CHAPTER IV

CLIQUE PARAMETERS OF THE K-POWER OF LADDERS AND GRIDS

In this chapter, we investigate the values or bounds of the clique covering numbers and the clique partition numbers of the k-power of ladders and grids. In the first and the second sections contain results of ladders, and the other sections contain results of grids.

4.1 Clique Coverings of the k-power of Ladders

First, we recall definitions of a grid and a ladder.

Definition 4.1.1. The *cartesian product* of G and H, written $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

Definition 4.1.2. The m-by-n *grid* is the cartesian product $P_m \times P_n$. In case $m = 2$, $P_2 \times P_n$ is called a *ladder*.

Next, we find the values of the clique covering numbers of the k-power of ladders.

Lemma 4.1.3. *For* $n, k \in \boldsymbol{N}$ *where* $2 \leq k < n$,

$$cc((P_2 \times P_n)^k) \geq 2(n-k) + 2.$$

Proof. Let $V(P_2 \times P_n) = \{(i, j) \mid i = 1, 2 \text{ and } j = 1, 2, ..., n\}$ and $E(P_2 \times P_n) = \{(i, j)(i, j+1) \mid i = 1, 2 \text{ and } j = 1, 2, ..., n-1\} \cup \{(1, j)(2, j) \mid j = 1, 2, ..., n\}$.

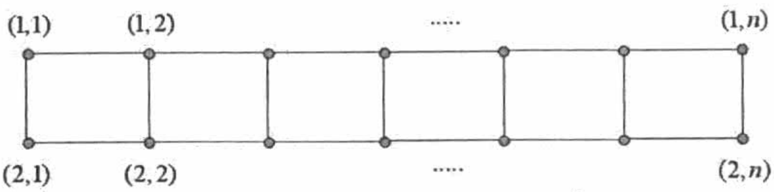

Figure 4.1: $P_2 \times P_n$

Let $I_k = \{(i,j)(i,j+k) \mid i = 1,2 \text{ and } j = 1,2,...,n-k\} \cup \{(1,1)(2,1),(1,n)(2,n)\}$. Then I_k is a subset of $E((P_2 \times P_n)^k)$ and $|I_k| = 2(n-k)+2$.

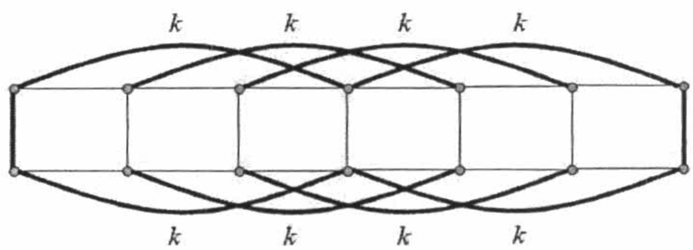

Figure 4.2: I_k in Lemma 4.1.3

We will next show that I_k is a clique-independent set of $(P_2 \times P_n)^k$. Let $e_1, e_2 \in I_k$ where $e_1 \neq e_2$.

Case 1 : $e_1, e_2 \in \{(i,j)(i,j+k) \mid i = 1,2 \text{ and } j = 1,2,...,n-k\}$.

Then $e_1 = (i_1,j_1)(i_1,j_1+k)$ and $e_2 = (i_2,j_2)(i_2,j_2+k)$ for some $i_1, i_2 \in \{1,2\}$ and $j_1, j_2 \in \{1,2,...,n-k\}$. WLOG, assume $j_1 \leq j_2$. We have that $d_{P_2 \times P_n}((i_1,j_1),(i_2,j_2+k)) \geq k+1 > k$. Thus (i_1,j_1) is not adjacent to (i_2,j_2+k) in $(P_2 \times P_n)^k$.

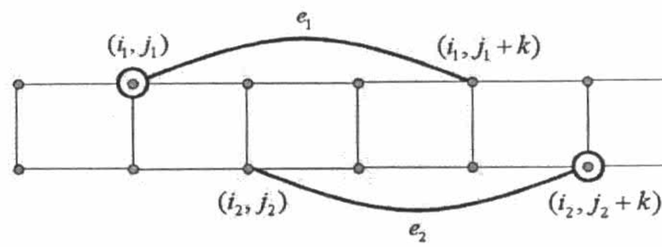

Figure 4.3: Case 1 in Lemma 4.1.3

Case 2 : $e_1, e_2 \in \{(1,1)(2,1), (1,n)(2,n)\}$.

WLOG, assume that $e_1 = (1,1)(2,1)$ and $e_2 = (1,n)(2,n)$. Since $k < n$, $(1,1)$ is not adjacent to $(2,n)$ in $(P_2 \times P_n)^k$.

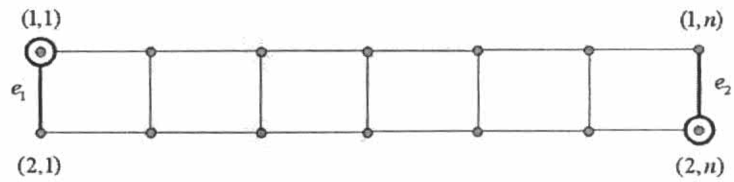

Figure 4.4: Case 2 in Lemma 4.1.3

Case 3 : $e_1 \in \{(i,j)(i,j+k) \mid i = 1,2 \text{ and } j = 1,2,...,n-k\}$ and $e_2 \in \{(1,1)(2,1), (1,n)(2,n)\}$.

Case 3.1 : $e_1 = (i,j)(i,j+k)$ for some $i \in \{1,2\}$ and $j \in \{1,2,...,n-k\}$, and $e_2 = (1,1)(2,1)$.

Then for $i' \in \{1,2\} \backslash \{i\}$, $d_{P_2 \times P_n}((i,j+k),(i',1)) = d_{P_2 \times P_n}((i,j+k),(i,1)) + d_{P_2 \times P_n}((1,1),(2,1)) \geq k+1 > k$. Thus $(i,j+k)$ is not adjacent to $(i',1)$ in $(P_2 \times P_n)^k$.

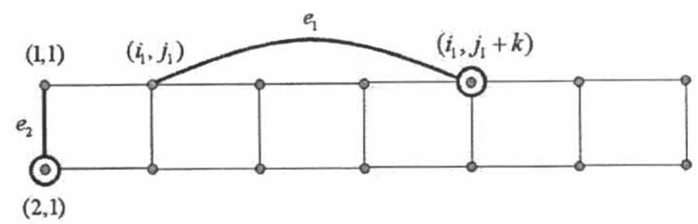

Figure 4.5: Case 3.1 in Lemma 4.1.3

Case 3.2 : $e_1 = (i,j)(i,j+k)$ for some $i \in \{1,2\}$ and $j \in \{1,2,...,n-k\}$, and $e_2 = (1,n)(2,n)$.

Similar to case 3.1, for $i' \in \{1,2\}\backslash\{i\}$, $d_{P_2 \times P_n}((i,j),(i',n)) > k$. Thus (i,j) is not adjacent to (i',n) in $(P_2 \times P_n)^k$.

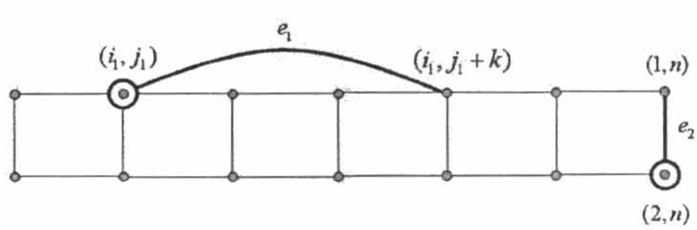

Figure 4.6: Case 3.2 in Lemma 4.1.3

From all cases, for $e_1, e_2 \in I_k$ where $e_1 \neq e_2$, we have that e_1 and e_2 are clique-independent edges of $(P_2 \times P_n)^k$. Thus I_k is a clique-independent set of $(P_2 \times P_n)^k$. Hence $cc((P_2 \times P_n)^k) \geq |I_k| = 2(n-k)+2$. $\square$

Lemma 4.1.4. *For $n, k \in \boldsymbol{N}$ where $2 \leq k < n$,*

$$cc((P_2 \times P_n)^k) \leq 2(n-k)+2.$$

Proof. Let $V(P_2 \times P_n) = \{(i,j) \mid i = 1,2 \text{ and } j = 1,2,...,n\}$ and $E(P_2 \times P_n) = \{(i,j)(i,j+1) \mid i = 1,2 \text{ and } j = 1,2,...,n-1\} \cup \{(1,j)(2,j) \mid j = 1,2,...,n\}$.

Consider subsets of the vertex set of $(P_2 \times P_n)^k$. For $i = 1, 2, .., n-k$, let

$$U_i = \{ (1,i), (1,i+1), (1,i+2), ..., (1,i+k-1), (1,i+k), (2,i+1), (2,i+2), ..., (2,i+k-1) \}.$$

Note that the distance between two vertices of U_i in $P_2 \times P_n$ is at most k. Thus $A_i := (P_2 \times P_n)^k[U_i]$, an induced subgraph of $(P_2 \times P_n)^k$, is a clique in $(P_2 \times P_n)^k$.

