \vdash_{sc} \phi \rightarrow \theta_k$. For the last case if θ_k is a conclusion from θ_j and $\theta_j \rightarrow \theta_k$ $(j < k)$ by MP., then $\theta_j \rightarrow \theta_k = \theta_m$ for some $m < k$ and by induction hypothesis we have $\Sigma \vdash_{sc} \phi \rightarrow \theta_j$ and $\Sigma \vdash_{sc} \phi \rightarrow (\theta_j \rightarrow \theta_k)$. Hence from A2, $\Sigma \vdash_{sc} \phi \rightarrow \theta_k$.

Therefore $\Sigma \vdash_{sc} \phi \rightarrow \theta_i$, $1 \leq i \leq n$. Consequently $\Sigma \vdash_{sc} \phi \rightarrow \theta_n$, i.e. $\Sigma \vdash_{sc} \phi \rightarrow \psi$.

2.5 Note. The following are theorems of SC :

(i) $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$

(ii) $\sim(p \wedge \sim p)$

(iii) $(p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow q \wedge r))$

(iv) $(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee (p \wedge r))$

(v) $((p \wedge q) \vee (p \wedge r)) \rightarrow (p \wedge (q \vee r))$

(vi) $(p \vee (q \wedge r)) \rightarrow ((p \vee q) \wedge (p \vee r))$

(vii) $((p \vee q) \wedge (p \vee r)) \rightarrow (p \vee (q \wedge r))$

(viii) $(\sim p \vee q) \rightarrow (p \rightarrow q)$

(ix) $(p \rightarrow q) \rightarrow (\sim p \vee q)$.

2.6 Definition. A realization of SC is a map

$$f : \{p, q, r, s, p_1, q_1, r_1, s_1, \ldots\ldots\ldots\} \rightarrow \mathcal{B}^2$$

where $\mathcal{B}^2$ is defined as in Appendix A.

Given a realization f of SC we extend it to a map

$$\bar{f} : \{\phi \mid \phi \text{ is a sentence of SC}\} \rightarrow \mathcal{B}^2$$

in the following manner :

For any sentences ϕ and ψ and sentence variable a_i,

(i) $\bar{f}(a_i) = f(a_i)$

(ii) $\bar{f}(\sim\phi) = \bar{f}(\phi)^*$

(iii) $\bar{f}(\phi \vee \psi) = \bar{f}(\phi) \vee \bar{f}(\psi)$

(iv) $\bar{f}(\phi \wedge \psi) = \bar{f}(\phi) \wedge \bar{f}(\psi)$

(v) $\bar{f}(\phi \rightarrow \psi) = \bar{f}(\phi)^* \vee \bar{f}(\psi)$.

2.7 Definition. If a realization f is extended in the way just described, and ϕ is a sentence of SC, then we say that $\bar{f}$ satisfies ϕ if and only if $\bar{f}(\phi) = 1$.

A sentence ϕ is said to be satisfiable if and only if it is satisfied by some extended realization.

A sentence ϕ is said to be a tautology if and only if it is satisfied by all extended realizations.

2.8 Theorem. If $\vdash_{SC} \phi$, then ϕ is a tautology.

Proof. The proof is by induction on the length of proofs in SC. We show that each axiom is a tautology and rules of inference preserve tautology. It follows that a proof consists of a sequence of tautologies, and that every provable sentence is a tautology.

We first show that all axioms of SC are tautologies. Let f be any realization of SC. Then

$$
\begin{aligned}
\bar{f}(p \rightarrow (q \rightarrow p)) &= \bar{f}(p)^* \vee \bar{f}(q \rightarrow p) \\
&= f(p)^* \vee (f(q)^* \vee f(p)) \\
&= 1 \vee f(q)^* \\
&= 1 \text{ ,}
\end{aligned}
$$

$$\bar{f}((s \rightarrow (p \rightarrow q)) \rightarrow ((s \rightarrow p) \rightarrow (s \rightarrow q)))$$

$$= \bar{f}(s \rightarrow (p \rightarrow q))^* \vee \bar{f}((s \rightarrow p) \rightarrow (s \rightarrow q))$$

$$= (f(s)^* \vee (f(p)^* \vee f(q)))^* \vee (f(s)^* \vee f(p))^* \vee (f(s)^* \vee f(q))$$

$$= (f(s) \wedge f(p) \wedge f(q)^*) \vee (f(s) \wedge f(p)^*) \vee (f(s)^* \vee f(q))$$

$$= (f(s) \wedge ((f(p) \wedge f(q)^*) \vee f(p)^*)) \vee (f(s)^* \vee f(q))$$

$$= (f(s) \wedge (f(q)^* \vee f(p)^*)) \vee (f(s)^* \vee f(q))$$

$$= (f(s) \wedge f(q)^*) \vee (f(s) \wedge f(p)^*) \vee (f(s) \wedge f(q)^*)^*$$

$$= 1 \vee (f(s) \wedge f(p)^*)$$

$$= 1 \quad ,$$

$$\bar{f}(p \wedge q \rightarrow p) = \bar{f}(p \wedge q)^* \vee \bar{f}(p)$$

$$= (\bar{f}(p) \wedge \bar{f}(q))^* \vee \bar{f}(p)$$

$$= (f(p) \wedge f(q))^* \vee f(p)$$

$$= f(p)^* \vee f(q)^* \vee f(p)$$

$$= 1 \vee f(q)^*$$

$$= 1 \quad ,$$

$$\bar{f}(p \wedge q \rightarrow q) = \bar{f}(p \wedge q)^* \vee \bar{f}(q)$$

$$= (\bar{f}(p) \wedge \bar{f}(q))^* \vee \bar{f}(q)$$

$$= (f(p) \wedge f(q))^* \vee f(q)$$

$$= f(p)^* \vee f(q)^* \vee f(q)$$

$$= f(P)^* \vee 1$$

$$= 1 \quad ,$$

$$\bar{f}(p \rightarrow (q \rightarrow p \wedge q)) = \bar{f}(p)^* \vee (\bar{f}(q)^* \vee \bar{f}(p \wedge q))$$

$$= f(p)^* \vee (f(q)^* \vee (f(p) \wedge f(q)))$$

$$= f(p)^* \vee ((f(q)^* \vee f(p)) \wedge (f(q)^* \vee f(q)))$$

$$= f(p)^* \vee ((f(q)^* \vee f(p)) \wedge 1)$$

$$= f(p)^* \vee (f(q)^* \vee f(p))$$

$$= 1 \vee f(q)^*$$

$$= 1 \quad ,$$

$$\begin{aligned} \bar{f}(p \rightarrow p \vee q) &= \bar{f}(p)^* \vee \bar{f}(p \vee q) \\ &= f(p)^* \vee (f(p) \vee f(q)) \\ &= (f(p)^* \vee f(p)) \vee f(q) \\ &= 1 \vee f(q) \\ &= 1 \quad , \end{aligned}$$

$$\begin{aligned} \bar{f}(q \rightarrow p \vee q) &= \bar{f}(q)^* \vee \bar{f}(p \vee q) \\ &= f(q)^* \vee (f(p) \vee f(q)) \\ &= 1 \vee f(p) \\ &= 1 \quad , \end{aligned}$$

$$\bar{f}((p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow (p \vee q \rightarrow r)))$$

$$\begin{aligned} &= (f(p)^* \vee f(r))^* \vee ((f(q)^* \vee f(r))^* \vee (f(p) \vee f(q))^* \vee f(r) \\ &= (f(p) \wedge f(r)^*) \vee (f(q) \wedge f(r)^*) \vee ((f(p) \vee f(q))^* \vee f(r)) \\ &= ((f(p) \vee f(q)) \wedge f(r)^*) \vee ((f(p) \vee f(q))^* \vee f(r)) \\ &= ((f(p) \vee f(q)) \wedge f(r)^*) \vee ((f(p) \vee f(q)) \wedge f(r)^*)^* \\ &= 1 \quad , \end{aligned}$$

$$\begin{aligned} \bar{f}(\sim p \rightarrow (p \rightarrow q)) &= (f(p)^*)^* \vee (f(p)^* \vee f(q)) \\ &= f(p) \vee f(p)^* \vee f(q) \\ &= 1 \vee f(q), \qquad = 1 \quad , \end{aligned}$$

$$\bar{f}((p \to q) \to ((p \to {\sim}q) \to {\sim}p))$$
$$= (f(p)^* \vee f(q))^* \vee (f(p)^* \vee f(q)^*)^* \vee f(p)^*$$
$$= (f(p) \wedge f(q)^*) \vee (f(p) \wedge f(q)) \vee f(p)^*$$
$$= (f(p) \wedge (f(q)^* \vee f(q))) \vee f(p)^*$$
$$= (f(p) \wedge 1) \vee f(p)^*$$
$$= f(p) \vee f(p)^*$$
$$= 1 \quad , \text{ and}$$

$$\bar{f}(p \vee {\sim} p) = f(p) \vee f(p)^*$$
$$= 1 \quad .$$

Hence all axioms of SC are tautologies.