For $i = 1, 2, .., n-k$, let

$$V_i = \{ (1,i+1), (1,i+2), ..., (1,i+k-1), (2,i), (2,i+1), (2,i+2), ..., (2,i+k-1), (2,i+k) \}.$$

Note that the distance between two vertices of V_i in $P_2 \times P_n$ is at most k. Thus $B_i := (P_2 \times P_n)^k[V_i]$, an induced subgraph of $(P_2 \times P_n)^k$, is a clique in $(P_2 \times P_n)^k$.

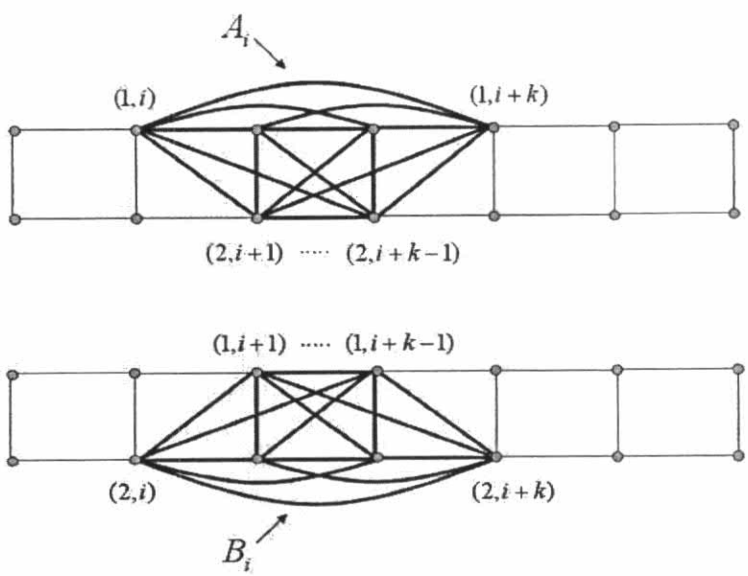

Figure 4.7: A_i and B_i in Lemma 4.1.4

Let $\mathcal{C} = \{A_i \mid i = 1, 2, ..., n-k\} \cup \{B_i \mid i = 1, 2, ..., n-k\} \cup \{(1,1)(2,1), (1,n)(2,n)\}$. Then $|\mathcal{C}| = 2(n-k)+2$. We claim that $\mathcal{C}$ is a clique covering of $(P_2 \times P_n)^k$. Let $e \in E((P_2 \times P_n)^k)$. Then $e = (i_1, j_1)(i_2, j_2)$ for some $i_1, i_2 \in \{1,2\}$ and $j_1, j_2 \in \{1, 2, .., n\}$.

Case 1 : $e = (1, j)(2, j)$ where $j \in \{1, 2, .., n\}$.

If $j = 1$ or n, then $e \in \{(1,1)(2,1), (1,n)(2,n)\}$.

If $2 \leq j \leq n - k$, then $e \in E(A_{j-1})$.

If $n - k + 1 \leq j \leq n - 1$, then $e \in E(A_{n-k})$.

Case 2 : $e = (i_1, j_1)(i_2, j_2)$ where $j_1 < j_2$.

Since $e \in E((P_2 \times P_n)^k)$, the distance between (i_1, j_1) and (i_2, j_2) in $P_2 \times P_n$ is at most k.

Case 2.1 : $i_1 = 1$.

If $1 \leq j_1 \leq n - k$, then $e \in E(A_{j_1})$.

If $n - k < j_1 \leq n - 1$ and $j_2 \neq n$, then $e \in E(A_{n-k})$.

If $n - k < j_1 \leq n - 1$ and $j_2 = n$, then $e \in E(B_{n-k})$.

Case 2.2 : $i_1 = 2$.

If $1 \leq j_1 \leq n - k$, then $e \in E(B_{j_1})$.

If $n - k < j_1 \leq n - 1$ and $j_2 \neq n$, then $e \in E(B_{n-k})$.

If $n - k < j_1 \leq n - 1$ and $j_2 = n$, then $e \in E(A_{n-k})$.

By all cases, we can conclude that $\mathcal{C}$ is a clique covering of $(P_2 \times P_n)^k$. Hence $cc((P_2 \times P_n)^k) \leq |\mathcal{C}| = 2(n - k) + 2$. □

In the next theorem, we conclude the values of the clique covering numbers of the k-power of ladders.

Theorem 4.1.5. *For $n, k \in \boldsymbol{N}$,*

$$cc((P_2 \times P_n)^k) = \begin{cases} 1 & \text{if } k \geq n, \\ 2(n-k)+2 & \text{if } 2 \leq k < n, \\ 3n-2 & \text{if } k = 1. \end{cases}$$

Proof. **Case 1** : $k \geq n$.

Since $diam(P_2 \times P_n) = n$, we have that $(P_2 \times P_n)^k$ is a complete graph. Hence $cc((P_2 \times P_n)^k) = 1$.

Case 2 : $2 \leq k < n$.

By Lemma 4.1.3 and Lemma 4.1.4, $cc((P_2 \times P_n)^k) = 2(n-k)+2$.

Case 3 : $k = 1$.

Since $P_2 \times P_n$ is K_3-free, $cc(P_2 \times P_n) = |E(P_2 \times P_n)| = 3n - 2$. □

4.2 Clique Partitions of the Square of Ladders

We give the number of edges of the square of ladders in the next proposition.

Proposition 4.2.1. *For $n \in \boldsymbol{N}$ where $n \geq 2$,*

$$|E((P_2 \times P_n)^2)| = 7n - 8.$$

Proof. Let $V(P_2 \times P_n) = \{(i,j) \mid i = 1,2 \text{ and } j = 1,2,...,n\}$ and $E(P_2 \times P_n) = \{(i,j)(i,j+1) \mid i = 1,2 \text{ and } j = 1,2,...,n-1\} \cup \{(1,j)(2,j) \mid j = 1,2,...,n\}$. Note that $|V((P_2 \times P_n)^2)| = |V((P_2 \times P_n))| = 2n$.

For $v \in V((P_2 \times P_n)^2)$,

$$d_{(P_2 \times P_n)^2}(v) = \begin{cases} 4, & \text{if} \quad v \in \{(1,1),(2,1),(1,n),(2,n)\}, \\ 6, & \text{if} \quad v \in \{(1,2),(2,2),(1,n-1),(2,n-1)\}, \\ 7, & \text{otherwise.} \end{cases}$$

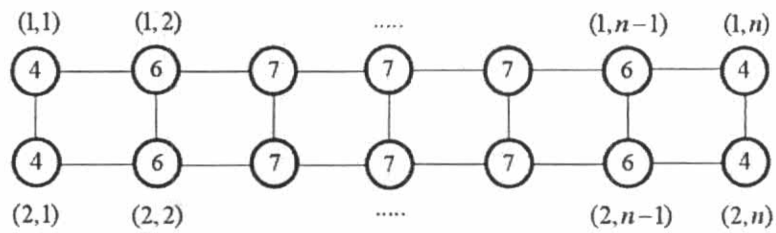

Figure 4.8: Degrees of vertices of $(P_2 \times P_n)^2$

Thus

$$\begin{aligned} \sum_{v \in V((P_2 \times P_n)^2)} d(v) &= 4(4) + 6(4) + 7(2n - 4 - 4) \\ &= 16 + 24 + 14n - 56 \\ &= 14n - 16. \end{aligned}$$

Hence

$$|E((P_2 \times P_n)^2)| = \frac{\sum d(v)}{2} = \frac{14n - 16}{2} = 7n - 8.$$

□

In Theorem 4.2.2, we show bounds of the clique partition numbers of the square of ladders.

Theorem 4.2.2. *For $n \in \boldsymbol{N}$.*

(i) If $n = 1$ or 2, then $cp((P_2 \times P_n)^2) = 1$.

(ii) If $n = 2r+1$ where $r \geq 1$, then

$$2n - 2 \leq cp((P_2 \times P_n)^2) \leq \frac{5n-3}{2}.$$

(iii) If $n = 2r$ where $r \geq 2$, then

$$2n - 2 \leq cp((P_2 \times P_n)^2) \leq \frac{5n-4}{2}.$$

Proof. Let $V(P_2 \times P_n) = \{(i,j) \mid i = 1,2 \text{ and } j = 1,2,...,n\}$ and $E(P_2 \times P_n) = \{(i,j)(i,j+1) \mid i = 1,2 \text{ and } j = 1,2,...,n-1\} \cup \{(1,j)(2,j) \mid j = 1,2,...,n\}$.

(i) Let $n = 1$ or 2.

Then $(P_2 \times P_n)^2$ is a complete graph. Hence $cp((P_2 \times P_n)^2) = 1$.

(ii) Let $n = 2r+1$ where $r \geq 1$.

For $i = 1,3,...,2r-1$, let $A_i = (P_2 \times P_n)^2[\{(1,i),(1,i+1),(1,i+2),(2,i+1)\}]$. Then A_i is a copy of K_4 and $|E(A_i)| = 6$.

For $i = 1,3,...,2r-1$, let $B_i = (P_2 \times P_n)^2[\{(2,i),(1,i+1),(2,i+2)\}]$. Then B_i is a copy of K_3 and $|E(B_i)| = 3$.

For $i = 2,4,...,2r-2$, let $C_i = (P_2 \times P_n)^2[\{(2,i),(2,i+1),(2,i+2)\}]$. Then C_i is a copy of K_3 and $|E(C_i)| = 3$.

By Proposition 4.2.1, $|E((P_2 \times P_n)^2)| = 7n - 8 = 7(2r+1) - 8 = 14r - 1$.

Let $H = (P_2 \times P_n)^2 \setminus [\ (A_1 + A_3 + ... + A_{2r-1}) \ + \ (B_1 + B_3 + ... + B_{2r-1})$
$+ \ (C_2 + C_4 + ... + C_{2r-2})\ ]$.

Then $|E(H)| = (14r-1) - [\ (6r) + (3r) + 3(r-1)\ ] = 2r + 2$.

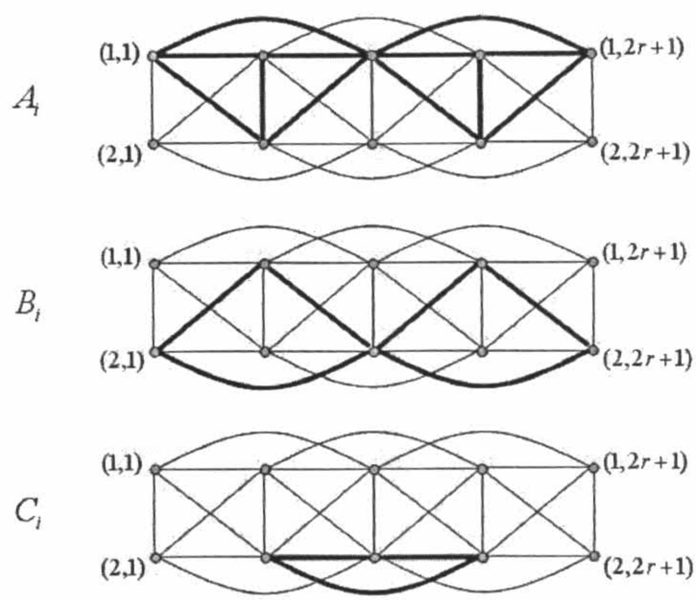

Figure 4.9: A_i, B_i and C_i in Theorem 4.2.2 (ii)

We have that $\{A_1, A_3, ..., A_{2r-1}\} \cup \{B_1, B_3, ..., B_{2r-1}\} \cup \{C_2, C_4, ..., C_{2r-2}\} \cup E(H)$ forms a clique partition $\mathcal{P}$ of $(P_2 \times P_n)^2$ such that

$$|\mathcal{P}| = r + r + (r-1) + (2r+2) = 5r + 1.$$

Since $r = \frac{n-1}{2}$, $|\mathcal{P}| = 5(\frac{n-1}{2}) + 1 = \frac{5n-3}{2}$. Thus $cp((P_2 \times P_n)^2) \leq \frac{5n-3}{2}$.

By Theorem 4.1.5, $cc((P_2 \times P_n)^2) = 2(n-2) + 2 = 2n - 2$. We have $cp((P_2 \times P_n)^2) \geq 2n - 2$. Therefore, $2n - 2 \leq cp((P_2 \times P_n)^2) \leq \frac{5n-3}{2}$.

(iii) Let $n = 2r$ where $r \geq 2$.

For $i = 1, 3, ..., 2r-3$, let $A_i = (P_2 \times P_n)^2[\{(1,i), (1,i+1), (1,i+2), (2,i+1)\}]$. Then A_i is a copy of K_4 and $|E(A_i)| = 6$.

For $i = 1, 3, ..., 2r-3$, let $B_i = (P_2 \times P_n)^2[\{(2,i), (1,i+1), (2,i+2)\}]$. Then B_i is a copy of K_3 and $|E(B_i)| = 3$.

For $i = 2, 4, ..., 2r-4$, let $C_i = (P_2 \times P_n)^2[\{(2,i), (2,i+1), (2,i+2)\}]$. Then C_i is a copy of K_3 and $|E(C_i)| = 3$.

Let $D = (P_2 \times P_n)^2[\{(1, 2r-1), (1, 2r), (2, 2r-1), (2, 2r)\}]$.

Then D is a copy of K_4 and $|E(D)| = 6$.

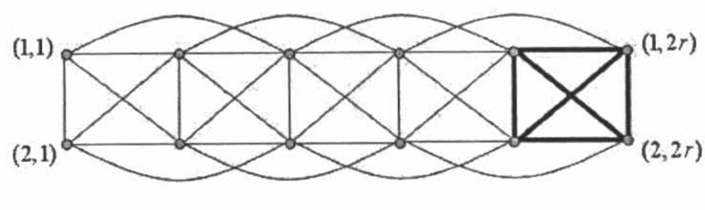

Figure 4.10: D in Theorem 4.2.2 (iii)

By Proposition 4.2.1, $|E((P_2 \times P_n)^2)| = 7n - 8 = 7(2r) - 8 = 14r - 8$.

Let $H = (P_2 \times P_n)^2 \setminus [\ (A_1 + A_3 + ... + A_{2r-3})\ +\ (B_1 + B_3 + ... + B_{2r-3})$
$+\ (C_2 + C_4 + ... + C_{2r-4})\ +\ D\ ]$.

Then $|E(H)| = (14r - 8) - 6(r-1) - 3(r-1) - 3(r-2) - 6 = 2r + 1$.

We have that $\{A_1, A_3, ..., A_{2r-3}\} \cup \{B_1, B_3, ..., B_{2r-3}\} \cup \{C_2, C_4, ..., C_{2r-4}\}$ $\cup\ \{D\} \cup E(H)$ forms a clique partition $\mathcal{P}$ of $(P_2 \times P_n)^2$ such that

$$|\mathcal{P}| = (r-1) + (r-1) + (r-2) + 1 + (2r+1) = 5r - 2.$$

Since $r = \frac{n}{2}$, $|\mathcal{P}| = 5(\frac{n}{2}) - 2 = \frac{5n-4}{2}$. Thus $cp((P_2 \times P_n)^2) \leq \frac{5n-4}{2}$.

By Theorem 4.1.5, $cc((P_2 \times P_n)^2) = 2(n-2) + 2 = 2n - 2$. Hence, $cp((P_2 \times P_n)^2) \geq 2n - 2$. Therefore, $2n - 2 \leq cp((P_2 \times P_n)^2) \leq \frac{5n-4}{2}$. □

In the next section, we investigate values and bounds of the clique covering numbers of the k-power of grids.

4.3 Clique Coverings of the k-power of Grids

In this chapter, we use $V(P_m \times P_n) = \{(i, j) \mid i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}$ and $E(P_m \times P_n) = \{(i, j)(i, j+1) \mid i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n-1\} \cup \{(i, j)(i+1, j) \mid i = 1, 2, ..., m-1 \text{ and } j = 1, 2, ..., n\}$.

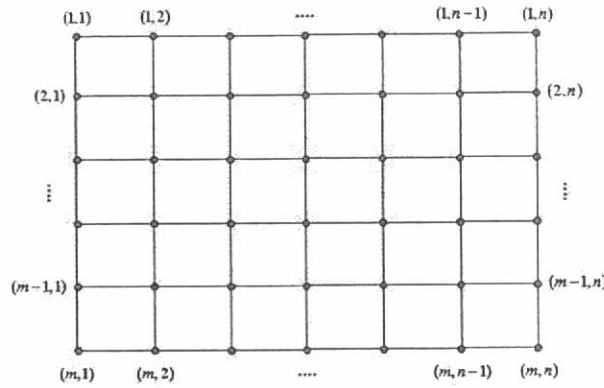

Figure 4.11: $P_m \times P_n$

Remark 4.3.1.

1. If $m = 1$ or $n = 1$, then $P_m \times P_n$ is a path. It is done by Theorem 2.1.5.
2. If $m = 2$ or $n = 2$, then $P_m \times P_n$ is a ladder. It is done by Theorem 4.1.5.

Remark 4.3.2. For $m, n, k \in \mathbf{N}$.

1. If $k = 1$, then $cc(P_m \times P_n) = |E(P_m \times P_n)| = m(n-1) + n(m-1)$ because $P_m \times P_n$ is K_3-free.

2. If $k \geq m + n - 2$, then $(P_m \times P_n)^k$ is a complete graph because $diam(P_m \times P_n) = (m-1) + (n-1) = m + n - 2$. Hence $cc((P_m \times P_n)^k) = 1$.

In Lemma 4.3.3 and Lemma 4.3.4, we give lower bounds of the clique covering numbers of the k-power of grids where $k < min\{m, n\}$.

Lemma 4.3.3. *For $m, n, k \in \boldsymbol{N}$ where $k < min\{m, n\}$ and k is odd,*

$$cc((P_m \times P_n)^k) \geq 2mn - k(m+n).$$

Proof. Let $A_k = \{(i,j)(i,j+k) \mid i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n-k\}$ and $B_k = \{(i,j)(i+k,j) \mid i = 1, 2, ..., m-k \text{ and } j = 1, 2, ..., n\}$. Let $I_k = A_k \cup B_k$. Then I_k is a subset of $E((P_m \times P_n)^k)$ and $|I_k| = m(n-k) + n(m-k) = 2mn - k(m+n)$. Next, we show that I_k is a clique-independent set of $(P_m \times P_n)^k$. Let $e_1, e_2 \in I_k$ where $e_1 \neq e_2$.