Next, will show that rules of inference preserve tautology. It is clear that Subs. preserves tautology. To show MP. preserves tautology, suppose that ϕ and $\phi \to \psi$ are tautologies and let f be any realization of SC. Then $\bar{f}(\phi) = \bar{f}(\phi \to \psi) = 1$, and so

$$\bar{f}(\psi) = (\bar{f}(\phi) \wedge \bar{f}(\phi)^*) \vee \bar{f}(\psi)$$
$$= (\bar{f}(\phi) \vee \bar{f}(\psi)) \wedge (f(\phi)^* \vee f(\psi))$$
$$= (1 \vee \bar{f}(\psi)) \wedge 1$$
$$= 1 \wedge 1$$
$$= 1 \quad .$$

Therefore ψ is a tautology.

2.9 Theorem. SC is consistent.

Proof. Since any sentence variable is not a tautology, it is not a theorem of SC. Thus SC is consistent.

Let S be the set of all sentences in SC. Define the relation $\sim$ on S as follows : For any sentences ϕ, ψ

$$\phi \sim \psi \text{ if and only if } \vdash_{SC} \phi \rightarrow \psi \text{ and } \vdash_{SC} \psi \rightarrow \phi.$$

Thus $\sim$ is an equivalence relation on S and denote the equivalence class containing ϕ by $[\phi]$.

2.10 Theorem. Let $\mathcal{A} = \{[\phi] \mid \phi \in S\}$. Define the relation $\leq$ on S by

$$[\phi] \leq [\psi] \quad \text{if and only if} \quad \vdash_{SC} \phi \rightarrow \psi.$$

Then $<\mathcal{A}, \leq>$ is a Boolean Algebra. (Call it the Lindenbaum Algebra.)

Proof. If $\phi \sim \psi$, then $\vdash_{SC} \phi \rightarrow \psi$ and $\vdash_{SC} \psi \rightarrow \phi$, so $[\phi] \leq [\psi]$ i.e. $[\phi] \leq [\phi]$. Then $\leq$ is reflexive. By Note 2.5 (i) we also have $\leq$ is transitive. Now suppose $[\phi] \leq [\psi]$ and $[\psi] \leq [\phi]$. Then $\vdash_{SC} \phi \rightarrow \psi$ and $\vdash_{SC} \psi \rightarrow \phi$, so $\phi \sim \psi$, i.e. $[\phi] = [\psi]$. Hence $\leq$ is antisymmetric. Thus $\leq$ is a partial ordering of $\mathcal{A}$.

Let $[\phi]$, $[\psi] \in \mathcal{A}$. Since from A3. and A4., $\vdash_{SC} \phi \wedge \psi \rightarrow \phi$ and $\vdash_{SC} \phi \wedge \psi \rightarrow \psi$ hence $[\phi \wedge \psi] \leq [\phi]$ and $[\phi \wedge \psi] \leq [\psi]$. Suppose $[\theta] \in \mathcal{A}$ such that $[\theta] \leq [\phi]$ and $[\theta] \leq [\psi]$, so $\vdash_{SC} \theta \rightarrow \phi$ and $\vdash_{SC} \theta \rightarrow \psi$. Hence by Note 2.5 (iii) and two applications of MP., $\vdash_{SC} \theta \rightarrow \phi \wedge \psi$, so $[\phi] \leq [\phi \wedge \psi]$. This shows that $[\phi] \wedge [\psi] = [\phi \wedge \psi]$. Similarly by using

A6., A7., and A8., we also have $[\phi] \vee [\psi] = [\phi \vee \psi]$. We have now that $<\mathcal{A}, \leq>$ is a lattice.

Let $[\phi], [\psi], [\theta] \in \mathcal{A}$. By Note 2.5 (iv) - (vii) we have $[(\phi \vee \psi) \wedge \theta] = [(\phi \wedge \theta) \vee (\psi \wedge \theta)]$, $[(\phi \wedge \psi) \vee \theta] = [(\phi \vee \theta) \wedge (\psi \vee \theta)]$, hence $([\phi] \vee [\psi]) \wedge [\theta] = ([\phi] \wedge [\theta]) \vee ([\psi] \wedge [\theta])$ and $([\phi] \wedge [\psi]) \vee [\theta] = ([\phi] \vee [\theta]) \wedge ([\psi] \vee [\theta])$. Therefore $<\mathcal{A}, \leq>$ is a distributive lattice.

By A1., if $\vdash_{sc} \phi$, then for any sentence ψ, $\vdash_{sc} \psi \rightarrow \phi$ and so $[\psi] \leq [\phi]$. Hence $[\phi] = 1$, the maximum element of $\mathcal{A}$. Similarly, using A9., if $\vdash_{sc} \sim\phi$, then for any sentence ψ, $\vdash_{sc} \phi \rightarrow \psi$ and so $[\phi] \leq [\psi]$. Hence $[\phi] = 0$, the minimum element of $\mathcal{A}$. Conversely if $[\phi] = 1$, then for any sentence ψ, $[\psi] \leq [\phi]$ and so $\vdash_{sc} \psi \rightarrow \phi$. Therefore by choosing ψ so that $\vdash_{sc} \psi$, we have $\vdash_{sc} \phi$ by MP. Similarly if $[\phi] = 0$, then for any sentence ψ, $[\phi] \leq [\psi]$, and so $\vdash_{sc} \phi \rightarrow \psi$. Then $\vdash_{sc} \phi \rightarrow p$ and $\vdash_{sc} \phi \rightarrow \sim p$ and so by A10., we have $\vdash_{sc} \sim\phi$.

Finally from A11., and Note 2.5 (ii) we have that for any sentence ϕ, $\vdash_{sc} \phi \vee \sim\phi$ and $\vdash_{sc} \sim(\phi \wedge \sim\phi)$, hence $[\phi] \vee [\sim\phi] = 1$ and $[\phi] \wedge [\sim\phi] = 0$. This shows that $<\mathcal{A}, \leq>$ is a complemented distributive lattice and the complement of $[\phi]$ is $[\sim\phi]$. Therefore $<\mathcal{A}, \leq>$ is a Boolean Algebra.

2.11 <u>Lemma</u>. Let h be a homomorphism of the Lindenbaum Algebra $\mathcal{A}$ into the Boolean Algebra $\mathcal{B}^2$. If $f : \{p, q, r, s, p_1, q_1, r_1, s_1, \ldots\} \longrightarrow \mathcal{B}^2$ is defined by $f(a) = h([a])$, where a is a sentence variable of SC, then f is a realization of SC such that for each ϕ

$$\overline{f}(\phi) \quad = \quad h([\phi]).$$

Proof. The proof is by induction on length of ϕ (number of connectives in ϕ). By hypothesis it is true for sentence variables standing by themselves.

Now suppose it is true for sentences of length less than k. Let ϕ be a sentence of length k. Then ϕ is of the forms $\psi \wedge \theta$, or $\psi \vee \theta$, or $\psi \longrightarrow \theta$ or $\sim\psi$ for some sentences ψ and θ. Then length of ψ and θ less than k. Then since h is a homomorphism,

$\overline{f}(\psi \wedge \theta) = \overline{f}(\psi) \wedge \overline{f}(\theta) = h([\psi]) \wedge h([\theta]) = h([\psi] \wedge [\theta]) = h([\psi \wedge \theta])$, and $\overline{f}(\sim\theta) = \overline{f}(\theta)^* = h([\theta])^* = h([\theta]^*) = h([\sim\theta])$, and similarly we also have $\overline{f}(\psi \vee \theta) = h([\psi \vee \theta])$ and $\overline{f}(\psi \longrightarrow \theta) = h([\psi \longrightarrow \theta])$, and hence the result is true for all sentences.

The realization f is called the realization induced by h.