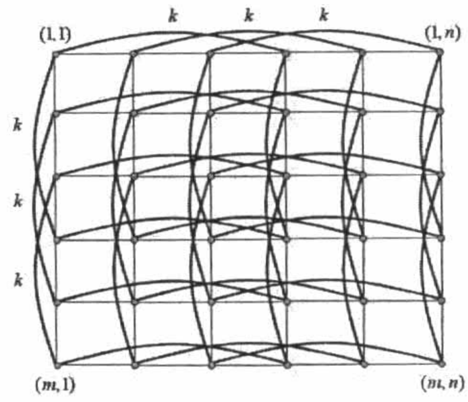

Figure 4.12: I_k in Lemma 4.3.3

Case 1 : $e_1, e_2 \in A_k$.

Then $e_1 = (i_1, j_1)(i_1, j_1 + k)$ and $e_2 = (i_2, j_2)(i_2, j_2 + k)$ for some $i_1, i_2 \in \{1, 2, ..., m\}$ and $j_1, j_2 \in \{1, 2, ..., n - k\}$. WLOG, assume $i_1 \leq i_2$.

Case 1.1 : $j_1 \leq j_2$.

We have $d_{P_m \times P_n}((i_1, j_1), (i_2, j_2 + k)) \geq k + 1 > k$. Thus (i_1, j_1) is not adjacent to $(i_2, j_2 + k)$ in $(P_m \times P_n)^k$. Hence e_1 and e_2 are clique-independent edges of $(P_m \times P_n)^k$.

Case 1.2 : $j_1 > j_2$.

We have $d_{P_m \times P_n}((i_1, j_1 + k), (i_2, j_2)) \geq k + 1 > k$. Thus $(i_1, j_1 + k)$ is not adjacent to (i_2, j_2) in $(P_m \times P_n)^k$. Hence e_1 and e_2 are clique-independent edges of $(P_m \times P_n)^k$.

Case 2 : $e_1, e_2 \in B_k$.

Then $e_1 = (i_1, j_1)(i_1 + k, j_1)$ and $e_2 = (i_2, j_2)(i_2 + k, j_2)$ for some $i_1, i_2 \in \{1, 2, ..., m - k\}$ and $j_1, j_2 \in \{1, 2, ..., n\}$. WLOG, assume $i_1 \leq i_2$. We have

$d_{P_m \times P_n}((i_1, j_1), (i_2+k, j_2)) \geq k+1 > k$. Thus (i_1, j_1) is not adjacent to (i_2+k, j_2) in $(P_m \times P_n)^k$. Hence e_1 and e_2 are clique-independent edges of $(P_m \times P_n)^k$.

Case 3 : $e_1 \in A_k$ and $e_2 \in B_k$.

Then $e_1 = (i_1, j_1)(i_1, j_1+k)$ for some $i_1 \in \{1, 2, ..., m\}$ and $j_1 \in \{1, 2, ..., n-k\}$ and $e_2 = (i_2, j_2)(i_2+k, j_2)$ for some $i_2 \in \{1, 2, ..., m-k\}$ and $j_2 \in \{1, 2, ..., n\}$.

Case 3.1 : $i_1 \leq i_2 + (\frac{k-1}{2})$.

If $j_1 < j_2 - (\frac{k-1}{2})$, then $d_{P_m \times P_n}((i_1, j_1), (i_2+k, j_2))$

$= d_{P_m \times P_n}((i_1, j_1), (i_1, j_2)) + d_{P_m \times P_n}((i_1, j_2), (i_2+k, j_2))$

$= (j_2 - j_1) + (i_2 + k - i_1) > (\frac{k-1}{2}) + (k - (\frac{k-1}{2})) = k$.

Thus (i_1, j_1) is not adjacent to (i_2+k, j_2) in $(P_m \times P_n)^k$.

If $j_1 \geq j_2 - (\frac{k-1}{2})$, then $d_{P_m \times P_n}((i_1, j_1+k), (i_2+k, j_2))$

$= d_{P_m \times P_n}((i_1, j_1+k), (i_1, j_2)) + d_{P_m \times P_n}((i_1, j_2), (i_2+k, j_2))$

$= (j_1 + k - j_2) + (i_2 + k - i_1) \geq (k - (\frac{k-1}{2})) + (k - (\frac{k-1}{2})) = k+1 > k$.

Thus (i_1, j_1+k) is not adjacent to (i_2+k, j_2) in $(P_m \times P_n)^k$.

Hence e_1 and e_2 are clique-independent edges of $(P_m \times P_n)^k$.

Case 3.2 : $i_1 > i_2 + (\frac{k-1}{2})$.

Similar to case 3.1, e_1 and e_2 are clique-independent edges of $(P_m \times P_n)^k$.

From all cases, we can conclude that I_k is a clique-independent set of $(P_m \times P_n)^k$. Hence, $cc((P_m \times P_n)^k) \geq |I_k| = 2mn - k(m+n)$. □

Next, we show lower bounds of the clique covering numbers of the k-power of grids where $k < min\{m, n\}$ and k is even.

Lemma 4.3.4. *For $m, n, k \in \boldsymbol{N}$ where $k < min\{m, n\}$ and k is even,*

$$cc((P_m \times P_n)^k) \geq mn - k^2.$$

Proof. Let $A_k = \{(i,j)(i,j+k) \mid i = 1,2,...,m \text{ and } j = 1,2,...,n-k\}$ and $B_k = \{(i,j)(i+k,j) \mid i = 1,2,...,m-k \text{ and } j = 1,2,...,\frac{k}{2}, n-\frac{k}{2}+1, n-\frac{k}{2}+2,...,n\}$. Let $I_k = A_k \cup B_k$. Then I_k is a subset of $E((P_m \times P_n)^k)$ and $|I_k| = m(n-k) + k(m-k) = mn - k^2$. We claim that I_k is a clique-independent set of $(P_m \times P_n)^k$. Let $e_1, e_2 \in I_k$ where $e_1 \neq e_2$.

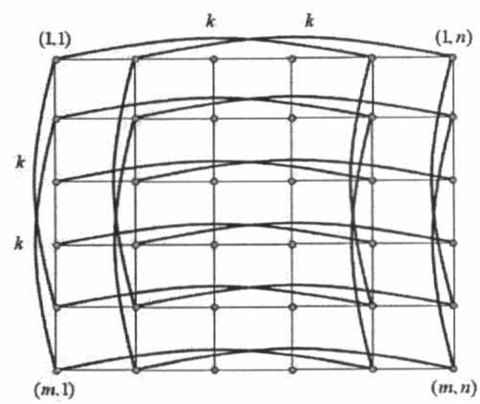

Figure 4.13: I_k in Lemma 4.3.4

Case 1 : $e_1, e_2 \in A_k$.

Similar to proof of case 1 in Lemma 4.3.3.

Case 2 : $e_1, e_2 \in B_k$.

Similar to proof of case 2 in Lemma 4.3.3.

Case 3 : $e_1 \in A_k$ and $e_2 \in B_k$.

Then $e_1 = (i_1, j_1)(i_1, j_1+k)$ for some $i_1 \in \{1,2,...,m\}$ and $j_1 \in \{1,2,...,n-k\}$ and $e_2 = (i_2, j_2)(i_2+k, j_2)$ for some $i_2 \in \{1,2,...,m-k\}$ and $j_2 \in \{1,2,...,\frac{k}{2}, n-\frac{k}{2}+1, n-\frac{k}{2}+2,...,n\}$.

Case 3.1 : $i_1 \le i_2 + \frac{k}{2}$.

If $j_2 \in \{1, 2, ..., \frac{k}{2}\}$, then $d_{P_m \times P_n}((i_1, j_1 + k), (i_2 + k, j_2))$

$= d_{P_m \times P_n}((i_1, j_1 + k), (i_1, j_2)) + d_{P_m \times P_n}((i_1, j_2), (i_2 + k, j_2))$

$= (j_1 + k - j_2) + (i_2 + k - i_1) \ge (1 + k - \frac{k}{2}) + (k - \frac{k}{2}) = k + 1 > k.$

Thus $(i_1, j_1 + k)$ is not adjacent to $(i_2 + k, j_2)$ in $(P_m \times P_n)^k$.

If $j_2 \in \{n - \frac{k}{2} + 1, n - \frac{k}{2} + 2, ..., n\}$, then $d_{P_m \times P_n}((i_1, j_1), (i_2 + k, j_2))$

$= d_{P_m \times P_n}((i_1, j_1), (i_1, j_2)) + d_{P_m \times P_n}((i_1, j_2), (i_2 + k, j_2))$

$= (j_2 - j_1) + (i_2 + k - i_1) \ge ((n - \frac{k}{2} + 1) - (n - k)) + (k - \frac{k}{2}) = k + 1 > k.$

Thus (i_1, j_1) is not adjacent to $(i_2 + k, j_2)$ in $(P_m \times P_n)^k$.

Hence, e_1 and e_2 are clique-independent edges of $(P_m \times P_n)^k$.

Case 3.2 : $i_1 > i_2 + \frac{k}{2}$.

Similar to case 3.1, we have that e_1 and e_2 are clique-independent edges of $(P_m \times P_n)^k$.

By all cases, we can conclude that I_k is a clique-independent set of $(P_m \times P_n)^k$. Hence $cc((P_m \times P_n)^k) \ge |I_k| = mn - k^2$. □

In the next theorem, we give the values of the clique covering numbers of the square of grids.

Theorem 4.3.5. *For $m, n \in \boldsymbol{N}$ where $m, n > 2$,*

$$cc((P_m \times P_n)^2) = mn - 4.$$

Proof. Recall that $V(P_m \times P_n) = \{(i, j) \mid i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}$ and $E(P_m \times P_n) = \{(i, j)(i, j + 1) \mid i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n - 1\} \cup \{(i, j)(i + 1, j) \mid i = 1, 2, ..., m - 1 \text{ and } j = 1, 2, ..., n\}$.

For $j = 1, 2, .., n-2$, let

$A_j = (P_m \times P_n)^2[\{(1,j), (1,j+1), (1,j+2), (2,j+1)\}]$ and

$B_j = (P_m \times P_n)^2[\{(m,j), (m,j+1), (m,j+2), (m-1,j+1)\}]$.

For $i = 1, 2, .., m-2$, let

$C_i = (P_m \times P_n)^2[\{(m,j), (m,j+1), (m,j+2), (m-1,j+1)\}]$ and

$D_i = (P_m \times P_n)^2[\{(m,j), (m,j+1), (m,j+2), (m-1,j+1)\}]$.

For $i = 2, 3, .., m-1$ and $j = 1, 2, .., n-2$, let

$F_{i,j} = (P_m \times P_n)^2[\{(i,j), (i,j+1), (i,j+2), (i-1,j+1), (i+1,j+1)\}]$.

Note that A_j, B_j, C_i, D_i and $F_{i,j}$ defined above are cliques in $(P_m \times P_n)^2$. Let $\mathcal{C} = \{A_j, B_j \mid j = 1, 2, ..., n-2\} \cup \{C_i, D_i \mid i = 1, 2, ..., m-2\} \cup \{F_{i,j} \mid i = 2, 3, .., m-1 \text{ and } j = 1, 2, .., n-2\}$. Then

$$
\begin{aligned}
|\mathcal{C}| &= 2(n-2) + 2(m-2) + (m-2)(n-2) \\
&= 2n - 4 + 2m - 4 + mn - 2m - 2n + 4 \\
&= mn - 4.
\end{aligned}
$$

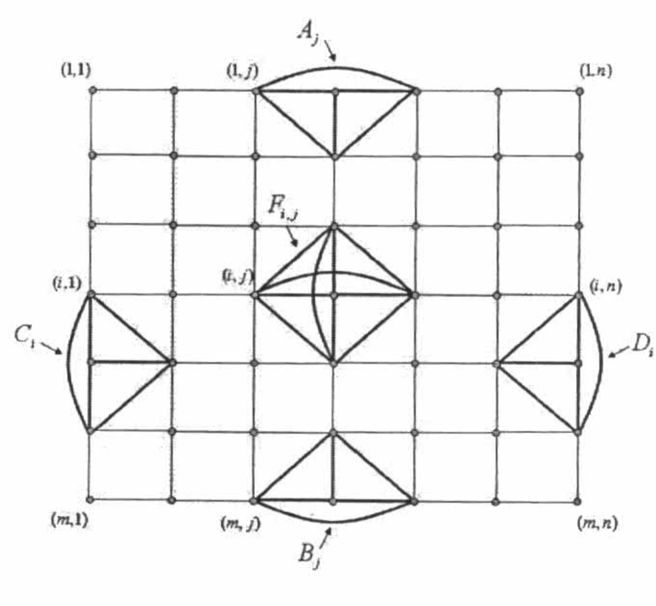

Figure 4.14: $\mathcal{C}$ in Theorem 4.3.5

We claim that $\mathcal{C}$ is a clique covering of $(P_m \times P_n)^2$. If $e \in E(P_m \times P_n)$. It is easy to see that e is covered by a clique in $\mathcal{C}$. We consider only edges in $(P_m \times P_n)^2 \backslash (P_m \times P_n)$. Let $e \in E((P_m \times P_n)^2 \backslash (P_m \times P_n))$. Then $e = (i_1, j_1)(i_2, j_2)$ for some $i_1, i_2 \in \{1, 2, ..., m\}$ and $j_1, j_2 \in \{1, 2, .., n\}$. WLOG, assume $i_1 \leq i_2$. Hence the distance between (i_1, j_1) and (i_2, j_2) in $P_m \times P_n$ is 2.

Case 1 : $i_1 = i_2$.

If $i_1 = 1$, then $e \in E(A_{j_1})$.

If $2 \leq i_1 \leq m-1$, then $e \in E(F_{i_1, j_1})$.

If $i_1 = m$, then $e \in E(B_{j_1})$.

Case 2 : $i_1 < i_2$.

Case 2.1 : $j_1 = j_2$.

If $j_1 = 1$, then $e \in E(C_{j_1})$.

If $2 \leq j_1 \leq n-1$, then $e \in E(F_{i_1+1, j_1-1})$.

If $j_1 = n$, then $e \in E(D_{j_1})$.

Case 2.2 : $j_1 > j_2$.

If $1 \leq i_1 \leq m-2$ and $2 \leq j_1 \leq n-1$, then $e \in E(F_{i_2, j_2})$.

If $1 \leq i_1 \leq m-2$ and $j_1 = n$, then $e \in E(D_{i_1})$.

If $i_1 = m-1$ and $2 \leq j_1 \leq n-1$, then $e \in E(B_{j_2})$.

If $i_1 = m-1$ and $j_1 = n$, then $e \in E(F_{m-1, n-2})$.

Case 2.3 : $j_1 < j_2$.

If $1 \le i_1 \le m-2$ and $j_1 = 1$, then $e \in E(C_{i_1})$.

If $1 \le i_1 \le m-2$ and $2 \le j_1 \le n-1$, then $e \in E(F_{i_1+1,j_1-1})$.

If $i_1 = m-1$ and $j_1 = 1$, then $e \in E(F_{m-1,1})$.

If $i_1 = m-1$ and $2 \le j_1 \le n-1$, then $e \in E(B_{j_1-1})$.

From all cases, we can conclude that $\mathcal{C}$ is a clique covering of $(P_m \times P_n)^2$. Hence $cc((P_m \times P_n)^2) \le |\mathcal{C}| = mn - 4$. By Lemma 4.3.4, $cc((P_m \times P_n)^2) \ge mn - 4$. Therefore, $cc((P_m \times P_n)^2) = mn - 4$. □

4.4 Clique Partitions of the Square of Grids

Remark 4.4.1.

1. If $m = 1$ or $n = 1$, then $P_m \times P_n$ is a path. It is done by Theorem 2.1.6.

2. If $m = 2$ or $n = 2$, then $P_m \times P_n$ is a ladder. Bounds of clique partition numbers of $(P_m \times P_n)^2$ follow from Theorem 4.2.2.

The number of edges of the square of grids is shown in the next proposition.

Proposition 4.4.2. *For $m, n \in \boldsymbol{N}$ where $m, n \ge 2$,*

$$|E((P_m \times P_n)^2)| = 6mn - 5m - 5n + 2.$$

Proof. Recall that $V((P_m \times P_n)^2) = V(P_m \times P_n) = \{(i,j) \mid i = 1, 2, ..., m$ and $j = 1, 2, ..., n\}$. For $v \in V((P_m \times P_n)^2)$,

$$d_{(P_m \times P_n)^2}(v) = \begin{cases} 5, & \text{if } v \in \{(1,1),(1,n),(m,1),(m,n)\}, \\ 7, & \text{if } v \in \{(1,2),(1,n-1),(2,1),(2,n), \\ & \qquad (m-1,1),(m-1,n),(m,2),(m,n-1)\}, \\ 8, & \text{if } v \in \{(i,1),(i,n) \mid i = 3,4,...,m-2\} \cup \\ & \qquad \{(1,j),(m,j) \mid j = 3,4,...,n-2\}, \\ 10, & \text{if } v \in \{(2,2),(2,n-1),(m-1,2),(m-1,n-1)\}, \\ 11, & \text{if } v \in \{(i,2),(i,n-1) \mid i = 3,4,...,m-2\} \cup \\ & \qquad \{(2,j),(m-1,j) \mid j = 3,4,...,n-2\}, \\ 12, & \text{otherwise.} \end{cases}$$