2.12 Theorem. Each tautology of SC is a theorem of SC.

Proof. Suppose $\not\vdash_{SC} \phi$. Then in the Lindenbaum Algebra $<\mathcal{A}, \leq>$, $[\phi] \neq 1$ and so $[\sim\phi] \neq 0$. Then by Theorem A7. in Appendix A, $[\sim\phi]$ is contained in an ultrafilter U in $\mathcal{A}$ and hence by Proposition A10. in Appendix A, $\mathcal{A}/U \cong \mathcal{B}^2$. Let f be the realization of SC induced by the canonical homomorphism h of $\mathcal{A}$ into $\mathcal{A}/U$. Since $[\sim\phi] \in U$, we have $|[\sim\phi]| = h([\sim\phi]) = 1$ in $\mathcal{A}/U$ by Lemma A9. in Appendix A, and so $\overline{f}(\sim\phi) = 1$. It follows that $\overline{f}(\phi) = 0$ and so ϕ is not a tautology of SC.

2.13 Theorem. SC is complete.

Proof. Let ϕ be any sentence of SC. Suppose that ϕ is not a theorem of SC. Then by Theorem 2.12, ϕ is not a tautology and so there exists a realization f such that $\bar{f}(\phi) \neq 1$, that is $\bar{f}(\phi) = 0$.

Let $a_1, \ldots\ldots, a_n$ be all sentence variables in ϕ and $t_i = f(a_i)$, $i = 1, \ldots, n$. Let $\phi_1 = S^{a_1, \ldots, a_n}_{\psi_1, \ldots, \psi_n} \phi \,|$ where $\psi_i = p \wedge {\sim}p$ if $t_i = 0$ and $\psi_i = p \vee {\sim}p$ if $t_i = 1$. Then for any realization f' of SC $f'(\phi_1) = 0$ and so

$$f'(\phi_1 \rightarrow p) = f'(\phi_1)^* \vee f(p) = 1 \vee f'(p) = 1.$$

Hence $\phi_1 \rightarrow p$ is a tautology and then again by Theorem 2.12, we have $\vdash_{SC} \phi_1 \rightarrow p$. Thus if we add ϕ as an axiom of SC, then $\vdash_{SC} p$ and this means every sentence in SC is a theorem of SC. Hence SC is complete.

2.14 Theorem. If X is a consistent set of sentences of SC which contains the following ten sentences as elements : $p \rightarrow {\sim\sim}p$, $q \rightarrow (p \rightarrow q)$, ${\sim}p \rightarrow (p \rightarrow q)$, $p \rightarrow ({\sim}q \rightarrow {\sim}(p \rightarrow q))$, $p \rightarrow p \vee q$, $q \rightarrow p \vee q$, ${\sim}p \rightarrow ({\sim}q \rightarrow {\sim}(p \vee q))$, $p \rightarrow (q \rightarrow p \wedge q)$, ${\sim}p \rightarrow {\sim}(p \wedge q)$, ${\sim}q \rightarrow {\sim}(p \wedge q)$, then SC is the only consistent and complete theory which includes the set X.

Proof. Let X_0 be the set of ten sentences : $p \rightarrow {\sim\sim}p$, $q \rightarrow (p \rightarrow q)$, ${\sim}p \rightarrow (p \rightarrow q)$, $p \rightarrow ({\sim}q \rightarrow {\sim}(p \rightarrow q))$, $p \rightarrow p \vee q$, $q \rightarrow p \vee q$, ${\sim}p \rightarrow ({\sim}q \rightarrow {\sim}(p \vee q))$, $p \rightarrow (q \rightarrow p \wedge q)$, ${\sim}p \rightarrow {\sim}(p \wedge q)$, ${\sim}q \rightarrow {\sim}(p \wedge q)$.

Will show that SC is the only consistent and complete theory which includes X_0.

Since every sentence in X_0 is a tautology, SC is a complete and consistent theory which includes X_0.

Next, let $Z = \{p \rightarrow \sim\sim p\}$ and $Y = \{\phi \in S \mid \phi \in Cn(X_0)$ or $\sim\phi \in Cn(X_0)\}$. We claim that $Z \subseteq Y$ and Y is closed under every connective of SC. Since $p \rightarrow \sim\sim p \in X_0 \subseteq Cn(X_0)$, $p \rightarrow \sim\sim p \in Y$ and so $Z \subseteq Y$. To show that Y is closed under $\sim$, let $\phi \in Y$. Then $\phi \in Cn(X_0)$ or $\sim\phi \in Cn(X_0)$. If $\phi \in Cn(X_0)$, then $\sim\sim\phi \in Cn(X_0)$ since $\phi \rightarrow \sim\sim\phi \in Cn(X_0)$. Hence $\sim\sim\phi \in Cn(X_0)$ or $\sim\phi \in Cn(X_0)$ and so $\sim\phi \in Y$. If $\sim\phi \in Cn(X_0)$, then $\sim\phi \in Y$. Next to show that Y is closed under $\rightarrow$, $\vee$, and $\wedge$, let ϕ, $\psi \in Y$. Here three cases are to be distinguished :