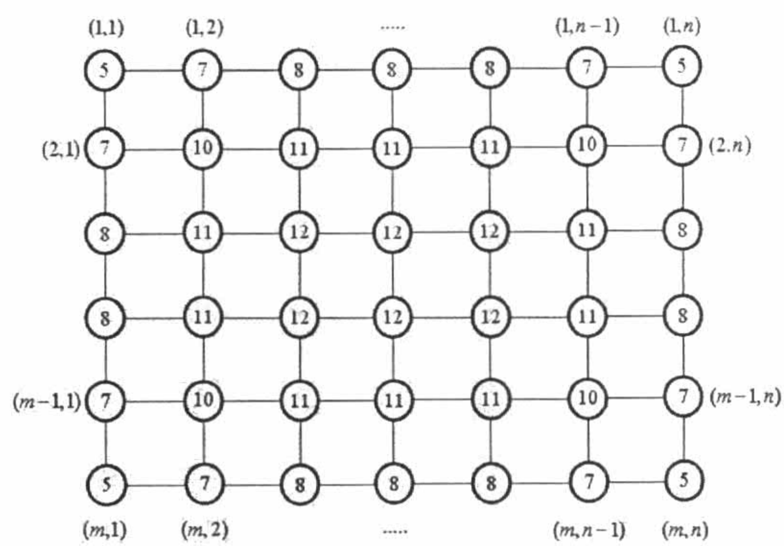

Figure 4.15: Degrees of vertices of $(P_m \times P_n)^2$

Thus

$$\sum_{v\in V((P_m\times P_n)^2)} d(v) = 5(4)+7(8)+8[2(m-4)+2(n-4)]+10(4)$$

$$+11[2(m-4)+2(n-4)]+12(m-4)(n-4)$$

$$= 20+56+40+19[2(m-4)+2(n-4)]+12(mn-4m-4m+16)$$

$$= 12mn-10m-10n+4.$$

Hence

$$|E((P_m\times P_n)^2)| = \frac{\sum d(v)}{2} = \frac{12mn-10m-10n+4}{2} = 6mn-5m-5n+2.$$

□

Next, we give bounds of the clique partition numbers of the square of grids.

Theorem 4.4.3. *For $m,n\in \boldsymbol{N}$ where $m,n>2$.*

(i) If $m=2r_1+1$ and $n=2r_2+1$ where $r_1,r_2\geq 1$, then

$$mn-4\leq cp((P_m\times P_n)^2)\leq 2mn-\frac{5m}{2}-2n+\frac{9}{2}.$$

(ii) If $m=2r_1+1$ and $n=2r_2$ where $r_1\geq 1$ and $r_2\geq 2$, then

$$mn-4\leq cp((P_m\times P_n)^2)\leq 2mn-\frac{3m}{2}-2n+\frac{5}{2}.$$

(iii) If $m=2r_1$ and $n=2r_2+1$ where $r_1\geq 2$ and $r_2\geq 1$, then

$$mn-4\leq cp((P_m\times P_n)^2)\leq 2mn-\frac{3n}{2}-2m+\frac{5}{2}.$$

(iv) If $m=2r_1$ and $n=2r_2$ where $r_1,r_2\geq 2$, then

$$mn-4\leq cp((P_m\times P_n)^2)\leq 2mn-\frac{3m}{2}-\frac{n}{2}-3.$$

Proof. Recall that $V(P_m \times P_n) = \{(i,j) \mid i = 1,2,...,m \text{ and } j = 1,2,...,n\}$ and $E(P_m \times P_n) = \{(i,j)(i,j+1) \mid i = 1,2,...,m \text{ and } j = 1,2,...,n-1\} \cup \{(i,j)(i+1,j) \mid i = 1,2,...,m-1 \text{ and } j = 1,2,...,n\}$.

(i) Let $m = 2r_1 + 1$ and $n = 2r_2 + 1$ where $r_1, r_2 \geq 1$.

For $i = 1,3,...,2r_1 - 1$ and $j = 2,4,...,2r_2$, let

$A_{i,j} = (P_m \times P_n)^2[\{(i,j),(i+1,j-1),(i+1,j),(i+1,j+1),(i+2,j)\}]$.

We have that $A_{i,j}$ is a copy of K_5 and $|E(A_{i,j})| = 10$.

For $i = 1,3,...,2r_1 - 1$ and $j = 3,5,...,2r_2 - 1$, let

$B_{i,j} = (P_m \times P_n)^2[\{(i,j),(i+1,j-1),(i+1,j+1),(i+2,j)\}]$.

We have that $B_{i,j}$ is a copy of K_4 and $|E(B_{i,j})| = 6$.

For $i = 1,3,...,2r_1 + 1$ and $j = 1,3,...,2r_2 - 1$, let

$C_{i,j} = (P_m \times P_n)^2[\{(i,j),(i,j+1),(i,j+2)\}]$.

We have that $C_{i,j}$ is a copy of K_3 and $|E(C_{i,j})| = 3$.

For $i = 1,3,...,2r_1 - 1$, let

$D1_i = (P_m \times P_n)^2[\{(i,1),(i+1,2),(i+2,1)\}]$ and

$D2_i = (P_m \times P_n)^2[\{(i,2r_2+1),(i+1,2r_2),(i+2,2r_2+1)\}]$.

We have that $D1_i$ and $D2_i$ are copies of K_3 and $|E(D1_i)| = |E(D2_i)| = 3$.

For $i = 2,4,...,2r_1 - 2$, let

$E1_i = (P_m \times P_n)^2[\{(i,1),(i+1,1),(i+2,1)\}]$ and

$E2_i = (P_m \times P_n)^2[\{(i,2r_2+1),(i+1,2r_2+1),(i+2,2r_2+1)\}]$.

We have that $E1_i$ and $E2_i$ are copies of K_3 and $|E(E1_i)| = |E(E2_i)| = 3$.

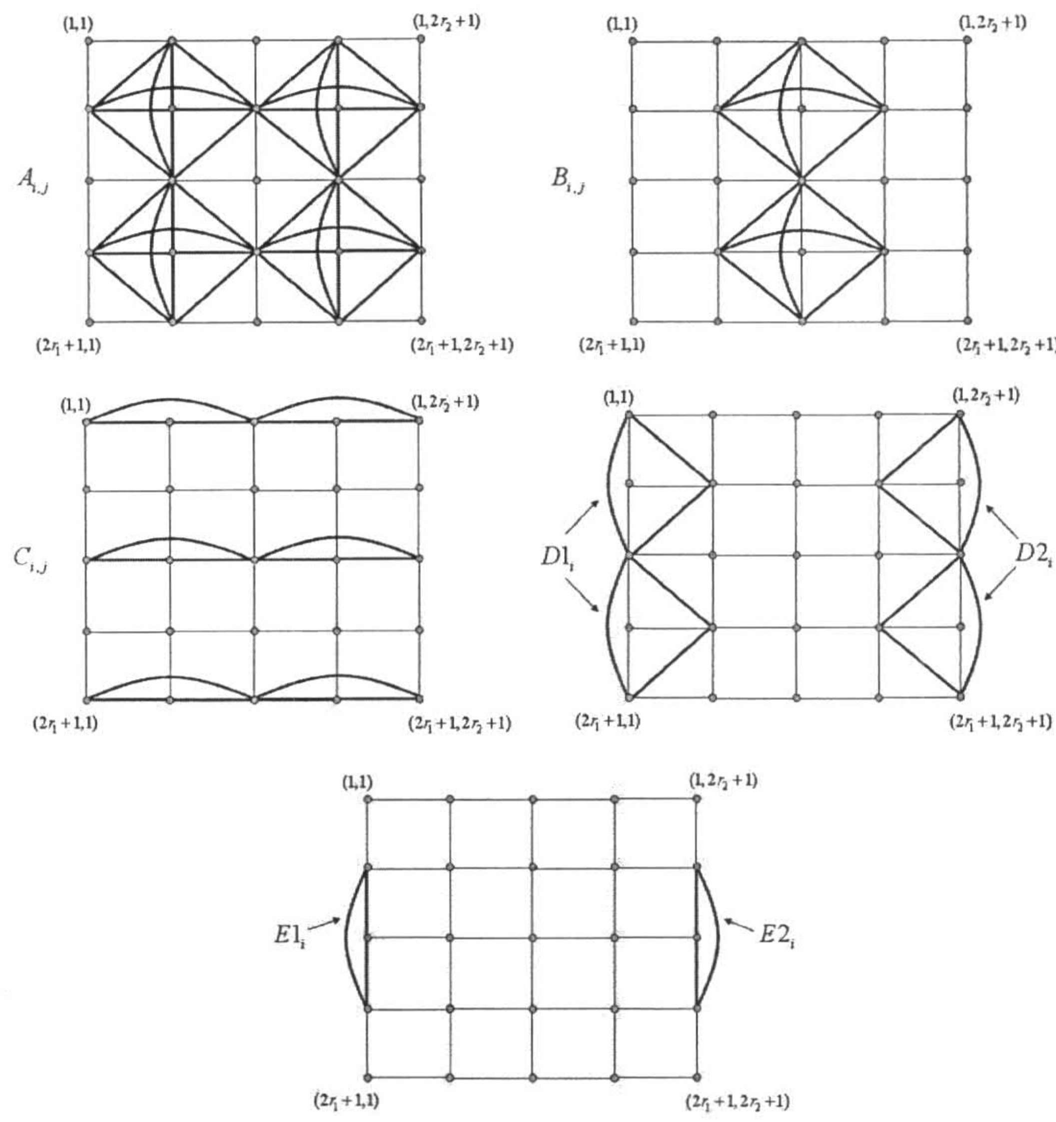

Figure 4.16: Cliques of $(P_m \times P_n)^2$ in Theorem 4.4.3 (i)

By Proposition 4.4.2,

$$\begin{aligned} |E((P_m \times P_n)^2)| &= 6mn - 5m - 5n + 2 \\ (\text{ since } m = 2r_1+1,\ n = 2r_2+1\ ) &= 6(2r_1+1)(2r_2+1) - 5(2r_1+1) - 5(2r_2+1) + 2 \\ &= 24r_1r_2 + 2r_1 + 2r_2 - 2. \end{aligned}$$

Let

$$\begin{aligned} H = (P_m \times P_n)^2 \setminus [\bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=2,4,\ldots,2r_2}} A_{i,j} + \bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=3,5,\ldots,2r_2-1}} B_{i,j} + \bigcup_{\substack{i=1,3,\ldots,2r_1+1 \\ j=1,3,\ldots,2r_2-1}} C_{i,j} \\ + \bigcup_{i=1,3,\ldots,2r_1-1} D1_i + \bigcup_{i=1,3,\ldots,2r_1-1} D2_i + \bigcup_{i=2,4,\ldots,2r_1-2} E1_i + \bigcup_{i=2,4,\ldots,2r_1-2} E2_i\ ]. \end{aligned}$$

Then

$$\begin{aligned} |E(H)| &= (24r_1r_2 + 2r_1 + 2r_2 - 2) - [\ (10r_1r_2) + 6(r_1)(r_2-1) + 3(r_1+1)(r_2) \\ &\quad + 2(3r_1) + 2(3(r_1-1))\ ] \\ &= (24r_1r_2 + 2r_1 + 2r_2 - 2) - (19r_1r_2 + 6r_1 + 3r_2 - 6) \\ &= 5r_1r_2 - 4r_1 - r_2 + 4. \end{aligned}$$

We have that

$$\bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=2,4,\ldots,2r_2}} \{A_{i,j}\} \cup \bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=3,5,\ldots,2r_2-1}} \{B_{i,j}\} \cup \bigcup_{\substack{i=1,3,\ldots,2r_1+1 \\ j=1,3,\ldots,2r_2-1}} \{C_{i,j}\} \cup \bigcup_{i=1,3,\ldots,2r_1-1} \{D1_i, D2_i\} \cup$$

$$\bigcup_{i=2,4,\ldots,2r_1-2} \{E1_i, E2_i\} \cup E(H)$$

forms a clique partition $\mathcal{P}$ of $(P_m \times P_n)^2$ such that

$$\begin{aligned} |\mathcal{P}| &= (r_1r_2) + (r_1)(r_2-1) + (r_1+1)(r_2) + (2r_1) + 2(r_1-1) + (5r_1r_2 - 4r_1 - r_2 + 4) \\ &= (3r_1r_2 + 3r_1 + r_2 - 2) + (5r_1r_2 - 4r_1 - r_2 + 4) \\ &= 8r_1r_2 - r_1 + 2. \end{aligned}$$

Since $r_1 = \frac{m-1}{2}$ and $r_2 = \frac{n-1}{2}$,

$$|\mathcal{P}| = 8(\frac{m-1}{2})(\frac{n-1}{2}) - (\frac{m-1}{2}) + 2 = 2mn - \frac{5m}{2} - 2n + \frac{9}{2}.$$

Thus $cp((P_m \times P_n)^2) \leq 2mn - \frac{5m}{2} - 2n + \frac{9}{2}$.

By Theorem 4.3.5, $cc((P_m \times P_n)^2) = mn - 4$. Hence $cp((P_m \times P_n)^2) \geq mn - 4$.

Therefore, $mn - 4 \leq cp((P_m \times P_n)^2) \leq 2mn - \frac{5m}{2} - 2n + \frac{9}{2}$.

(ii) Let $m = 2r_1 + 1$ and $n = 2r_2$ where $r_1 \geq 1$ and $r_2 \geq 2$.

For $i = 1, 3, ..., 2r_1 - 1$ and $j = 2, 4, ..., 2r_2 - 2$, let

$A_{i,j} = (P_m \times P_n)^2[\{(i,j), (i+1, j-1), (i+1, j), (i+1, j+1), (i+2, j)\}]$.

We have that $A_{i,j}$ is a copy of K_5 and $|E(A_{i,j})| = 10$.

For $i = 1, 3, ..., 2r_1 - 1$ and $j = 3, 5, ..., 2r_2 - 1$, let

$B_{i,j} = (P_m \times P_n)^2[\{(i,j), (i+1, j-1), (i+1, j+1), (i+2, j)\}]$.

We have that $B_{i,j}$ is a copy of K_4 and $|E(B_{i,j})| = 6$.

For $i = 1, 3, ..., 2r_1 + 1$ and $j = 1, 3, ..., 2r_2 - 3$, let

$C_{i,j} = (P_m \times P_n)^2[\{(i,j), (i, j+1), (i, j+2)\}]$.

We have that $C_{i,j}$ is a copy of K_3 and $|E(C_{i,j})| = 3$.

For $i = 1, 3, ..., 2r_1 - 1$, let

$D1_i = (P_m \times P_n)^2[\{(i,1), (i+1, 2), (i+2, 1)\}]$ and

$D2_i = (P_m \times P_n)^2[\{(i, 2r_2), (i+1, 2r_2 - 1), (i+1, 2r_2), (i+2, 2r_2)\}]$.

We have that $D1_i$ and $D2_i$ are copies of K_3 and K_4, respectively.

Thus $|E(D1_i)| = 3$ and $|E(D2_i)| = 6$.

For $i = 2, 4, ..., 2r_1 - 2$, let

$E_i = (P_m \times P_n)^2[\{(i,1), (i+1, 1), (i+2, 1)\}]$ and

We have that E_i is a copy of K_3 and $|E(E_i)| = 3$.

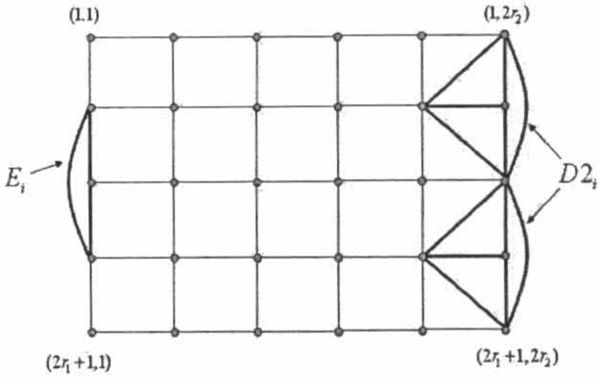

Figure 4.17: E_i and $D2_i$ in Theorem 4.4.3 (ii)

By Proposition 4.4.2,

$$\begin{aligned} |E((P_m \times P_n)^2)| &= 6mn - 5m - 5n + 2 \\ (\text{ since } m = 2r_1 + 1,\ n = 2r_2\ ) &= 6(2r_1+1)(2r_2) - 5(2r_1+1) - 5(2r_2) + 2 \\ &= 24r_1r_2 - 10r_1 + 2r_2 - 3. \end{aligned}$$

Let

$$\begin{aligned} H &= (P_m \times P_n)^2 \setminus [\bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=2,4,\ldots,2r_2-2}} A_{i,j} + \bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=3,5,\ldots,2r_2-1}} B_{i,j} + \bigcup_{\substack{i=1,3,\ldots,2r_1+1 \\ j=1,3,\ldots,2r_2-3}} C_{i,j} \\ &\quad + \bigcup_{i=1,3,\ldots,2r_1-1} D1_i + \bigcup_{i=1,3,\ldots,2r_1-1} D2_i + \bigcup_{i=2,4,\ldots,2r_1-2} E_i\ ]. \end{aligned}$$

Then

$$\begin{aligned} |E(H)| &= (24r_1r_2 - 10r_1 + 2r_2 - 3) - [\ 10(r_1)(r_2-1) + 6(r_1)(r_2-1) \\ &\quad + 3(r_1+1)(r_2-1) + (3r_1) + (6r_1) + 3(r_1-1)\ ] \\ &= (24r_1r_2 - 10r_1 + 2r_2 - 3) - (19r_1r_2 - 7r_1 + 3r_2 - 6) \\ &= 5r_1r_2 - 3r_1 - r_2 + 3. \end{aligned}$$

We have that

$$\bigcup_{\substack{i=1,3,\dots,2r_1-1\\ j=2,4,\dots,2r_2-2}} \{A_{i,j}\} \cup \bigcup_{\substack{i=1,3,\dots,2r_1-1\\ j=3,5,\dots,2r_2-1}} \{B_{i,j}\} \cup \bigcup_{\substack{i=1,3,\dots,2r_1+1\\ j=1,3,\dots,2r_2-3}} \{C_{i,j}\} \cup \bigcup_{i=1,3,\dots,2r_1-1} \{D1_i, D2_i\} \cup$$

$$\bigcup_{i=2,4,\dots,2r_1-2} \{E_i\} \cup E(H)$$

forms a clique partition $\mathcal{P}$ of $(P_m \times P_n)^2$ such that

$$\begin{aligned}|\mathcal{P}| &= (r_1)(r_2-1) + (r_1)(r_2-1) + (r_1+1)(r_2-1) + 2(r_1) \\ &\quad + (r_1-1) + (5r_1r_2 - 3r_1 - r_2 + 3) \\ &= (3r_1r_2 + r_2 - 2) + (5r_1r_2 - 3r_1 - r_2 + 3) \\ &= 8r_1r_2 - 3r_1 + 1.\end{aligned}$$

Since $r_1 = \frac{m-1}{2}$ and $r_2 = \frac{n}{2}$,

$$|\mathcal{P}| = 8(\frac{m-1}{2})(\frac{n}{2}) - 3(\frac{m-1}{2}) + 1 = 2mn - \frac{3m}{2} - 2n + \frac{5}{2}.$$

Thus $cp((P_m \times P_n)^2) \leq 2mn - \frac{3m}{2} - 2n + \frac{5}{2}$.