Case 1. $\phi \in Cn(X_0)$, $\psi \in Cn(X_0)$. Since $\phi \rightarrow (\psi \rightarrow \phi)$, $\phi \rightarrow \phi \vee \psi$ and $\phi \rightarrow (\psi \rightarrow \phi \wedge \psi)$ are in $Cn(X_0)$, we have $\psi \rightarrow \phi$, $\phi \vee \psi$ and $\phi \wedge \psi$ are also in $Cn(X_0)$. Hence $\psi \rightarrow \phi$, $\phi \wedge \psi$ and $\phi \vee \psi$ are in Y.

Case 2. $\phi \in Cn(X_0)$, $\sim\psi \in Cn(X_0)$. Since $\phi \rightarrow \phi \vee \psi$, $\phi \rightarrow (\sim\psi \rightarrow \sim(\phi \rightarrow \psi))$, $\sim\psi \rightarrow (\psi \rightarrow \phi)$, $\sim\psi \rightarrow \sim(\phi \wedge \psi)$ are in $Cn(X_0)$, we have $\phi \vee \psi$, $\sim(\phi \rightarrow \psi)$, $\psi \rightarrow \phi$ and $\sim(\phi \wedge \psi)$ are also in $Cn(X_0)$. Hence $\psi \rightarrow \phi$, $\phi \rightarrow \psi$, $\phi \vee \psi$ and $\phi \wedge \psi$ are in Y.

Case 3. $\sim\phi \in Cn(X_0)$, $\sim\psi \in Cn(X_0)$. Since $\sim\phi \rightarrow \sim(\phi \wedge \psi)$, $\sim\phi \rightarrow (\sim\psi \rightarrow \sim(\phi \vee \psi))$, and $\sim\phi \rightarrow (\phi \rightarrow \psi)$ are in $Cn(X_0)$, we have $\sim(\phi \wedge \psi)$, $\sim(\phi \vee \psi)$, $\phi \rightarrow \psi$ are also in $Cn(X_0)$. Hence $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \rightarrow \psi$ are in Y.

Therefore by Theorem 1.11, $Sb_Z(S) \subseteq Y$.

Next show that X_0 is complete with respect to $Sb_Z(S)$. Let $\phi \varepsilon Sb_Z(S)$. Suppose $\phi \notin Cn(X_0)$. Since $Sb_Z(S) \subseteq Y$, $\sim\phi \varepsilon Cn(X_0)$ and since further for any sentence ψ, $\sim\phi \rightarrow (\phi \rightarrow \psi) \varepsilon Cn(X_0) \subseteq Cn(X_0 \cup \{\phi\})$, and so $\psi \varepsilon Cn(X_0 \cup \{\phi\})$. Then $X_0 \cup \{\phi\}$ is inconsistent. Therefore X_0 is complete with respect to $Sb_Z(S)$.

Now will show that $X_0 \cup Sb_Z(S)$ is inconsistent. Let ψ be any sentence of SC. Since $\sim(p \rightarrow \sim\sim p) \rightarrow ((p \rightarrow \sim\sim p) \rightarrow \psi) \varepsilon Cn(X_0 \cup Sb_Z(S))$ and $\sim(p \rightarrow \sim\sim p)$, $p \rightarrow \sim\sim p$ are in $Sb_Z(S) \subseteq Cn(X_0 \cup Sb_Z(S))$, $\psi \varepsilon Cn(X_0 \cup Sb_Z(S))$. Hence $Cn(X_0 \cup Sb_Z(S)) = S$ and so $X_0 \cup Sb_Z(S)$ is inconsistent.

Then now we apply Theorem 1.12 and in this way conclude that SC is the only consistent and complete theory which includes X_0. If X is any consistent set of sentences such that $X_0 \subseteq X$, then by Lindenbaum's Theorem, X can be extended to a consistent and complete theory Y. Since $X_0 \subseteq Y$, Y can not be distinct from SC. Consequently SC is the only consistent and complete theory which includes X.