By Theorem 4.3.5, $cc((P_m \times P_n)^2) = mn - 4$. Hence $cp((P_m \times P_n)^2) \geq mn - 4$.

Therefore, $mn - 4 \leq cp((P_m \times P_n)^2) \leq 2mn - \frac{3m}{2} - 2n + \frac{5}{2}$.

(iii) Let $m = 2r_1$ and $n = 2r_2 + 1$ where $r_1 \geq 2$ and $r_2 \geq 1$.

Similar to case *(ii)*.

(iv) Let $m = 2r_1$ and $n = 2r_2$ where $r_1, r_2 \geq 2$.

For $i = 1, 3, \dots, 2r_1 - 3$ and $j = 2, 4, \dots, 2r_2 - 2$, let

$A_{i,j} = (P_m \times P_n)^2[\{(i,j), (i+1, j-1), (i+1, j), (i+1, j+1), (i+2, j)\}]$.

We have that $A_{i,j}$ is a copy of K_5 and $|E(A_{i,j})| = 10$.

For $i = 1, 3, \dots, 2r_1 - 3$ and $j = 3, 5, \dots, 2r_2 - 1$, let

$B_{i,j} = (P_m \times P_n)^2[\{(i,j), (i+1, j-1), (i+1, j+1), (i+2, j)\}]$.

We have that $B_{i,j}$ is a copy of K_4 and $|E(B_{i,j})| = 6$.

For $i = 1, 3, ..., 2r_1 - 1$ and $j = 1, 3, ..., 2r_2 - 3$, let

$C_{i,j} = (P_m \times P_n)^2[\{(i,j), (i,j+1), (i,j+2)\}]$.

We have that $C_{i,j}$ is a copy of K_3 and $|E(C_{i,j})| = 3$.

For $i = 1, 3, ..., 2r_1 - 3$, let

$D1_i = (P_m \times P_n)^2[\{(i,1), (i+1,2), (i+2,1)\}]$ and

$D2_i = (P_m \times P_n)^2[\{(i,2r_2), (i+1,2r_2-1), (i+1,2r_2), (i+2,2r_2)\}]$.

We have that $D1_i$ and $D2_i$ are copies of K_3 and K_4, respectively.

Thus $|E(D1_i)| = 3$ and $|E(D2_i)| = 6$.

For $i = 2, 4, ..., 2r_1 - 2$, let

$E_i = (P_m \times P_n)^2[\{(i,1), (i+1,1), (i+2,1)\}]$ and

We have that E_i is a copy of K_3 and $|E(E_i)| = 3$.

For $j = 1, 3, ..., 2r_2 - 3$, let

$F_j = (P_m \times P_n)^2[\{(2r_1, j), (2r_1 - 1, j+1), (2r_1, j+1), (2r_1, j+2)\}]$

We have that F_j is a copy of K_4 and $|E(F_j)| = 6$.

Let $Z = (P_m \times P_n)^2[\{(2r_1-1, 2r_2-1), (2r_1-1, 2r_2), (2r_1, 2r_2-1), (2r_1, 2r_2)\}]$.

We have that Z is a copy of K_4 and $|E(Z)| = 6$.

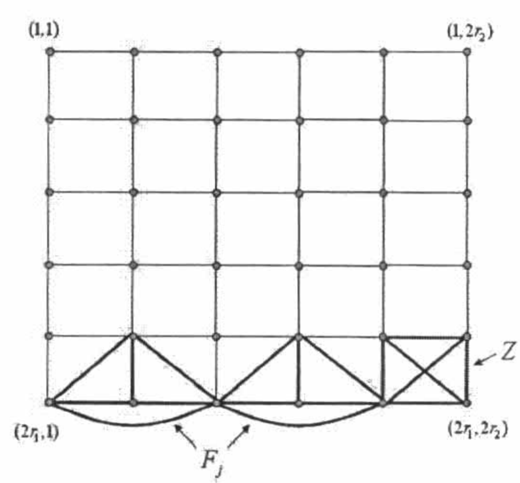

Figure 4.18: F_j and Z in Theorem 4.4.3 (iv)

By Proposition 4.4.2,

$$\begin{aligned} |E((P_m \times P_n)^2)| &= 6mn - 5m - 5n + 2 \\ (\text{ since } m = 2r_1,\ n = 2r_2\ ) &= 6(2r_1)(2r_2) - 5(2r_1) - 5(2r_2) + 2 \\ &= 24r_1r_2 - 10r_1 - 10r_2 + 2. \end{aligned}$$

Let

$$H = (P_m \times P_n)^2 \setminus [\bigcup_{\substack{i=1,3,\ldots,2r_1-3 \\ j=2,4,\ldots,2r_2-2}} A_{i,j} + \bigcup_{\substack{i=1,3,\ldots,2r_1-3 \\ j=3,5,\ldots,2r_2-1}} B_{i,j} + \bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=1,3,\ldots,2r_2-3}} C_{i,j}$$

$$+ \bigcup_{i=1,3,\ldots,2r_1-3} D1_i + \bigcup_{i=1,3,\ldots,2r_1-3} D2_i + \bigcup_{i=2,4,\ldots,2r_1-2} Ei + \bigcup_{j=1,3,\ldots,2r_2-3} F_j + Z\ ].$$

Then

$$\begin{aligned} |E(H)| &= (24r_1r_2 - 10r_1 - 10r_2 + 2) - [\ 10(r_1-1)(r_2-1) + 6(r_1-1)(r_2-1) \\ &\quad + 3(r_1)(r_2-1) + 3(r_1-1) + 6(r_1-1) + 3(r_1-1) + 6(r_2-1) + 6\ ] \\ &= (24r_1r_2 - 10r_1 - 10r_2 + 2) - (19r_1r_2 - 7r_1 - 10r_2 + 4) \\ &= 5r_1r_2 - 3r_1 - 2. \end{aligned}$$

We have that

$$\bigcup_{\substack{i=1,3,\ldots,2r_1-3 \\ j=2,4,\ldots,2r_2-2}} \{A_{i,j}\} \cup \bigcup_{\substack{i=1,3,\ldots,2r_1-3 \\ j=3,5,\ldots,2r_2-1}} \{B_{i,j}\} \cup \bigcup_{\substack{i=1,3,\ldots,2r_1-1 \\ j=1,3,\ldots,2r_2-3}} \{C_{i,j}\} \cup \bigcup_{i=1,3,\ldots,2r_1-3} \{D1_i, D2_i\} \cup$$

$$\bigcup_{i=2,4,\ldots,2r_1-2} \{E_i\} \cup \bigcup_{j=1,3,\ldots,2r_2-3} \{F_j\} \cup \{Z\} \cup E(H)$$

forms a clique partition $\mathcal{P}$ of $(P_m \times P_n)^2$ such that

$$\begin{aligned} |\mathcal{P}| &= (r_1-1)(r_2-1) + (r_1-1)(r_2-1) + (r_1)(r_2-1) + 2(r_1-1) \\ &\quad + (r_1-1) + (r_2-1) + 1 + (5r_1r_2 - 3r_1 - 2) \\ &= (3r_1r_2 - r_2 - 1) + (5r_1r_2 - 3r_1 - 2) \\ &= 8r_1r_2 - 3r_1 - r_2 - 3. \end{aligned}$$

Since $r_1 = \frac{m}{2}$ and $r_2 = \frac{n}{2}$,

$$|\mathcal{P}| = 8(\frac{m}{2})(\frac{n}{2}) - 3(\frac{m}{2}) - \frac{n}{2} - 3 = 2mn - \frac{3m}{2} - \frac{n}{2} - 3.$$

Thus $cp((P_m \times P_n)^2) \leq 2mn - \frac{3m}{2} - \frac{n}{2} - 3$.

By Theorem 4.3.5, $cc((P_m \times P_n)^2) = mn - 4$. Hence $cp((P_m \times P_n)^2) \geq mn - 4$.

Therefore, $mn - 4 \leq cp((P_m \times P_n)^2) \leq 2mn - \frac{3m}{2} - \frac{n}{2} - 3$. □

In conclusion, we get the complete results of the values of the clique covering numbers of the k-power of ladders for all $k \in \mathbf{N}$. For grids, we obtain the values of the clique covering numbers of the square of grids and lower bounds of the clique covering numbers of the k-power of grids where $k < min\{m, n\}$. Finding the values of the clique covering numbers of the k-power of grids where $k \geq 3$ is still an open problem. In the section 4.2 and the section 4.4, we obtain bounds of the clique partition numbers of the square of ladders and grids, respectively.